Plan

Elements

QCD fields

Composite operators

Correlation functions

Basic properties

Spectral representation

QCD ↔ hadron properties

Dynamics

Computing correlation functions

Perturbation theory at short distances

Vacuum fields and their effects

Gluon and quark condensate

Method I: Vacuum condensates

Operator product expansion

Shifman, Vainshtein, Zakharov 1979

Vacuum condensates

QCD ↔ hadron matching ("QCD sum rules")

Applications to heavy and light mesons

Limitations of method

Method II: Vacuum fields ←

QFT at imaginary time ("Euclidean")

Vacuum fields in "cooled" lattice QCD

Topological landscape and tunneling

Instanton ensemble

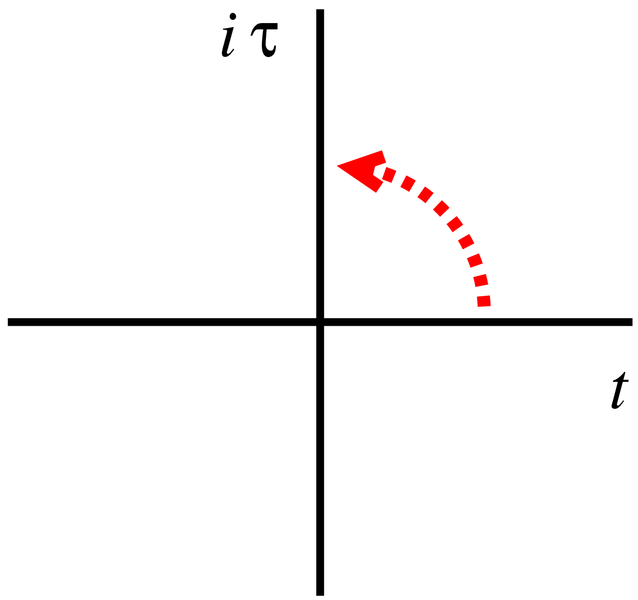
Shuryak 1982; Diakonov, Petrov 1984

Chiral symmetry breaking

Meson and baryon correlation functions

Current developments

Imaginary time



Transition to imaginary time

Enabled by complex-analytic properties of amplitudes and correlation functions

Can be applied to individual amplitudes (Wick rotation) or entire functional integral = generating function

$$e^{iE_h t} \quad \rightarrow \quad e^{-E_h \tau}$$

Time dependence = exponential decay

Imaginary time: Correlation functions

$$\langle 0 \,|\, J(\tau, \mathbf{x}) \; J(0,\mathbf{0}) \,|\, 0 \rangle \quad = \quad \sum_h \underbrace{\langle 0 \,|\, J(\tau, \mathbf{x}) \,|\, h \rangle}_{e^{-E_h \tau} \; \langle 0 \,|\, J(0, \mathbf{x}) \,|\, h \rangle} \; \langle h \,|\, J(0,\mathbf{0}) \,|\, 0 \rangle$$

Limit $\tau \to \infty$:

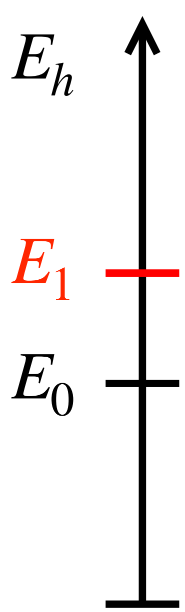
$$= \quad e^{-E_0 \tau} \; \langle 0 \,|\, J(0, \mathbf{x}) \,|\, h_0 \rangle \langle h_0 \,|\, J(0, \mathbf{0}) \,|\, 0 \rangle$$

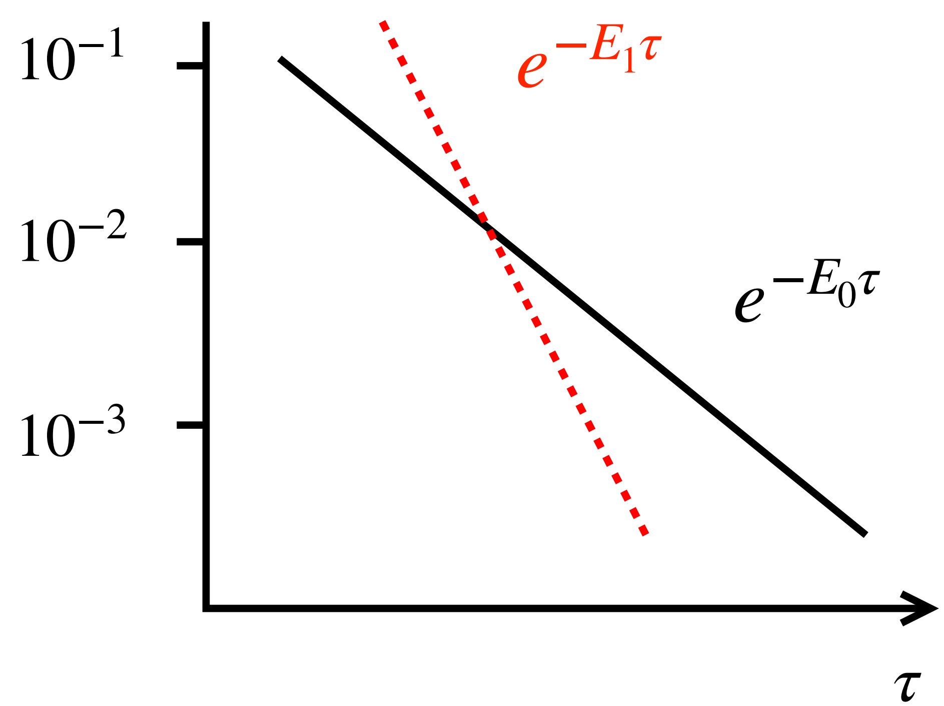
lowest-mass hadron

$$+ \quad e^{-E_1 \tau} \; \langle 0 \,|\, J(0, \mathbf{x}) \,|\, h_1 \rangle \langle h_1 \,|\, J(0, \mathbf{0}) \,|\, 0 \rangle$$

higher-mass states suppressed

$$+ \quad \cdots$$





Properties of lowest-mass hadron states can be obtained from large-time limit of correlation functions

Masses, couplings to currents

Practical calculations: Trade-offs

Techniques beyond lowest-mass: “Distillation” Lecture Dudek

Imaginary time: Functional integral

$$\int DA \;\; e^{-i\, S[A]} \qquad \to \qquad \int DA \;\; \underbrace{e^{-S_E[A]}}_{}$$

real, positive definite!

$$A(t, \mathbf{x}) \quad \to \quad A(\tau, \mathbf{x})$$

QFT $\quad \longrightarrow \quad$ statistical mechanics

→ Numerical simulations, Monte-Carlo methods

→ Concept of “contribution” of certain field configurations to correlation functions

→ Semiclassical methods: Saddle point approximation

Imaginary-time representation limited to calculation of static properties
(= independent of real time), cannot be applied to real time dependent properties

Recent developments: Light-cone correlation functions

Imaginary time: Euclidean metric

$$x_\mu \;=\; (x_1, x_2, x_3, x_4) \;=\; (\mathbf{x}, \tau)$$ Euclidean 4-vector

$$x_\mu x_\mu \;=\; x_1^2 + x_2^2 + x_3^2 + x_4^2 \;=\; |\,\mathbf{x}\,|^2 + \tau^2$$ Euclidean metric

same for momenta, other 4-vectors

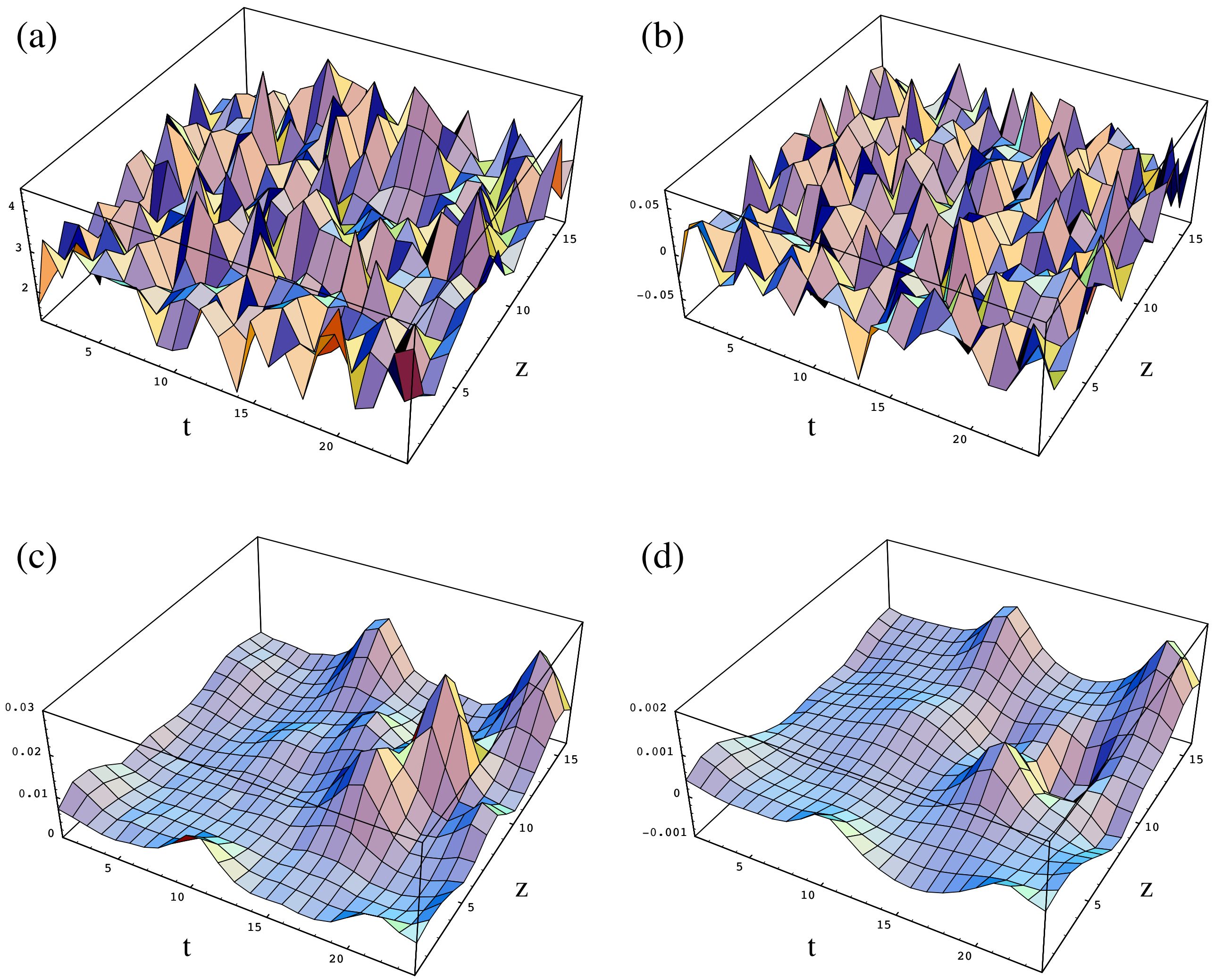
For computation of imaginary-time correlation functions,
QFT is formulated in 4D Euclidean space

4D rotational invariance O(4)
No difference between “space” and “time” directions!

All Euclidean distances/vectors are space-like

Euclidean 4-vectors correspond to spacelike Minkowskian 4-vectors

Vacuum fields: Cooled lattice QCD configurations



$G_{\mu\nu}G_{\mu\nu}(x)$ $\qquad$ $G_{\mu\nu}\tilde{G}_{\mu\nu}(x)$

Fig: Chu, Grandy, Huang, Negele 1993

What are the gauge field configurations giving rise to non-perturbative structure of QCD vacuum?

Inspect lattice QCD configurations!

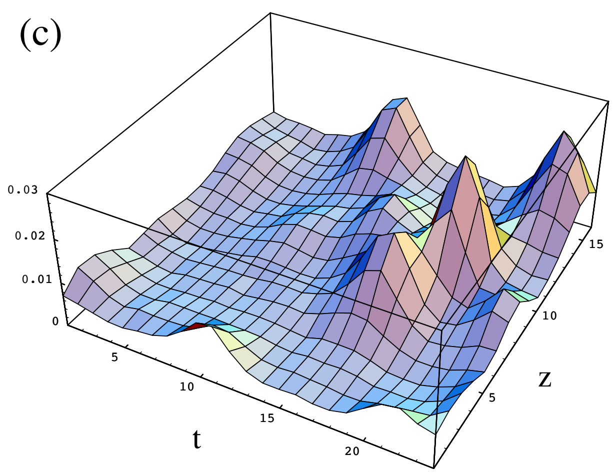
Usual field configurations are “rough”: Contain quantum fluctuations of any wavelength

Cooling of lattice QCD configurations identifies “smooth” field configurations

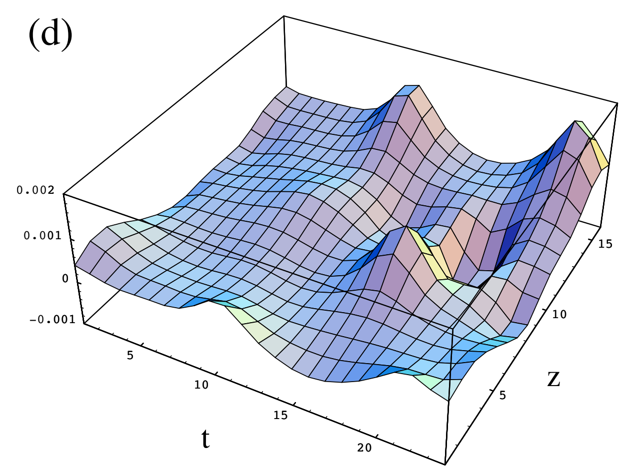
Strong features: Concentrations of action and topological charge density

Alt technique: Gradient flow = systematic “smoothing” transformation of lattice QCD configurations. Controlled by parameter, reversible

see e.g. Bonati, D’Elia 2014; Athenodorou et al. 2018; Alexandrou et al. 2020



$G_{\mu\nu}G_{\mu\nu}(x)$



$G_{\mu\nu}\tilde{G}_{\mu\nu}(x)$

Vacuum populated by localized gauge fields

Typical size $\bar{\rho}$ ~ 0.3 fm $\quad \ll$ hadronic size ~ 1 fm

Typical 4D separation $\bar{R}$ ~ 1 fm

Fraction of 4D space occupied by fields:
$\pi^2 \bar{\rho}^4 / \bar{R}^4 \approx 0.1$

Strong fields: $$\int_{\text{vol.field}} d^4x \, G_{\mu\nu}G_{\mu\nu} \approx \frac{16\pi^2}{g^2} \approx 20$$ large action $\gg 1$

[In this estimate:
g at scale $\mu = \bar{\rho}^{-1}$]

Topologically charged: $$\underbrace{\frac{g^2}{16\pi^2} \int_{\text{vol.field}} d^4x \, G_{\mu\nu}\tilde{G}_{\mu\nu}}_{\text{topological charge}} = \pm \underbrace{\frac{g^2}{16\pi^2} \int_{\text{vol.field}} d^4x \, G_{\mu\nu}G_{\mu\nu}}_{\text{action}} = \pm 1$$

Vacuum fields: Interpretation

Vacuum fields observed in cooled lattice QCD configurations are fluctuations with local topological charge ± 1

QCD instantons: Classical solutions of Yang-Mills equations with topological charge ± 1

Belavin, Polyakov, Shvarts, Tyupkin 1975, ‘tHooft 1976

Physical interpretation: Tunneling processes in topological landscape of gauge theory

Important for chiral symmetry breaking: Topologically charged gauge fields induce chirality-changing interactions between fermions, cause chiral symmetry breaking

Program:

- Learn about topological structure of gauge theory and tunneling processes = instantons
- Construct effective description of QCD vacuum based on instanton fields using semiclassical approximation
- Explain/describe dynamics of chiral symmetry breaking
- Compute hadronic correlation functions and extract hadron structure

Topological structure: Gauge fields

Space of gauge potentials has topological structure

$$A_i(\mathbf{x}) \xrightarrow{\text{gauge tf}} U^{-1}(\mathbf{x})\, A_i(\mathbf{x})\, U(\mathbf{x}) \;+\; ig^{-1} U^{-1}(\mathbf{x})\, \partial_i U(\mathbf{x})$$

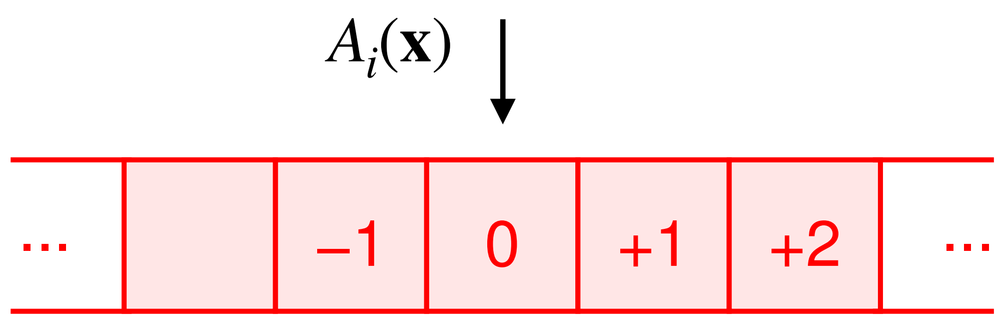
Gauge transformation
Here: $A_0 = 0$ gauge, fixed time τ

$U(\mathbf{x})$: $\quad R^3 \;\rightarrow\; SU(2) \;\subset\; SU(3)$

Mapping with topological characteristics
$U \rightarrow 1$ for $|\mathbf{x}| \rightarrow \infty$

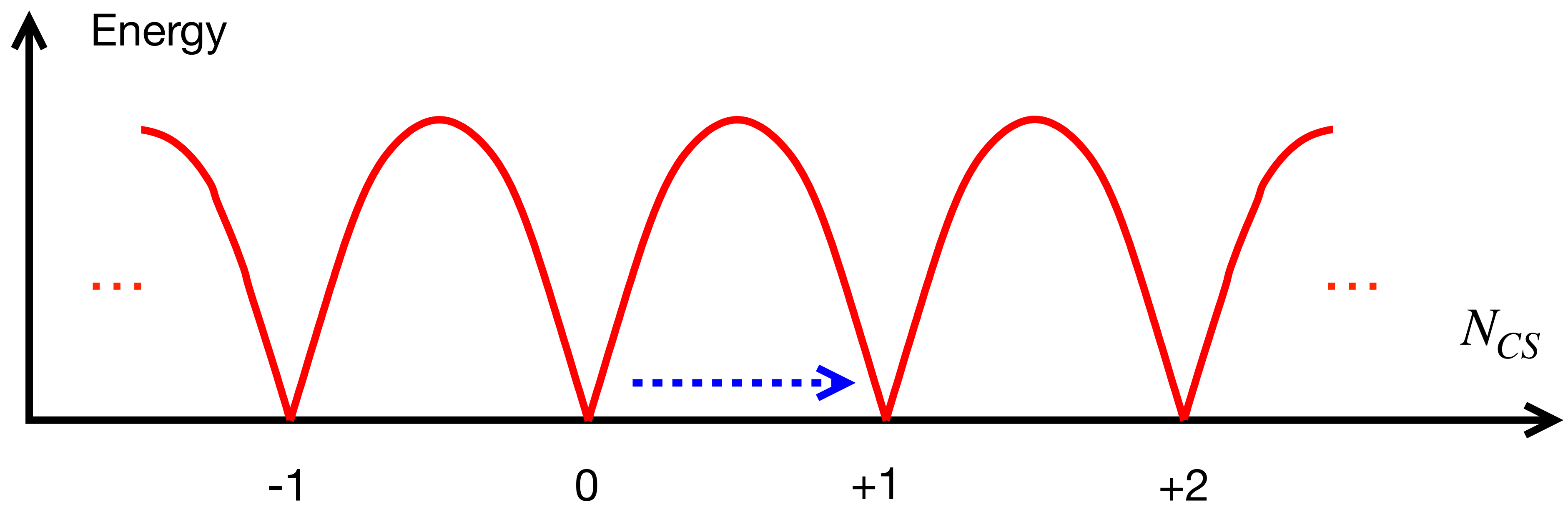
Winding number N_{CS} (Chern-Simons number): How many times U covers $SU(2)$ group while going over R^3 space

Topological sectors labeled by N_{CS}



Quantum-mechanical motion of gauge fields extends over all topological sectors

Topological structure: Tunneling



Energy of gauge field configurations is periodic function in N_{CS}

For every configuration A_i with energy $E[A_i]$, there are gauge-equivalent configurations with the same energy in all the N_{CS} sectors

Minima $E[A_i] = 0$ periodic in N_{CS}, separated by finite barriers

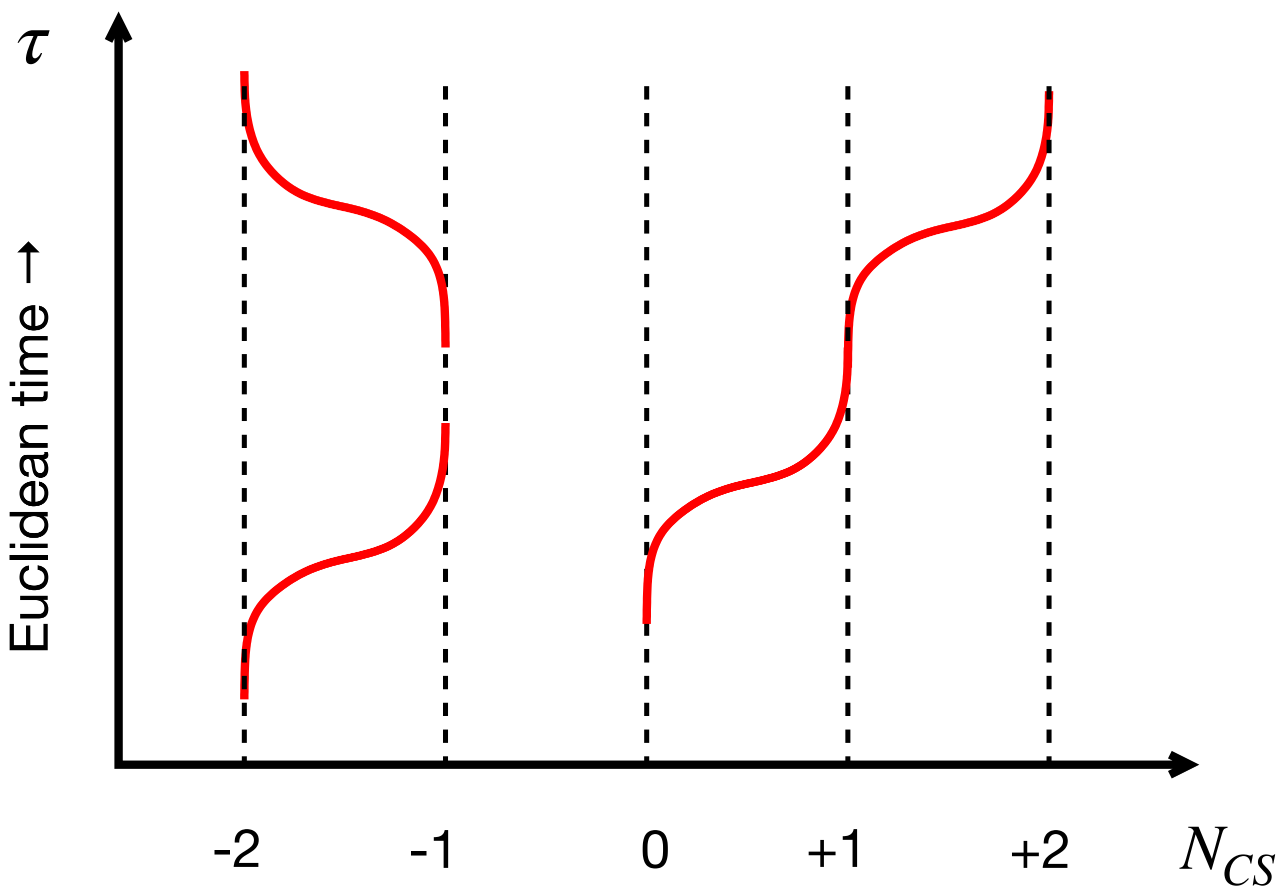
What is the ground state?

Analog: QM particle in periodic 1D potential

Ground state: Particle not localized in one minimum, but in periodic state involving coherent superposition of all minima

Tunneling: QM transitions between configurations localized in different minima

Topological structure: Ground state



QCD ground state = coherent superposition of gauge fields in all topological sectors

Functional integral: Trajectories involve tunneling between topological sectors

Elementary tunneling process $\Delta N_{CS} = \pm 1$

Described by classical trajectory (semiclassical approximation): Instanton

$A_{\text{inst}\pm}(\mathbf{x}, \tau)$:

$$\mathbf{A}_{\text{inst}\pm}(\mathbf{x}, \tau = -\infty) = \mathbf{A}'(\mathbf{x})$$

$$\mathbf{A}_{\text{inst}\pm}(\mathbf{x}, \tau = +\infty) = \mathbf{A}''(\mathbf{x})$$

$$\mathbf{A}'(\mathbf{x}) \xrightarrow{U(N_{CS} = \pm 1)} \mathbf{A}''(\mathbf{x})$$

Topological structure: Instanton

$$A_{\mu,\text{inst}\pm}(x) = \frac{i}{g}\,\frac{\eta^a_{\mu\nu}\,x_\nu}{|x^2|}\,f\left(\frac{|x|}{\rho}\right)$$

Explicit form of instanton gauge potential in covariant gauge*

e.g. $$f = \frac{1}{1+x^2/\rho^2}$$

Localized field. Profile function f depends on gauge

$$D^{ab}_\mu G^b_{\mu\nu,\text{inst}\pm}(x) = 0$$

Solution of Yang-Mills equation

$$\tilde{G}^a_{\mu\nu,\text{inst}\pm} = \pm\, G^a_{\mu\nu,\text{inst}\pm}$$

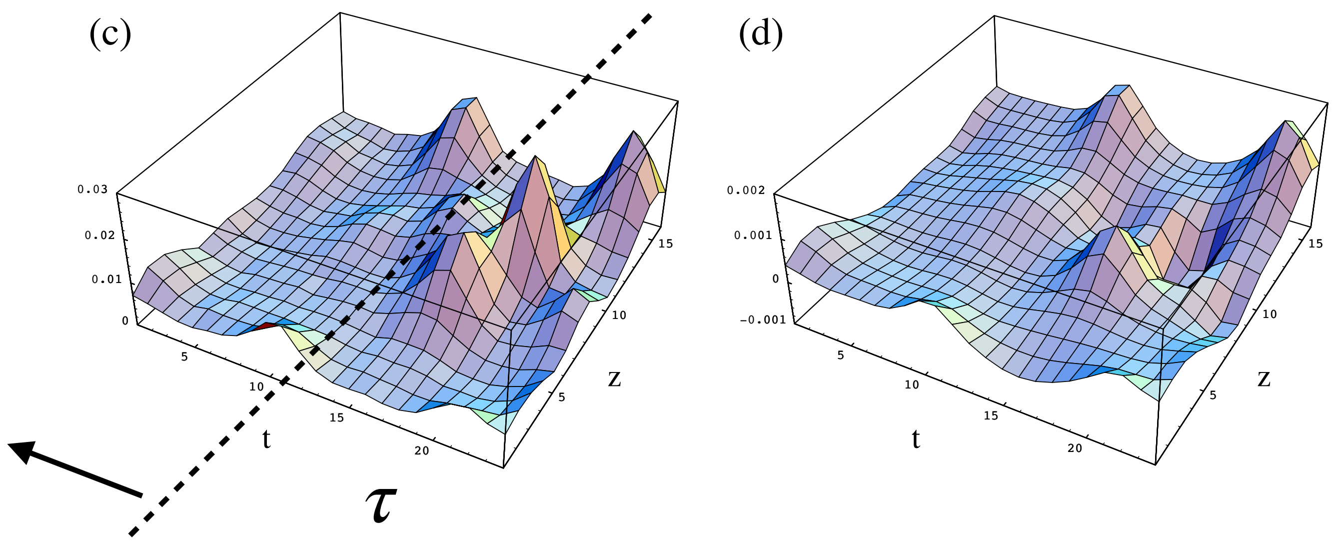
Field is (anti-) self-dual

$$\frac{g^2}{16\pi^2}\int d^4x\; G_{\mu\nu,\text{inst}\pm}(x)\;\tilde{G}_{\mu\nu,\text{inst}\pm}(x) \;=\pm 1$$

Topological charge ± 1 (= ΔN_{CS} in tunneling process)

*Position of center arbitrary, field can be shifted: $x \to x - z,\;\; z =$ center coordinate

Topological structure: Summary



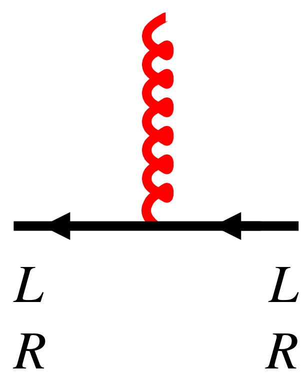
Euclidean time

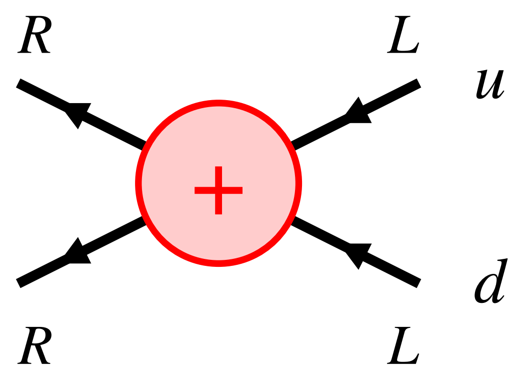
Strong localized gauge fields in QCD vacuum with topological charge ± 1

Tunneling trajectories between topological sectors with $\Delta N_{CS} = \pm\, 1$

Instantons — localized Classical solutions of Yang-Mills equations with topological charge ± 1, entangle space-time and color/spin dependence

Instantons: Fermion interactions





average over
color orientation

$\det_{\text{flav}}\, \bar\psi_R(x)\, \psi_L(x)$

$$= \bar u_R u_L \bar d_R d_R - \bar u_R d_L \bar d_R u_R$$

$\times$ form factor with spatial range ρ
(size of instanton field)

For antiinstanton: $+ \to -\,,\, L \leftrightarrow R$

$$\psi_{L,R} = \frac{1 \pm \gamma_5}{2}\psi$$

Reminder: Chiral components of quark fields (handedness)

Perturbative QCD interactions conserve chirality

Interaction with instanton field changes chirality

Instanton field entangles spin and color DoF, interaction rotates quark spin from L to R

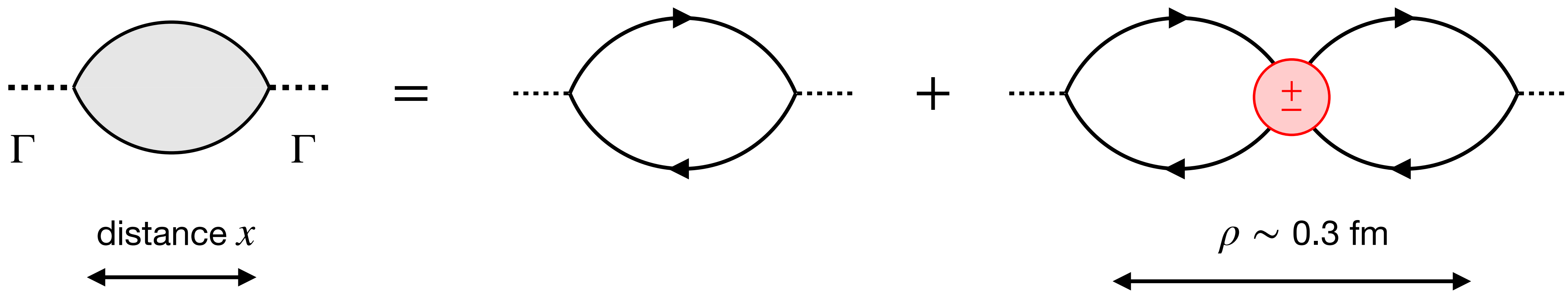
Formal: Zero-virtuality mode of fermion field in instanton background

Flavor structure

Determinant in quark flavor indices
'tHooft 1976

Multifermion interaction, involves all light flavors at the same time

Instantons: Nonperturbative effects



Single instantons cause nonperturbative effects in correlation functions

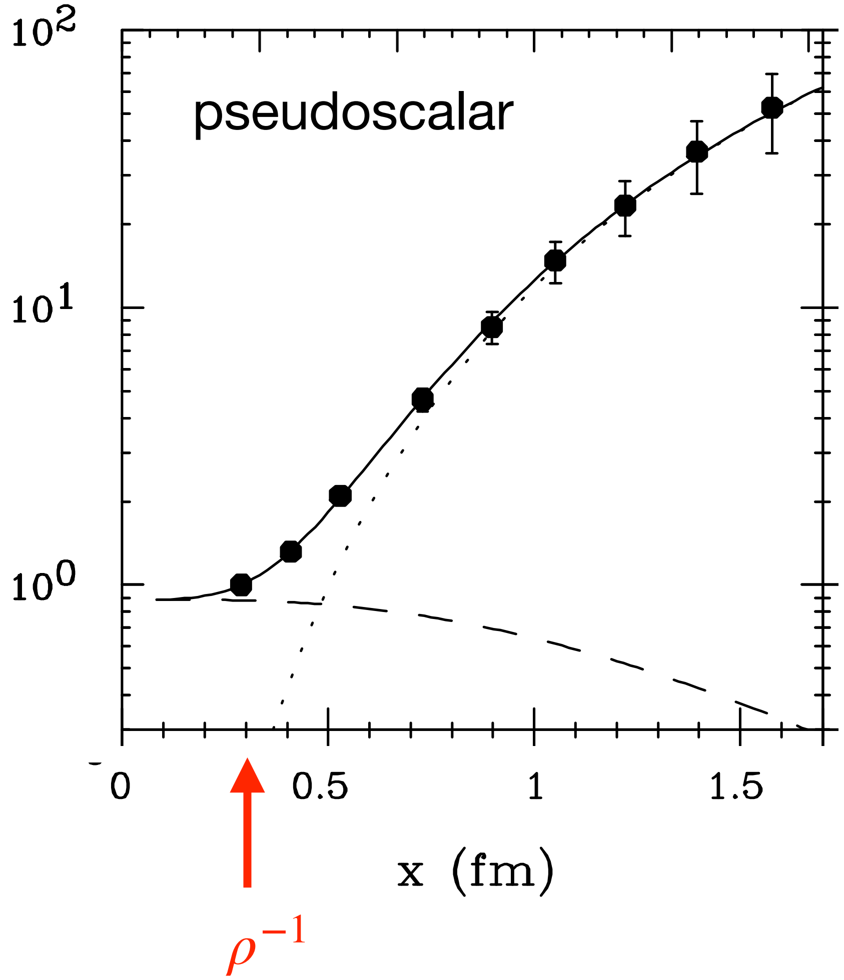
Strong attractive interactions in channels $\Gamma = 1 =$ scalar 0^+, $\Gamma = i\gamma_5 =$ pseudoscalar 0^-

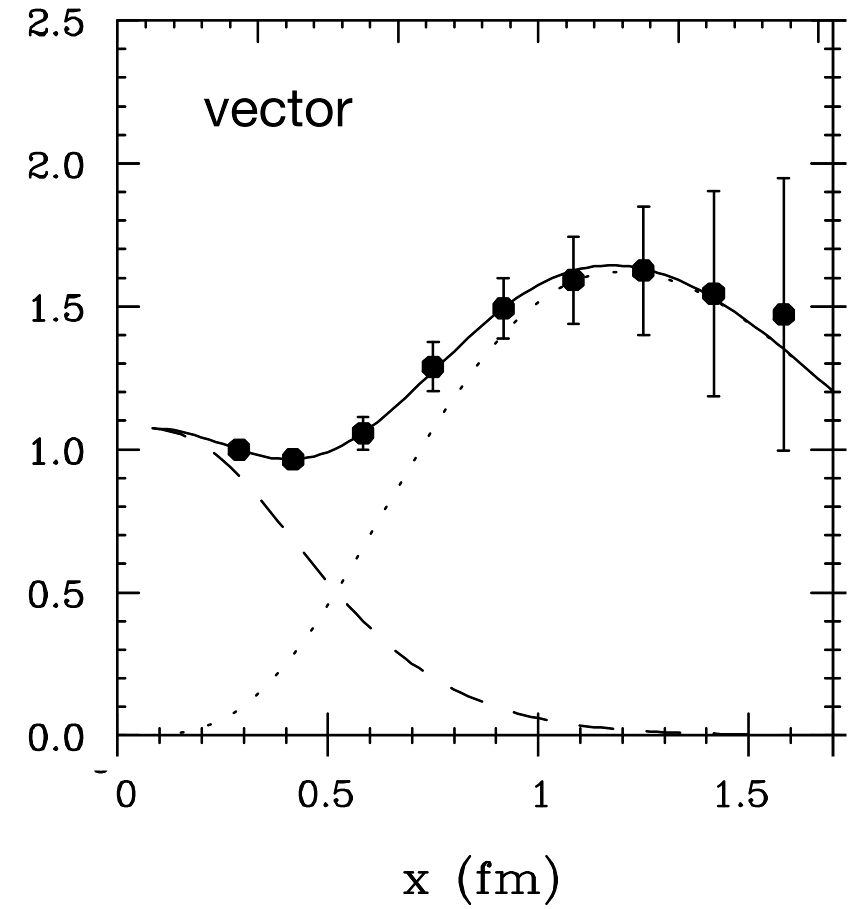
Relevant at distances $|x| \sim \rho^{-1} \sim$ 0.3 fm

→ Sets quantitative limit for applicability of perturbation theory in correlation functions

Direct evidence for instantons in LQCD correlation function

Instantons: Nonperturbative effects





note different scale!

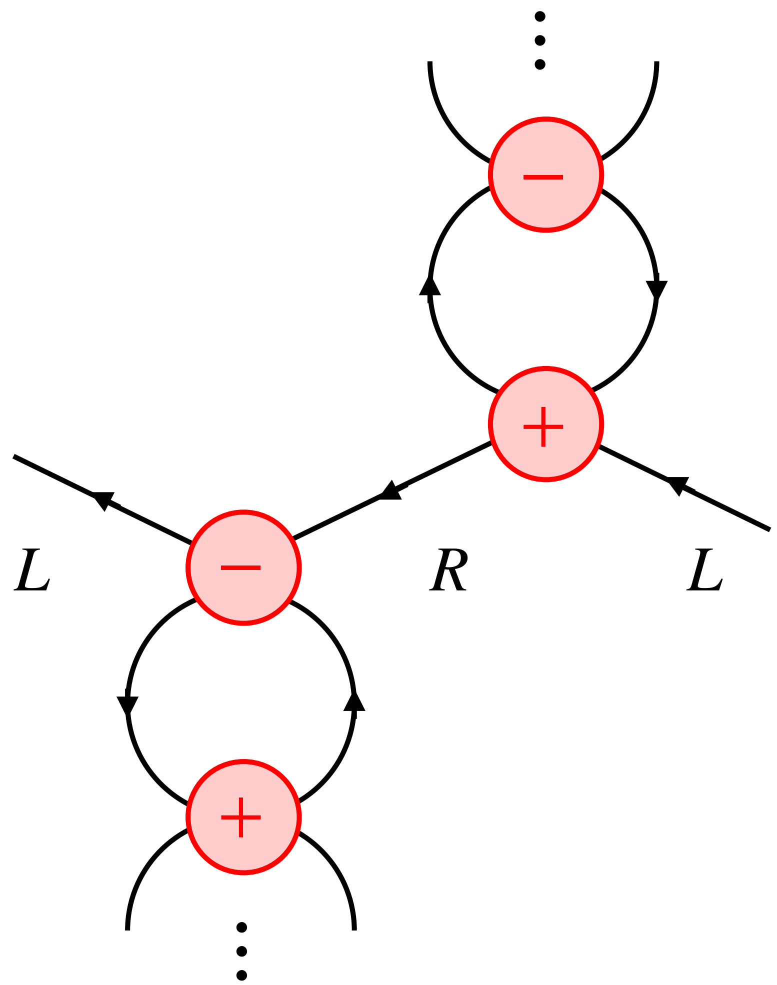
Lattice QCD results for correlation function ratio $\dfrac{\Pi(x)\,\text{[full]}}{\Pi(x)\,\text{[pert]}}$

Results and figures:
Chu, Grandy, Huang, Negele 1993

Pseudoscalar: Ratio becomes $\gg$ 1 for $x >$ 0.3 fm due to direct instanton effect

Vector: Ratio remains $\sim$ 1 up to $x \sim$ 1 fm, no direct instanton effect

Instantons: Chiral symmetry breaking



Finite density of instantons in vacuum causes chiral symmetry breaking

Quarks propagating in vacuum experience chirality flips $L \to R \to L \to \ldots$ at constant rate

Equivalent to having dynamical mass $M \neq 0$

Relevant at distances $\sim$ 1 fm
= separation of instantons in vacuum

Vacuum condensate $\langle 0 | \bar{\psi}\psi | 0 \rangle \neq 0$

Instanton vacuum: Program

- Abstract from lattice QCD results

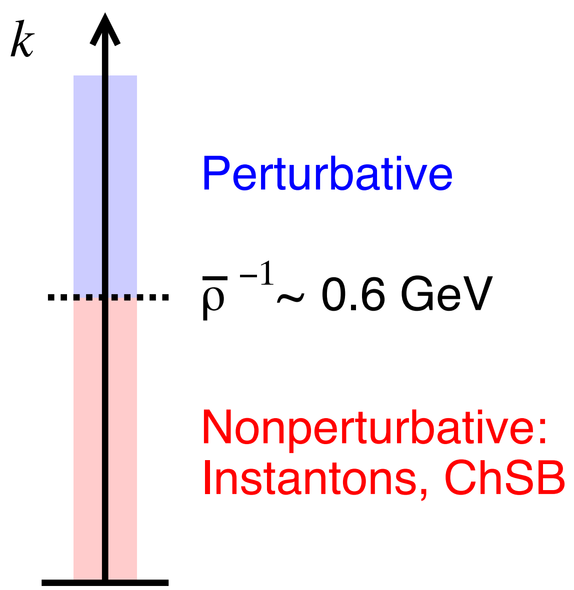
- Construct effective description of QCD vacuum based on topological fields = instantons

- Compute chiral symmetry breaking and hadronic correlation functions

- Use framework of variational approximation to Yang-Mills partition function: Self-consistent, gauge/ansatz dependence contained in "choice of trial function"

- Use smallness of instanton density in 4D space $(\pi^2 \bar{\rho}^4 / \bar{R}^4) \ll 1$ for systematic approximation: Parametric expansion

Instanton vacuum: Integration over modes



Separate modes

$k > \bar{\rho}^{-1}$: Integrate perturbatively:
Renormalization, $\bar{\rho}^{-2} \gg \Lambda^2_{\mathrm{QCD}}$

$k < \bar{\rho}^{-1}$: Integrate nonperturbatively:
Instantons + massive fermions

Integrate over modes

$$\underbrace{\int [DA]_{\text{low}}} \int [DA]_{\text{high}} \; \exp(-S_{\text{YM}}) \;\; [\times \text{ fermions}] \; \ldots$$

$$A(x) \; = \; \sum_{I}^{N_+} A_+(x \,|\, z_I, \rho_I, O_I) \; + \; \sum_{\bar{I}}^{N_-} A_-(x \,|\, z_{\bar{I}}, \rho_{\bar{I}}, O_{\bar{I}})$$

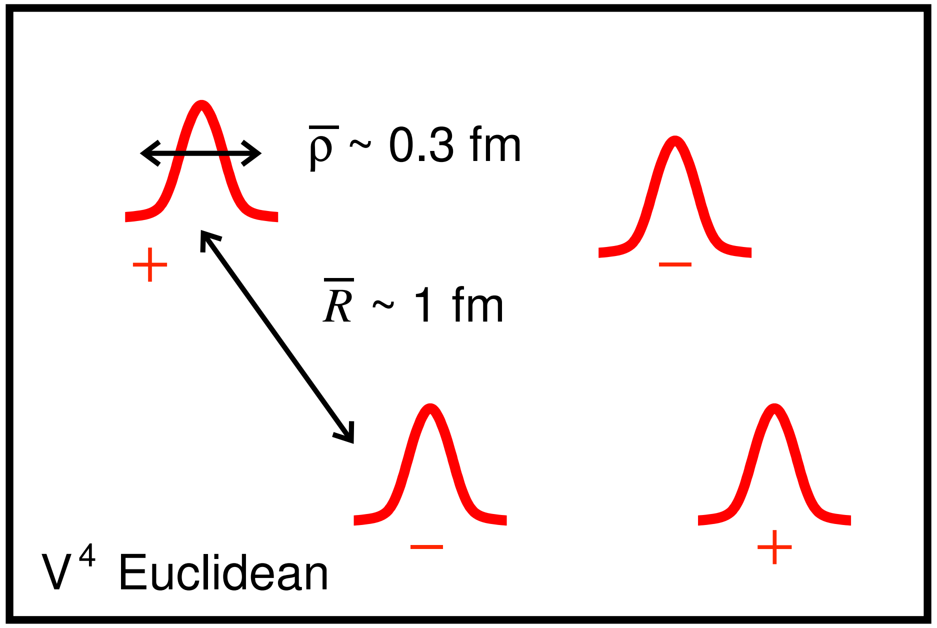
superposition of instantons and antiinstantons (in singular gauge)

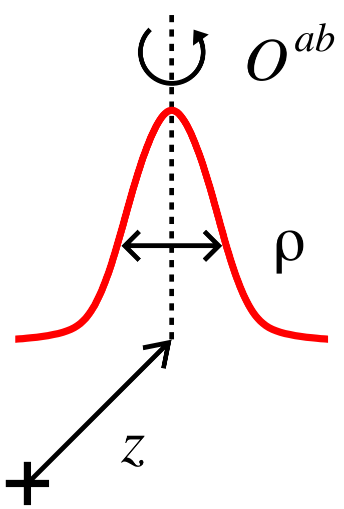
$$\int [DA]_{\text{low}} \to \int \prod_{I,\bar{I}}^{N_\pm} dz_I \, d\rho_I \, dO_I \;\, d_0(\rho_I) \; \ldots$$

integration over instanton collective coordinates: position z, size ρ, color orientation O

weight resulting from high-momentum integration ['tHooft 1976]

Instanton vacuum: Instanton ensemble





$$Z_{\text{inst}} = \int \prod_{I,\bar{I}}^{N_\pm} dz_I \, d\rho_I \, dO_I \; d_{\text{eff}}(\rho_I)$$

Stable system emerges due to instanton interactions

Different implementations: Variational principle, numerical simulations

Diakonov, Petrov 1984; Shuryak 1988+

Robust solutions, do not depend on details

Effective instanton size distribution with average size $\bar{\rho} \sim 0.3$ fm

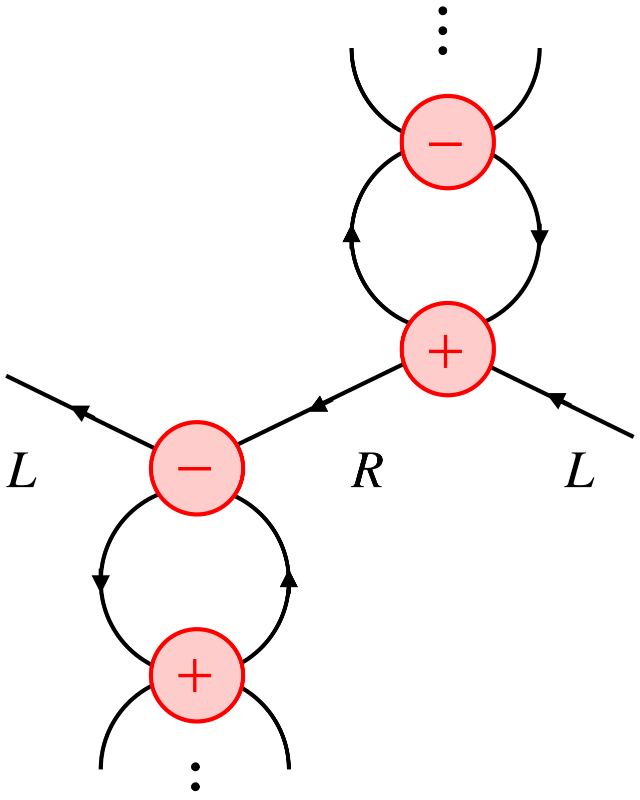
System reproduces small instanton density $\pi^2 \bar{\rho}^4 / \bar{R}^4 \approx 0.1$ observed in LQCD: Approximations are self-consistent

Approximations reduce QFT to statistical ensemble

All dynamical scales "emerge" from Λ_{QCD} in renormalized QCD coupling

No dynamical scales injected by approximations!

Instanton vacuum: Chiral symmetry breaking



Quarks in instanton ensemble experience multi-fermion interactions

Ground state of interacting system exhibits ChSB

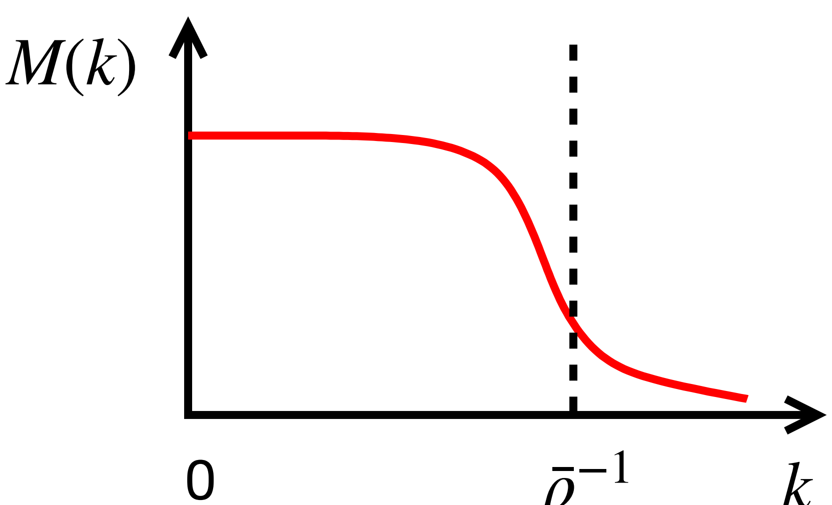
Can be determined using various methods:
Green function, functional integral — bosonization

Vacuum condensate $\langle 0 | \bar\psi \psi | 0 \rangle \neq 0$

Dynamical quark mass M

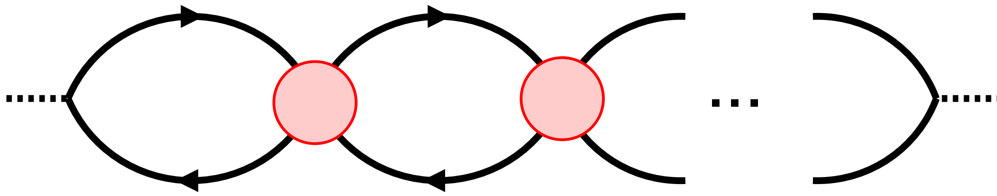
$M \sim$ 0.3-0.4 GeV — "constituent quark" mass

$M \rightarrow 0$ for momenta $k \gtrsim \bar\rho^{-1} \sim$ 0.6 GeV



Chiral symmetry breaking connected with dynamical mass generation

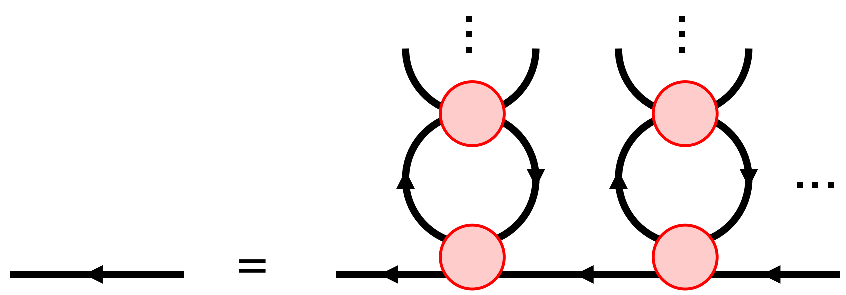
Instanton vacuum: Meson correlation functions



Pseudoscalar correlation functions $\Gamma = i\gamma_5$

Strong effect from direct instantons

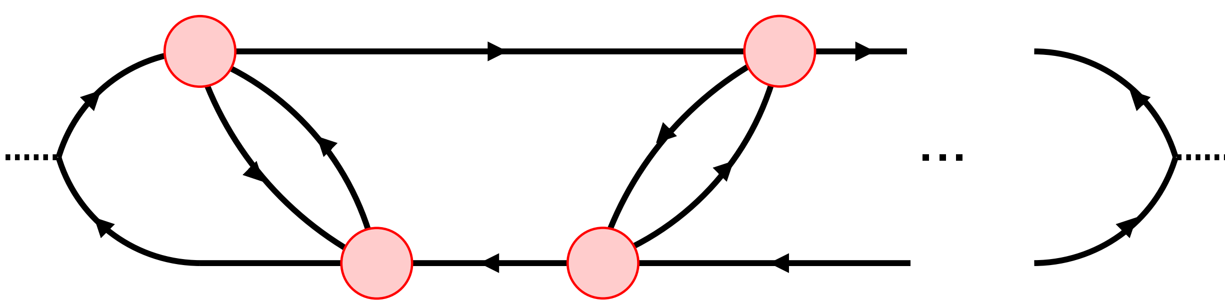
Isovector: π massless $M_\pi^2 = 0 \; + \mathcal{O}(m_{u,d})$
$\leftrightarrow$ chiral symmetry breaking



Pion decay constant F_π predicted

Isoscalar u, d, s: η' massive

Vector correlation functions $\Gamma = \gamma^\mu$

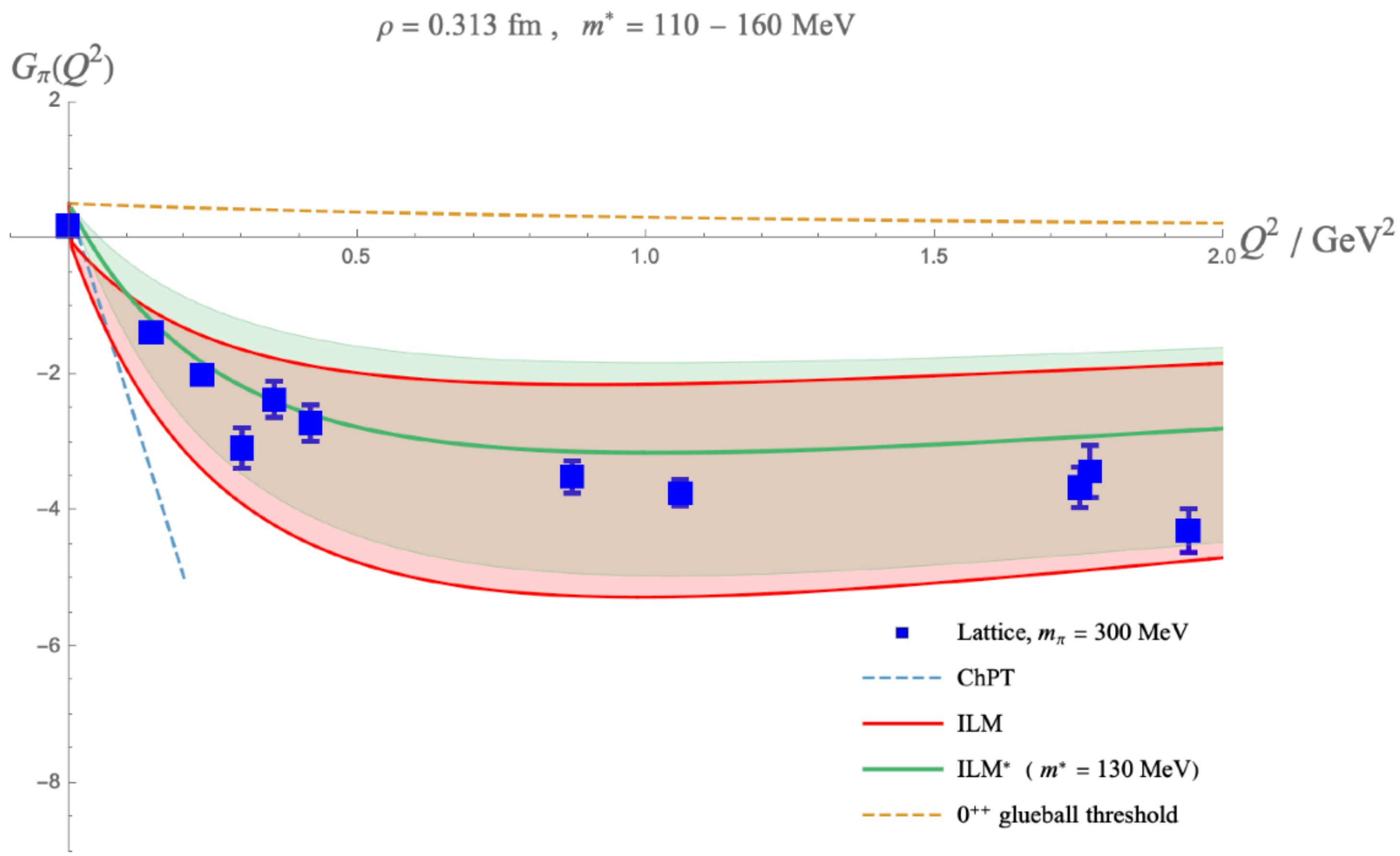


No direct instantons

ρ meson mass $M_\rho \approx 2M$,
weakly bound

Explains/predicts many structures: Meson couplings, form factors, …

Instanton vacuum: Pion structure



Example: Pion scalar gluon form factor

$$\langle \pi(p') \,|\, g^2 G^2_{\mu\nu}(0) \,|\, \pi(p) \rangle$$

$$= -\frac{32\pi^2}{b} \,(2M_\pi^2)\, G_\pi(Q^2)$$

Extracted from 3-point correlation function in instanton vacuum

Liu, Shuryak, Weiss, Zahed, PRD 110, 054021 (2024)

Describes response of pion state to change of local scalar gluon field

Good agreement with LQCD results

χQCD Collab: B. Wang et al., Phys.Rev.D 109 (2024) 094504; see also Hackett et al. 2023

Instanton vacuum: Pion structure

QCD energy-momentum tensor

Gluon fields (trace anomaly)

Quark fields and mass (sigma term)

$$M_\pi = \frac{\langle \pi | T^\mu{}_\mu | \pi \rangle}{2M_\pi} = \underbrace{-\frac{b}{32\pi^2}\frac{\langle \pi | g^2 G^2 | \pi \rangle}{2M_\pi}}_{\frac{M_\pi}{2}} + \underbrace{\frac{\langle \pi | m_q \bar{\psi}\psi | \pi \rangle}{2M_\pi}}_{\frac{M_\pi}{2}} \pm \mathcal{O}(m_q)$$

Here: Forward matrix element $p' = p$

Pion mass decomposition predicted by instanton vacuum

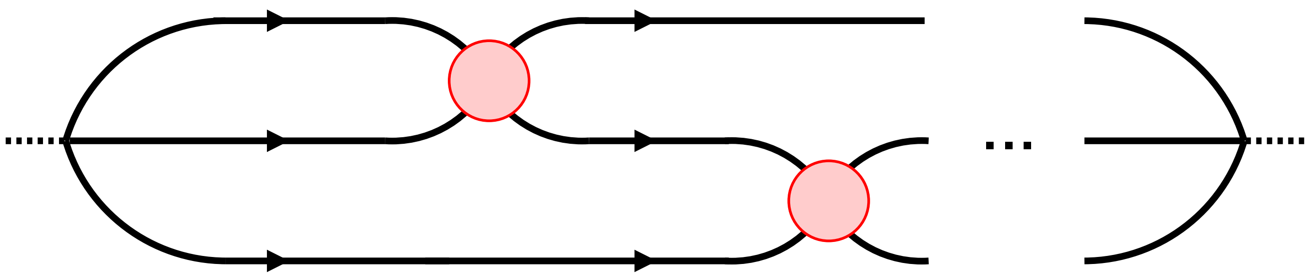
Pion mass arises in equal parts from gluon fields and quark fields

Gluon field contribution to pion mass = change of vacuum gluon condensate in pion state

Instanton vacuum: Fluctuations of instanton density — “topological compressibility” of QCD vacuum

Liu, Shuryak, Weiss, Zahed, PRD 110, 054021 (2024)

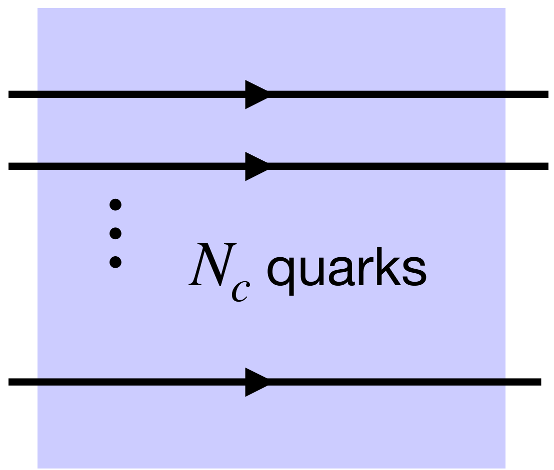
Instanton vacuum: Baryon correlation functions



Attraction in qq channel — “diquark”

3q, 5q, 7q… components:
Needs systematic treatment

Large-N_c limit: Mean-field picture



classical meson field

Stationary state at finite Euclidean times

Quarks move independently in single-particle orbits
↔ quark model picture of baryons

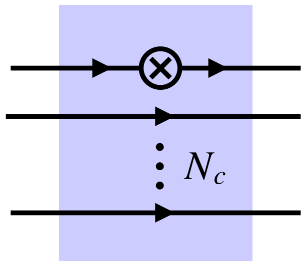
Classical meson field develops: Vacuum “deformed" by presence of valence quarks ↔ soliton picture

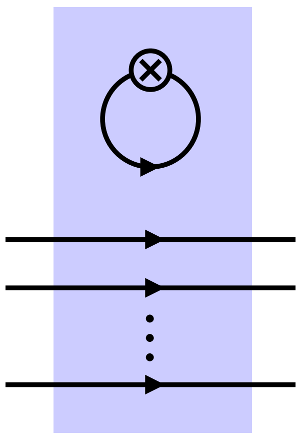
Systematic calculation of baryon observables:
$1/N_c$ expansion

Particular realization of the general “mean field picture” of baryons in large-N_c limit of QCD

Instanton vacuum: Nucleon structure

3-point functions include disconnected quark diagrams…
connected by meson field

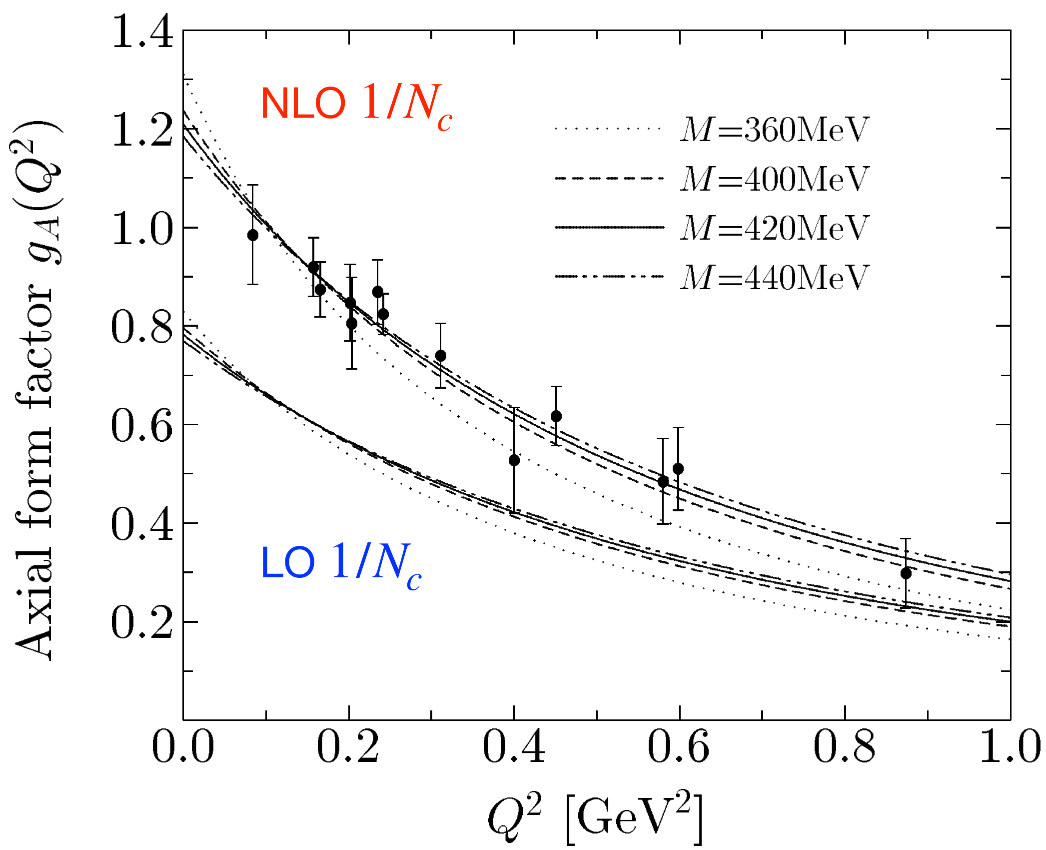


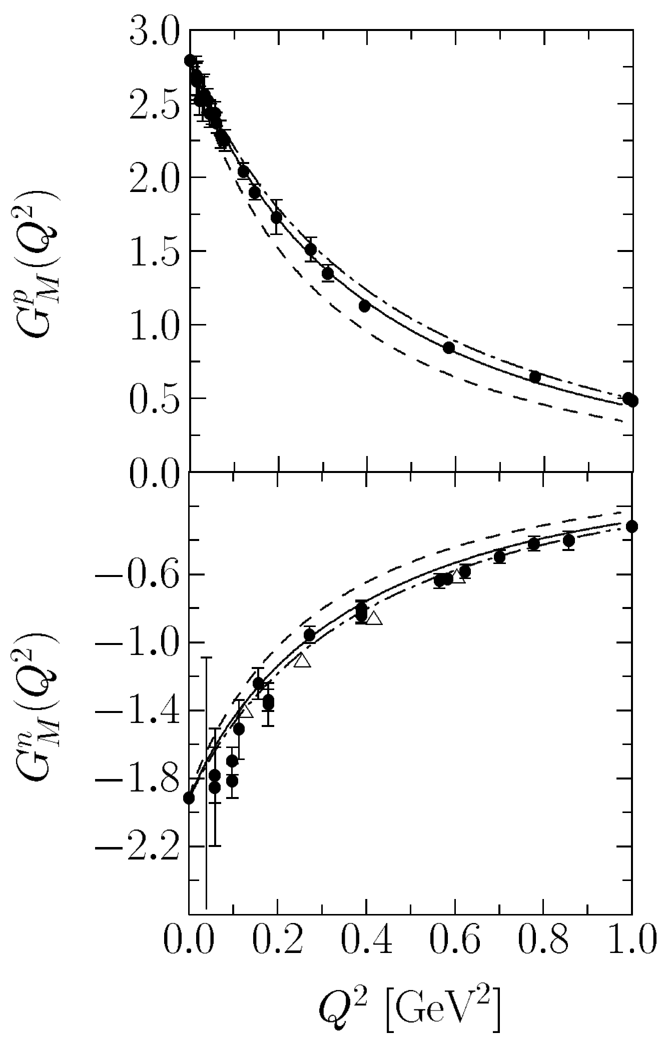


Example: Axial and vector and current matrix elements for SU(2) flavor

[Review: Christov et al 1995. Many more results: Sigma term, $N \to \Delta$, SU(3), …]

Systematic $1/N_c$ expansion

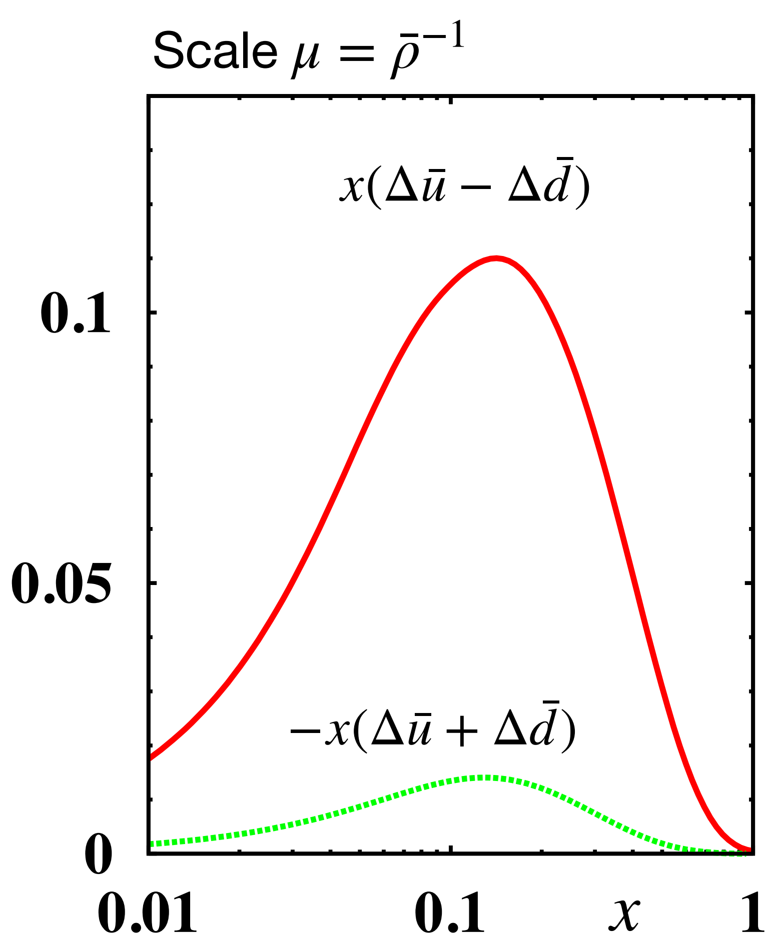


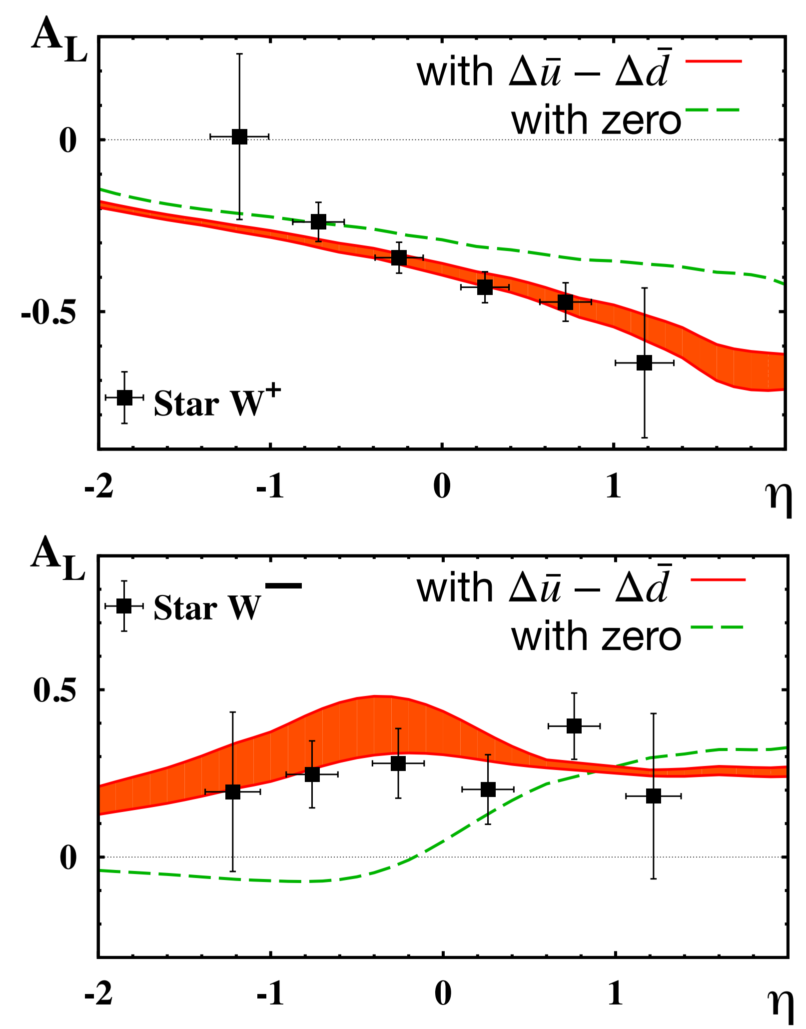


Instanton vacuum: Nucleon structure

Quark/antiquark PDFs/GPDs computed in instanton vacuum

Nucleon's antiquark content $\mathcal{O}(1)$, exhibits rich spin-flavor dependence





Large polarized flavor asymmetry
$\Delta\bar{u} - \Delta\bar{d}$ predicted ($1/N_c$ expansion)

[Diakonov, Petrov, Pobylitsa, Polyakov, Weiss, 1996]

RHIC $W^{\pm}$ production data
with effect of $\Delta\bar{u} - \Delta\bar{d}$

[Adamczyk et al (STAR) 2014+, Adare et al. (PHENIX) 2016+]

Instanton vacuum: Summary

Effective description of QCD vacuum based on instantons describes chiral symmetry breaking and basic features of light hadron structure

Mesons: $\pi - \rho$ qualitative differences, η' mass, meson structure $\leftrightarrow$ instantons

Baryons: Mean-field picture at large N_c, systematic treatment of antiquark content, many predictions

Current research directions

Partonic structure of baryons: PDFs, GPDs, gluonic structure, chiral-odd quark distributions (transversity), $N \to \Delta$ transition GPDs, higher-twist quark-gluon correlations J-Y Kim, Weiss + collaborators

Mechanical properties of hadrons: Hadron form factors of QCD energy-momentum tensor, hadron mass decomposition, connection with vacuum structure, interpretation

Heavy quarkonia: NRQCD effective theory, heavy-quark potential from vacuum fields, exotic states Shuryak, Zahed 2023+

Beyond instantons: Broader view of tunneling processes in topological landscape, instanton-antiinstanton molecules, finite-energy tunneling, effect on hadron structure Shuryak, Zahed, Liu 2023+

Instanton vacuum: Summary

Effective description of QCD vacuum based on instantons describes chiral symmetry breaking and basic features of light hadron structure

Mesons: $\pi - \rho$ qualitative differences, η' mass, meson structure $\leftrightarrow$ instantons

Baryons: Mean-field picture at large N_c, systematic treatment of antiquark content, many predictions

Current research directions

Partonic structure of baryons: PDFs, GPDs, gluonic structure, chiral-odd quark distributions (transversity), $N \to \Delta$ transition GPDs, higher-twist quark-gluon correlations J-Y Kim, Weiss + collaborators

Mechanical properties of hadrons: Hadron form factors of QCD energy-momentum tensor, hadron mass decomposition, connection with vacuum structure, interpretation

Heavy quarkonia: NRQCD effective theory, heavy-quark potential from vacuum fields, exotic states Shuryak, Zahed 2023+

Beyond instantons: Broader view of tunneling processes in topological landscape, instanton-antiinstanton molecules, finite-energy tunneling, effect on hadron structure Shuryak, Zahed, Liu 2023+

Summary: Nonperturbative methods in QCD

Correlation functions of composite operators: Basic objects, connect QCD — hadron spectrum/structure

Dynamics changes with distance: pQCD at short distances (< 0.1 fm), nonperturbative vacuum fields at larger distances

Methods for including nonperturbative vacuum fields

Operator product expansion with vacuum condensates (“QCD sum rules”):
Systematic, but effective only in certain channels

Semiclassical description of topological fluctuations of gauge fields (“Instanton vacuum”):
Variational approximation, self-consistent, describes ChSB and resulting spin-flavor dynamics

Analytic methods complementary to lattice QCD: Use LQCD input, validate results with LQCD, perform efficient computations in domains presently not accessible to LQCD

Many similarities/commonalities with condensed matter physics: Complex ground state, low-energy excitations conditioned by structure of ground state - symmetry breaking, scales

Much there to be explored/developed!

References

Reviews of the instanton vacuum and its applications

T. Schäfer and E. V. Shuryak, "Instantons in QCD," Rev. Mod. Phys. 70, 323-426 (1998)
doi:10.1103/RevModPhys.70.323 [arXiv:hep-ph/9610451 [hep-ph]] [INSPIRE]

D. Diakonov, "Instantons at work," Prog. Part. Nucl. Phys. 51, 173-222 (2003)
doi:10.1016/S0146-6410(03)90014-7 [arXiv:hep-ph/0212026 [hep-ph]] [INSPIRE]

Some classic works

C.G. Callan, Jr., R.F. Dashen and D.J. Gross, "The Structure of the Gauge Theory Vacuum," Phys. Lett. B 63, 334-340 (1976)
doi:10.1016/0370-2693(76)90277-X https://inspirehep.net/literature/3673

C.G. Callan, Jr., R.F. Dashen and D.J. Gross, "Toward a Theory of the Strong Interactions," Phys. Rev. D 17, 2717 (1978)
doi:10.1103/PhysRevD.17.2717 https://inspirehep.net/literature/121225

A.I. Vainshtein, V.I. Zakharov, V.A. Novikov and M.A. Shifman, "ABC's of Instantons," Sov. Phys. Usp. 25, 195 (1982)
doi:10.1070/PU1982v025n04ABEH004533 https://inspirehep.net/literature/165126

E.V. Shuryak, "The Role of Instantons in Quantum Chromodynamics. 1. Physical Vacuum," Nucl. Phys. B 203, 93 (1982)
doi:10.1016/0550-3213(82)90478-3 https://inspirehep.net/literature/169642

D. Diakonov and V.Y. Petrov, "Instanton Based Vacuum from Feynman Variational Principle," Nucl. Phys. B 245, 259-292 (1984)
doi:10.1016/0550-3213(84)90432-2 https://inspirehep.net/literature/193768

D. Diakonov and V. Y. Petrov, "A Theory of Light Quarks in the Instanton Vacuum," Nucl. Phys. B \textbf{272}, 457-489 (1986)
doi:10.1016/0550-3213(86)90011-8 https://inspirehep.net/literature/214407