Perturbative Renormalization and Hopf Algebras

Paul Anderson[1, 2]

[1]University of Virginia, [2]Jefferson Lab

Abstract

When it comes to extracting actual testable predictions from theoretical models, perturbative QFT is one of the most successful frameworks, with Feynman diagrams being one of the key bookkeeping devices. However, as one increases the loop order, it becomes a daunting task to overcome the increasingly rampant subdivergences that appear. The BPHZ renormalization scheme provides a way of organizing these divergences, but even then, the amount of effort needed to handle 5-loop diagrams demonstrates that renormalization, in its current form, is still a monumental task. Here, we present a different perspective on perturbative renormalization, based in Hopf algebras, which has been used to recontextualize Ward identities as well as automate counterterm calculations out to as much as 200 loops..

Review of Renormalization

Consider the following 5-loop Feynman diagram in the φ^4 QFT theory in $3+1$ dimensions.

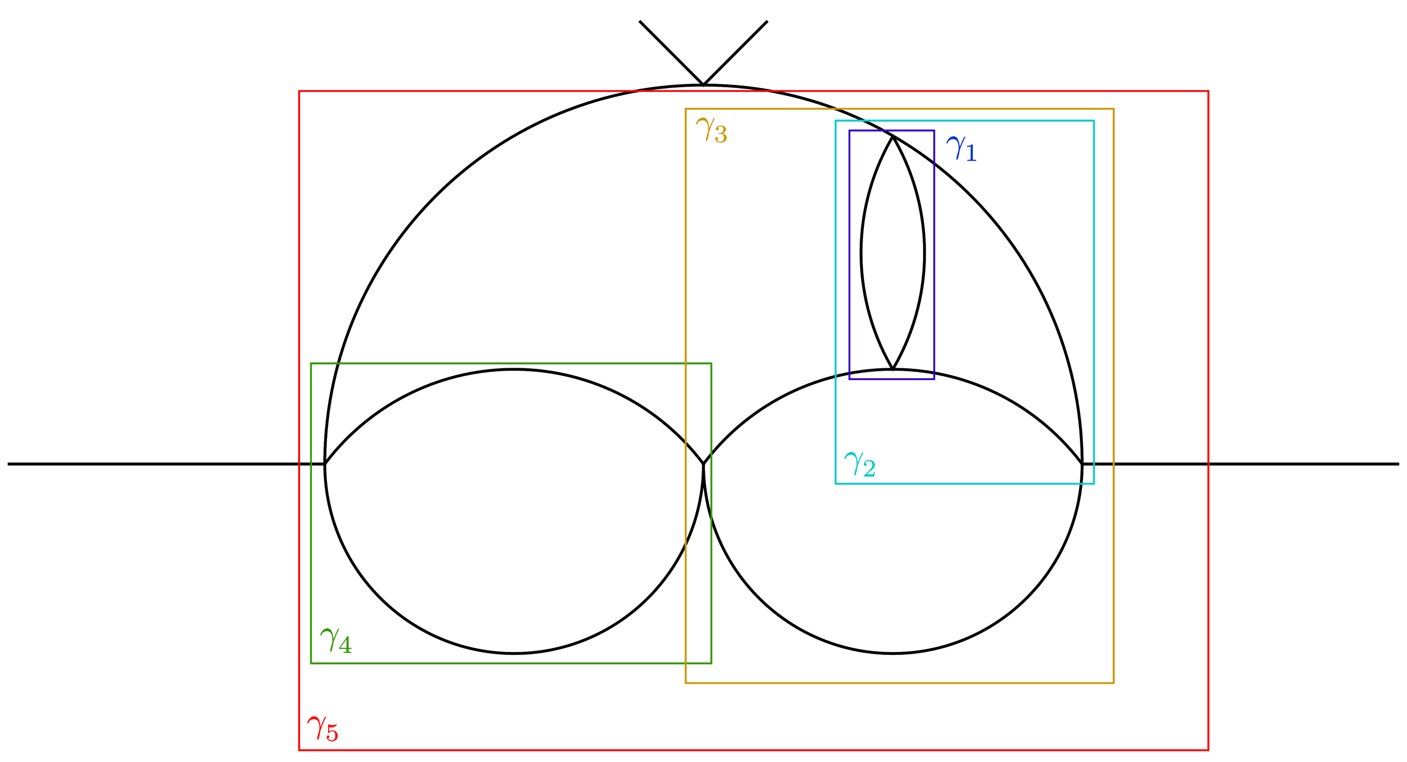


Figure 1: Feynman diagram G with 1PI UV-divergent subdiagrams.

Through naive powercounting, one can identify that this diagram diverges $\frac{(\mathrm{d}^4\ell)^5}{(\ell^2)^{10}} \sim \log(\ell)$ and so needs to be renormalized to obtain physically meaningful quantities. However, if the diagram has subdiagrams that also diverge, as in Figure 1, then the subdivergences must also be dealt with. With nested subdiagrams like $\gamma_1 \subset \gamma_2 \subset \gamma_3 \subset \gamma_5$, one must avoid “overcancelling” subdivergences. Handling this task at all loop orders is a long story, but culminated in one form with Zimmerman’s forest formula:

$$Z(G) = -R(G) - \sum_{\gamma \in G} R[Z(\gamma)G/\gamma]$$

One poster is not enough to explain the notation, but in short, one finds all subdiagrams (including disjoint ones) that contribute subdivergences and then replaces them with their counterterms; the terms that, upon subtraction from the overall diagram, will cancel the subdivergence. Do this for all subdiagrams and the overall diagram, and one has the term one must add to G to get a finite quantity; that is,

$$G + Z(G)$$

is what one wants.

Where’s the Forest?

Okay, so Zimmerman’s formula cancels all subdivergences, but where’s the forest? The answer lies in how the subdivergences are organized. Notice that, from G, it is possible to have nested subdivergences or disjoint diagrams. One can organize all these subdivergences in the form of a tree, as seen in Figure 2, where the main diagram G forms the root, disjoint subdiagrams form different branches, while nested subdiagrams form lower branches (the one shown is actually technically wrong, can you spot why?).

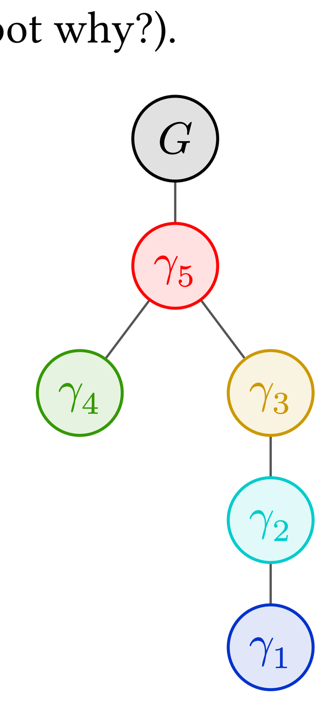


Figure 2: Divergence tree structure of Feynman diagram G.

One gets forests from trees because cancelling subdivergences comes down to cutting the tree into pieces “every possible way” and replacing the piece without the root with its counterterm. An example is shown in the next box.

Hopf What Now?

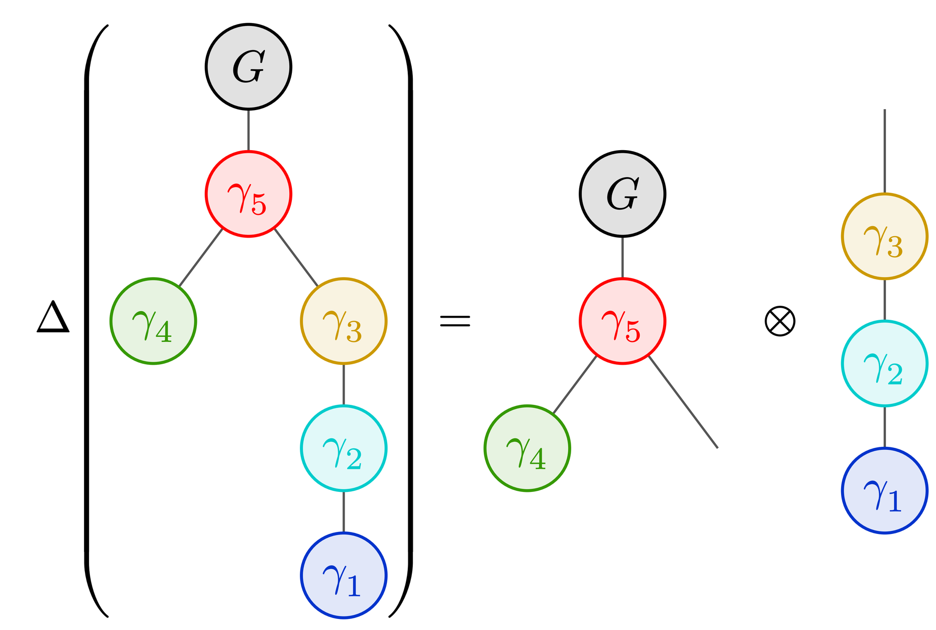
Okay, so we have renormalization via Zimmerman’s forest formula, why bring in more math? The answer is that, while the forest formula is correct, the practicality of using it for high loop order calculations is less than ideal. Hopf algebras provide a more natural framework to understand how to manipulate the forests. The heart of the naturality is that, while an algebra $\mathcal{A}$ only allow one to take two operators $A, B \in \mathcal{A}$ and multiply them, sending two objects into one via

$$m(A, B) = AB$$

a Hopf algebra $\mathcal{H}$ allows one to split one operator $C \in \mathcal{H}$ into two, via its *comultiplication function* Δ and the tensor product:

$$\Delta(A) \overset{\text{Ex.}}{=} B \otimes C$$

This might seem arbitrary, but remember how the forest formula cancels subdivergences by cutting trees into pieces? Well, for the Hopf algebra of Feynman diagrams, the *Connes-Kreimer Hopf algebra*, we have (this expression is actually oversimplified):



With this, it should be a bit more clear why the comultiplication of Hopf algebras is relevant for renormalization. The real clincher to this structure is that the Hopf algebra possesses a unique function, called the *antipode*, which “sort of” maps an element to its inverse; that is, the antipode S acts via:

$$S(A) \approx A^{-1}$$

For the Hopf algebra of Feynman graphs, the antipode map is Zimmerman’s forest formula.

Consequences of Hopf Algebra Approach

Model Tested	1998	2025
Model: BPHZ; 12 loops	7466.264 sec	113.64 sec
Model: QFT; 11 loops	1846.880 sec	121.32 sec
Model: HQ; 11 loops	2381.448 sec	36.49 sec

Table 1: Performance Comparison of Hopf Algebra REDUCE program

So far, we have only seen a neat repackaging of things that were already known, and that trend does continue in [Sui07] and [Sui09], where the Ward/Slavnov-Taylor identities are recontextualized using Hopf algebras. However, because of this more natural organization, it has become much easier to perform higher-order calculations, and some aspects of renormalization have been automated as well, as seen in [BK99] and [BS08]. In particular, the results by Kreimer and Broadhurst handled the analytical aspects of multiple field theories for 10-12 loop calculations, back in 1998. The calculations used REDUCE and took a couple hours. Rough benchmarks using modern hardware are given in Table 1. As for the paper by by Bellon and Schaposnik, through some optimizations and understanding of their supersymmetric theory of interest, they were able to renormalize diagrams up to 200 loops.

References

[BK99] Broadhurst, D.J. ; Kreimer, D.: Renormalization Automated by Hopf Algebra. In: *Journal of Symbolic Computation* vol. 27 (1999), Nr. 6, pp. 581–600

[BS08] Bellon, Marc P. ; Schaposnik, Fidel A.: Renormalization group functions for the Wess-Zumino model: up to 200 loops through Hopf algebras. In: *Nuclear Phys. B* vol. 800 (2008), Nr. 3, pp. 517–526

[Sui07] Suijlekom, Walter D. van: Renormalization of Gauge Fields: A Hopf Algebra Approach. In: *Communications in Mathematical Physics* vol. 276, Springer Science, Business Media LLC (2007), Nr. 3, pp. 773–798

[Sui09] Suijlekom, Walter D. van: Renormalization of Gauge Fields using Hopf Algebras.. In: *Quantum Field Theory* : Birkhäuser Basel, 2009 — ISBN 9783764387365, pp. 137–154