Plan

Elements

QCD fields

Composite operators

Correlation functions

Basic properties

Spectral representation

QCD ↔ hadron properties

Dynamics

Computing correlation functions

Perturbation theory at short distances

Vacuum fields and their effects

Gluon and quark condensate

Method I: Vacuum condensates

Operator product expansion

Shifman, Vainshtein, Zakharov 1979

Vacuum condensates

QCD ↔ hadron matching (“QCD sum rules”)

Applications to heavy and light mesons

Limitations of method

Method II: Vacuum fields ←

QFT at imaginary time (“Euclidean”)

Vacuum fields in “cooled” lattice QCD

Topological landscape and tunneling

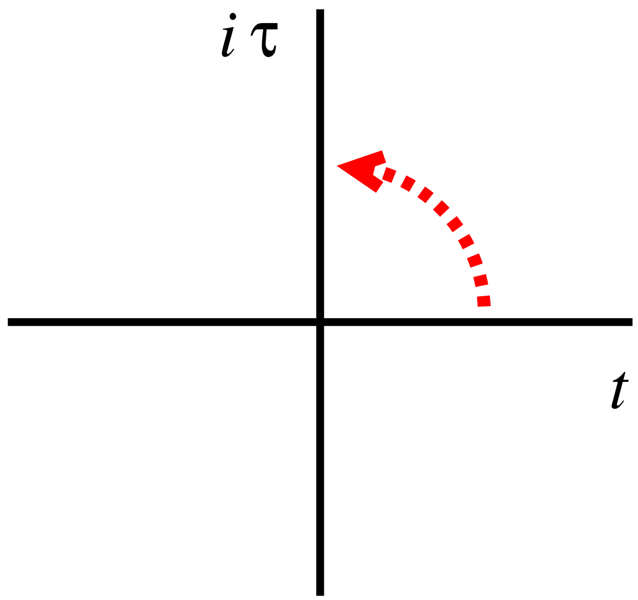
Instanton ensemble

Shuryak 1982; Diakonov, Petrov 1984

Chiral symmetry breaking

Meson and baryon correlation functions

Current developments



Transition to imaginary time

Enabled by complex-analytic properties of amplitudes and correlation functions

Can be applied to individual amplitudes (Wick rotation) or entire functional integral = generating function

$$e^{iE_h t} \quad \rightarrow \quad e^{-E_h \tau}$$

Time dependence = exponential decay

Imaginary time: Correlation functions

$$\langle 0 \,|\, J(\tau, \mathbf{x})\; J(0,\mathbf{0}) \,|\, 0 \rangle \;=\; \sum_h \underbrace{\langle 0 \,|\, J(\tau, \mathbf{x}) \,|\, h \rangle}_{e^{-E_h \tau}\;\langle 0 \,|\, J(0, \mathbf{x}) \,|\, h \rangle} \langle h \,|\, J(0,\mathbf{0}) \,|\, 0 \rangle$$

Limit $\tau \to \infty$:

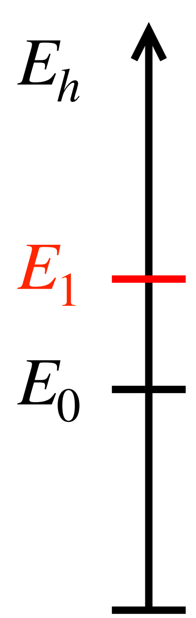
$$= \; e^{-E_0 \tau} \;\langle 0 \,|\, J(0, \mathbf{x}) \,|\, h_0 \rangle \langle h_0 \,|\, J(0, \mathbf{0}) \,|\, 0 \rangle$$

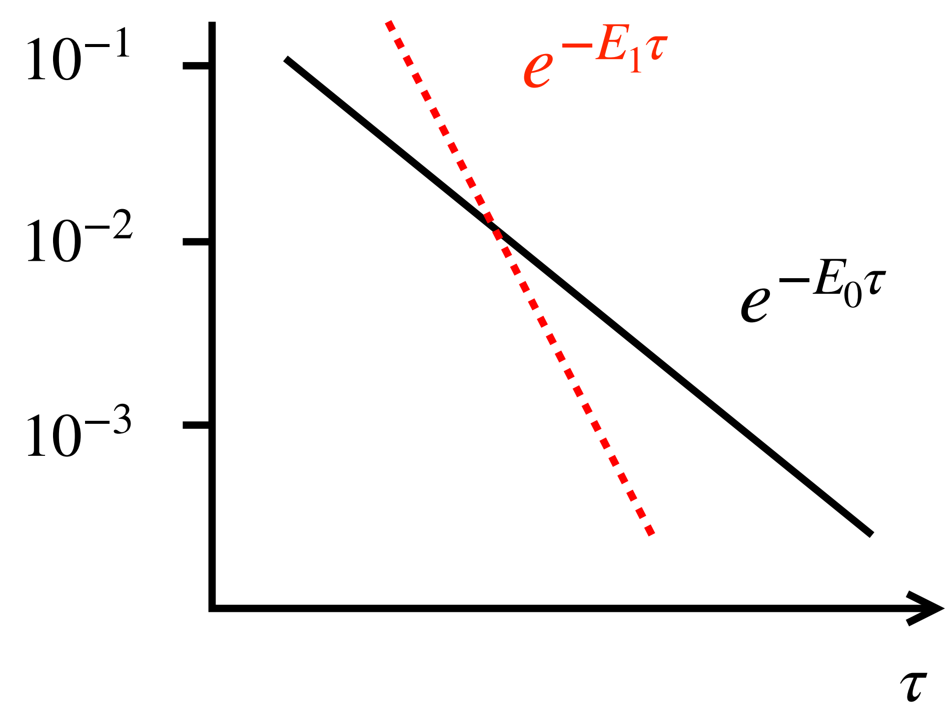
lowest-mass hadron

$$+ \; e^{-E_1 \tau} \;\langle 0 \,|\, J(0, \mathbf{x}) \,|\, h_1 \rangle \langle h_1 \,|\, J(0, \mathbf{0}) \,|\, 0 \rangle$$

higher-mass states suppressed

$$+ \; \cdots$$





Properties of lowest-mass hadron states can be obtained from large-time limit of correlation functions

Masses, couplings to currents

Practical calculations: Trade-offs

Techniques beyond lowest-mass: "Distillation" Lecture Dudek

Imaginary time: Functional integral

$$\int DA \;\; e^{-i\,S[A]} \qquad \to \qquad \int DA \;\; \underbrace{e^{-S_E[A]}}_{\text{real, positive definite!}}$$

$$A(t, \mathbf{x}) \quad \to \quad A(\tau, \mathbf{x})$$

QFT $\quad \to \quad$ statistical mechanics

$\to$ Numerical simulations, Monte-Carlo methods

$\to$ Concept of "contribution" of certain field configurations to correlation functions

$\to$ Semiclassical methods: Saddle point approximation

Imaginary-time representation limited to calculation of static properties
(= independent of real time), cannot be applied to real time dependent properties

Recent developments: Light-cone correlation functions

Imaginary time: Euclidean metric

$$x_\mu \;=\; (x_1, x_2, x_3, x_4) \;=\; (\mathbf{x}, \tau)$$ Euclidean 4-vector

$$x_\mu x_\mu \;=\; x_1^2 + x_2^2 + x_3^2 + x_4^2 \;=\; |\mathbf{x}|^2 + \tau^2$$ Euclidean metric

same for momenta, other 4-vectors

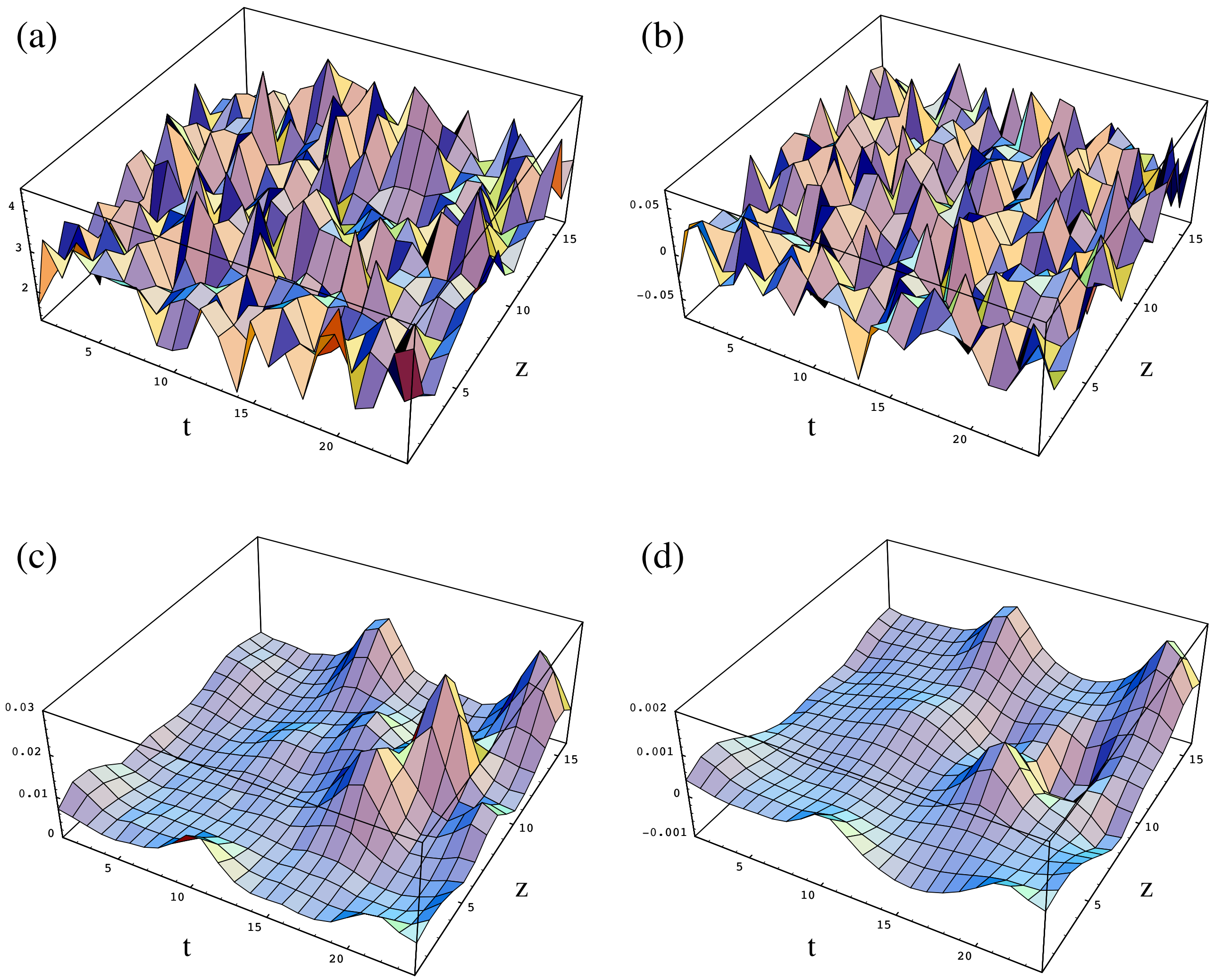
For computation of imaginary-time correlation functions,
QFT is formulated in 4D Euclidean space

4D rotational invariance O(4)
No difference between “space” and “time” directions!

All Euclidean distances/vectors are space-like

Euclidean 4-vectors correspond to spacelike Minkowskian 4-vectors

Vacuum fields: Cooled lattice QCD configurations



$G_{\mu\nu}G_{\mu\nu}(x)$ $\quad$ $G_{\mu\nu}\tilde{G}_{\mu\nu}(x)$

Fig: Chu, Grandy, Huang, Negele 1993

What are the gauge field configurations giving rise to non-perturbative structure of QCD vacuum?

Inspect lattice QCD configurations!

Usual field configurations are “rough”: Contain quantum fluctuations of any wavelength

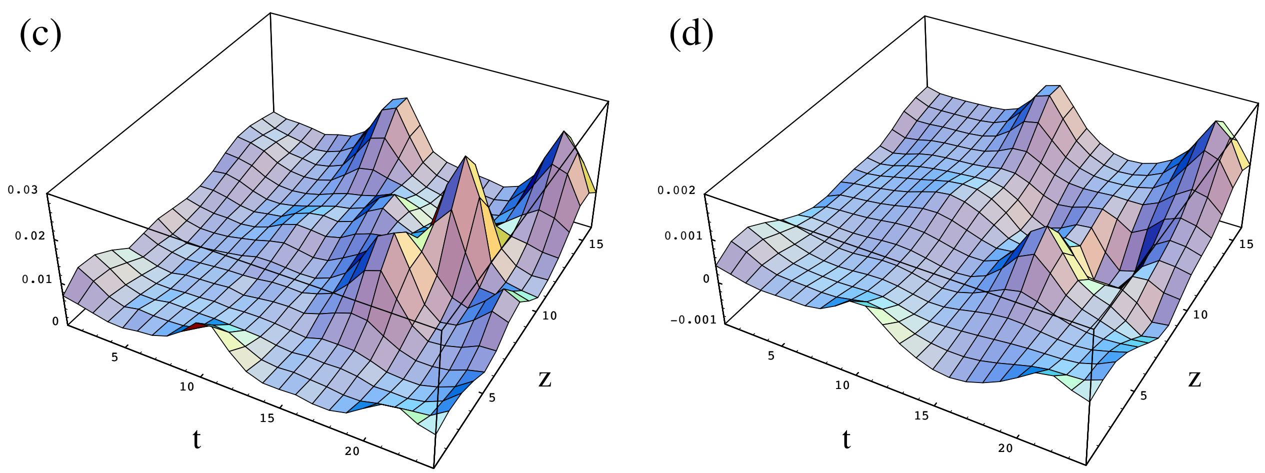
Cooling of lattice QCD configurations identifies “smooth” field configurations

Strong features: Concentrations of action and topological charge density

Alt technique: Gradient flow = systematic “smoothing” transformation of lattice QCD configurations. Controlled by parameter, reversible

see e.g. Bonati, D’Elia 2014; Athenodorou et al. 2018; Alexandrou et al. 2020

Vacuum fields: Nonperturbative fields



$G_{\mu\nu}G_{\mu\nu}(x)$ $\quad$ $G_{\mu\nu}\tilde{G}_{\mu\nu}(x)$

Vacuum populated by localized gauge fields

Typical size $\bar{\rho}$ ~ 0.3 fm $\quad \ll$ hadronic size ~ 1 fm

Typical 4D separation $\bar{R}$ ~ 1 fm

Fraction of 4D space occupied by fields:
$\pi^2\bar{\rho}^4/\bar{R}^4 \approx 0.1$

Strong fields: $$\int_{\text{vol.field}} d^4x\, G_{\mu\nu}G_{\mu\nu} \approx \frac{16\pi^2}{g^2} \approx 20$$ large action $\gg 1$

[In this estimate:
g at scale $\mu = \bar{\rho}^{-1}$]

Topologically charged: $$\underbrace{\frac{g^2}{16\pi^2}\int_{\text{vol.field}} d^4x\, G_{\mu\nu}\tilde{G}_{\mu\nu}}_{\text{topological charge}} = \pm \underbrace{\frac{g^2}{16\pi^2}\int_{\text{vol.field}} d^4x\, G_{\mu\nu}G_{\mu\nu}}_{\text{action}} = \pm 1$$

Vacuum fields: Interpretation

Vacuum fields observed in cooled lattice QCD configurations are fluctuations with local topological charge ± 1

QCD instantons: Classical solutions of Yang-Mills equations with topological charge ± 1

Belavin, Polyakov, Shvarts, Tyupkin 1975, ‘tHooft 1976

Physical interpretation: Tunneling processes in topological landscape of gauge theory

Important for chiral symmetry breaking: Topologically charged gauge fields induce chirality-changing interactions between fermions, cause chiral symmetry breaking

Program:

Learn about topological structure of gauge theory and tunneling processes = instantons

Construct effective description of QCD vacuum based on instanton fields using semiclassical approximation

Explain/describe dynamics of chiral symmetry breaking

Compute hadronic correlation functions and extract hadron structure

Topological structure: Gauge fields

Space of gauge potentials has topological structure

$$A_i(\mathbf{x}) \xrightarrow{\text{gauge tf}} U^{-1}(\mathbf{x})\,A_i(\mathbf{x})\,U(\mathbf{x}) \;+\; ig^{-1}U^{-1}(\mathbf{x})\,\partial_i U(\mathbf{x})$$

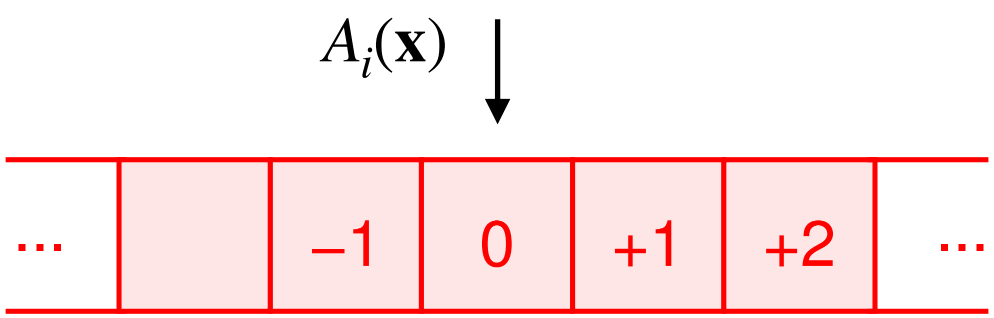
Gauge transformation
Here: $A_0 = 0$ gauge, fixed time τ

$U(\mathbf{x})$: $\quad R^3 \;\rightarrow\; SU(2) \;\subset\; SU(3)$

Mapping with topological characteristics
$U \rightarrow 1$ for $|\mathbf{x}| \rightarrow \infty$

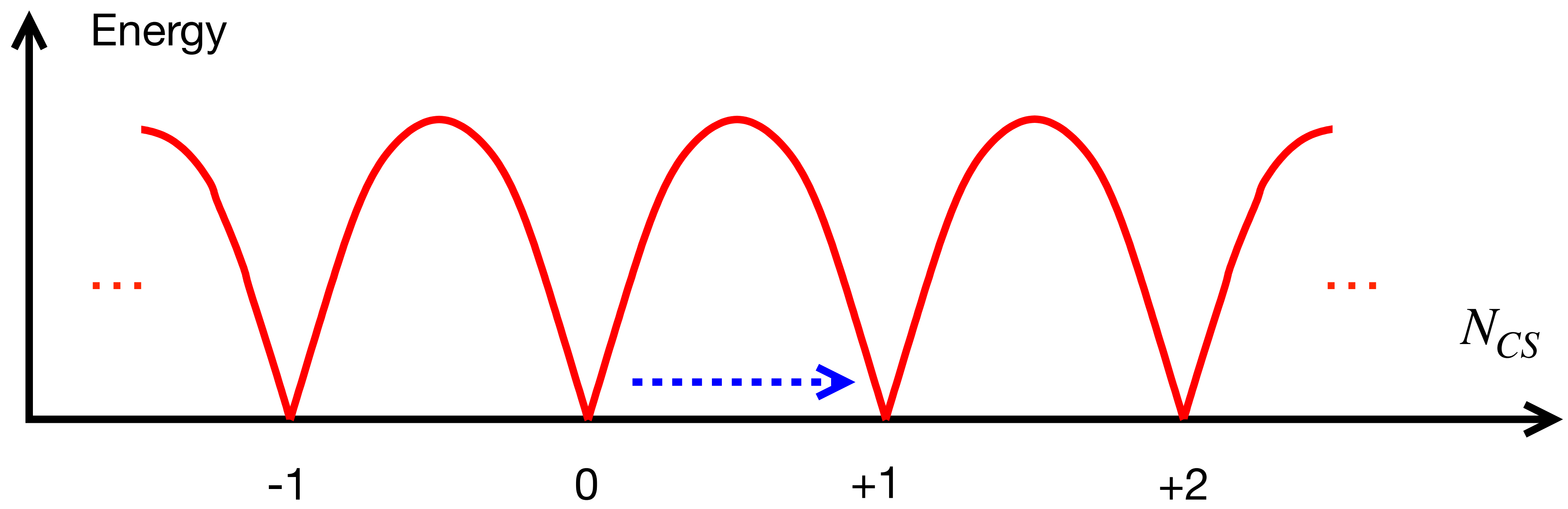
Winding number N_{CS} (Chern-Simons number): How many times U covers $SU(2)$ group while going over R^3 space

Topological sectors labeled by N_{CS}



Quantum-mechanical motion of gauge fields extends over all topological sectors

Topological structure: Tunneling



Energy of gauge field configurations is periodic function in N_{CS}

For every configuration A_i with energy $E[A_i]$, there are gauge-equivalent configurations with the same energy in all the N_{CS} sectors

Minima $E[A_i] = 0$ periodic in N_{CS}, separated by finite barriers

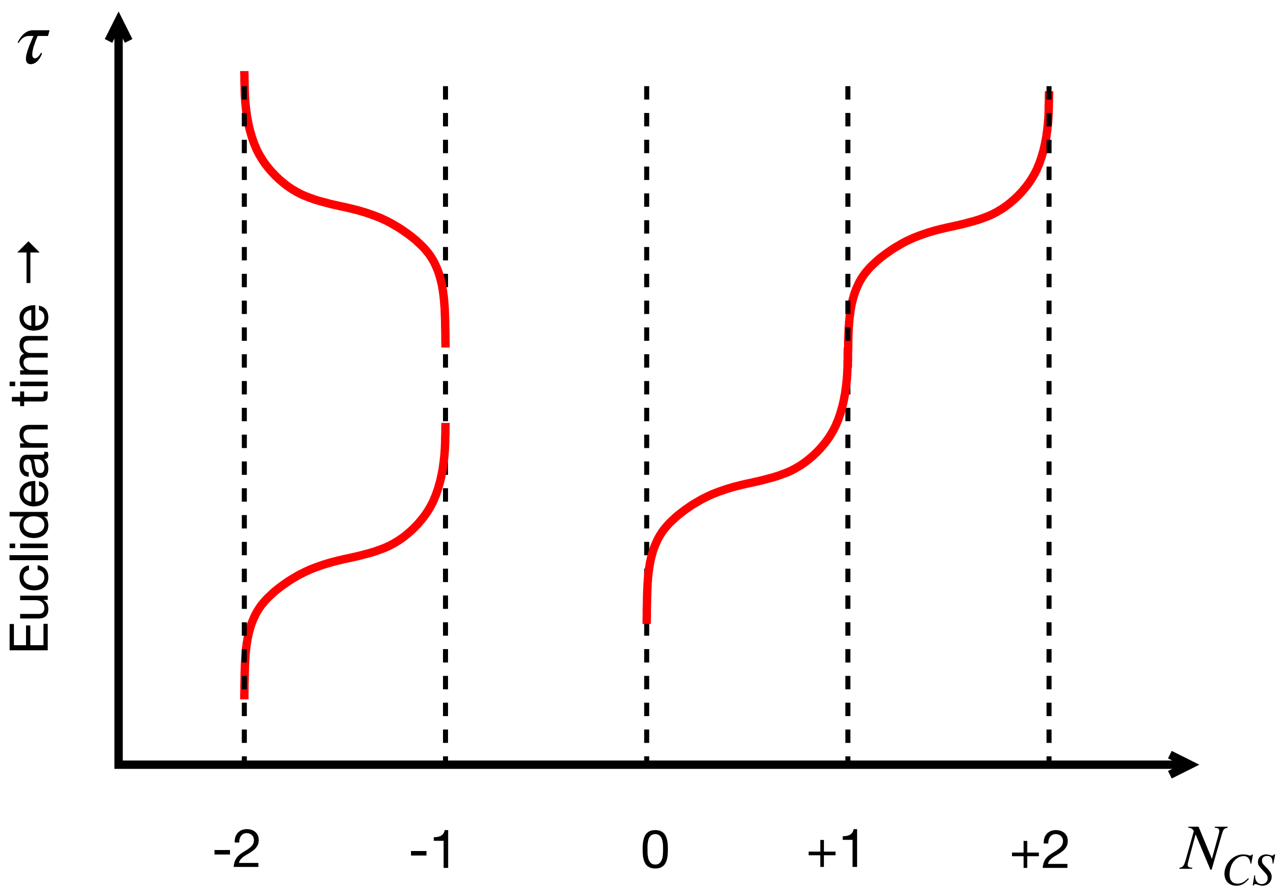
What is the ground state?

Analog: QM particle in periodic 1D potential

Ground state: Particle not localized in one minimum, but in periodic state involving coherent superposition of all minima

Tunneling: QM transitions between configurations localized in different minima

Topological structure: Ground state



QCD ground state = coherent superposition of gauge fields in all topological sectors

Functional integral: Trajectories involve tunneling between topological sectors

Elementary tunneling process $\Delta N_{CS} = \pm 1$

Described by classical trajectory (semiclassical approximation): Instanton

$A_{\text{inst}\pm}(\mathbf{x}, \tau)$:

$$\mathbf{A}_{\text{inst}\pm}(\mathbf{x}, \tau = -\infty) = \mathbf{A}'(\mathbf{x})$$

$$\mathbf{A}_{\text{inst}\pm}(\mathbf{x}, \tau = +\infty) = \mathbf{A}''(\mathbf{x})$$

$$\mathbf{A}'(\mathbf{x}) \xrightarrow{U(N_{CS} = \pm 1)} \mathbf{A}''(\mathbf{x})$$

Topological structure: Instanton

$$A_{\mu,\mathrm{inst}\pm}(x) = \frac{i}{g}\,\frac{\eta^a_{\mu\nu}\,x_\nu}{|x^2|}\,f\left(\frac{|x|}{\rho}\right)$$

Explicit form of instanton gauge potential in covariant gauge*

e.g. $$f = \frac{1}{1 + x^2/\rho^2}$$

Localized field. Profile function f depends on gauge

$$D^{ab}_\mu G^b_{\mu\nu,\mathrm{inst}\pm}(x) = 0$$

Solution of Yang-Mills equation

$$\tilde{G}^a_{\mu\nu,\mathrm{inst}\pm} = \pm\, G^a_{\mu\nu,\mathrm{inst}\pm}$$

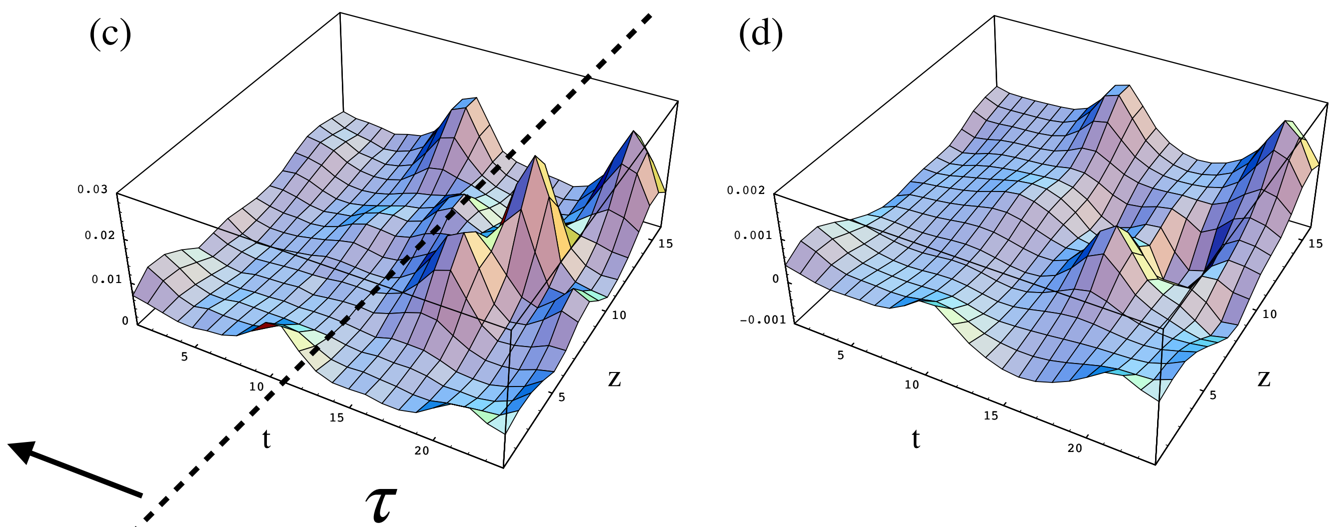
Field is (anti-) self-dual

$$\frac{g^2}{16\pi^2}\int d^4x\; G_{\mu\nu,\mathrm{inst}\pm}(x)\;\tilde{G}_{\mu\nu,\mathrm{inst}\pm}(x) \;= \pm\, 1$$

Topological charge ± 1 (= ΔN_{CS} in tunneling process)

*Position of center arbitrary, field can be shifted: $x \to x - z,\; z =$ center coordinate

Topological structure: Summary



Euclidean time

Strong localized fields in QCD vacuum = tunneling events

Semiclassical approximation: Describe vacuum fields as superposition of instantons and antiinstantons