Nonperturbative methods in QCD

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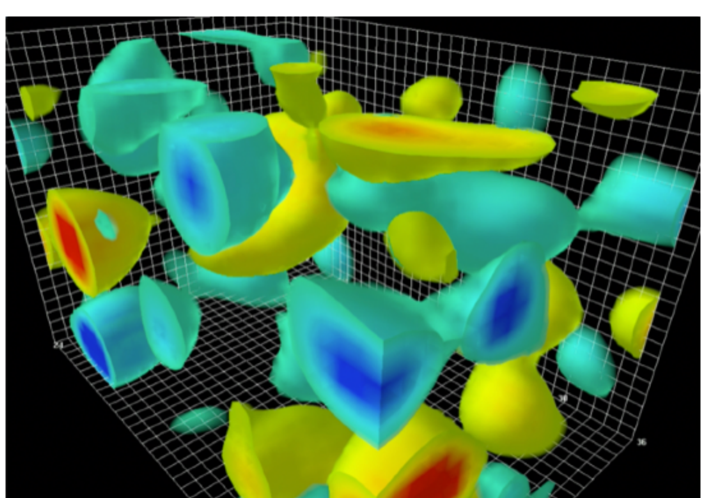
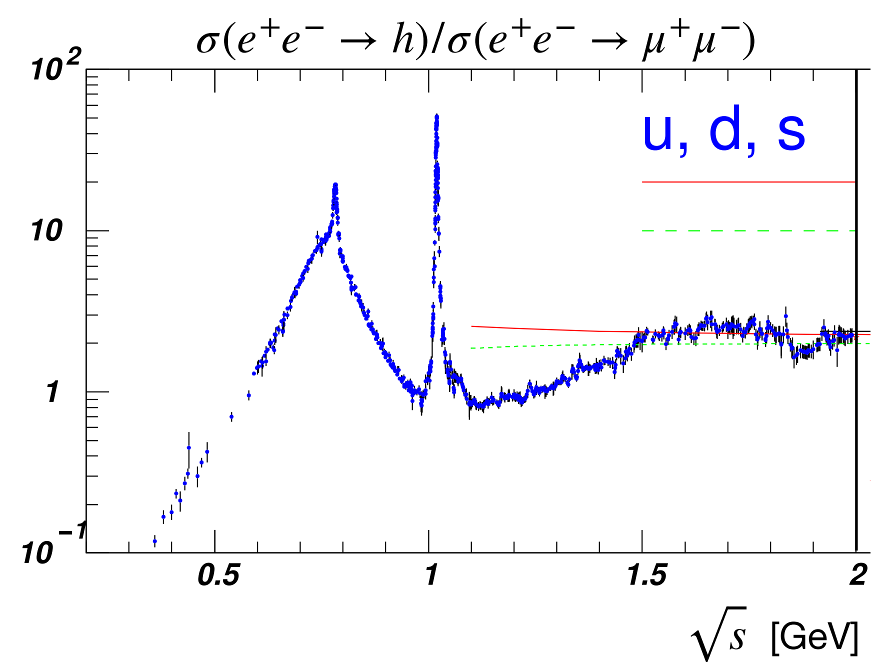
How do hadrons emerge from QCD?

How to compute hadron spectra and structure from QCD?

Here: Analytic methods of nonperturbative QCD

- Correlation functions: QCD ↔ hadron spectrum
- Vacuum structure: Condensates, symmetry breaking
- Computational methods: Operator product expansion, semiclassical methods





Why interesting

Understand “how QCD works” in nonperturbative regime

Analytic methods are synergistic with lattice QCD:
Explain lattice results, use input from lattice simulations

Analytic methods are very efficient: Calculations simple, high ratio output/input

Link up with current research: Semiclassical methods, instantons

Connections with condensed matter physics:
Complex ground state, study “small” excitations above ground state

Connections with data science: Extraction of information on hadrons from
QCD correlation functions has characteristics of inverse problem

Plan

Elements

QCD fields

Composite operators

Correlation functions

Basic properties

Spectral representation

QCD ↔ hadron properties

Dynamics

Computing correlation functions

Perturbation theory at short distances

Vacuum fields and their effects

Gluon and quark condensate

Method I: Vacuum condensates

Shifman, Vainshtein, Zakharov 1979

Operator product expansion

Vacuum condensates

QCD ↔ hadron matching (“QCD sum rules”)

Applications to heavy and light mesons

Limitations of method

Method II: Vacuum fields

Vacuum fields in “cooled” lattice QCD

Topological landscape and tunneling

Instanton ensemble

Shuryak 1982; Diakonov, Petrov 1984

Chiral symmetry breaking

Meson and baryon correlation functions

Current developments

This lecture

Focus on concepts - what they mean, how they are connected and applied

This will include some mathematics, but don’t be afraid…

Skip most calculations but explain basic steps

Go over material at uniform level with aim to understand “what it is about”.
There are many more aspects, but they can be explored later

Please ask questions and give feedback at any time!
There will be summaries after each topic - good time for questions.

Notice: References to literature still missing or incomplete

Elements: Fields and gauge symmetry

$A_\mu(x) = A^a_\mu(x)\,\frac{\lambda^a}{2}$ gauge potential

$G_{\mu\nu}(x) = \partial_\mu A_\nu - \partial_\nu A_\mu + g[A_\mu, A_\nu]$

$= G^a_{\mu\nu}(x)\,\frac{\lambda^a}{2}$ gauge field

$\psi(x),\ \bar{\psi}(x)$ matter fields

Degrees of freedom = fields, functions of space-time

$SU(N_c)$ gauge group, $N_c = 3$ (“color”)

Matter fields in fundamental representation, gauge fields as matrices in fundamental or vectors in adjoint rep

Use compact notation: Spinor/flavor/color

Gauge transformations

$\psi(x) \to U(x)\,\psi(x)$

$A_\mu \to UA_\mu U^\dagger - ig^{-1}U\partial_\mu U^\dagger$

$G_{\mu\nu} \to UG_{\mu\nu}U^\dagger$

$G^a_{\mu\nu} \to O^{ab}G^b_{\mu\nu}, \quad O^{ab} = \mathrm{tr}\,[\frac{\lambda^a}{2}U\frac{\lambda^b}{2}U^\dagger]$

$U(x)$ rotation in color, space-time dependent

A transforms non-covariantly, G covariantly

Dynamics invariant under gauge transformation

Degrees of freedom = orbits under gauge transformations

Elements: Composite operators

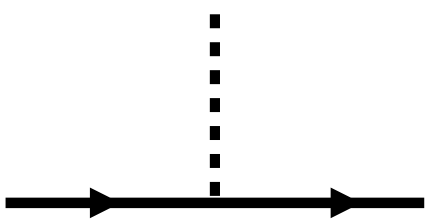
$\bar{\psi}(x)\,\Gamma\,\psi(x) \;\equiv J_\Gamma(x)$

Physical quantities carried by fields:
Gauge-invariant composite operators, color-singlet

$\Gamma =$	$\gamma^\mu,\ \gamma^\mu\gamma^5$	vector, axial vector
	$1,\ i\gamma^5$	scalar, pseudoscalar
	$\sigma^{\mu\nu}$	tensor

Operators characterized by spin,
discrete quantum numbers C, P, T

Quark flavor combinations:
$\bar{u}\,..\,u \pm \bar{d}\,..\,d$ isoscalar/isovector
$\bar{u}\,..\,d, \bar{d}\,..\,u$ charged



Γ

diagrammatic
representation

Here: Bilinear operators,
“meson” quantum numbers

Other quark operators

Multilinear operators: $\bar{\psi}\Gamma'\psi\bar{\psi}\Gamma\psi,\ \bar{\psi}\Gamma'\frac{\lambda^a}{2}\psi\ \bar{\psi}\Gamma\frac{\lambda^a}{2}\psi$

Baryon operators $\psi\psi\psi, \bar{\psi}\bar{\psi}\bar{\psi}\ \to$ later

Elements: Composite operators

Gluon operators

$$\text{tr}[G^{\mu\nu}(x)G_{\mu\nu}(x)] \;=\; G^{a,\mu\nu}(x)G^{a}_{\mu\nu}(x) \;\equiv\; G^{\mu\nu}(x)G_{\mu\nu}(x)$$ scalar

$$G^{\mu\nu}(x)\tilde{G}^{a}_{\mu\nu}(x)$$ pseudoscalar

$$\tilde{G}^{a}_{\mu\nu}(x) \equiv \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}G^{\rho\sigma}(x)$$ dual field strength

Higher-dimension gluon operators

$f^{abc}(G^a)^{\mu}{}_{\nu}(G^b)^{\nu}{}_{\rho}(G^c)^{\rho}{}_{\mu}$ dim-6 etc.

Gluon operators with derivatives $G^{\mu\nu}D_{\alpha}\ldots G_{\rho\sigma}$ etc.

Mixed quark-gluon operators $\bar{\psi}G^{\mu\nu}\sigma_{\mu\nu}\psi$ etc.

Quark and gluon operators characterized by

Dimension: (mass)^N

Spin: Rank and symmetry of Lorentz tensor

Discrete symmetries: C, P, T

Correlation functions: Definition

$$\langle 0 \,|\, T\, J_\Gamma(x)\, J_\Gamma(y) \,|\, 0 \rangle$$

Correlation functions: Vacuum expectation value of products of gauge-invariant operators

Time-ordered product: Analytic properties (later)

$$T: \quad \begin{array}{ll} J_\Gamma(x)\, J_\Gamma(y) & x^0 > y^0 \\ J_\Gamma(y)\, J_\Gamma(x) & y^0 > x^0 \end{array}$$

Basic physical objects of QCD as gauge theory
→ hadrons, observables

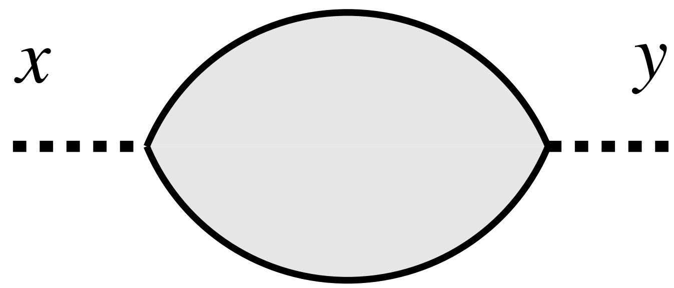


Illustration of concept, not Feynman diagram

Vacuum state $|\,0\rangle$: Ground state of QCD. Complex structure with vacuum fields (condensates), dynamical scales

Translational invariance:
Function depends only relative coordinate $x - y$

Correlation functions: Momentum representation

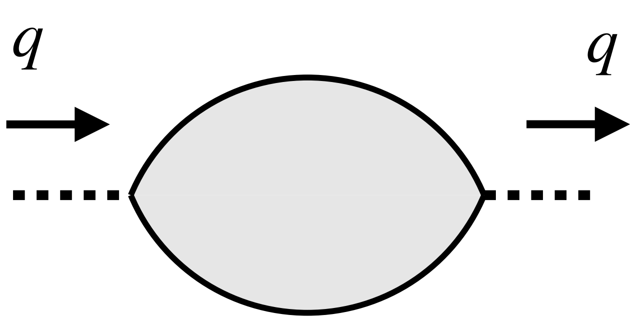
$$i \int d^4x e^{iq(x-y)} \langle 0 | T J_\Gamma(x) J_\Gamma(y) | 0 \rangle$$

Momentum representation: 4D Fourier $(x - y) \to q$

$$= \sum_n T_n^{\mu\nu\ldots}(q)\, \Pi_n(q^2)$$

4D tensors × scalar functions of q^2

Specific form depends on Γ



$\Pi_n(q^2)$ functions of single variable q^2

Example: Vector operator

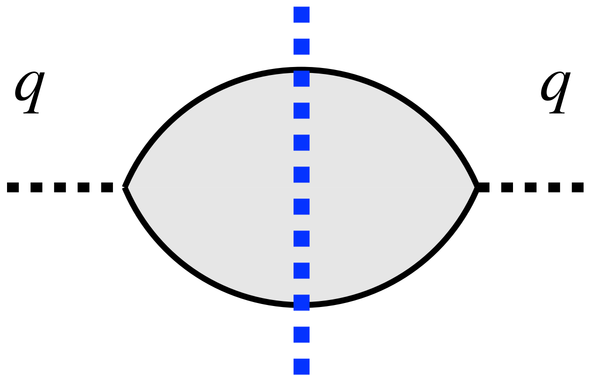
$$i \int d^4x e^{iq(x-y)} \langle 0 | T J_V^\mu(x) J_V^\nu(y) | 0 \rangle$$

$J_V(x) = \bar{\psi}(x)\gamma^\mu\psi(x)$ vector current

$$= \left(q^\mu q^\nu - \frac{q^2}{4} g^{\mu\nu} \right) \Pi_V(q^2)$$

Current conservation $\partial_\mu J_V^\mu(x) = 0$
requires $q_\mu(\ldots)^{\mu\nu} = 0$

Correlation functions: Connection with hadrons



$q^2 < 0$ 4-momentum spacelike, cannot produce hadrons

$q^2 > 0$ 4-momentum timelike, can produce hadrons

$$i \int d^4x e^{iq(x-y)} \langle 0 | T J_\Gamma(x) J_\Gamma(y) | 0 \rangle$$

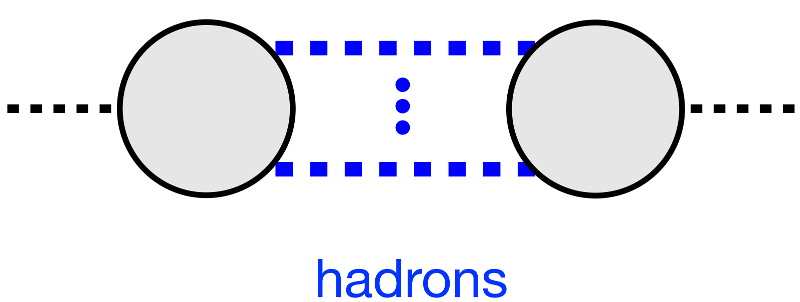
$$\sum_h |h\rangle\langle h| \uparrow$$

Evaluate correlation function for $q^2 > 0$

- Insert complete set of hadronic states
- Use translational invariance to move operators to $x, y = 0$
- Combine terms from the two time orderings

$$\text{Im}\,\Pi(q^2) = \sum_h (2\pi)^4 \delta^{(4)}(q - P_h) \langle 0 | J_\Gamma(0) | h \rangle \langle h | J_\Gamma(0) | 0 \rangle \quad \equiv \rho(q^2) > 0$$

spectral density

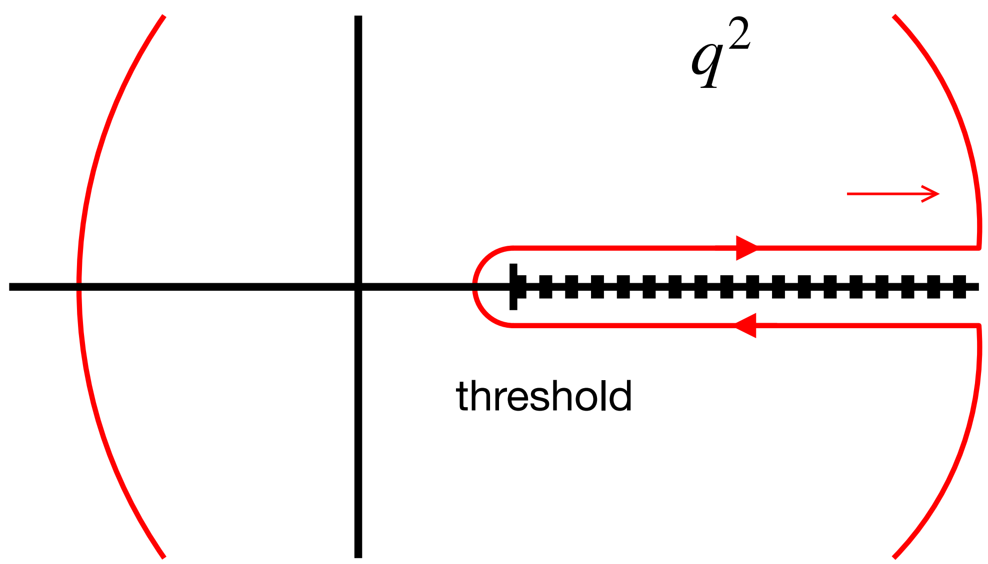


Correlation function acquires imaginary part from hadronic states

Allowed states depend on operator quantum numbers and q^2

Single-hadron or multi-hadron states

Correlation functions: Spectral representation



$\Pi(q^2)$ analytic function of q^2

No singularities at $q^2 < 0$

Singularities (poles, cuts) at real $q^2 > 0$

Dispersion relation

$$\Pi(q^2) = \int_{\text{thr}}^{\infty} ds \, \frac{\operatorname{Im}\Pi(s)}{s - q^2 - i\epsilon}$$

Correlation function expressed as integral over imaginary part at $q^2 >$ threshold

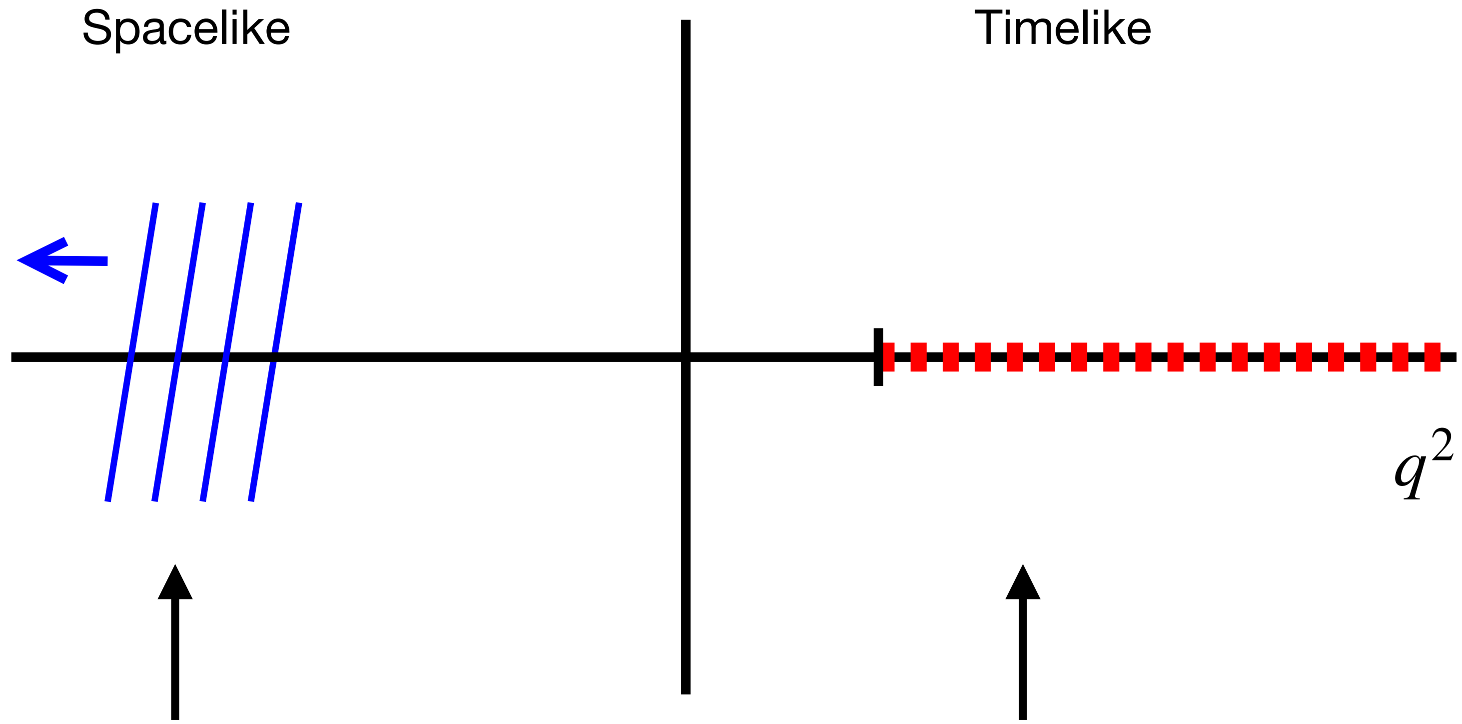
$$\operatorname{Im}\Pi(s) = \rho(s) \quad \leftrightarrow \text{hadrons}$$

Imaginary part = spectral density from hadronic states:
Spectral representation

Representation valid at all q^2: Spacelike, timelike, even complex

Depending on asymptotic behavior of $\Pi(q^2)$:
Write dispersion relation for $\Pi(q^2) - \Pi(0)$ etc. ("subtractions")

Correlation functions: Basic situation



Here we can compute

Perturbation theory $-q^2 \to \infty$

Nonperturbative methods:
Short-distance expansion, semiclassics

Lattice QCD

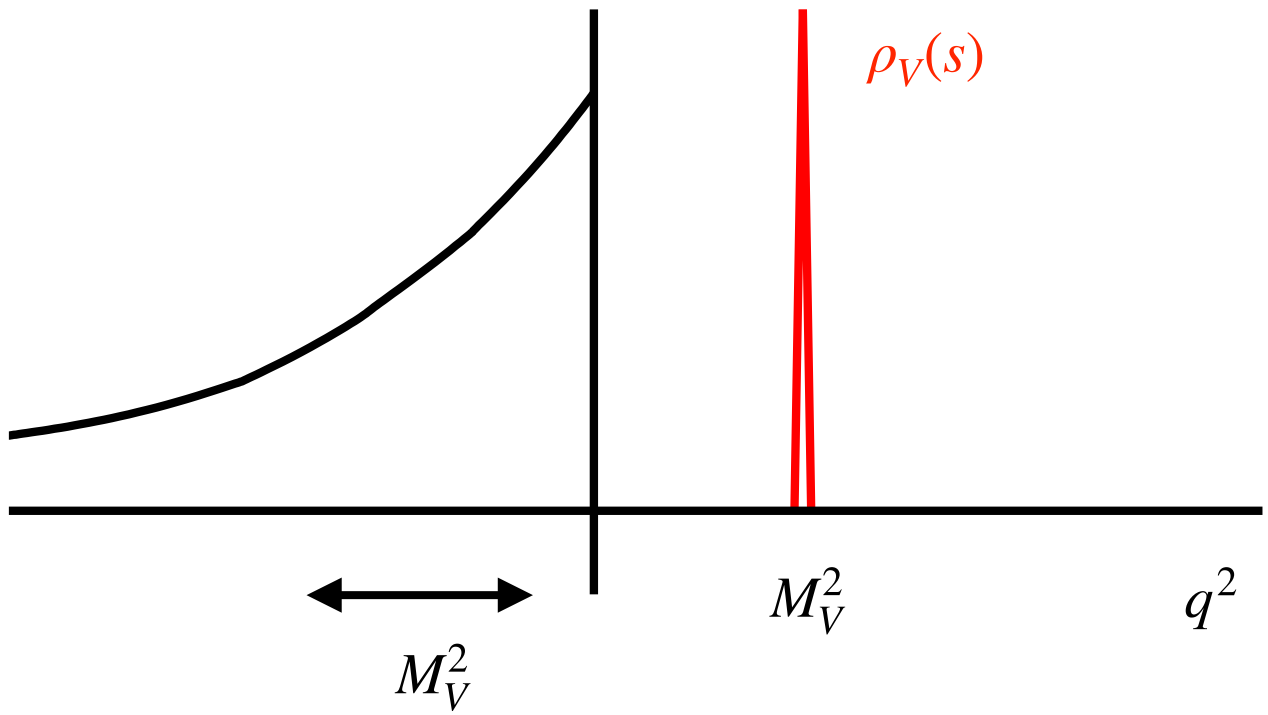
Here is the information on hadrons

Spectral density $\rho = \sum_h |\langle 0 | J_\Gamma | h \rangle|^2$

connected by spectral representation

In the following we will put
“numbers” on this graph…

Correlation functions: Spectral density

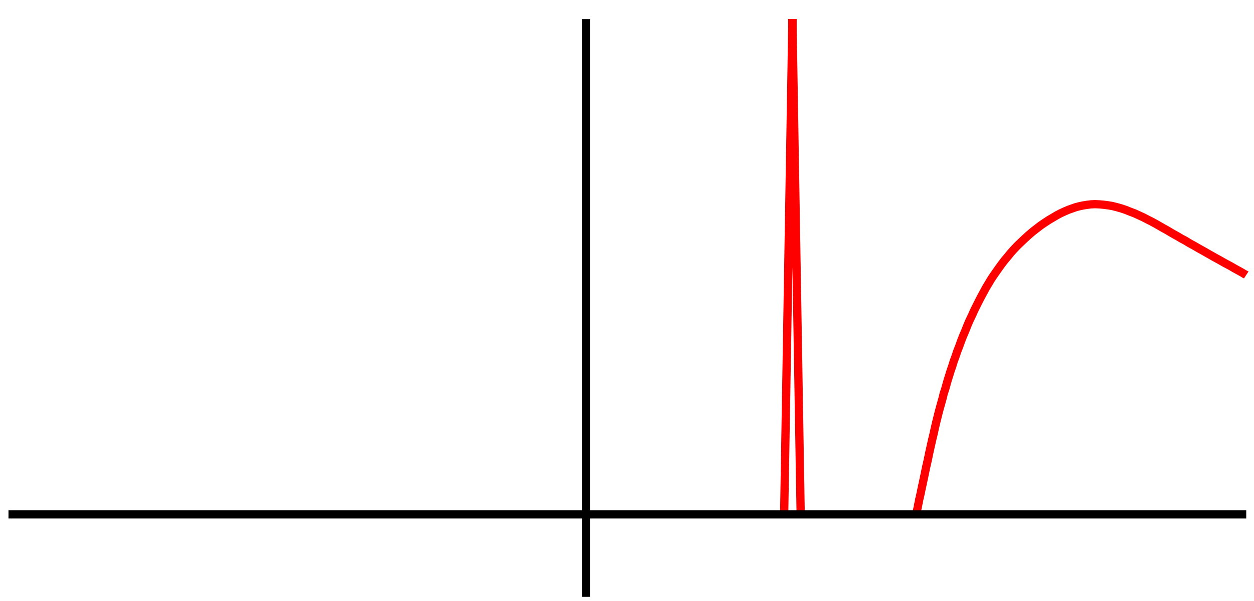


Spectral density with single pole

$$\rho_V(s) \;=\; f_V^2\, \delta(s - M_V^2)$$

$$\Pi_V(q^2) \;=\; \frac{f_V^2}{M_V^2 - q^2}$$

To extract meson mass and coupling, need to compute correlation function at spacelike $-q^2 \sim M_V^2$



Realistic: Pole + continuum

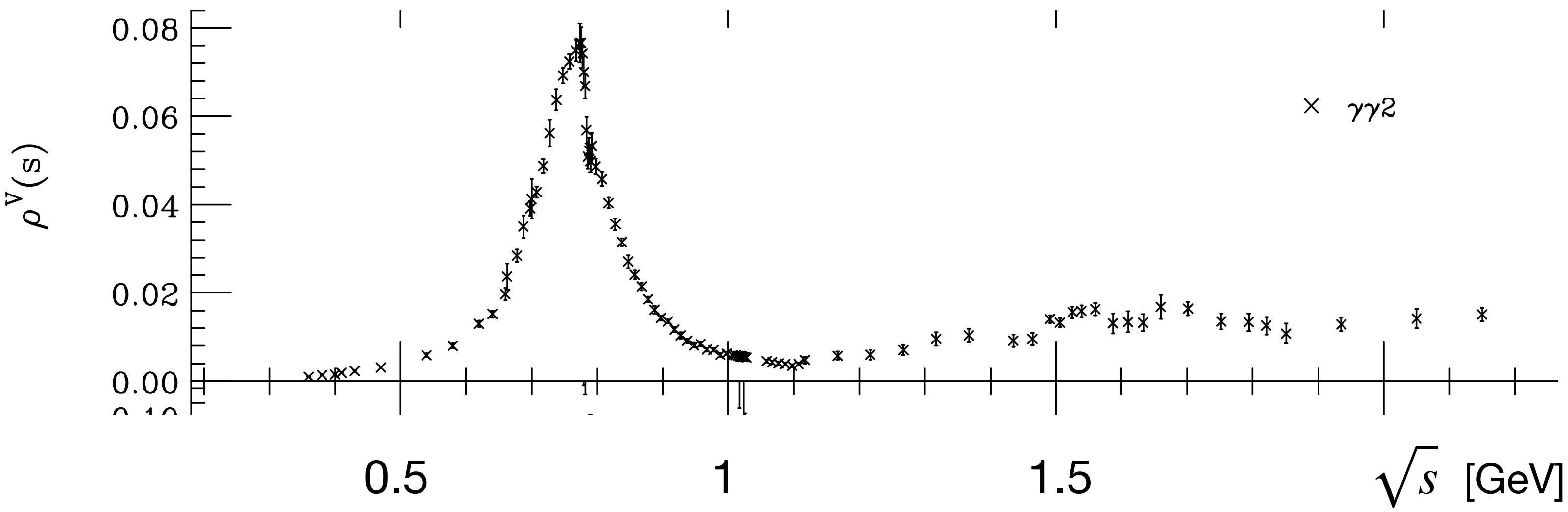
Need to include continuum:
Various techniques → following

Inverse problem: Information loss between spectral density and spacelike correlation function

Correlation functions: Empirical spectral densities

Vector and axial vector currents couple to leptons through electromagnetic and weak interactions

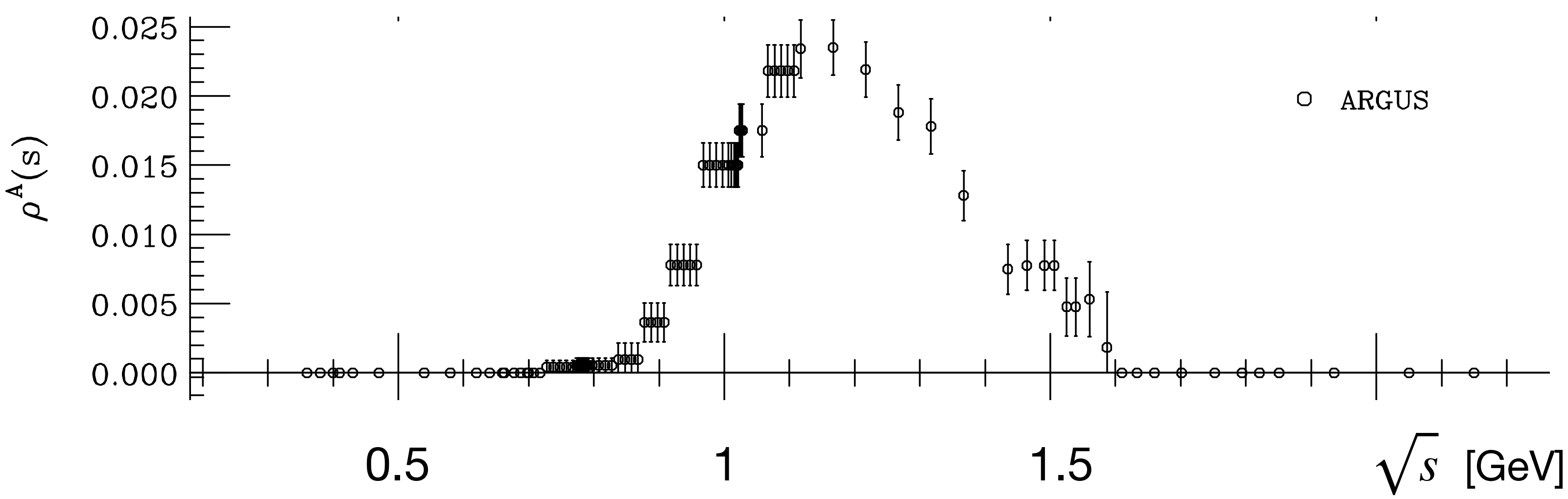
Spectral densities can be measured in lepton annihilation/decay into hadrons



Vector spectral density from e^+e^- annihilation into hadrons

$$\rho_V(s) \propto \sigma(e^+e^- \to \sum h)$$

Rho meson pole at ~0.77 GeV



Axial vector spectral density from τ lepton decay into hadrons

$$\sigma(\tau \to \nu + \sum h)$$

Also pion pole at $\sqrt{s} = M_\pi$

Fig: Rapp, Wambach 1999

Vector and axial vector spectral functions very different: Chiral symmetry breaking (later)

Correlation functions: More

Baryon correlation functions

$$B_\Gamma(x) = \epsilon^{\alpha\beta\gamma}\ \psi_i^\alpha(x)\psi_j^\beta(x)\psi_k^\gamma(x)\ \Gamma_{ijk}$$

$$B_\Gamma^\dagger(x) \leftrightarrow \bar{\psi}(x)$$

Baryon operators

Totally antisymmetric in color

Spin structure determined by spinor matrix Γ

$$\langle 0 \,|\, T\, B_\Gamma^\dagger(x) B_\Gamma(y) \,|\, 0 \rangle$$

Hadronic states: Baryon number 1

3-point functions

$$\langle 0 \,|\, T\, J_\Gamma(x)\, \mathcal{O}(z)\, J_\Gamma(y) \,|\, 0 \rangle$$

Spectral density more complex,
describes coupling of hadrons to operator $\mathcal{O}$

Summary: Correlation functions

Correlation functions of gauge-invariant operators are the basic physical objects of quantum field theory $\leftrightarrow$ spectrum, observables

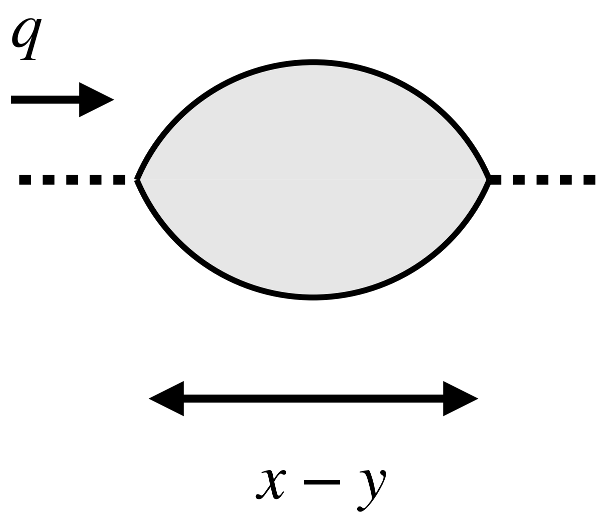
Hadronic states appear in spectral density: Masses, couplings

Spectral representation (dispersion relation) connects timelike and spacelike regions

Extraction of spectral information from computed spacelike correlation functions is inverse problem

Need methods for computation of correlation functions!

Dynamics: Computing correlation functions

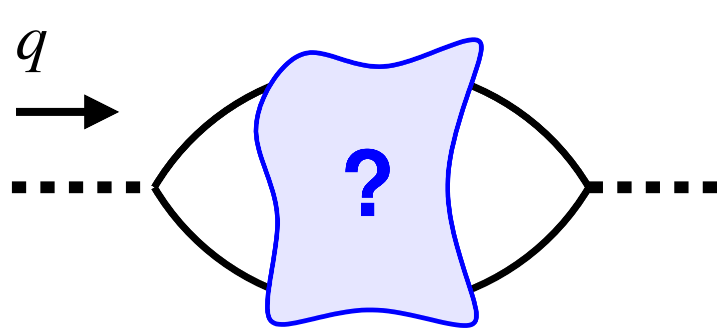


Compute correlation function

Spacelike distances $x - y$ $(x-y)^2 < 0$

Spacelike momenta q $q^2 < 0$

Dynamics changes with scale

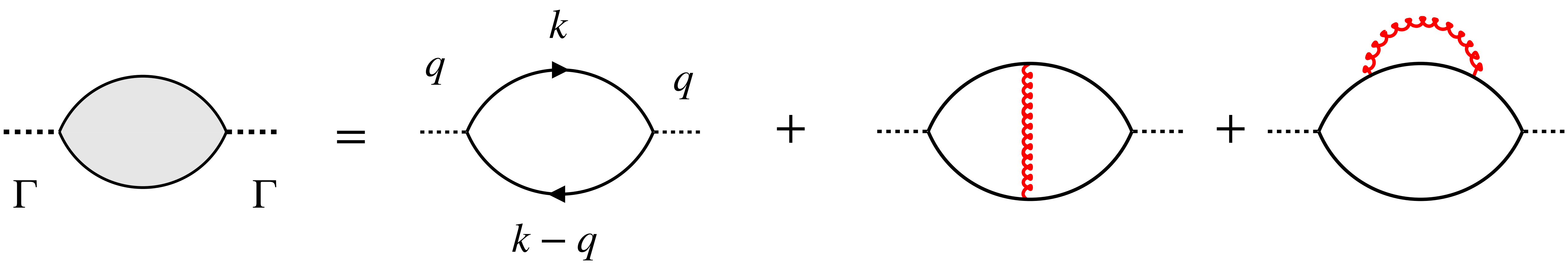


$|-q^2| \to \infty$ Asymptotic freedom
Perturbative dynamics

$|-q^2| \sim \mu^2_{\text{nonpert}}$ Nonperturbative dynamics
→ need new methods

$\mu_{\text{nonpert}} \sim 1$ GeV discussed in following

Dynamics: Perturbation theory



$$\Pi(q^2) = i \int \frac{d^4k}{(2\pi)^4} \, \mathrm{tr} \left[\Gamma \, G(k) \, \Gamma \, G(k-q) \right]$$

Feynman integral in momentum representation
Divergent - regularization, renormalization

$$\Pi(q^2) - \Pi(0) - q^2 \frac{d}{dq^2} \Pi(0) + \ldots$$

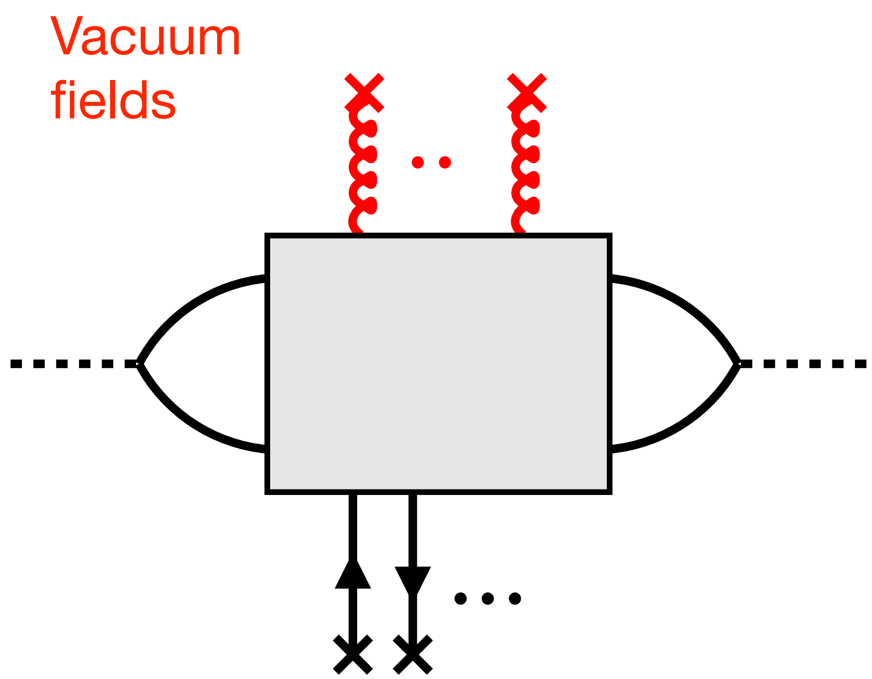
Regularization by subtraction

Alt.: Calculation in coordinate representation $i \, \mathrm{tr} \left[\Gamma \, G(x-y) \, \Gamma \, G(y-x) \right]$
Finite expressions as long as $x \neq y$

Vector correlation function $\Gamma = \gamma^\mu$: Im $\Pi(s) \propto \sigma(e^+e^- \to$ hadrons).
Perturbative result can be compared directly with inclusive annihilation data

$\to$ Discussion

Dynamics: Vacuum fields



QCD vacuum not “empty:”
Quantum fluctuations of fields

Fluctuations exist independently of external probes:
“Vacuum structure”

Propagation of quarks/gluons in correlation functions in presence of “vacuum fluctuations."
Need to take them into account

→ Characterize the vacuum fluctuations

→ Compute correlation functions in their presence

Dynamics: Characterizing vacuum fields

Two main approaches:

I) Characterize quantum averages of the vacuum fields — vacuum condensates

$\langle F^{\mu\nu} F_{\mu\nu} \rangle$, $\langle \bar{\psi} \psi \rangle$, higher-dimensional…

Vacuum expectation values of gauge-invariant local operators

II) Characterize form of fields of certain important vacuum fluctuations

A_μ(fluct), $F_{\mu\nu}$(fluct)

Physical nature of vacuum fluctuations: Tunneling processes, topological fields

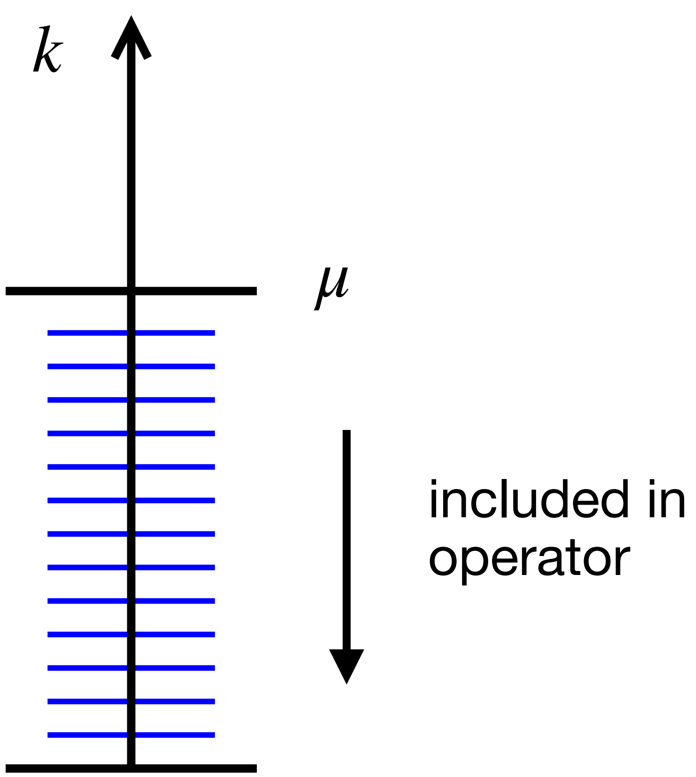
[Lattice QCD: Average over “all” field configurations without distinction.]

Dynamics: Vacuum condensates

Vacuum expectation values of gauge-invariant local operators
$\mathcal{O}(x) \;=\; F^{\mu\nu}F_{\mu\nu}(x),\;\; \bar{\psi}\psi(x),$ higher-dim

Translational invariance: VEV is independent of position of operator $\langle \mathcal{O}(x) \rangle \equiv \langle \mathcal{O} \rangle$.
Constant "density" filling the vacuum

Operators and VEV depend on renormalization scale μ:
Controls which modes of the fields are included in operator and condensate



$\langle \mathcal{O} \rangle(\mu)$ typically $\mu \sim 1$ GeV

Scale dependence governed by renormalization group equation for operator ("anomalous dimension").
Not needed in following

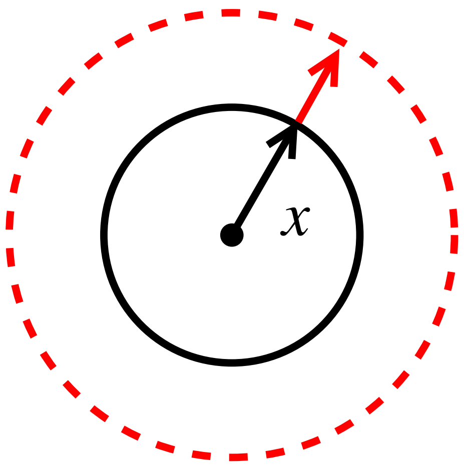
Dynamics: Gluon condensate

$\langle G^{\mu\nu} G_{\mu\nu} \rangle \equiv \langle G^2 \rangle \neq 0$ VEV of scalar density of gluon field

$\langle \frac{\alpha_s}{\pi} G^2 \rangle = (0.36 \pm 0.02 \,\text{GeV})^4$ at $\mu = 1$ GeV

Empirical value (→ following), depends strongly on renormalization scale and definition

Emergence of mass scale in QCD



QCD has no "intrinsic" mass scale!

Classical action invariant under space-time rescaling (dilatation):

$x^\mu \to \lambda x^\mu, \quad A_\mu \to \lambda^{-1} A_\mu$

Mass scale appears only due to quantum fluctuations:
UV cutoff → renormalization → scale in running coupling

Gluon condensate represents "emergent" mass scale:
Scalar density in the vacuum

$T^\mu{}_\mu = \frac{\beta(g)}{2g} G^{\mu\nu} G_{\mu\nu}$ trace of energy momentum tensor ("trace anonaly")

Dynamics: Quark condensate

$$\sum_{f=u,d} \langle 0 | \bar{\psi}\psi | 0 \rangle = \langle 0 | \bar{u}u + \bar{d}d | 0 \rangle \neq 0$$

Scalar density of quark/antiquark field

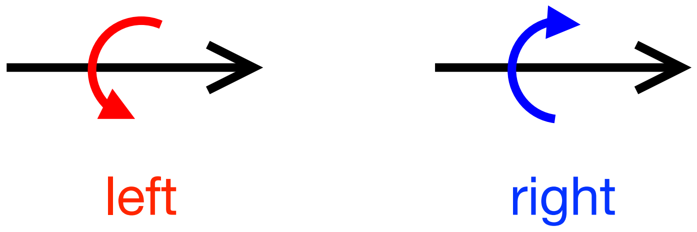
$$\approx 2 \times (0.22 \pm 0.02\,\text{GeV})^3$$

Empirical value at $\mu = 1$ GeV

Chiral symmetry breaking in QCD

$$\psi_{L,R}(x) \equiv \frac{1 \pm \gamma_5}{2} \psi(x)$$

Left/right-handed components of quark field (chirality)



In QCD action: L and R components of field decouple (if quark masses $m = 0$)

$$S = S[\psi_L] + S[\psi_R]$$

In ground state (vacuum): L and R components are “locked”

$$\langle 0 | \bar{\psi}\psi | 0 \rangle = \langle 0 | \bar{\psi}_L \psi_R + \bar{\psi}_R \psi_L | 0 \rangle \neq 0$$

Dynamics: Quark condensate

Dynamics: $S[\psi_L] + S[\psi_R]$ $\quad SU(2)_L \times SU(2)_R$ independent flavor rotations of L and R components

$\downarrow$

Ground state: $\langle 0 | \bar{\psi}_L \psi_R + \bar{\psi}_R \psi_L | 0 \rangle \neq 0$ $\quad SU(2)_{L=R}$ identical flavor rotations of L and R components

Spontaneous symmetry breaking: Symmetry of ground states "less" than symmetry of dynamics

Examples in condensed matter physics: Spontaneous magnetization in spin systems

Theory: Order parameter, massless excitations — Goldstone bosons

Symmetry of ground state determines symmetry of emergent effective dynamics:
Hadron spectrum, hadron interactions

Summary: Dynamics

The dynamics governing QCD correlation functions changes with distance/momentum

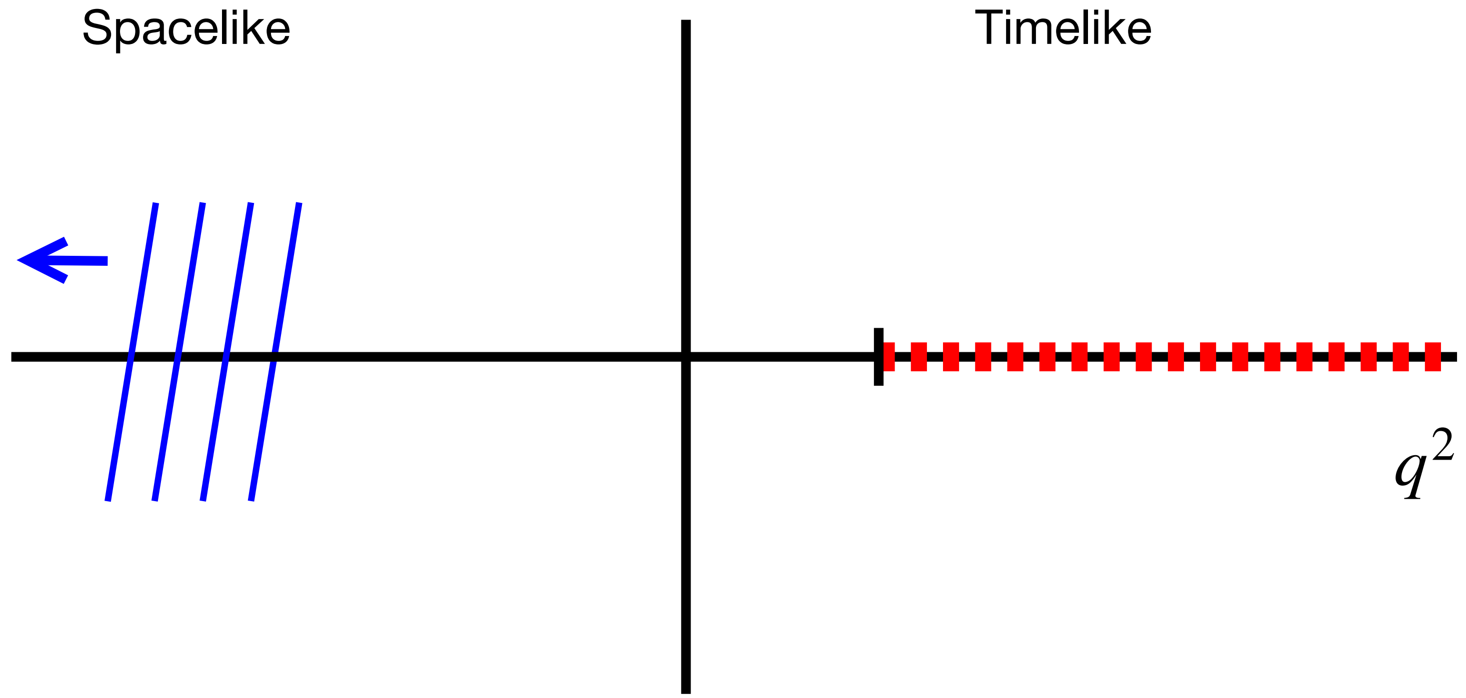
At momenta $|-q^2| \gg \mu^2_{\text{nonpert}}$ correlation functions can be computed using perturbation theory

At momenta $|-q^2| \sim \mu^2_{\text{nonpert}}$ the correlation functions functions are essentially modified by the coupling to vacuum fluctuations of the fields

Vacuum condensate of gluon field represents dynamical mass scale in QCD arising from quantum fluctuations

Vacuum condensate of quark-antiquark fields connected with spontaneous breaking of chiral symmetry in QCD

Method: Computing correlation functions



Compute correlation functions at spacelike momenta $q^2 < 0$ in presence of vacuum fields

→ Compute down to lower momenta $|-q^2| \lesssim 1$ GeV^2

→ Extract information on hadrons

Idea: Perform asymptotic expansion of correlation function for large spacelike momenta $q^2 < 0$

$$\Pi(q^2) = \underbrace{[\text{perturbative}]}_{\alpha_s \sim \frac{1}{\log(-q^2/\Lambda^2_{\text{QCD}})}} + \frac{A_4}{(-q^2)^2} + \frac{A_6}{(-q^2)^3} + \ldots$$

Language: “Power corrections”

Perturbative part: Logarithmic q^2 dependence

A_4: Dimension-4, proportional to dimension-4 vacuum condensates $\langle 0 | G^2 | 0 \rangle$, $\langle 0 | m\bar{\psi}\psi | 0 \rangle$

A_6: Dimension-6, proportional to dimension-6 vacuum condensates

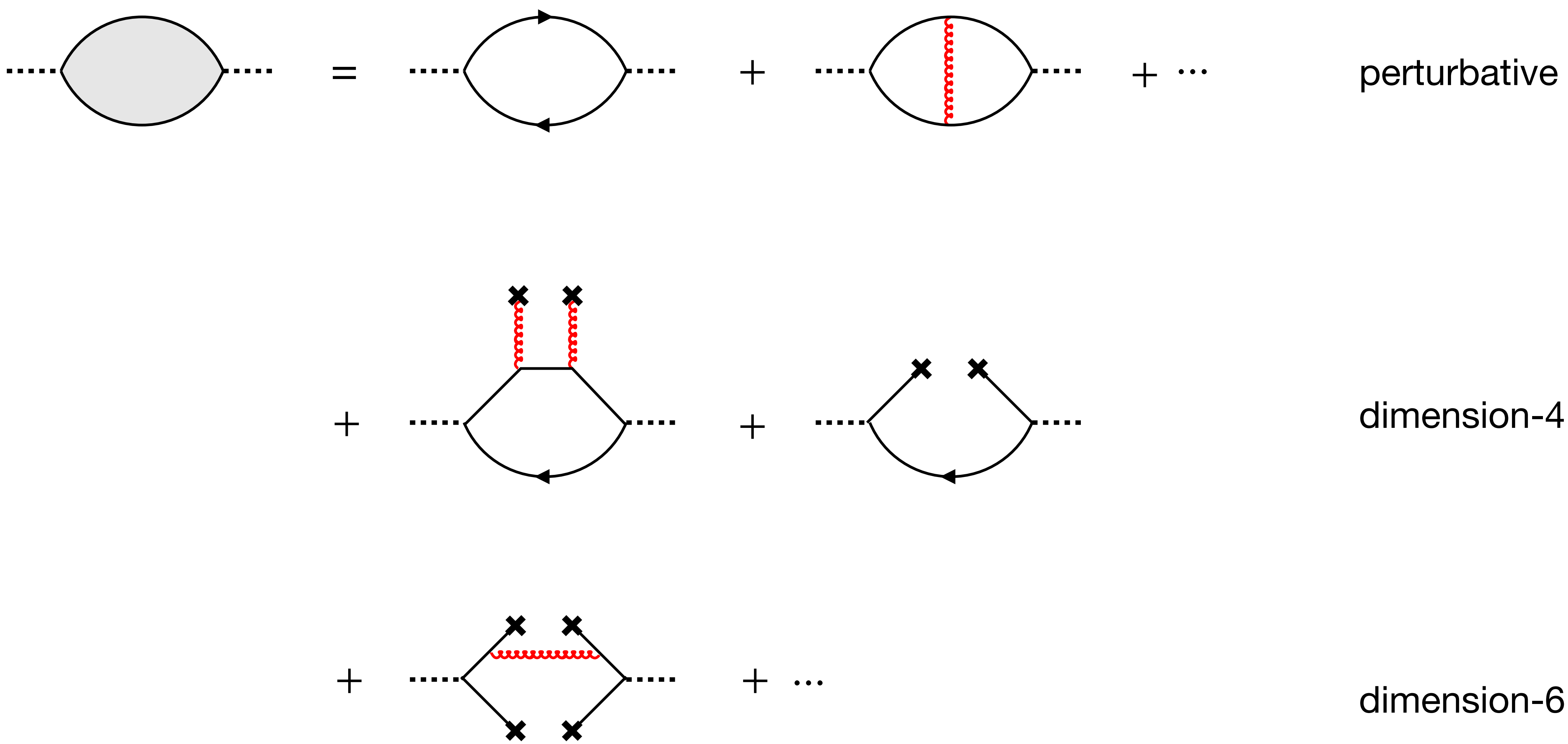
⋮

Expansion in powers of $1/(-q^2)$ = Expansion in dimension of vacuum condensates

Systematic approach. Combines perturbative and nonperturbative dynamics

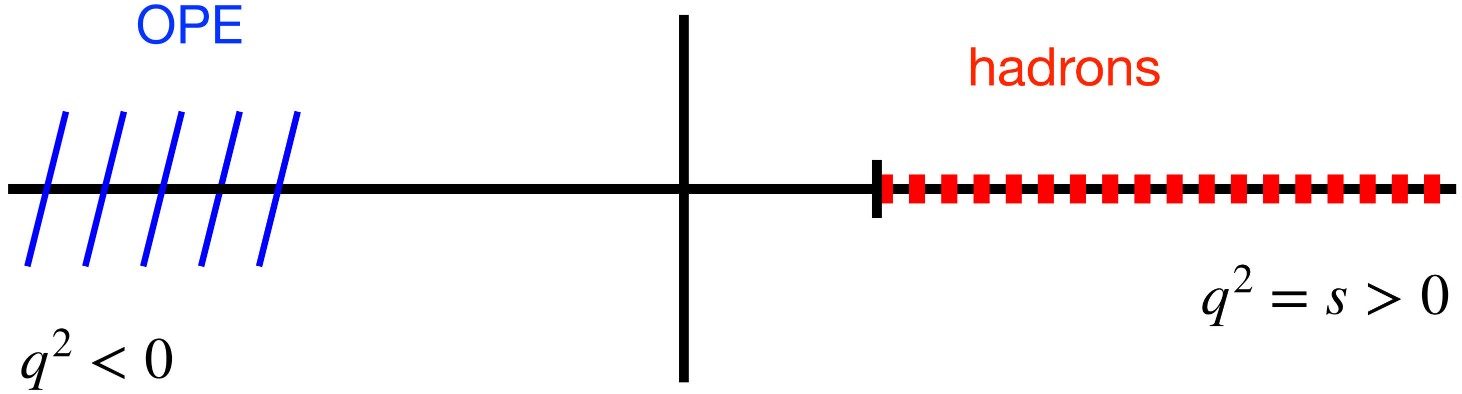
Method: Operator product expansion

Operator product in correlation function expanded in insertions of background field



$$T J_\Gamma J_\Gamma = C_{\text{pert}}(q)\,\hat{1} + \sum_{d=4,6,\ldots} C_d(q)\,\frac{\hat{O}_d}{(q^2)^{d/2}}$$

Method: Matching QCD and hadrons



Techniques for spacelike-timelike comparison

$$\Pi(q^2) = \int_{\text{thr}}^{\infty} ds \, \frac{\rho(s)}{s - q^2 - i\epsilon}$$

spectral representation

$$\frac{1}{n!}\left(\frac{d}{dq^2}\right)^n \Pi(q^2)\Big|_{q_0^2} = \int_{\text{thr}}^{\infty} ds \, \frac{\rho(s)}{(s - q_0^2)^{n+1}}$$

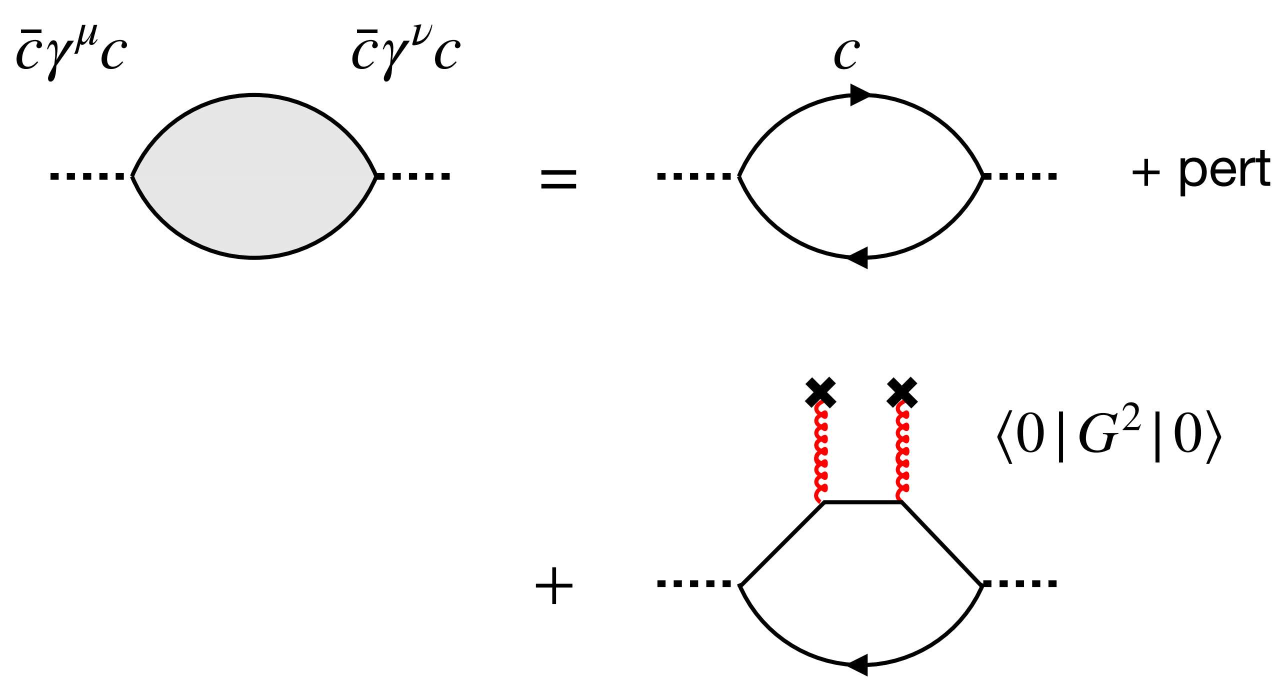
differentiation suppresses high masses in spectral representation (moments)

Alt: Borel transform:

$n \to \infty, \; -q_0^2/n = M^2$ fixed $\quad \to \int_{\text{thr}}^{\infty} ds \, e^{-s/M^2} \rho(s)$

suppresses high masses exponentially

Method: Charmed vector meson J/ψ



Correlation function of charm quark vector current $\bar{c}\gamma^\mu c$

Charm quark couples only to gluon condensate $\langle 0|G^2|0\rangle$ at LO

Moment sum rules predict mass (= binding energy) of charmed vector meson

Input to be determined: Charm quark mass, gluon condensate

Simplest example of “QCD sum rules” method

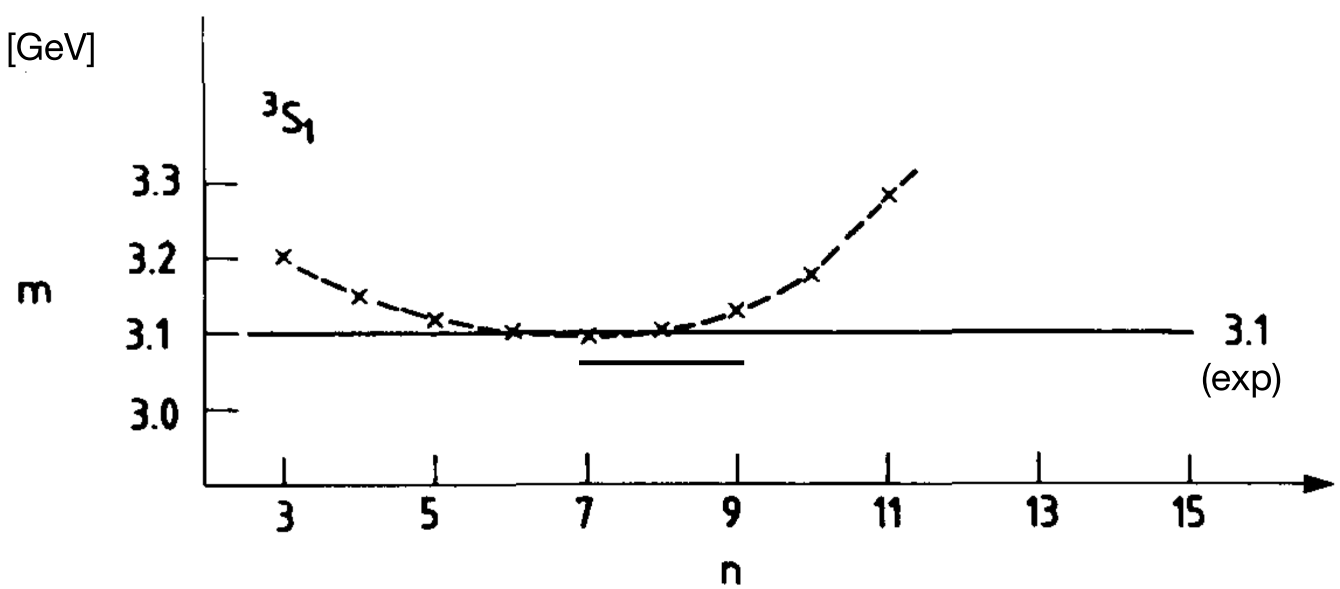


Fig: Reinders, Rubinstein, Yazaki 1985

Extensions

Higher-order perturbative, higher dimension OPE

Charmed pseudoscalar meson η_c, excited states

Method: ρ meson

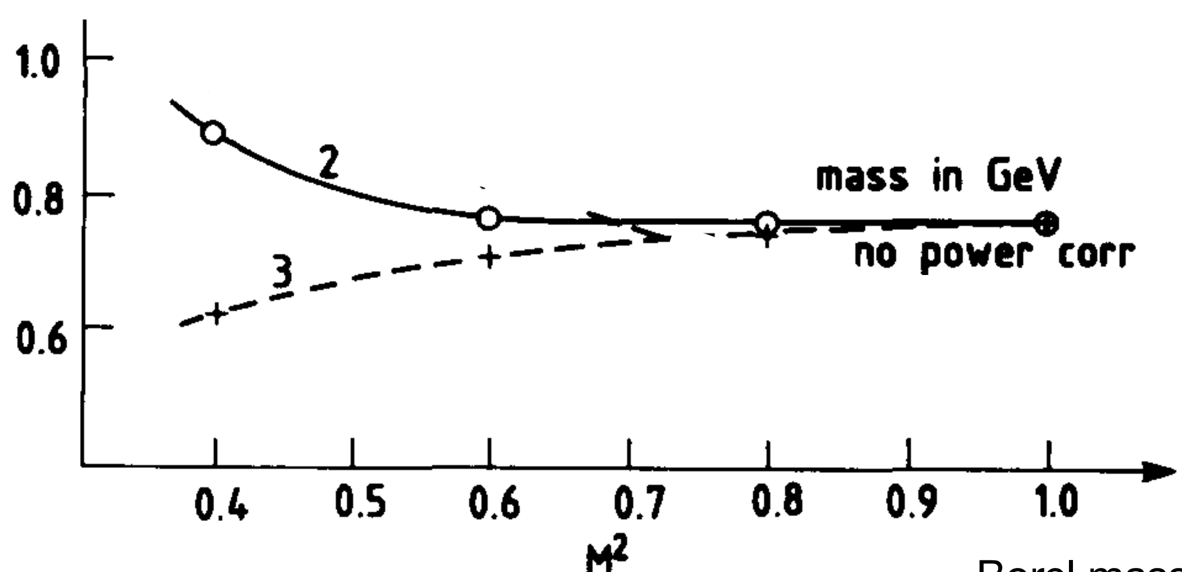
Correlation function of isovector vector current $J^\mu = \bar{u}\gamma^\mu u - \bar{d}\gamma^\mu d$

Operator product expansion involves gluon and light quark condensates

Spectral density parametrized as ρ meson pole + continuum: $\rho(s) = f_\rho^2\,\delta(s - m_\rho^2) + \text{cont.}$

Spacelike-timelike comparison allows to determine ρ meson mass and coupling

m_ρ [GeV]



Borel mass

Fig: Reinders, Rubinstein, Yazaki 1985

Here: Borel transform technique

Window of stability

Method: Extensive applications

Light meson masses and couplings

Light baryons

Heavy mesons (quarkonia), including exotics (tetraquarks)

3-point functions: Meson/baryon form factors

Finite temperature and density: Condensates depend on temperature/density

Couplings to external fields, e.g. chromomagnetic fields

[References to be provided]

Parameters

Condensates determined empirically in simple correlation functions, used in calculations of more complex functions

Condensates estimated using other methods: Semiclassical methods (instantons), lattice QCD, model assumptions

Method: Limitations

Expansion in dimension of condensates poorly convergent in some channels:
Reason understood - instanton effects, discussed later

Extraction of hadron properties from spacelike function limited by “inverse problem” difficulties

Summary: Sum rule method

Correlation function computed beyond perturbative regime using operator product expansion: Systematic parametric expansion

Effect of vacuum fields included through vacuum condensates of increasing dimension

Hadron information extracted by applying functional transforms “filtering” spectral density, various method

Successful description of vector correlation function and heavy quarkonia

Extensive applications