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Introduction to Factorization Theorems

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The Infrared problem

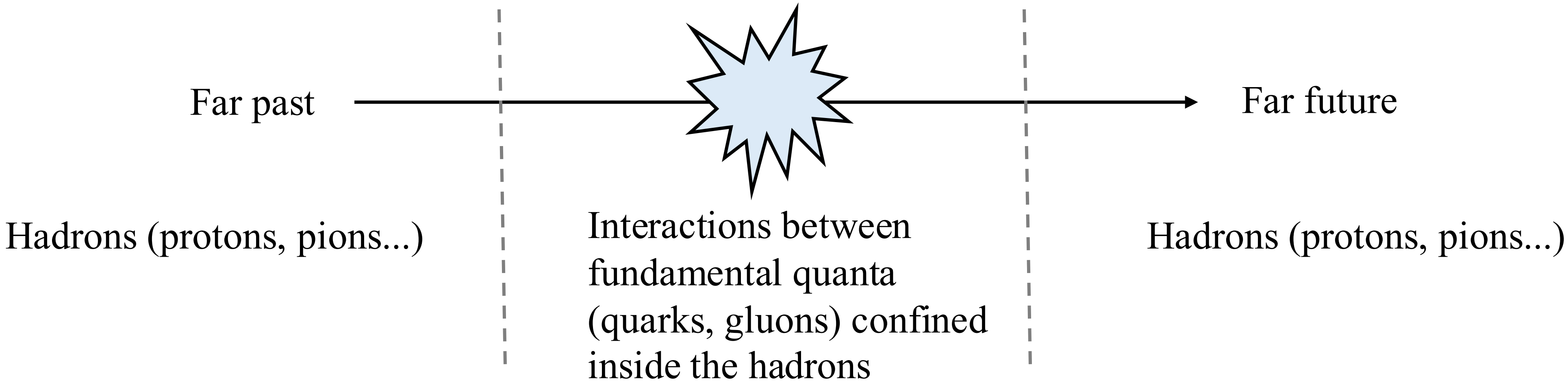
How make reliable predictions in a theory with asymptotic freedom?

A complicated picture

QCD operators are written in terms of quarks and gluon fields, but the observables depend only on color neutral hadronic states

$$\mathcal{L}_{\mathrm{QCD}} = -\frac{1}{4}G^2(x) + \overline{\psi}\,(i\gamma^\mu D_\mu - m)\,\psi$$

What we have:

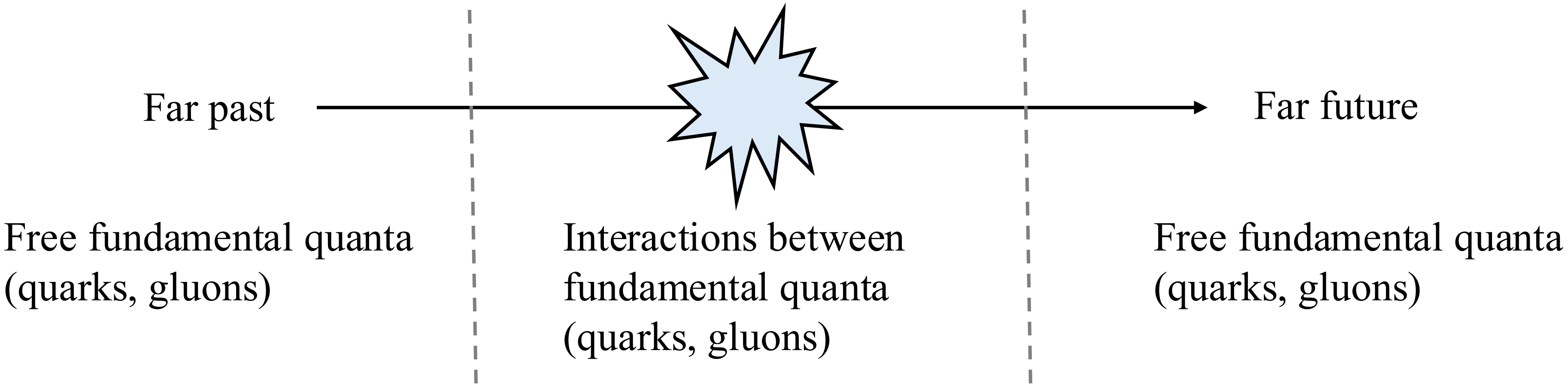


A complicated picture

QCD operators are written in terms of quarks and gluon fields, but the observables depend only on color neutral hadronic states

$$\mathcal{L}_{\text{QCD}} = -\frac{1}{4}G^2(x) + \overline{\psi}\,(i\gamma^\mu D_\mu - m)\,\psi$$

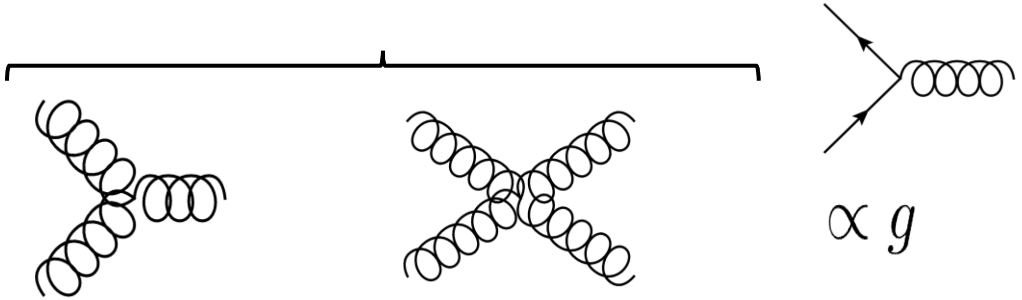
What we wish we had:



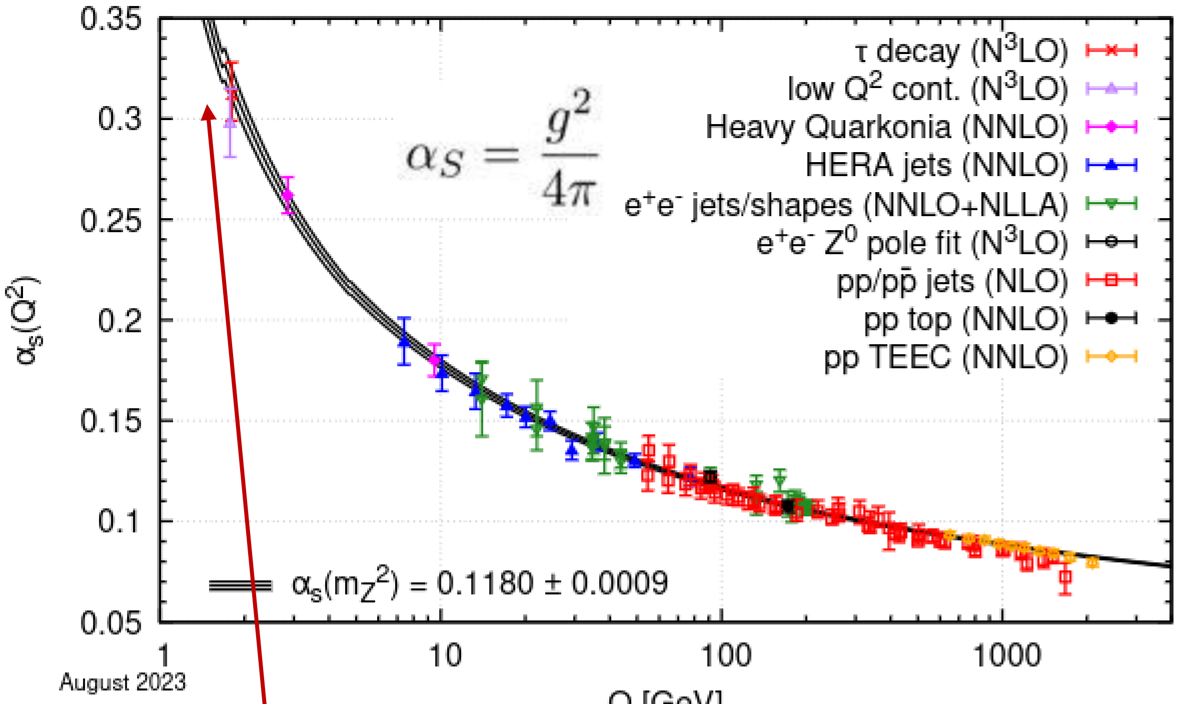
Such that we can use perturbative methods.

A complicated picture

$$\mathcal{L}_{\rm QCD} = -\frac{1}{4}G^2(x) + \overline{\psi}\,(i\gamma^\mu D_\mu - m)\,\psi$$



$\propto g$



PDG (2023)

The strong coupling is not constant, but it *evolves* with energy:

$$\frac{d\alpha_S}{d\log\mu^2} = \beta(\alpha_S) \;\rightarrow\; \int_{\alpha_S(Q_0)}^{\alpha_S(Q)} \frac{da}{\beta(a)} = \log\left(\frac{Q^2}{Q_0^2}\right)$$

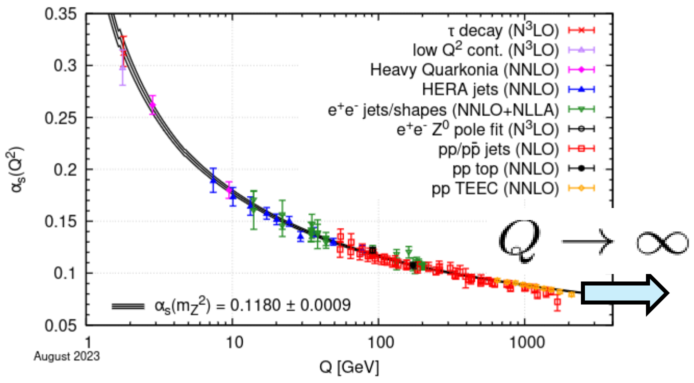
$$\alpha_S(Q) = \frac{\alpha_S(Q_0)}{1-\alpha_S(Q_0)\beta_0\log\left(\frac{Q_0^2}{Q^2}\right)} + \dots$$

Landau pole

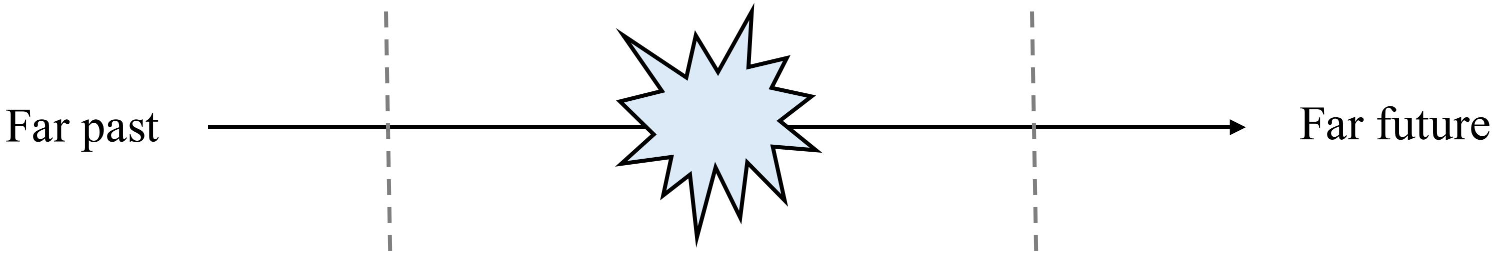
$$Q = Q_0\, e^{-\frac{1}{2\beta_0\alpha_S(Q_0)}}$$

Failure of perturbative methods

The high energy limit



It seems that our idealized scenario could only happen in the high energy limit $Q \to \infty$



Free fundamental quanta (quarks, gluons)

Interactions between fundamental quanta (quarks, gluons)

Free fundamental quanta (quarks, gluons)

In this limit, the theory is almost free, and the operators/observables can be computed *analytically* expanding them as a power series:

$$\mathcal{O}(Q,\dots) = \underbrace{\underbrace{\mathcal{O}^{[0]}(Q,\dots)}_{\text{LO}} + \alpha_S(Q)\mathcal{O}^{[1]}(Q,\dots)}_{\text{NLO}} + \dots$$

$$\frac{\text{NLO} - \text{LO}}{\text{LO}} \ll 1$$ Fixed order is reliable

The high energy limit

But where did all the hadron complexity go?

The idealized scenario only appears *exactly* at the limit $Q = \infty$. This is clearly never reached!
Therefore, in general:

$$\mathcal{O}(Q, \dots) = \mathcal{O}^{[0]}(Q, \dots) + \alpha_S(Q)\mathcal{O}^{[1]}(Q, \dots) + \dots \quad ?$$

Hadronic features

What do we do? The asymptotic freedom gives us a handle:

High energy limit		Massless limit
$Q \to \infty$	$\Longleftrightarrow$	$m \to 0$

Therefore, we can *safely* compute all those operators/observables that are well-defined in the massless limit.

Infrared safety

What masses? In general, what we mean is: $\text{masses} \approx m_{\text{quarks}};\ \Lambda_{QCD};\ m_{\text{hadrons}} \ll Q$

Any quantity which does not depend on masses in the high energy limit is *insensitive* to the infrared details of the theory ⟷ Infrared safety

$$\mathcal{O}(Q,\ldots) = \mathcal{O}^{[0]}(Q,\ldots) + \alpha_S(Q)\mathcal{O}^{[1]}(Q,\ldots) + \ldots$$

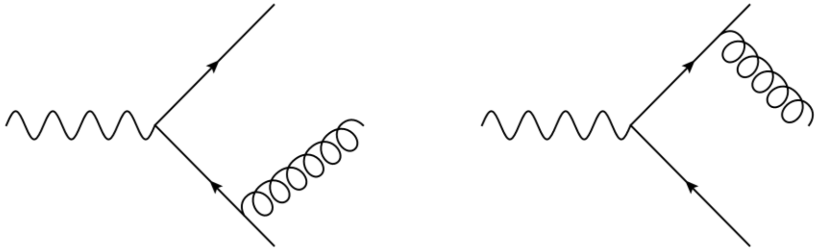
Only quarks and gluons ✓

The problem is that there are very few quantities that are known to be infrared safe!

An infrared safe example: $e^+e^- \to \mathrm{hadrons}$

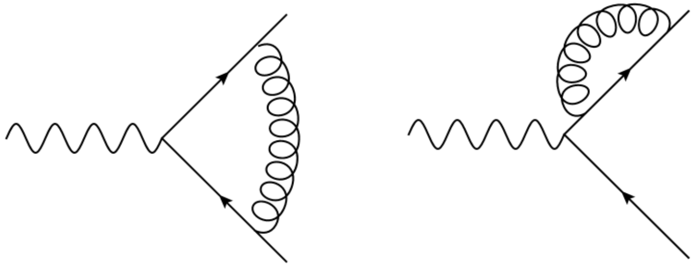
Set all fermion masses to zero. Then the first non-trivial terms are:

- Real gluon emission



$$W_{q\bar{q}g} = W^{[0]} \frac{\alpha_S}{4\pi} C_F (4\pi e^{-\gamma_E})^\varepsilon \Big[\frac{4}{\varepsilon^2} + \frac{1}{\varepsilon}\big(6 - 4\log\frac{Q^2}{\mu^2}\big) + 2\log^2\frac{Q^2}{\mu^2} - 6\log\frac{Q^2}{\mu^2} + 19 - \frac{7\pi^2}{3} \Big]$$

- Virtual gluon emission



$$W_{q\bar{q}} = W^{[0]} \frac{\alpha_S}{4\pi} C_F (4\pi e^{-\gamma_E})^\varepsilon \Big[-\frac{4}{\varepsilon^2} - \frac{1}{\varepsilon}\big(+6 - 4\log\frac{Q^2}{\mu^2}\big) - 2\log^2\frac{Q^2}{\mu^2} + 6\log\frac{Q^2}{\mu^2} - 16 + \frac{7\pi^2}{3} \Big]$$

All poles cancels out. This is the power of the infrared safety!

Infrared safe and inclusivity

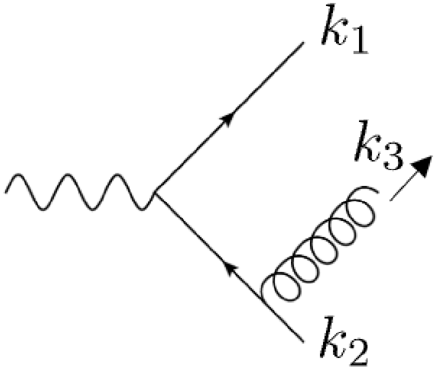
Why this magic?

Infrared safety is closely related to *inclusivity*: the more details we probe, the less infrared safe the observable becomes.

Consider again the real gluon emission contribution. It actually comes from the expression:

$$W_{q\bar{q}g} = W^{[0]} \frac{\alpha_S C_F}{4\pi\Gamma(1-\varepsilon)} \left(\frac{4\pi\mu^2}{Q^2}\right)^{\varepsilon} \int_0^1 dy_1 \int_0^{1-y_1} dy_2 \, (y_1 y_2 y_3)^{-\varepsilon} \frac{4(y_3+y_1)+2(1-\varepsilon)(y_1^2+y_2^2)}{y_1\, y_2}$$

Where:



$$y_i = 1 - \frac{2|\vec{k}_i|^2}{Q}$$

$$\begin{cases} y_1 = 0 \leftrightarrow \text{gluon} \parallel \text{antiquark} \\ y_2 = 0 \leftrightarrow \text{gluon} \parallel \text{quark} \\ y_1 = y_2 = 0 \leftrightarrow \text{gluon soft} \end{cases}$$

Infrared safe and inclusivity

Suppose that we *measure* the fractional energy of the quark in the center of mass frame: $z = \frac{E_{\text{quark}}}{Q/2}$

The variables can be written as:

$$\begin{cases} y_1 = 1 - z \\ y_2 = z\alpha \\ y_3 = z(1-\alpha) \end{cases}$$

$y_1 = 1 - z$ ⟶ The integral over y_1 disappears $\delta(y_1 - (1-z))$

$$W_{q\bar{q}g} = W^{[0]} \frac{\alpha_S C_F}{4\pi\Gamma(1-\varepsilon)} \left(\frac{4\pi\mu^2}{Q^2}\right)^{\varepsilon} z^{1-2\varepsilon}(1-z)^{-\varepsilon} \int_0^z d\alpha \alpha^{-\varepsilon}(1-\alpha)^{-1-\varepsilon}$$
$$\times \left[4(z(1-\alpha) + \varepsilon z(1-z)\alpha) + 2(1-\varepsilon)((1-z)^2 + z^2\alpha^2)\right]$$

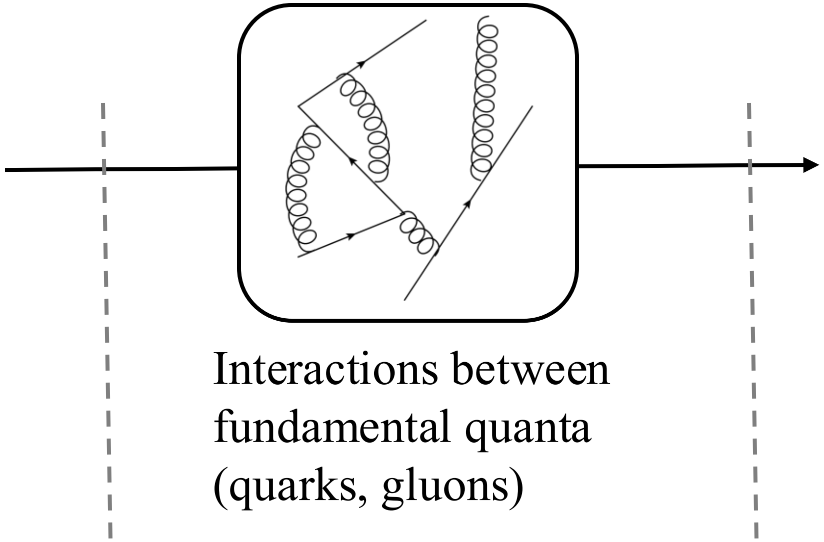
Once solved, this only has one single pole (the gluon can only be collinear to the quark).
The cancellation of the divergences with the virtual term does not hold anymore!

How to treat quantities that are not IR safe

Clearly, we cannot proceed *entirely* perturbatively.

However, if we could *isolate* and *classify* all the infrared divergences, we would still be able to compute all the remaining contributions perturbatively.
Where do these divergences come from?

Let's classify the internal states according to their **virtuality** k^2



HARD

High virtuality $k^2 \approx Q^2$

These states propagate across short distances (UV).
Purely quantum contributions.

Low virtuality $k^2 \approx \lambda^2 \ll Q^2$

These states propagate across long distances (IR).
Semi-classical contributions.

SOFT

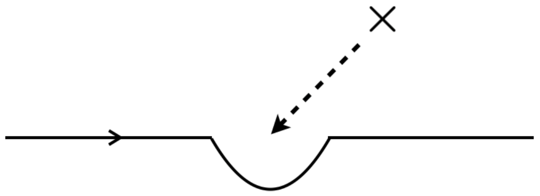
COLLINEAR

How to treat quantities that are not IR safe

What is the difference?
Two cases:

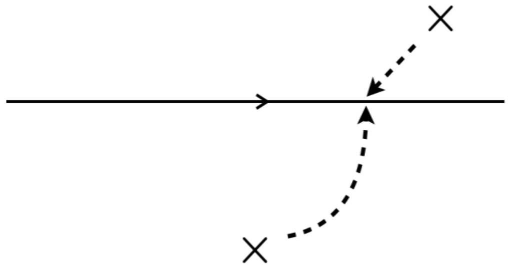
$$\int dk \cdots \frac{1}{k^2 - m^2} \cdots \frac{1}{(P-k)^2 - m^2} \cdots$$

1) Single pole on integration path



HARD

2) Pinch and endpoint singularity

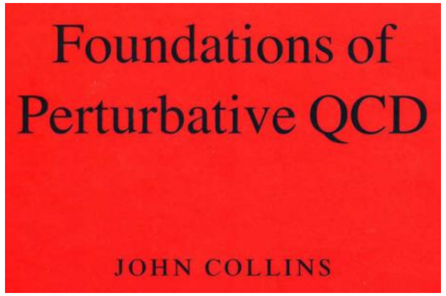


SOFT and COLLINEAR

The problem has been reformulated as the task of cataloguing all pinch singularities.

Factorization

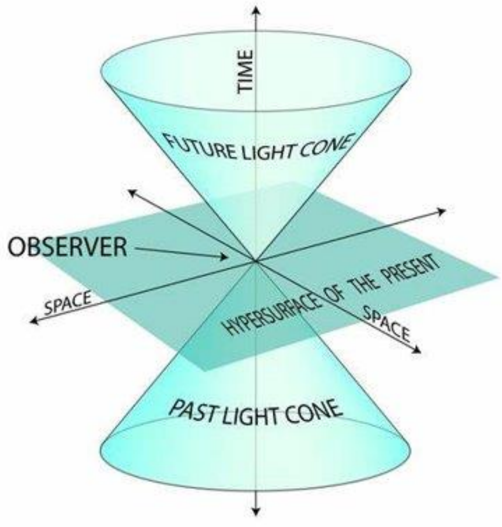
How we rescue our predictive power



Finding pinches: Landau criterion

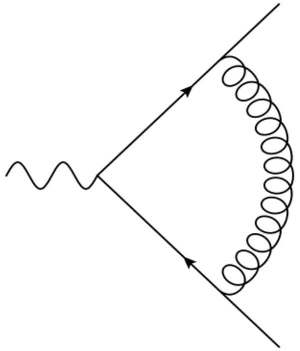
Pinch singularities appear when internal states go simultaneously on-shell.
This happens on certain surfaces (Pinch Singular Surfaces – PSSs) in loop momentum space.

In general, the PSSs are found solving sets of coupled equations: the Landau equations.
However, in the massless limit, these task heavily simplifies into the following simple criterion:



> The PSSs are where the on-shell propagators and momenta correspond to *classically allowed* scattering processes treated in coordinate space

The PSSs in the massless limit constitute the skeleton of the spacetime structure of a certain hadron process.

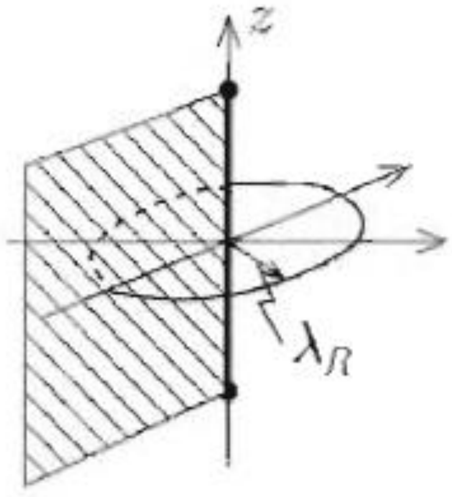


⇒ REDUCED GRAPHS

Weighing pinches: Libby-Sterman power counting

Not all PSSs count the same and in fact each of them has a certain weight.

$$Q^p \left(\frac{\lambda}{Q}\right)^{p_C+2p_S}$$



Classical power
(dimensional analysis)

$$p = 4 - \#\text{ext. lines}$$

Quantum corrections

$$p_C = \#(\text{CH})^* - \#\text{ext. lines}$$

$$p_S = \#(0,1;\text{SH}) + \frac{3}{2}\#(1/2;\text{SH}) + \frac{1}{2}\#(0,1/2;\text{SC})$$

Those PSSs that leave the classical power unchanged define the LEADING POWER.

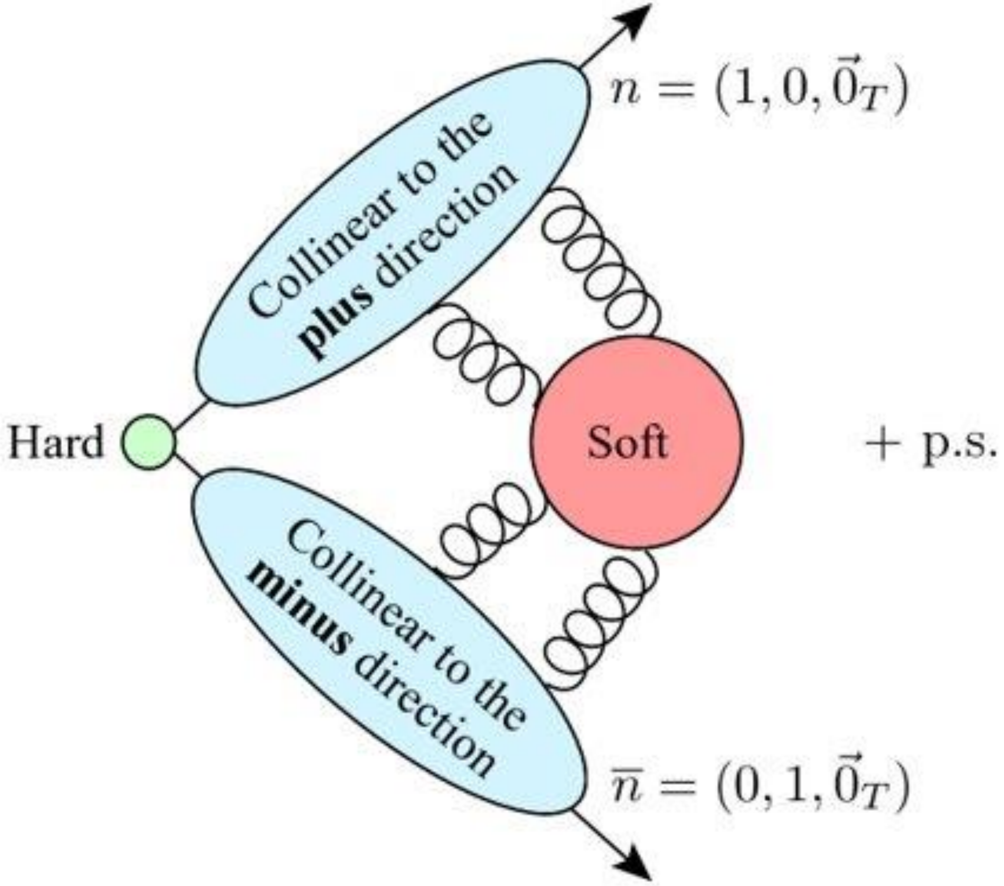
Weighing pinches: Libby-Sterman power counting

There are simple *rules of thumb* to determine whether a PSS is leading.
We get a power suppression when we do any of the following:

1. Attach extra collinear lines to the hard subgraph, except for scalar polarized gluons.

2. Attach any soft line to the hard subgraph.

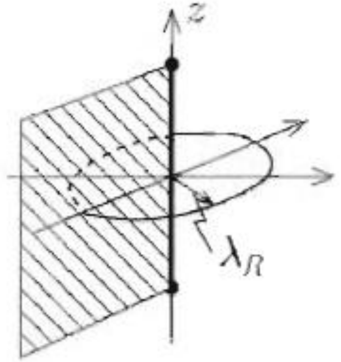
3. Attach the soft subgraph to the collinear subgraph by anything but gluons.



Disentangling pinches: factorization

The factorization procedure is essentially based onto two cornerstones:

1) Kinematic approximations



Taylor expansion of the Feynman graphs around the PSSs in the limit

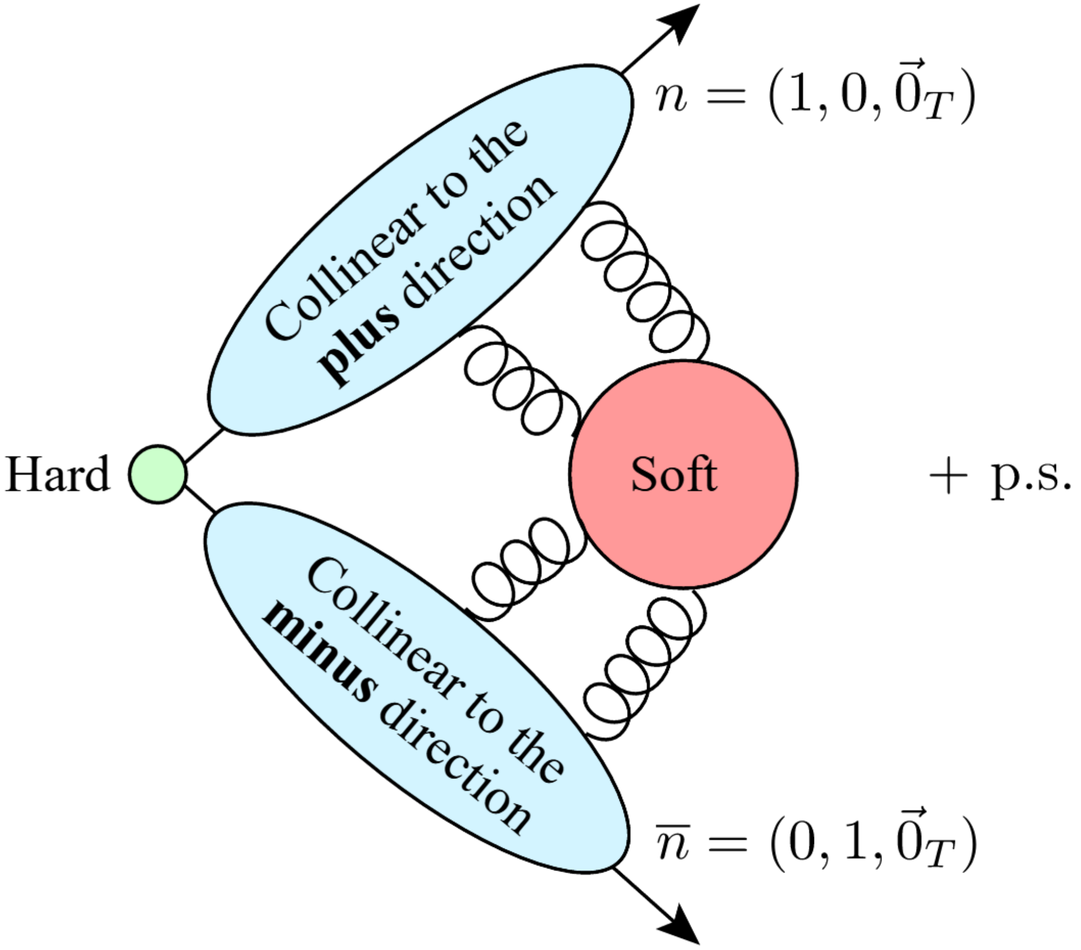
$$\lambda \to 0$$

$$\begin{cases} k_H \approx (Q, Q, Q) \\ k_C \approx (Q, \frac{\lambda^2}{Q}, \lambda) \\ k_S \approx (\frac{\lambda^2}{Q}, \frac{\lambda^2}{Q}, \frac{\lambda^2}{Q}) \end{cases}$$

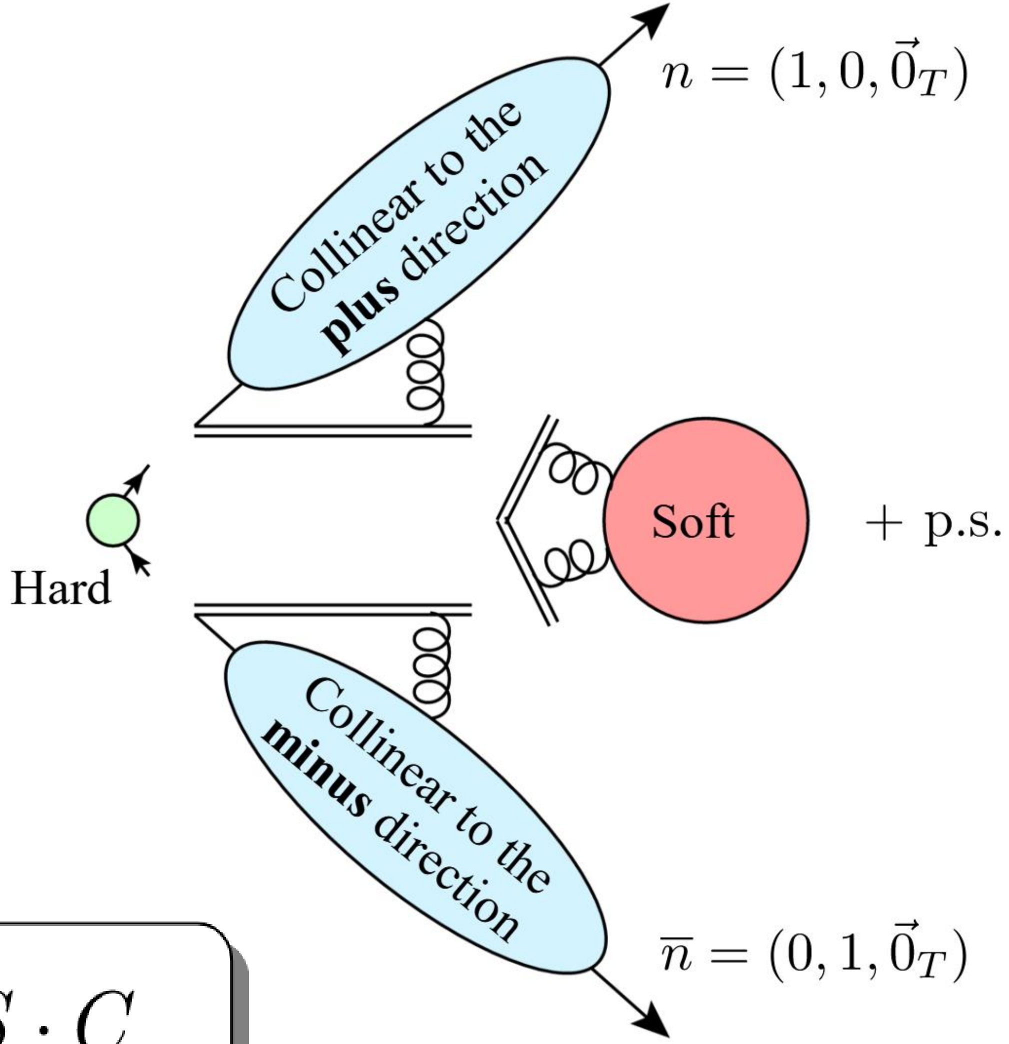
2) Exploit of Ward identities (gauge invariance)

$$W_{x\to y;\gamma} = \mathcal{P}\exp\left\{-ig\int_\gamma dx^\mu A_\mu(x)\right\} = \mathcal{P}\exp\left\{-i\frac{g}{2}\int_\Sigma d\sigma^{\mu\nu}G_{\mu\nu}(x)\right\}$$

Disentangling pinches: factorization



- Kinematic approximations
- Ward identities
- Subtractions



$$H \cdot C_+ \cdot S \cdot C_-$$

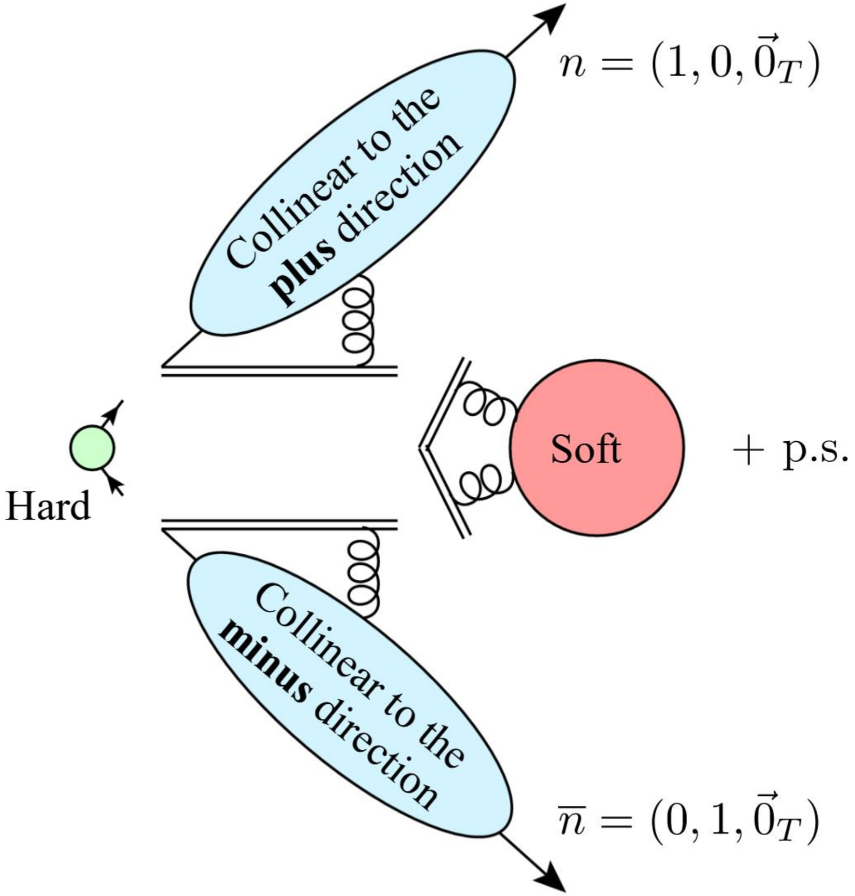
Universality

Soft and Collinear operators cannot be evaluated perturbatively, since they are sensitive to the infrared limit of the theory (they are not infrared safe quantities).
However, they are not sensitive to the details of the specific process, which instead are encoded into the hard factor.

Thus, why factorization allows us to make predictions?

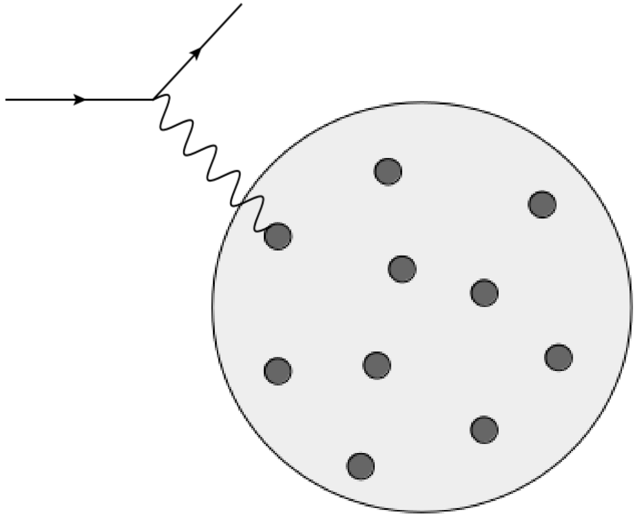
The answer lies in **universality**.

We settle for isolating the non-calculable parts into operators that are independent of the specific process.
Once these operators are determined from one observable (necessarily through non-perturbative methods), they can be reused in other processes, thus rescuing the predictive power of QCD.



Back to Deep Inelastic Scattering

Feynman's parton model



The scattering off the proton is seen as an incoherent (i.e. independent) sum of elastic scattering processes.

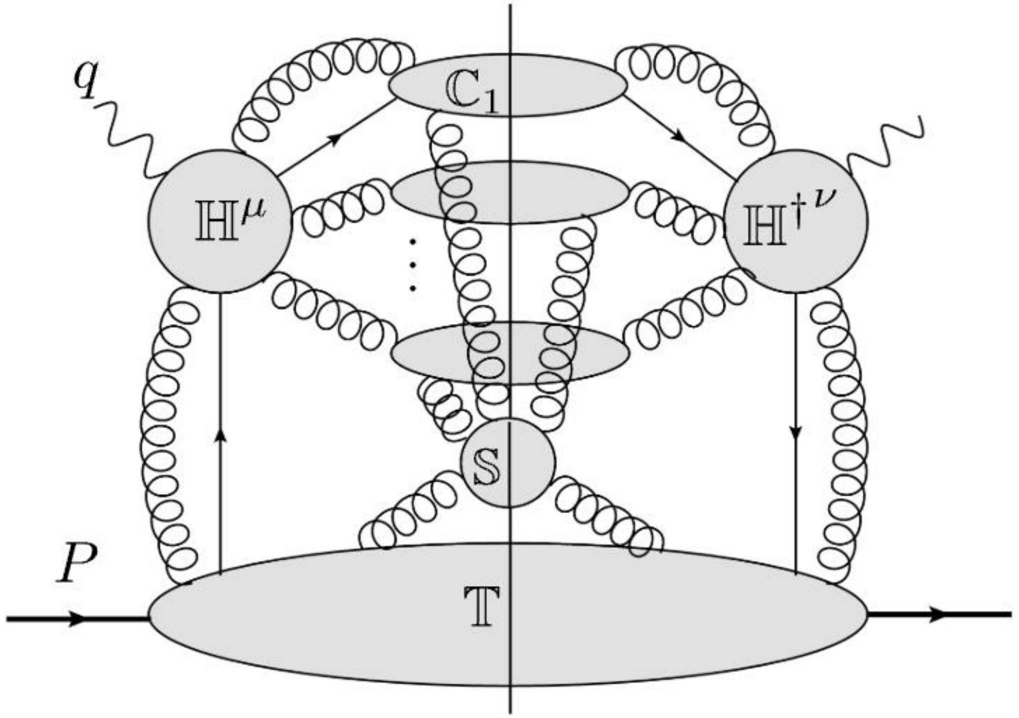
Sum over all partons inside the proton

$$\frac{d\sigma}{dQ^2\,dx} = \int_x^1 \frac{d\rho}{\rho} \sum_j C_j(\rho, Q^2)\, f_j(x/\rho)$$

Electromagnetic interaction with a point-like parton

Probability density of finding a point-like parton inside the proton

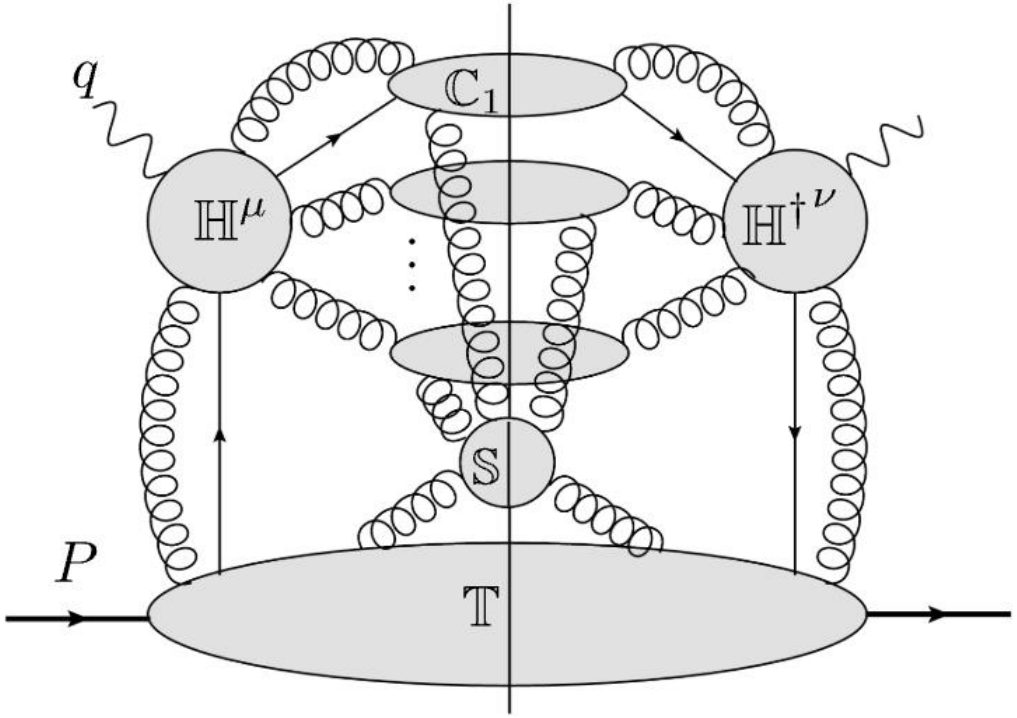
Back to Deep Inelastic Scattering



Many leading terms:

- Hard vertex (photon – quark interaction)
- Jets in the final state
- Soft radiation
- Target contribution

Back to Deep Inelastic Scattering



Many leading terms:

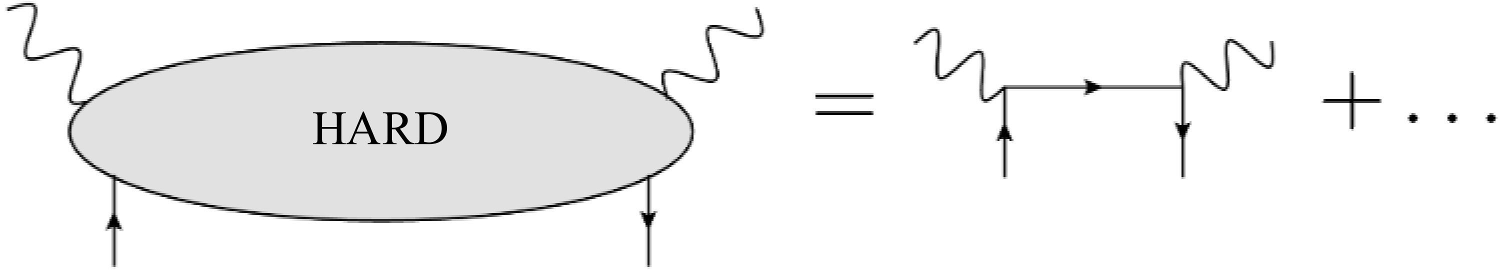
- Hard vertex (photon – quark interaction)
- Jets in the final state
- Soft radiation
- Target contribution

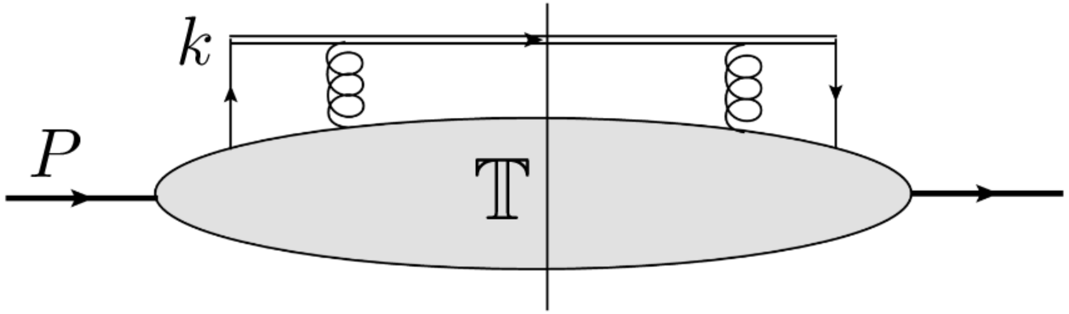
The whole soft correlation term is trivial.
This is due to DIS being "enough" inclusive

Luckily, we have a huge simplification in this case!

Back to Deep Inelastic Scattering

After factorization:



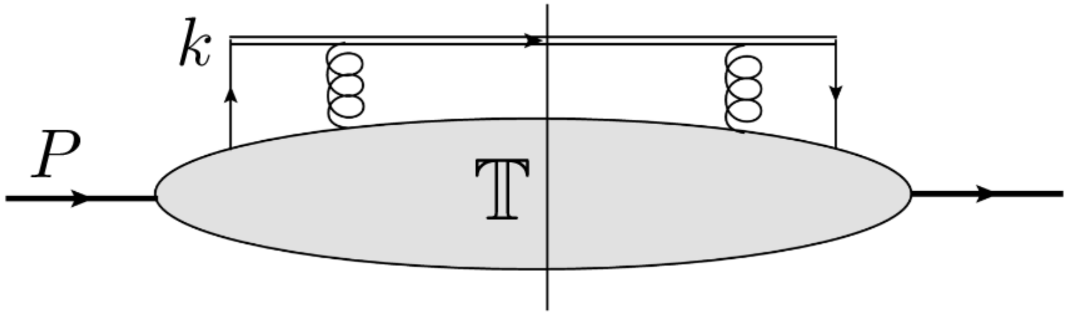


Which looks a lot like the parton model!

$$\frac{d\sigma}{dQ^2 dx} = \sum_j \int_x^1 \frac{d\rho}{\rho} C_{j\,j'}\left(\rho, \alpha_S(\mu), Q/\mu\right) f_{j'}(x/\rho, \mu)$$

Parton Density Functions (PDFs)

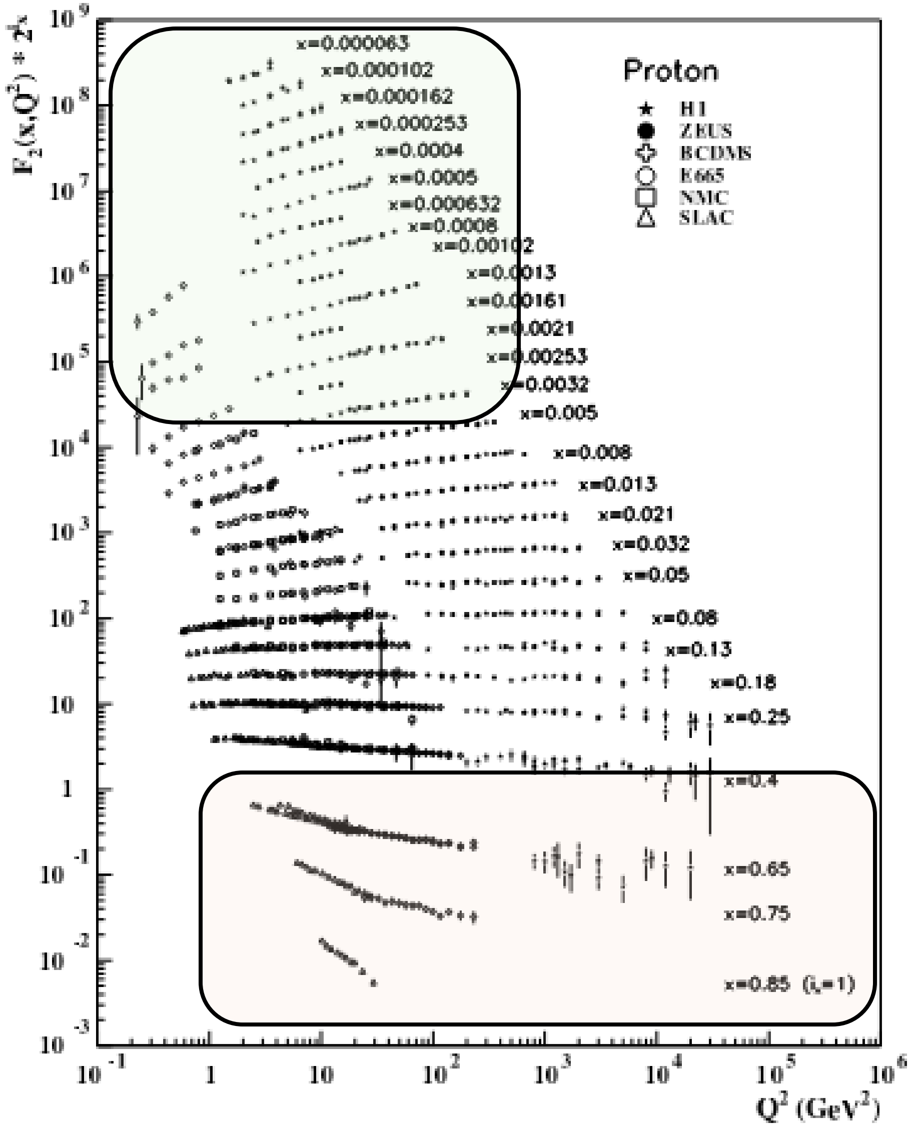
"Probability" of finding parton j (quark, antiquark, gluon) with momentum fraction x inside the Proton:



$$f\,(x;\mu) = Z_{\mathrm{UV}}(x;\mu) \int \frac{d\sigma^-}{2\pi} e^{-i\xi P^+ \sigma^-}$$

$$\times \, \mathrm{Tr}_{\mathrm{C,D}} \langle P |\, T\, \overline{\psi}_j^{(0)}(\sigma) W[0,\sigma] \frac{\gamma^+}{2} \psi_j^{(0)}(0) | P \rangle$$

Bjorken scaling



The violation to the Bjorken scaling is predicted in QCD through the renormalization of the PDFs

$$\frac{df_j(x,\mu)}{d\log\mu^2} = \int_x^1 \frac{d\rho}{\rho} P_{jj'}(\rho, \alpha_S(\mu))\, f_{j'}(x/\rho, \mu)$$

Which is the famous DGLAP evolution.

...this was just the tip of the iceberg!

- Subtractions and Rapidity divergences
- Processes sensitive to soft radiation
- Glauber gluons and violation to factorization
- Spin and polarization
- Sum rules
- Global analyses
- ...