Gravitational form factors on the lattice

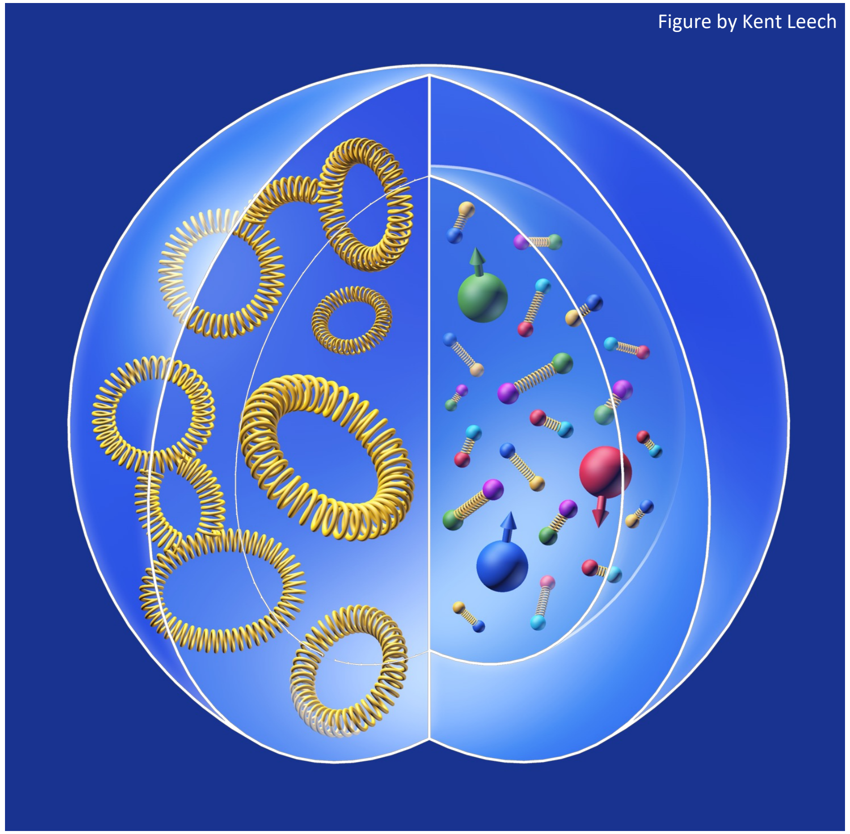


Figure by Kent Leech

2025 Summer Hall A/C
Collaboration Meeting
JLab

Jun 17, 2025

Dan Hackett (FNAL)

Patrick Oare (BNL)

Dimitra Pefkou (Berkeley)

Phiala Shanahan (MIT)

Outline

Gravitational structure of the nucleon

- Gravitational form factors (GFFs)?
- Why are GFFs interesting?

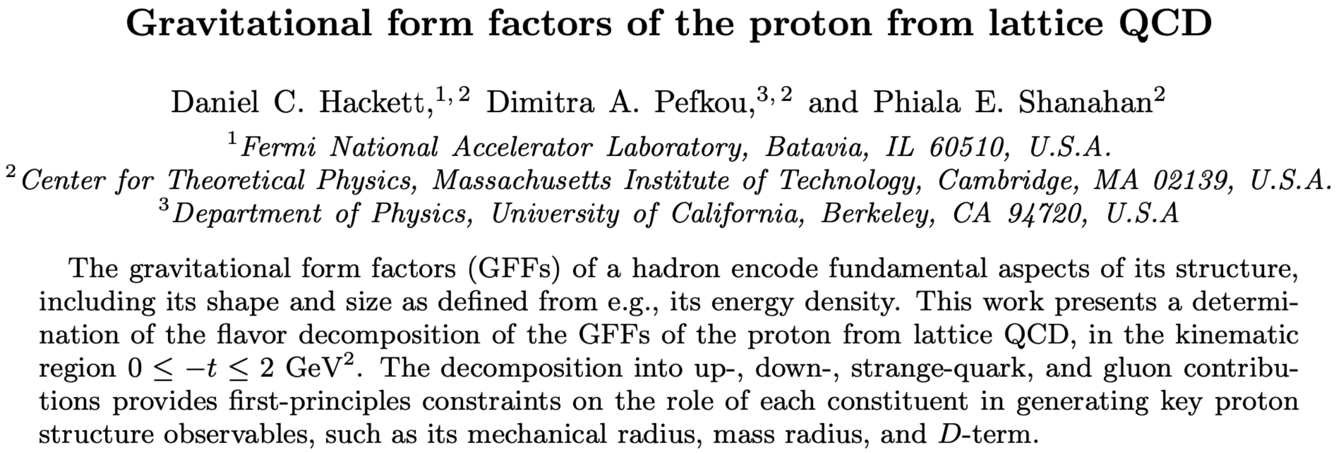
GFFs on the lattice

- Overview of calculation

Results

- Proton GFFs (w/ flavor decomp)
- Experimental comparison
- Mechanical densities & radii
- Pion GFFs
- Glueball GFFs (prelim)

2310.08484

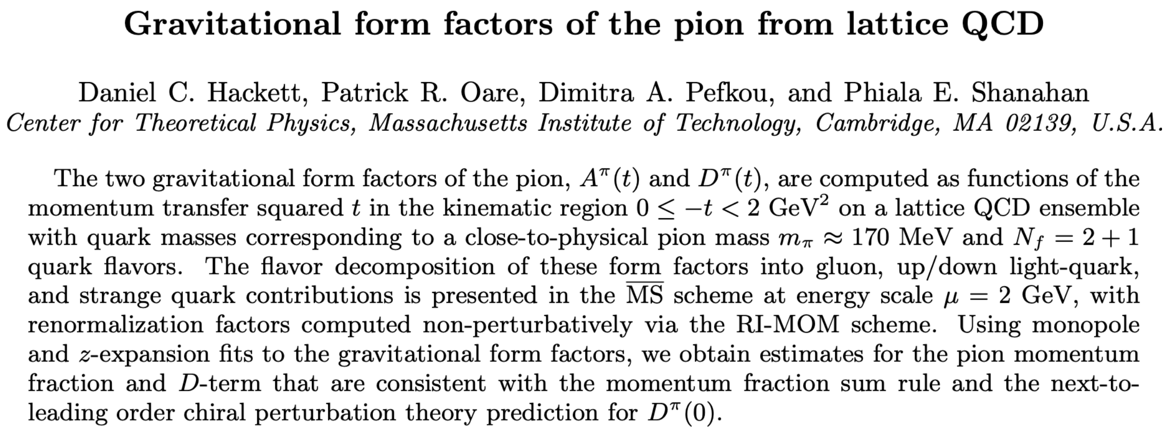


Gravitational form factors of the proton from lattice QCD

Daniel C. Hackett,[1,2] Dimitra A. Pefkou,[3,2] and Phiala E. Shanahan[2]
[1]*Fermi National Accelerator Laboratory, Batavia, IL 60510, U.S.A.*
[2]*Center for Theoretical Physics, Massachusetts Institute of Technology, Cambridge, MA 02139, U.S.A.*
[3]*Department of Physics, University of California, Berkeley, CA 94720, U.S.A*

The gravitational form factors (GFFs) of a hadron encode fundamental aspects of its structure, including its shape and size as defined from e.g., its energy density. This work presents a determination of the flavor decomposition of the GFFs of the proton from lattice QCD, in the kinematic region $0 \leq -t \leq 2$ GeV^2. The decomposition into up-, down-, strange-quark, and gluon contributions provides first-principles constraints on the role of each constituent in generating key proton structure observables, such as its mechanical radius, mass radius, and D-term.

2307.11707



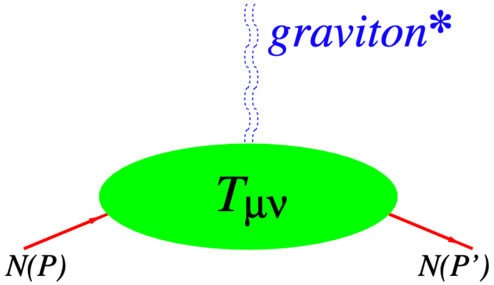
Gravitational form factors of the pion from lattice QCD

Daniel C. Hackett, Patrick R. Oare, Dimitra A. Pefkou, and Phiala E. Shanahan
Center for Theoretical Physics, Massachusetts Institute of Technology, Cambridge, MA 02139, U.S.A.

The two gravitational form factors of the pion, $A^{\pi}(t)$ and $D^{\pi}(t)$, are computed as functions of the momentum transfer squared t in the kinematic region $0 \leq -t < 2$ GeV^2 on a lattice QCD ensemble with quark masses corresponding to a close-to-physical pion mass $m_{\pi} \approx 170$ MeV and $N_f = 2+1$ quark flavors. The flavor decomposition of these form factors into gluon, up/down light-quark, and strange quark contributions is presented in the $\overline{\text{MS}}$ scheme at energy scale $\mu = 2$ GeV, with renormalization factors computed non-perturbatively via the RI-MOM scheme. Using monopole and z-expansion fits to the gravitational form factors, we obtain estimates for the pion momentum fraction and D-term that are consistent with the momentum fraction sum rule and the next-to-leading order chiral perturbation theory prediction for $D^{\pi}(0)$.

Gravitational structure of the nucleon

Gravitational form factors (GFFs)



GFFs are EMT form factors

$$T^{\{\mu\nu\}} = 2\,\mathrm{Tr}\left[-G^{\alpha\mu}G^{\nu}_{\alpha} + \frac{1}{4}g^{\mu\nu}G^{\alpha\beta}G_{\alpha\beta}\right] + \bar{q}\,\gamma^{\{\mu}i\overleftrightarrow{D}^{\nu\}}\,q$$

Nucleon:

$$\langle N(p')|T^{\{\mu\nu\}}|N(p)\rangle = \bar{U}(p')\left[A(t)\,\frac{P^{\{\mu}P^{\nu\}}}{M} + J(t)\,\frac{i\,P^{\{\mu}\sigma^{\nu\}\rho}\Delta_\rho}{2M} + D(t)\,\frac{\Delta^{\{\mu}\Delta^{\nu\}} - g^{\mu\nu}\Delta^2}{4M}\right]U(p)$$

Why are these interesting?

$a^{\{\mu}b^{\nu\}} \equiv \frac{1}{2}(a^\mu b^\nu + \mathrm{a}^\nu \mathrm{b}^\mu)$

$\overleftrightarrow{D} = (\overrightarrow{D} - \overleftarrow{D})/2$

$U, \bar{U}$ = Dirac spinors

$P = (p' + p)/2$

$\Delta = p' - p$

$t = \Delta^2$

Global properties

$$\langle N(p')|T^{\{\mu\nu\}}|N(p)\rangle = \bar{U}(p')\left[A(t)\frac{P^{\{\mu}P^{\nu\}}}{M} + J(t)\frac{i\,P^{\{\mu}\sigma^{\nu\}\rho}\Delta_\rho}{2M} + D(t)\frac{\Delta^{\{\mu}\Delta^{\nu\}} - g^{\mu\nu}\Delta^2}{4M}\right]U(p)$$

$\partial_\mu T^{\mu\nu} = 0$ → GFFs are scale- and scheme-independent

Forward GFFs are fundamental, global properties:

$A(0) = 1 \Leftrightarrow \langle p|T^{tt}|p\rangle = M$

$J(0) = \frac{1}{2}$ = Total spin

$B(0) = 2J(0) - A(0) = 0$ “vanishing of the anomalous gravitomagnetic moment”

$D(0) =$???* (internal forces)

Flavor decomposition

Gluons $T_g^{\{\mu\nu\}} = 2\,\mathrm{Tr}\left[-G^{\alpha\mu}G_\alpha^\nu + \frac{1}{4}g^{\mu\nu}G^{\alpha\beta}G_{\alpha\beta}\right]$ Quarks $T_q^{\{\mu\nu\}} = \bar{q}\,\gamma^{\{\mu}i\overleftrightarrow{D}^{\nu\}}\,q$

$$\left\langle N(p')\middle|T_{g,q}^{\{\mu\nu\}}\middle|N(p)\right\rangle = \bar{u}(p')\left[A_{g,q}(t)\frac{P^{\{\mu}P^{\nu\}}}{M} + J_{g,q}(t)\frac{i\,P^{\{\mu}\sigma^{\nu\}\rho}\Delta_\rho}{2M} + D_{g,q}(t)\frac{\Delta^{\{\mu}\Delta^{\nu\}} - g^{\mu\nu}\Delta^2}{4M} + \bar{c}_{g,q}(t)Mg^{\mu\nu}\right]u(p)$$

Flavor decomposition

Gluons $T_g^{\{\mu\nu\}} = 2\,\mathrm{Tr}\left[-G^{\alpha\mu}G_\alpha^\nu + \frac{1}{4}g^{\mu\nu}G^{\alpha\beta}G_{\alpha\beta}\right]$ Quarks $T_q^{\{\mu\nu\}} = \bar{q}\,\gamma^{\{\mu}i\overleftrightarrow{D}^{\nu\}}\,q$

$$\left\langle N(p')\middle|T_{g,q}^{\{\mu\nu\}}\middle|N(p)\right\rangle = \bar{u}(p')\left[A_{g,q}(t)\frac{P^{\{\mu}P^{\nu\}}}{M} + J_{g,q}(t)\frac{i\,P^{\{\mu}\sigma^{\nu\}\rho}\Delta_\rho}{2M} + D_{g,q}(t)\frac{\Delta^{\{\mu}\Delta^{\nu\}} - g^{\mu\nu}\Delta^2}{4M} + \bar{c}_{g,q}(t)Mg^{\mu\nu}\right]u(p)$$

Not conserved $\sum_q \bar{c_q} + \bar{c_g} = 0$

Power-divergent mixing

Flavor decomposition

Gluons $T_g^{\{\mu\nu\}} = 2\,\mathrm{Tr}\left[-G^{\alpha\mu}G_\alpha^\nu + \frac{1}{4}g^{\mu\nu}G^{\alpha\beta}G_{\alpha\beta}\right]$ Quarks $T_q^{\{\mu\nu\}} = \bar{q}\,\gamma^{\{\mu}i\overleftrightarrow{D}^{\nu\}}\,q$

Momentum fraction

$A_{q,g}(0) = \langle x\rangle_{q,g}$

$A_g(0) + \sum_q A_q(0) = 1$

Spin fraction

$J_g(0) + \sum_q J_q(0) = \frac{1}{2}$

$$\left\langle N(p')\middle|T_{g,q}^{\{\mu\nu\}}\middle|N(p)\right\rangle = \bar{u}(p')\left[A_{g,q}(t)\frac{P^{\{\mu}P^{\nu\}}}{M} + J_{g,q}(t)\frac{i\,P^{\{\mu}\sigma^{\nu\}\rho}\Delta_\rho}{2M} + D_{g,q}(t)\frac{\Delta^{\{\mu}\Delta^{\nu\}} - g^{\mu\nu}\Delta^2}{4M} + \bar{c}_{g,q}(t)Mg^{\mu\nu}\right]u(p)$$

Internal forces

$D(0) = D_g(0) + \sum_q D_q(0)$

Not conserved $\sum_q \bar{c}_q + \bar{c}_g = 0$

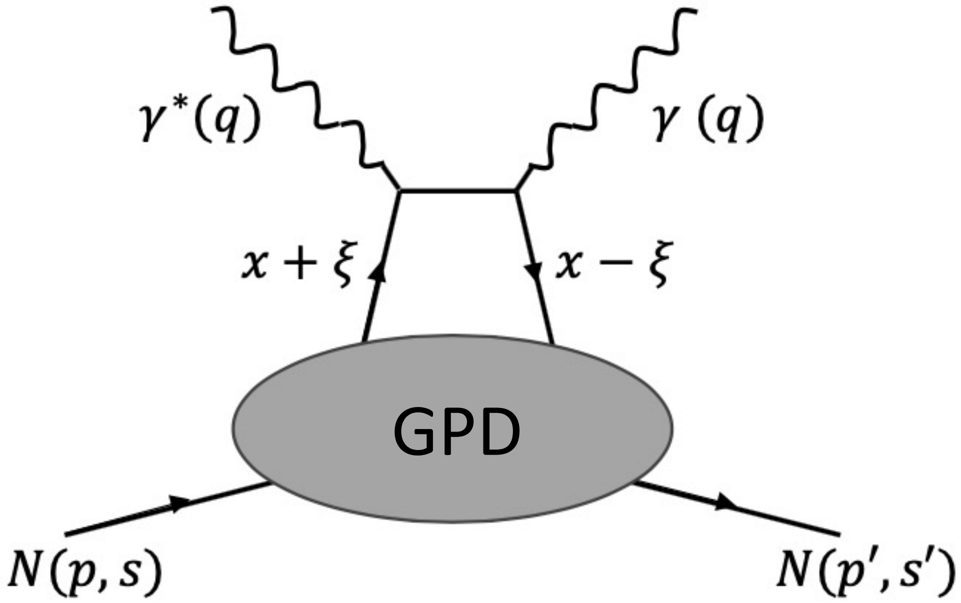
Power-divergent mixing

Relation to GPDs

GFFs ⊂ GFFs

Gravitational Form Factors

Generalized Form Factors



GFFs are Mellin moments of GPDs, e.g.

$$\int dx\, x\, H_q(x,\xi,t) = A_q(t) + \xi^2 D_q(t) \qquad \int dx\, H_g(x,\xi,t) = A_g(t) + \xi^2 D_g(t)$$

$$\int dx\, x\, E_q(x,\xi,t) = B_q(t) - \xi^2 D_q(t) \qquad \int dx\, E_g(x,\xi,t) = B_g(t) - \xi^2 D_g(t)$$

→ relate to experiment via factorization

GFFs on the lattice

Prior art

Early work: Generalized FFs / GPD moments

- Quark only
- Neglecting disconnected contributions (isovector ok)
- e.g. [LHPC 0705.4295, QCDSF/UKQCD hep-lat/0509133]

Momentum & spin fractions

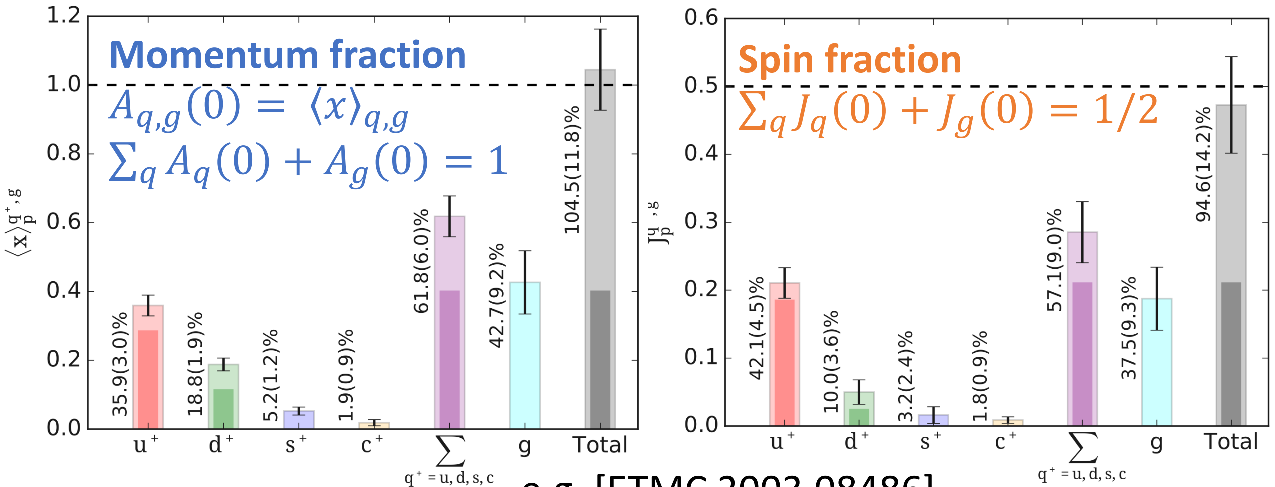
- Forward ($t = 0$) moments of GFFs
- Many, more recently including glue & disconnected

Previous determinations of gluon GFFs w/ unphysically heavy quark masses ($M_\pi \approx 450$ MeV)

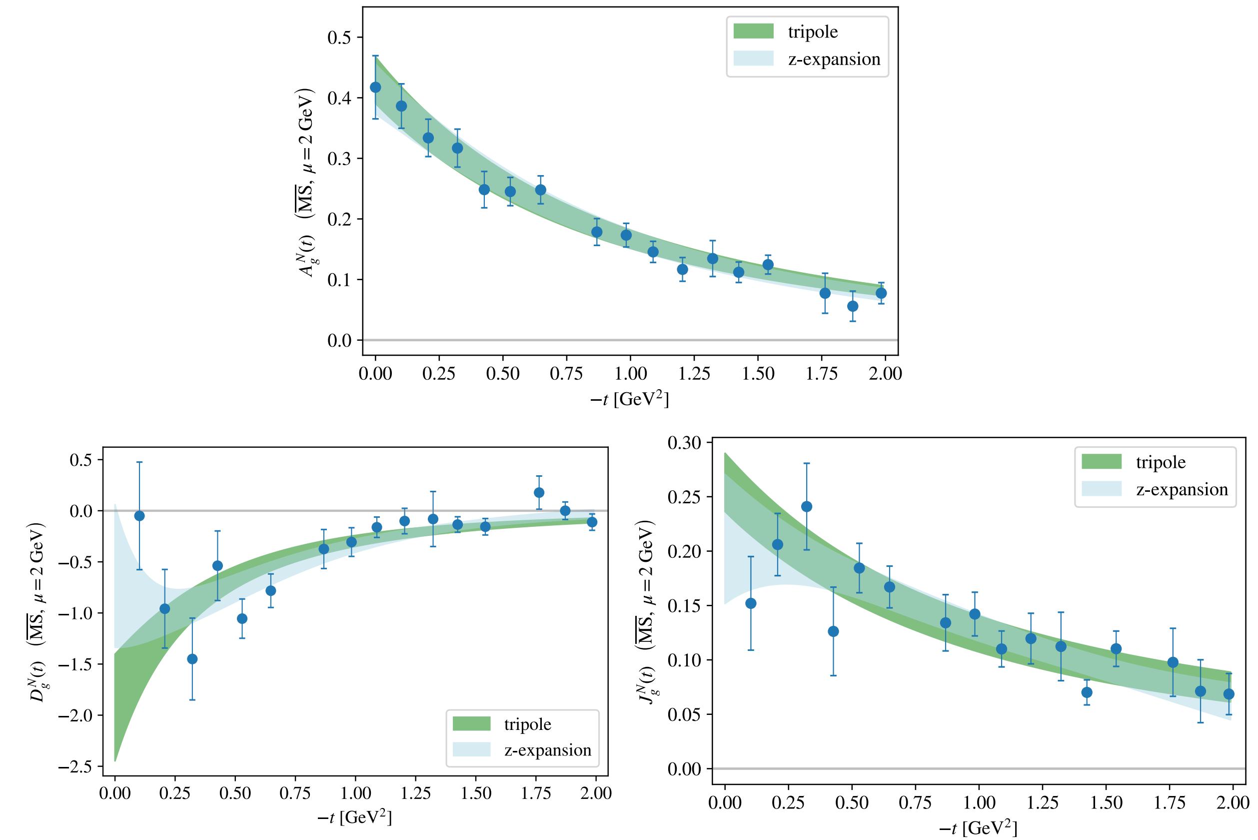
- e.g. [Shanahan Detmold 1810.04626] (π, N)
- e.g. [Pefkou Hackett Shanahan 2107.10368] (π, N, ρ, Δ)

This calculation:

- Close-to-physical ($M_\pi \approx 170$ MeV)
- Full flavor decomposition



e.g. [ETMC 2003.08486]



Overview of calculation

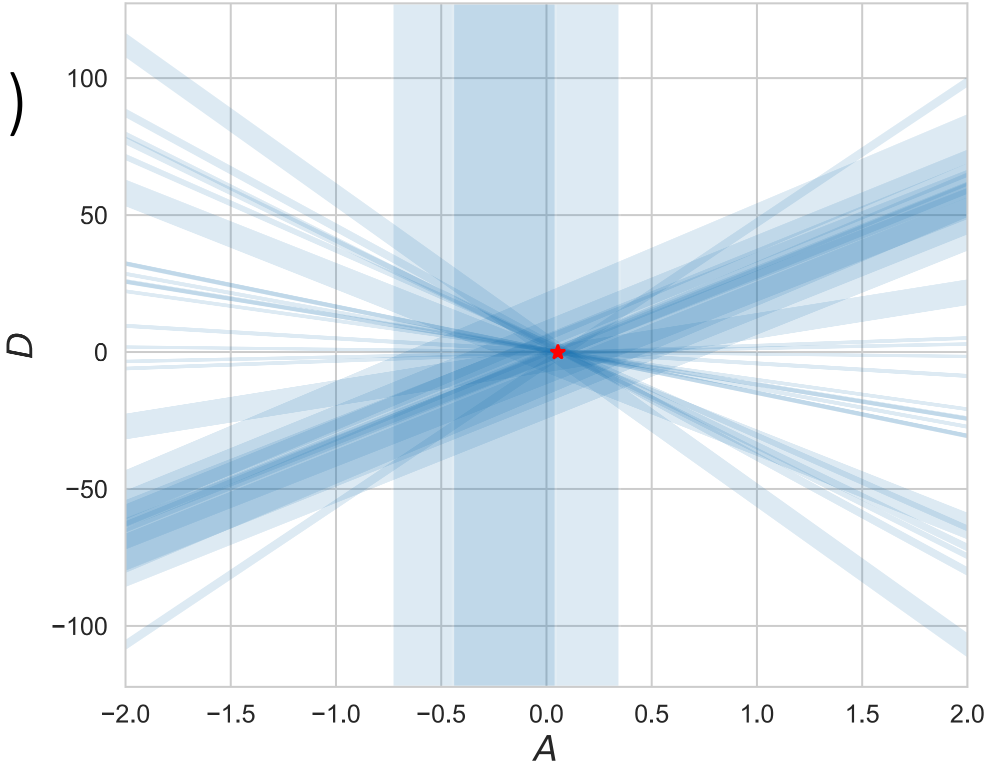
Need to compute:

1. Bare matrix elements for $f \in \{g, u, d, s\}$ to constrain bare GFFs

$$\langle p' | T_f^{\mathrm{b}}(\Delta) | p \rangle = c_A\, A_f^{b}(t) + c_J\, J_f^{\mathrm{b}}(t) + c_D\, D_f^{\mathrm{b}}(t)$$

2. Isosinglet mixing matrix (+ non-singlet Z_{u+d-2s})

$$\begin{bmatrix} T_q^{\overline{MS}} \\ T_g^{\overline{MS}} \end{bmatrix} = \begin{bmatrix} Z_{qq}^{\overline{MS}} & Z_{qg}^{\overline{MS}} \\ Z_{gq}^{\overline{MS}} & Z_{gg}^{\overline{MS}} \end{bmatrix} \begin{bmatrix} T_q^{\mathrm{bare}} \\ T_g^{\mathrm{bare}} \end{bmatrix}$$



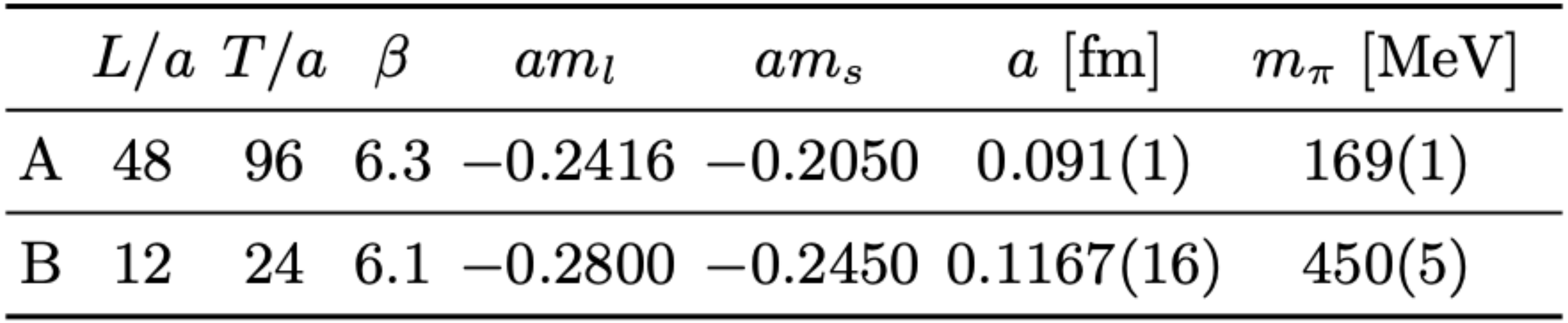
→ Renormalized linear constraints on GFFs at different values of $t = \Delta^2 = (p' - p)^2$

→ Fit to extract GFFs(t)

Ensembles

Gauge action: tadpole-improved Luscher-Weisz

Fermion action: $2+1$ flavors, stout-smeared clover



	L/a	T/a	β	am_l	am_s	a [fm]	m_π [MeV]
A	48	96	6.3	-0.2416	-0.2050	0.091(1)	169(1)
B	12	24	6.1	-0.2800	-0.2450	0.1167(16)	450(5)

Bare matrix elements

Glue: 2511 configs

Quarks: 1381 configs (subset)

[“a091m170” (JLab/W&M/MIT/LANL)]

Renormalization

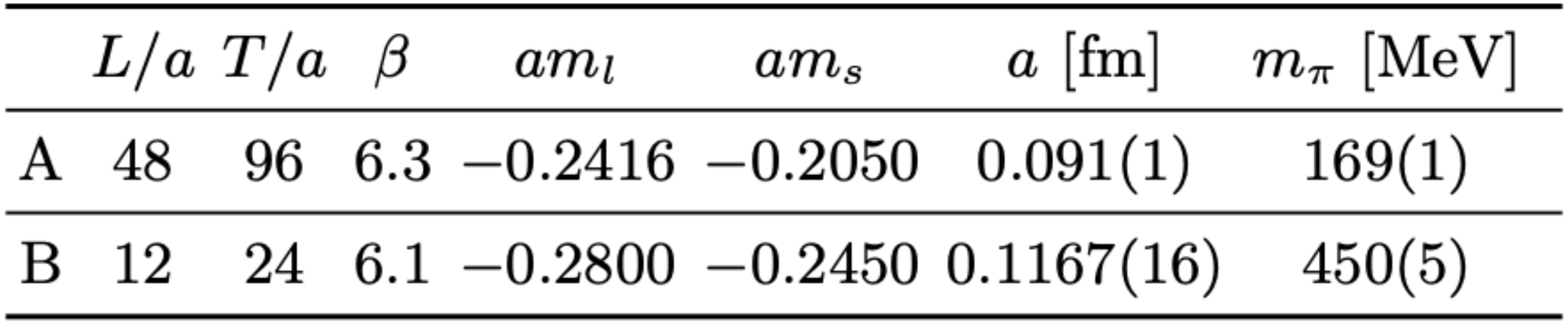
Conn. quark: 240 configs

Disco./glue: 20000 configs

Ensembles

Gauge action: tadpole-improved Luscher-Weisz

Fermion action: $2 + 1$ flavors, stout-smeared clover



	L/a	T/a	β	am_l	am_s	a [fm]	m_π [MeV]
A	48	96	6.3	-0.2416	-0.2050	0.091(1)	169(1)
B	12	24	6.1	-0.2800	-0.2450	0.1167(16)	450(5)

Bare matrix elements

Glue: 2511 configs

Quarks: 1381 configs (subset)

[“a091m170” (JLab/W&M/MIT/LANL)]

Renormalization

Conn. quark: 240 configs

Disco./glue: 20000 configs

“Single”-ensemble calculation: can’t quantify remaining artifacts due to discretization, unphysical quark masses, finite volume

Bare matrix elements from three-point functions

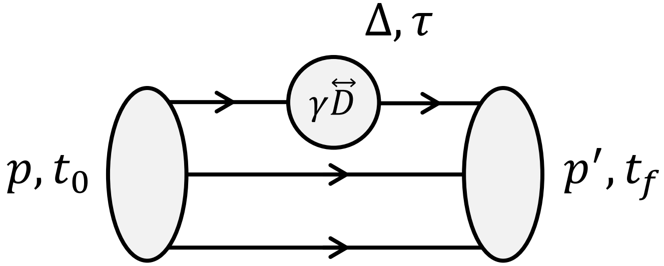
Can’t compute matrix elements directly, must extract from

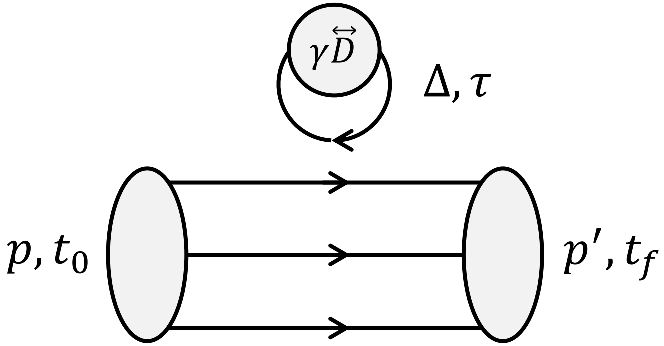
$$\left\langle \chi\left(p', t_f\right) T^{\mathrm{b}}(\Delta, \tau)\, \bar{\chi}(p, 0)\right\rangle \sim Z_{p'}\, Z_p \boxed{\left\langle p' \middle| T^{\mathrm{b}}(\Delta) \middle| p \right\rangle} e^{-E'\left(t_f-\tau\right)-E\tau} + (\text{excited states})$$

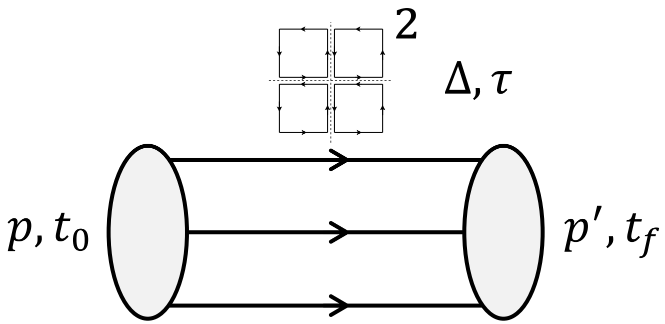
Bare matrix elements from three-point functions

Can’t compute matrix elements directly, must extract from

$$\langle \chi(p', t_f)\, T^{\mathrm{b}}(\Delta, \tau)\, \bar{\chi}(p, 0)\rangle \sim Z_{p'}\, Z_p \boxed{\langle p' | T^{\mathrm{b}}(\Delta) | p\rangle} e^{-E'(t_f - \tau) - E\tau} + (\text{excited states})$$







Connected Quark (u, d)

Sequential source (thru sink)

- 3 sink momenta
- 1 spin channel
- Sources / cfg varies w/ t_f

Disconnected Quark $(u = d, s)$

- 1024 sources / cfg
- 4 spin channels
- Hierarchical probing w/ 512 Hadamard vectors
- 2 Z_4 noise shots / cfg

Glue (disconnected)

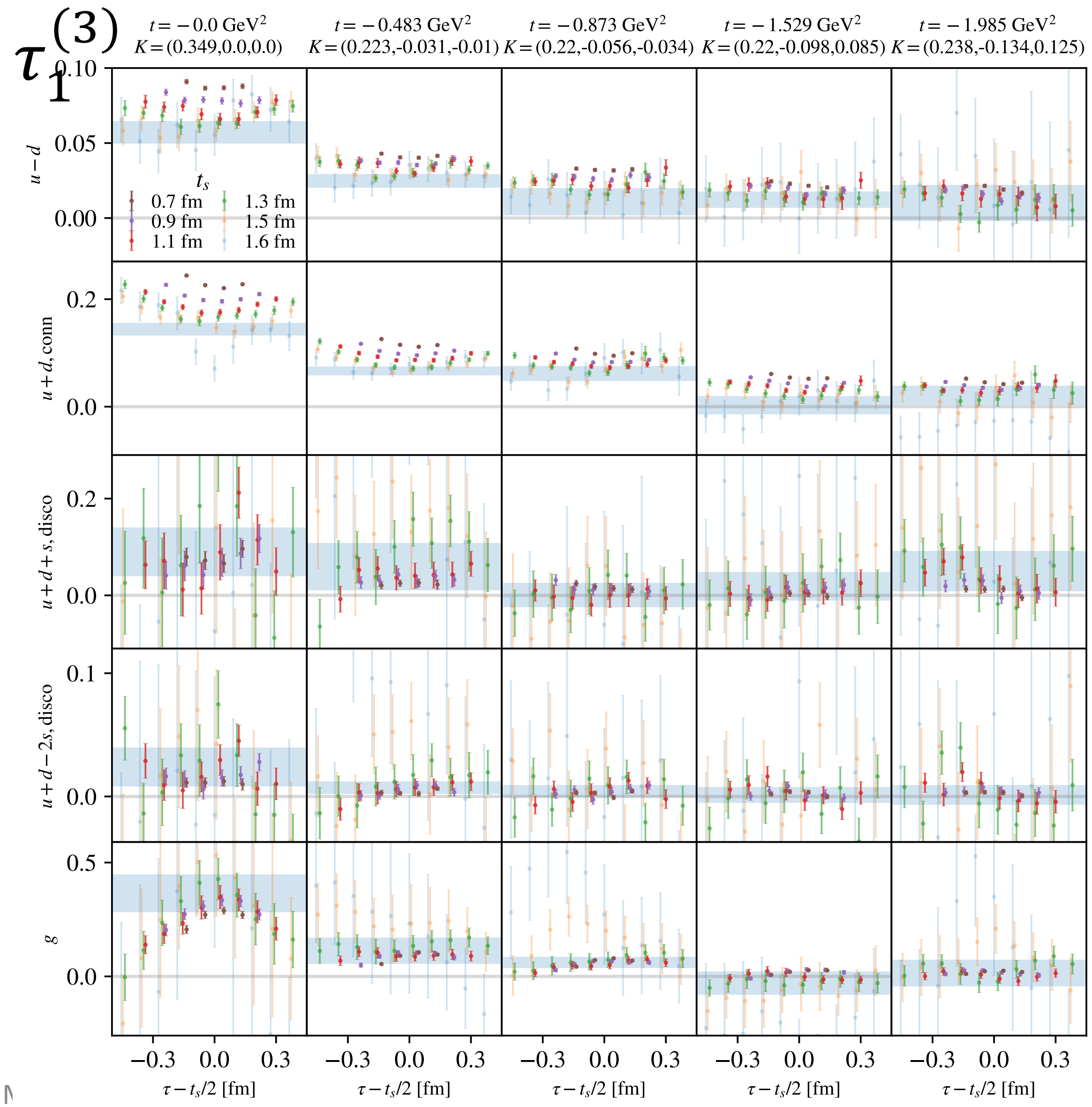
- 1024 sources / cfg
- 4 spin channels

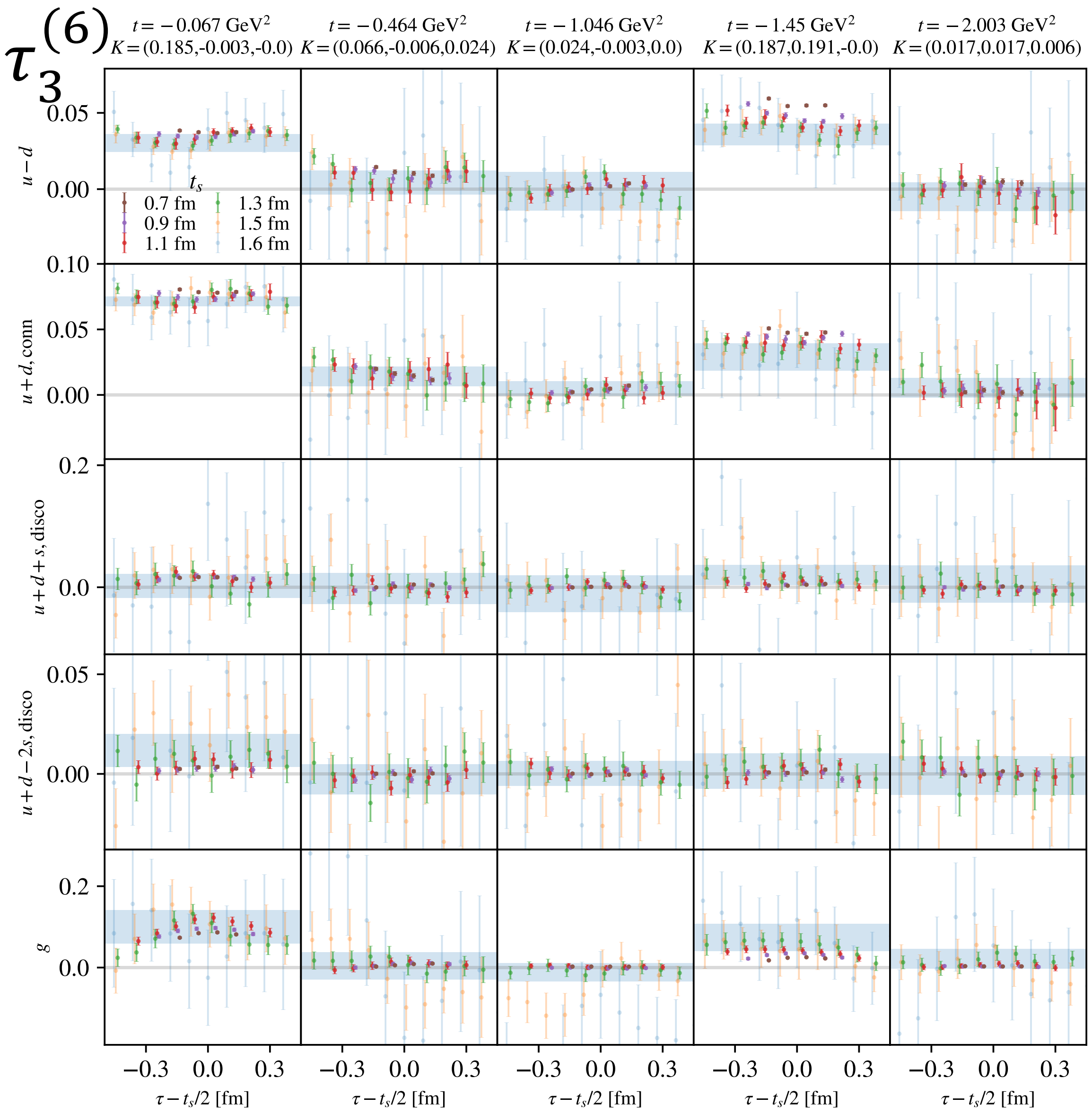
Analysis

Large-scale automated analysis – fit data to extract

$\sim$ 40k matrix elements for u, d, s, g channels (conn/disc)

Example nucleon data w/ fits





Renormalization

Assert RI-MOM conditions at scale $\mu^2 = p^2$

$$\langle q(p)\, T_f(0)\, \bar{q}(p) \rangle_{\text{lattice}} = Z_q\, R^{\text{RI}}_{fq}\, \langle q(p)\, T_f(0)\, \bar{q}(p) \rangle_{\text{tree}}$$

$$\langle A(p)\, T_f(0)\, A(p) \rangle_{\text{lattice}} = Z_g\, R^{\text{RI}}_{fg}\, \langle A(p)\, T_f(0)\, A(p) \rangle_{\text{tree}}$$

…in Landau gauge

…flow T_g to $t/a^2 = 1.2$ to match operator in bare matrix elements

Apply perturbative matching to $\overline{\text{MS}}$ and run to $\mu = 2\text{ GeV}$

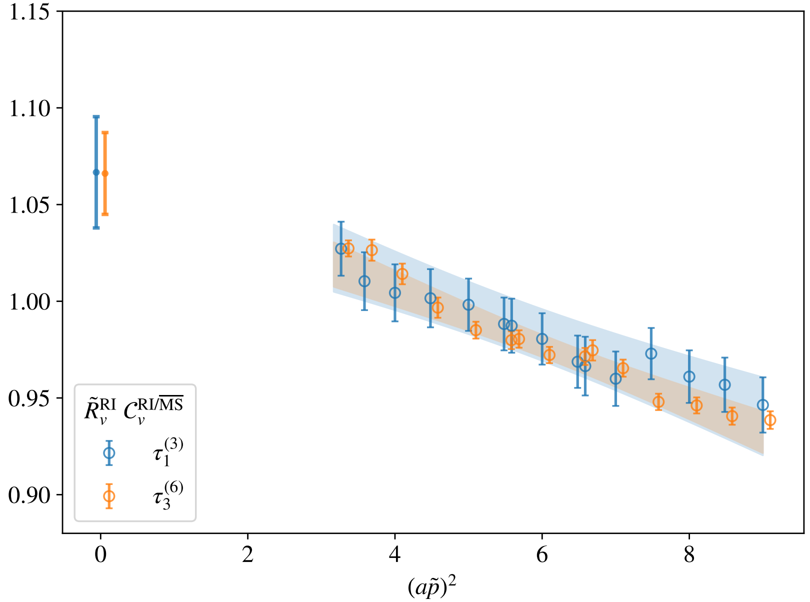
$$\left(Z^{\overline{MS}}_{u-d}\right)^{-1}(\mu^2) = C^{\text{RI}/\overline{MS}}_{u-d}(\mu^2, \mu_R^2)\, R^{\text{RI}}_{u-d}(\mu_R^2)$$

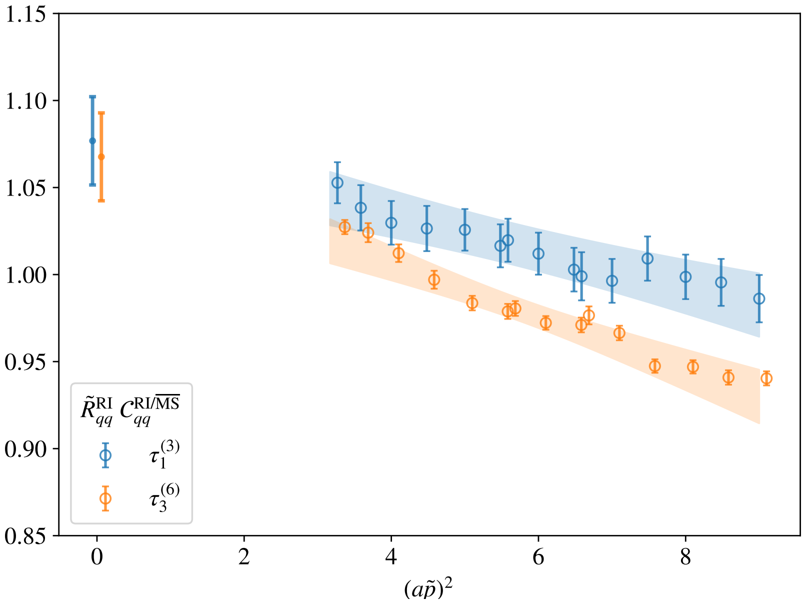
$$\begin{bmatrix} Z^{\overline{MS}}_{qq} & Z^{\overline{MS}}_{qg} \\ Z^{\overline{MS}}_{gq} & Z^{\overline{MS}}_{gg} \end{bmatrix}^{-1}(\mu^2) = \begin{bmatrix} R^{\text{RI}}_{qq} & R^{\text{RI}}_{qg} \\ R^{\text{RI}}_{gq} & R^{\text{RI}}_{gg} \end{bmatrix}(\mu_R^2) \begin{bmatrix} C^{\text{RI}/\overline{MS}}_{qq} & C^{\text{RI}/\overline{MS}}_{qg} \\ C^{\text{RI}/\overline{MS}}_{gq} & C^{\text{RI}/\overline{MS}}_{gg} \end{bmatrix}(\mu^2, \mu_R^2)$$

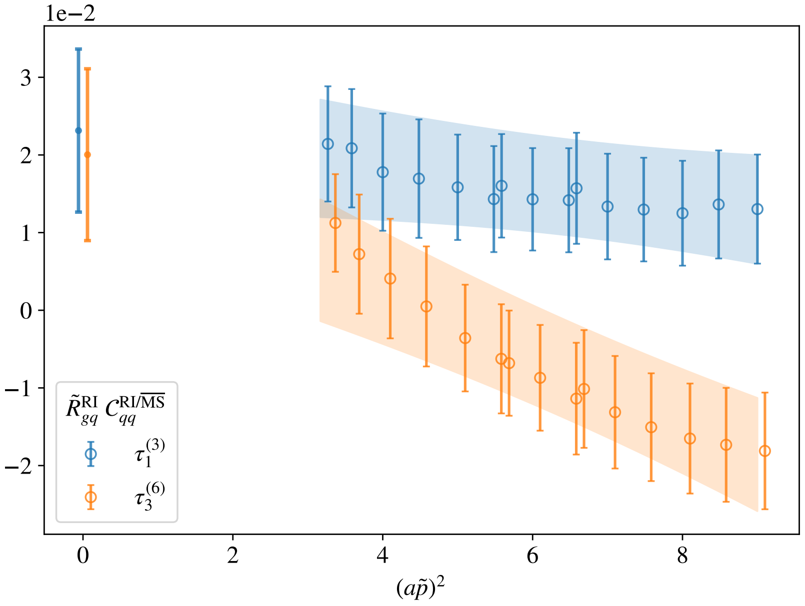
Model and fit residual $(ap)^2$ dependence in each of product $R^{\text{RI}}\, C^{\text{RI}/\overline{MS}}$

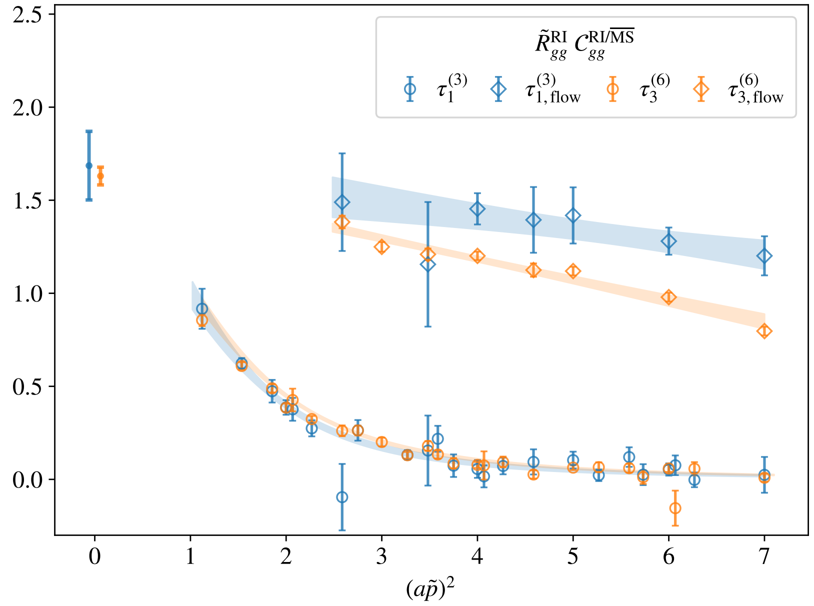
Renormalization: removing discretization artifacts

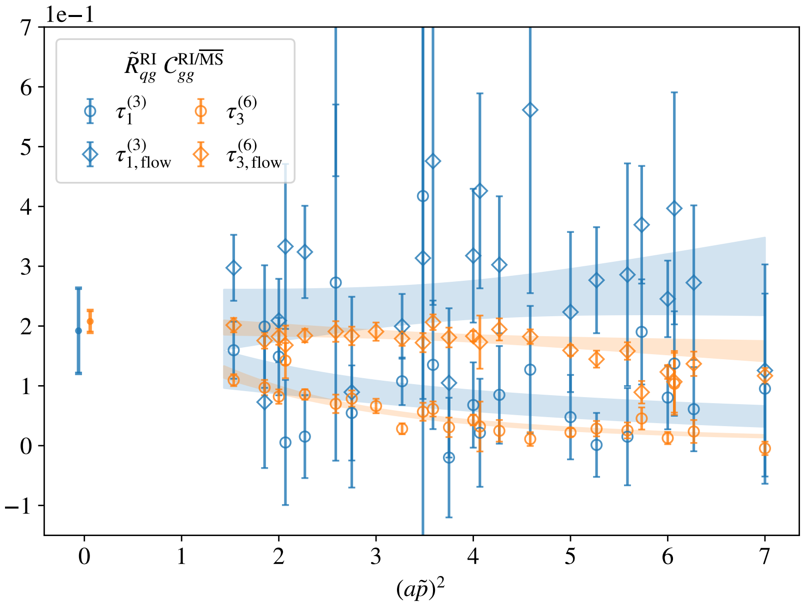
Model discretization artifacts as polynomials, inverse polynomials







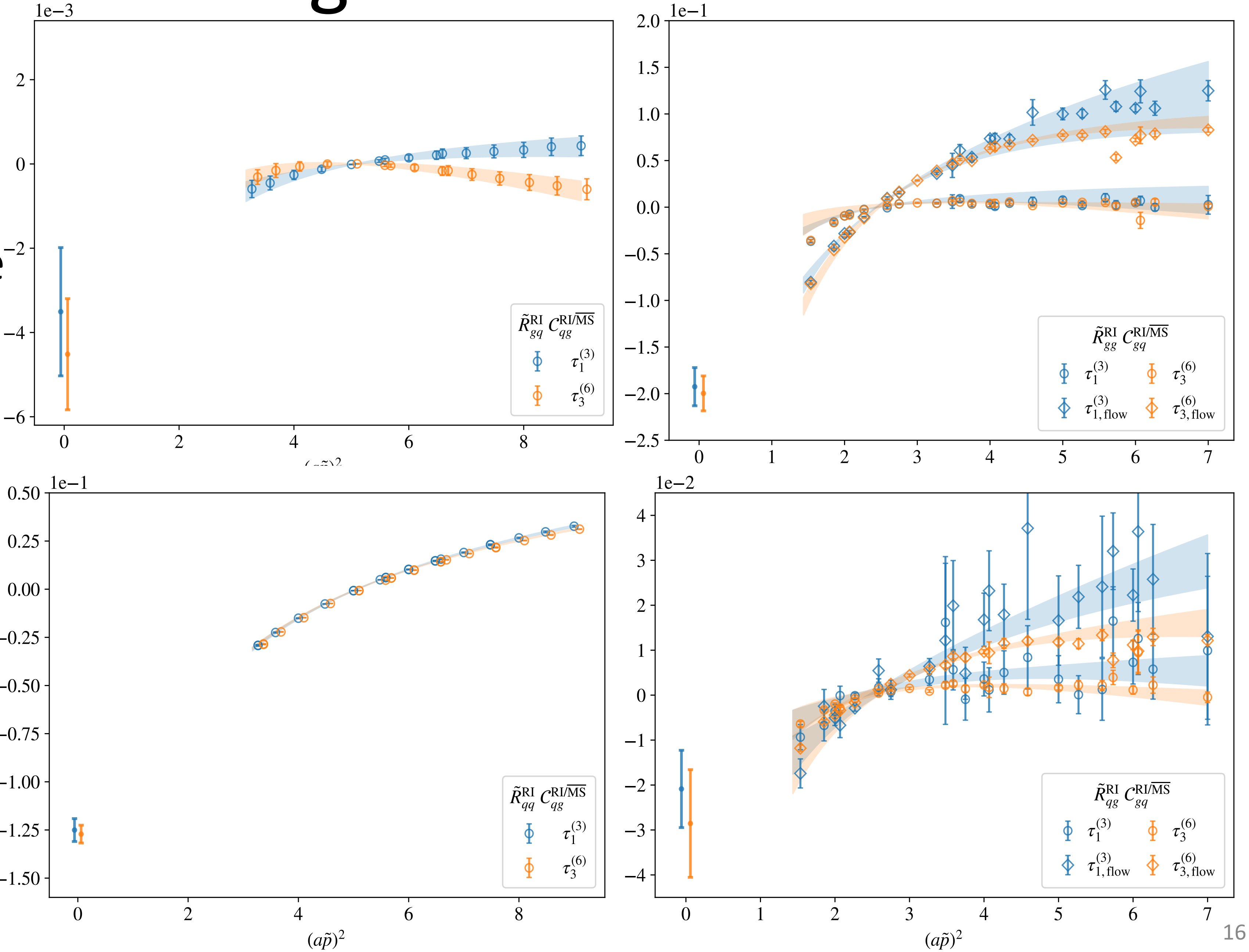




Renormalization: removing discretization artifacts

Model discretization artifacts as polynomials, inverse polynomials

+ logs for nonperturbative effects



Results

Nucleon GFFs

Dark bands: dipole

$$G(t) \sim \frac{\alpha}{(1-t/\Lambda^2)^2}$$

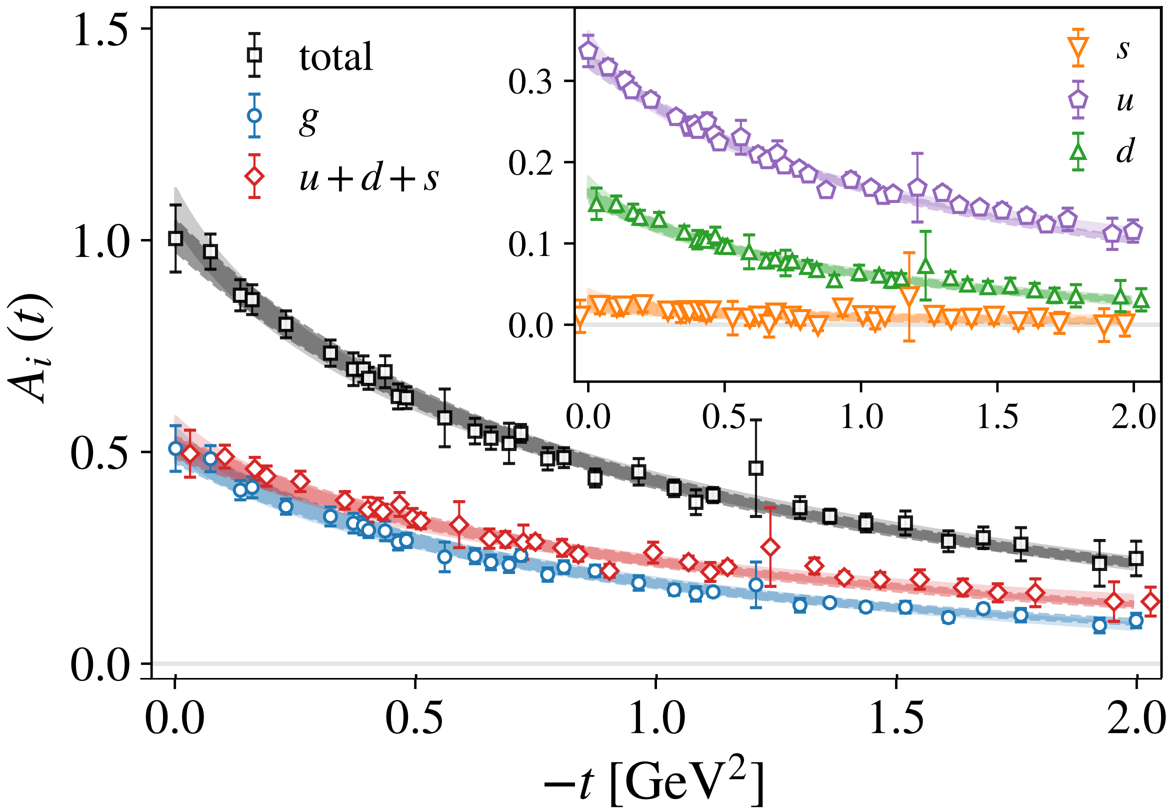
Light bands: z-expansion

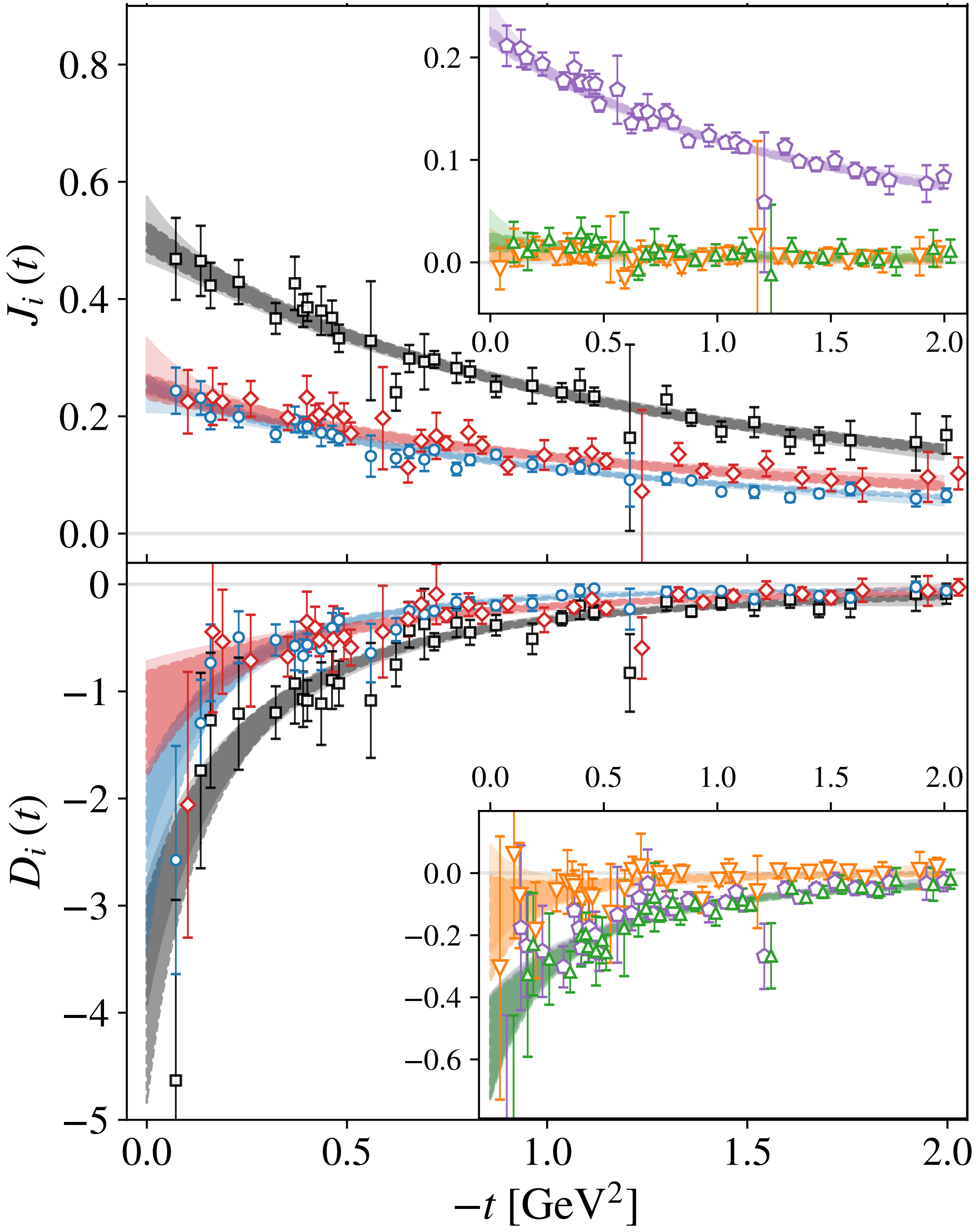
$$G(t) \sim \sum_{k=0}^{k_{\max}=2} \alpha_k [z(t)]^k$$

$$z(t) = \frac{\sqrt{t_{\mathrm{cut}}-t} - \sqrt{t_{\mathrm{cut}}-t_0}}{\sqrt{t_{\mathrm{cut}}-t} + \sqrt{t_{\mathrm{cut}}-t_0}}$$

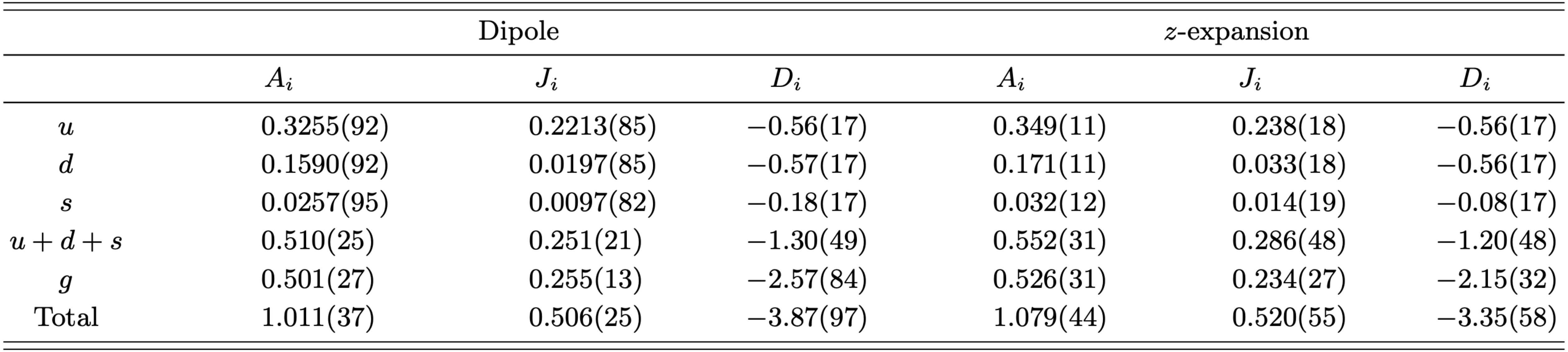
$$t_{\mathrm{cut}} = 4M_\pi^2$$

$$t_0 = t_{\mathrm{cut}}\left(1 - \sqrt{1 + (2\ \mathrm{GeV}^2)/t_{\mathrm{cut}}}\right)$$





Forward limits



	Dipole			z-expansion		
	A_i	J_i	D_i	A_i	J_i	D_i
u	0.3255(92)	0.2213(85)	−0.56(17)	0.349(11)	0.238(18)	−0.56(17)
d	0.1590(92)	0.0197(85)	−0.57(17)	0.171(11)	0.033(18)	−0.56(17)
s	0.0257(95)	0.0097(82)	−0.18(17)	0.032(12)	0.014(19)	−0.08(17)
$u+d+s$	0.510(25)	0.251(21)	−1.30(49)	0.552(31)	0.286(48)	−1.20(48)
g	0.501(27)	0.255(13)	−2.57(84)	0.526(31)	0.234(27)	−2.15(32)
Total	1.011(37)	0.506(25)	−3.87(97)	1.079(44)	0.520(55)	−3.35(58)

Forward limits

	Dipole			z-expansion		
	A_i	J_i	D_i	A_i	J_i	D_i
u	0.3255(92)	0.2213(85)	−0.56(17)	0.349(11)	0.238(18)	−0.56(17)
d	0.1590(92)	0.0197(85)	−0.57(17)	0.171(11)	0.033(18)	−0.56(17)
s	0.0257(95)	0.0097(82)	−0.18(17)	0.032(12)	0.014(19)	−0.08(17)
$u+d+s$	0.510(25)	0.251(21)	−1.30(49)	0.552(31)	0.286(48)	−1.20(48)
g	0.501(27)	0.255(13)	−2.57(84)	0.526(31)	0.234(27)	−2.15(32)
Total	1.011(37)	0.506(25)	−3.87(97)	1.079(44)	0.520(55)	−3.35(58)

Sum rules (consistency check)

Forward limits

	Dipole			z-expansion		
	A_i	J_i	D_i	A_i	J_i	D_i
u	0.3255(92)	0.2213(85)	−0.56(17)	0.349(11)	0.238(18)	−0.56(17)
d	0.1590(92)	0.0197(85)	−0.57(17)	0.171(11)	0.033(18)	−0.56(17)
s	0.0257(95)	0.0097(82)	−0.18(17)	0.032(12)	0.014(19)	−0.08(17)
$u+d+s$	0.510(25)	0.251(21)	−1.30(49)	0.552(31)	0.286(48)	−1.20(48)
g	0.501(27)	0.255(13)	−2.57(84)	0.526(31)	0.234(27)	−2.15(32)
Total	1.011(37)	0.506(25)	−3.87(97)	1.079(44)	0.520(55)	−3.35(58)

Sum rules (consistency check)

cf. global fit result

$$A_g(0) = 0.414(8)$$

[Hou et al. 1912.10053]

Forward limits

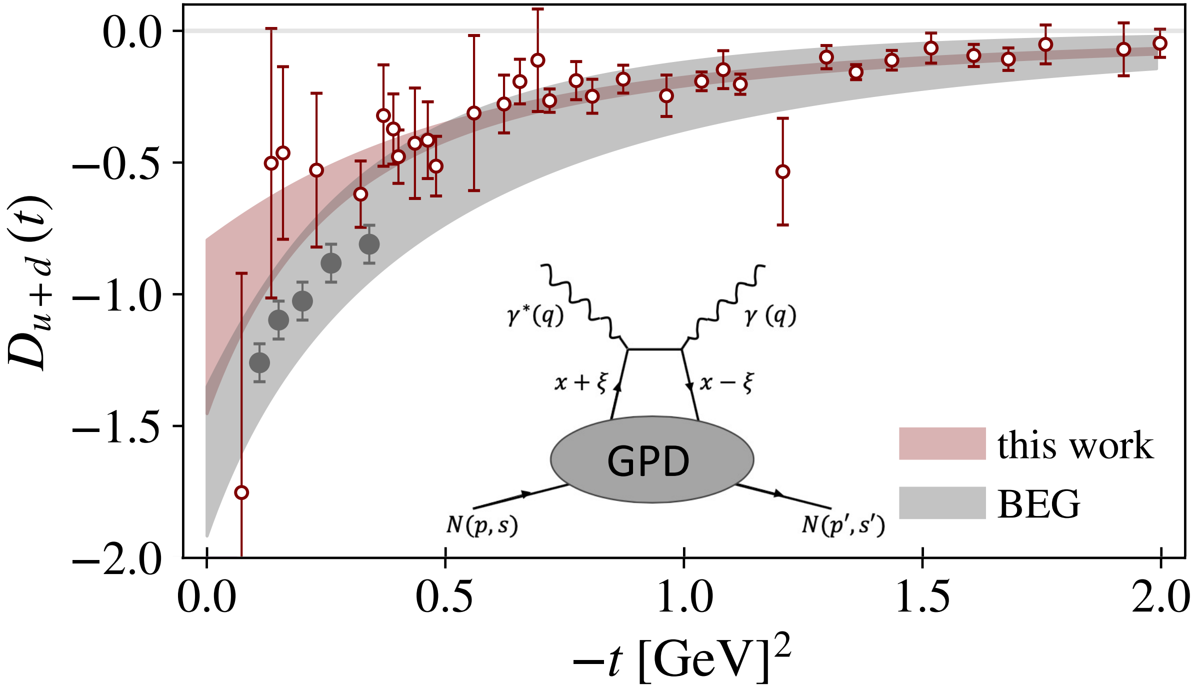
	Dipole			z-expansion		
	A_i	J_i	D_i	A_i	J_i	D_i
u	0.3255(92)	0.2213(85)	$-0.56(17)$	0.349(11)	0.238(18)	$-0.56(17)$
d	0.1590(92)	0.0197(85)	$-0.57(17)$	0.171(11)	0.033(18)	$-0.56(17)$
s	0.0257(95)	0.0097(82)	$-0.18(17)$	0.032(12)	0.014(19)	$-0.08(17)$
$u+d+s$	0.510(25)	0.251(21)	$-1.30(49)$	0.552(31)	0.286(48)	$-1.20(48)$
g	0.501(27)	0.255(13)	$-2.57(84)$	0.526(31)	0.234(27)	$-2.15(32)$
Total	1.011(37)	0.506(25)	$-3.87(97)$	1.079(44)	0.520(55)	$-3.35(58)$

Sum rules (consistency check)

cf. global fit result
$A_g(0) = 0.414(8)$
[Hou et al. 1912.10053]

First determination!
Satisfies χPT bound
$D(0)/M \leq -1.1(1)\ \mathrm{GeV}^{-1}$

Nucleon vs. experiment



BEG = [Burkert Elouadrhiri Girod 2018] (DVCS)

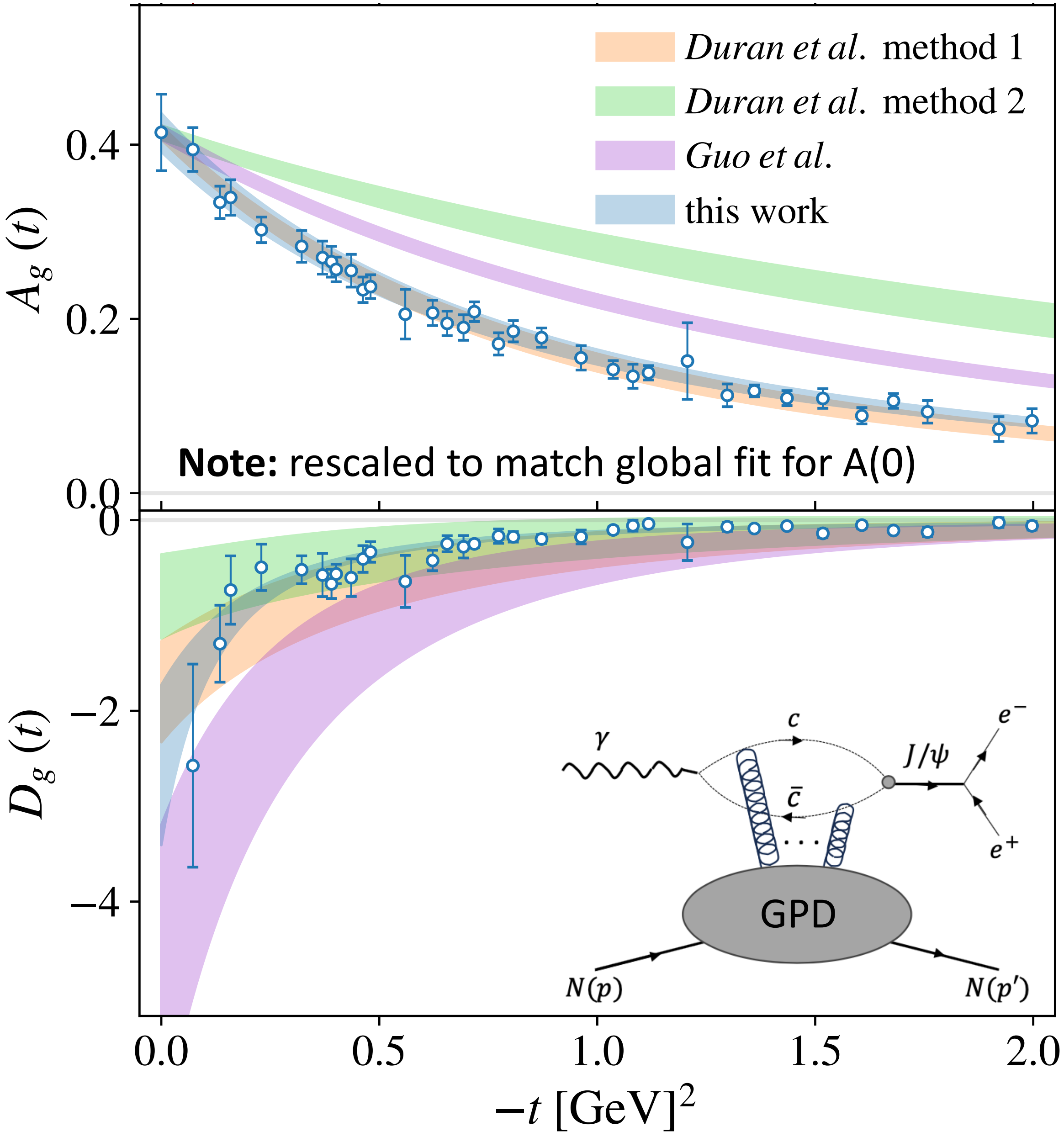
[Duran et al. 2207.05212] (J/ψ)

Method 1: holographic QCD (Mamo Jahed, PRD 21,22)

Method 2: GPD (Guo Ji Liu, PRD 2021)

[Guo et al. 2305.06992]

Updated GPD analysis + GlueX data



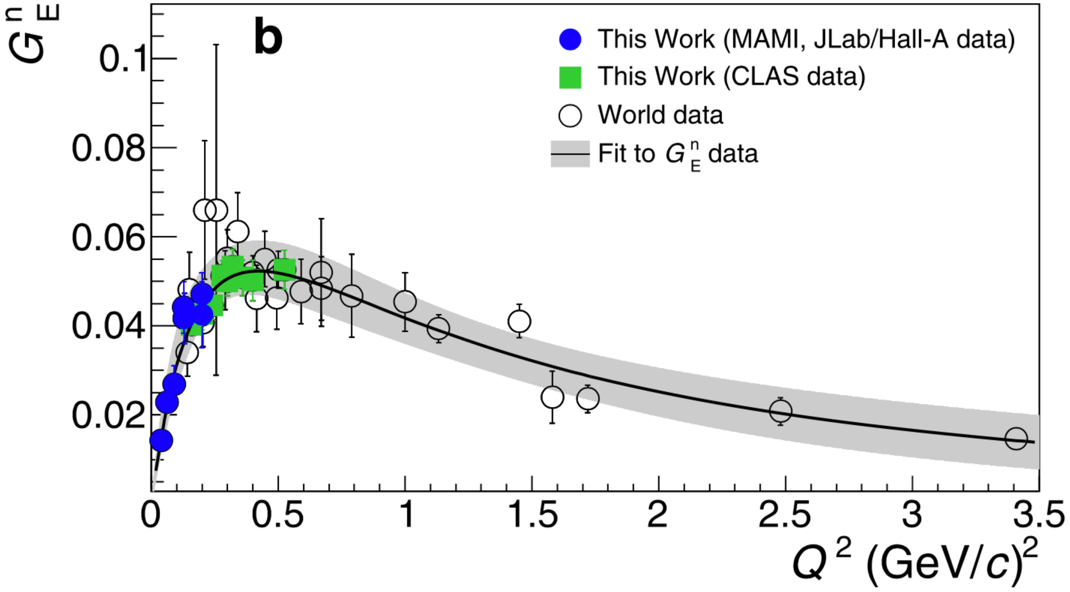
Note: rescaled to match global fit for A(0)

(G)FFs and Tomography

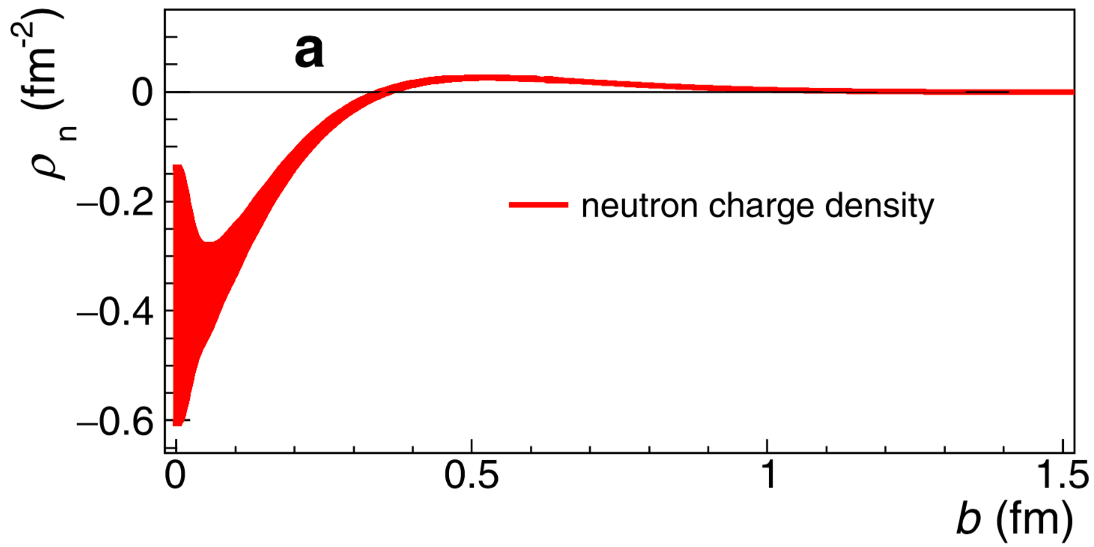
Fourier-transformed form factors provide information about spatial densities

Example: electric charge density in the neutron from G_E^n

[Atac, Constantinou, Meziani, Paolone, Sparveris 2103.10840]



Fourier transform →



Applies also for GFFs → mechanical densities

Mechanical densities from GFFs

$$T_{\mu\nu}(r) = \begin{bmatrix} T_{tt}(r) & T_{tj}(r) \\ T_{it}(r) & T_{ij}(r) \end{bmatrix} = \begin{bmatrix} \epsilon(r) & \\ & \left(\frac{r_i r_j}{r^2} - \frac{1}{d}\delta_{ij}\right) s(r) + \delta_{ij} p(r) \end{bmatrix}$$

Mechanical densities from GFFs

$$T_{\mu\nu}(r) = \begin{bmatrix} T_{tt}(r) & T_{tj}(r) \\ T_{it}(r) & T_{ij}(r) \end{bmatrix} = \begin{bmatrix} \epsilon(r) & \\ & \left(\frac{r_i r_j}{r^2} - \frac{1}{d}\delta_{ij}\right) s(r) + \delta_{ij} p(r) \end{bmatrix}$$

1. Parametrize with GFFs (choose kinematics)

$$T_{\mu\nu}(t) \sim A(t)\, k^A_{\mu\nu}(t) + J(t)\, k^J_{\mu\nu}(t) + D(t)\, k^D_{\mu\nu}(t)$$

Mechanical densities from GFFs

$$T_{\mu\nu}(r) = \begin{bmatrix} T_{tt}(r) & T_{tj}(r) \\ T_{it}(r) & T_{ij}(r) \end{bmatrix} = \begin{bmatrix} \epsilon(r) & \\ & \left(\frac{r_i r_j}{r^2} - \frac{1}{d}\delta_{ij}\right) s(r) + \delta_{ij} p(r) \end{bmatrix}$$

1. Parametrize with GFFs (choose kinematics)

$$T_{\mu\nu}(t) \sim A(t)\, k^A_{\mu\nu}(t) + J(t)\, k^J_{\mu\nu}(t) + D(t)\, k^D_{\mu\nu}(t)$$

2. Fourier transform $T_{\mu\nu}(t) \to T_{\mu\nu}(r)$

$$[f(t)]_{\mathrm{FT}} = \int \frac{d^3\boldsymbol{\Delta}}{(2\pi)^3} e^{-i\boldsymbol{\Delta}\cdot\mathbf{r}} f(t)$$

Mechanical densities from GFFs

$$T_{\mu\nu}(r) = \begin{bmatrix} T_{tt}(r) & T_{tj}(r) \\ T_{it}(r) & T_{ij}(r) \end{bmatrix} = \begin{bmatrix} \epsilon(r) & \\ & \left(\frac{r_i r_j}{r^2} - \frac{1}{d}\delta_{ij}\right) s(r) + \delta_{ij} p(r) \end{bmatrix}$$

1. Parametrize with GFFs (choose kinematics)

$$T_{\mu\nu}(t) \sim A(t)\, k^A_{\mu\nu}(t) + J(t)\, k^J_{\mu\nu}(t) + D(t)\, k^D_{\mu\nu}(t)$$

2. Fourier transform $T_{\mu\nu}(t) \rightarrow T_{\mu\nu}(r)$

$$[f(t)]_{\mathrm{FT}} = \int \frac{d^3\mathbf{\Delta}}{(2\pi)^3} e^{-i\mathbf{\Delta}\cdot\mathbf{r}} f(t)$$

3. Identify components → Spatial densities (3D Breit frame)

energy $\epsilon(r) = M\left[A(t) - \frac{t}{4M^2}\left(D(t) + A(t) - 2J(t)\right)\right]_{FT}$

pressure $p(r) = \frac{1}{6M}\frac{1}{r^2}\frac{d}{dr} r^2 \frac{d}{dr} [D(t)]_{FT}$

shear forces $s(r) = -\frac{1}{4M}\, r \frac{d}{dr}\frac{1}{r}\frac{d}{dr} [D(t)]_{FT}$

longitudinal force $F^{\parallel}(r) = p(r) + 2s(r)/3$

Mechanical densities from GFFs

$$T_{\mu\nu}(r) = \begin{bmatrix} T_{tt}(r) & T_{tj}(r) \\ T_{it}(r) & T_{ij}(r) \end{bmatrix} = \begin{bmatrix} \epsilon(r) & \\ & \left(\frac{r_i r_j}{r^2} - \frac{1}{d}\delta_{ij}\right) s(r) + \delta_{ij} p(r) \end{bmatrix}$$

1. Parametrize with GFFs (choose kinematics)

$$T_{\mu\nu}(t) \sim A(t)\, k^A_{\mu\nu}(t) + J(t)\, k^J_{\mu\nu}(t) + D(t)\, k^D_{\mu\nu}(t)$$

2. Fourier transform $T_{\mu\nu}(t) \rightarrow T_{\mu\nu}(r)$

$$[f(t)]_{\mathrm{FT}} = \int \frac{d^3\mathbf{\Delta}}{(2\pi)^3} e^{-i\mathbf{\Delta}\cdot\mathbf{r}} f(t)$$

3. Identify components → Spatial densities (3D Breit frame)

energy $\epsilon(r) = M\left[A(t) - \frac{t}{4M^2}\left(D(t) + A(t) \; - 2J(t)\right)\right]_{FT}$

pressure $p(r) = \frac{1}{6M}\frac{1}{r^2}\frac{d}{dr}\, r^2 \frac{d}{dr}\, [D(t)]_{FT}$

shear forces $s(r) = -\frac{1}{4M}\, r \frac{d}{dr}\frac{1}{r}\frac{d}{dr}\, [D(t)]_{FT}$

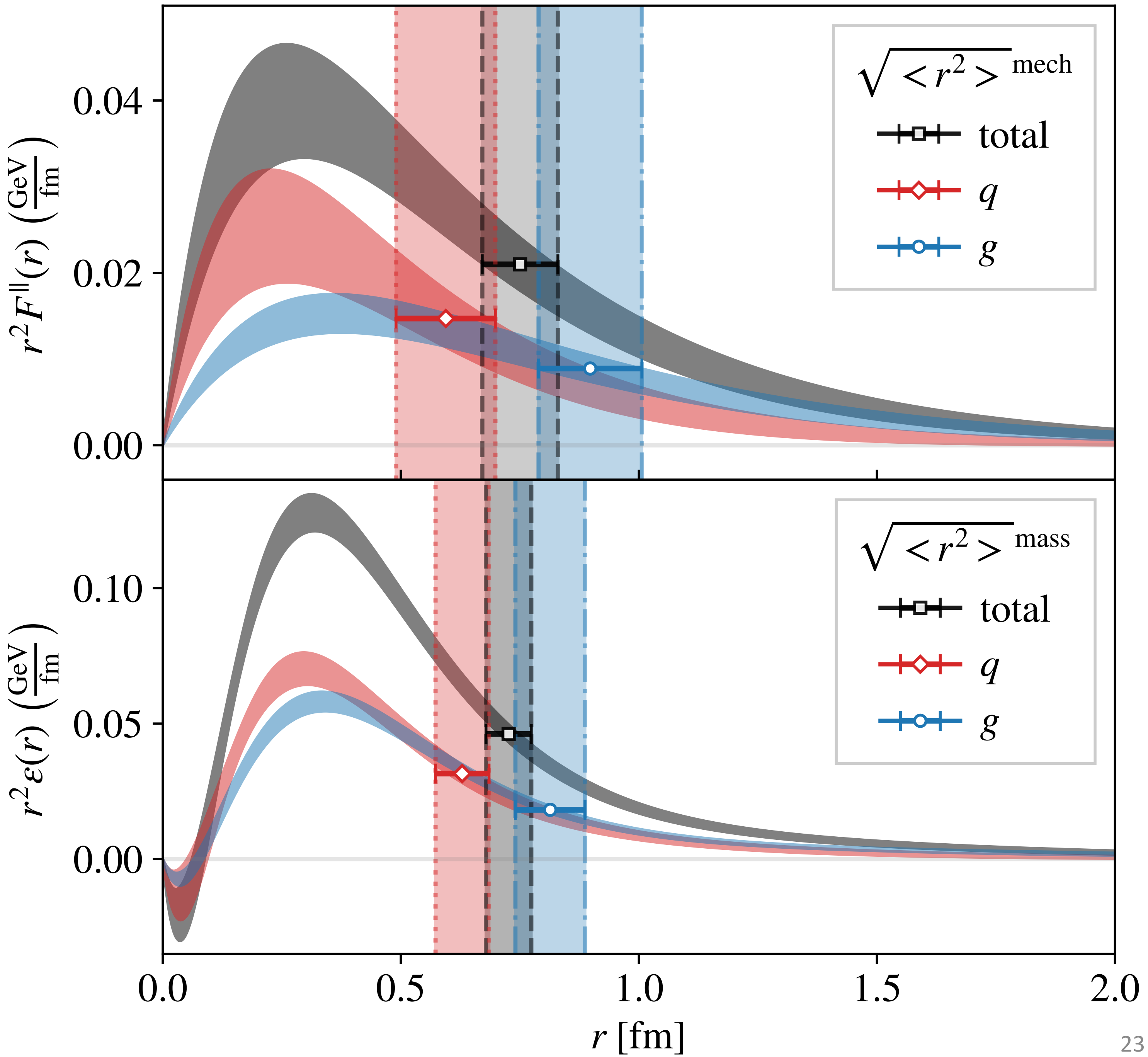
longitudinal force $F^{\parallel}(r) = p(r) + \; 2s(r)/3$

Caveat: physical significance of these analogies is under debate!

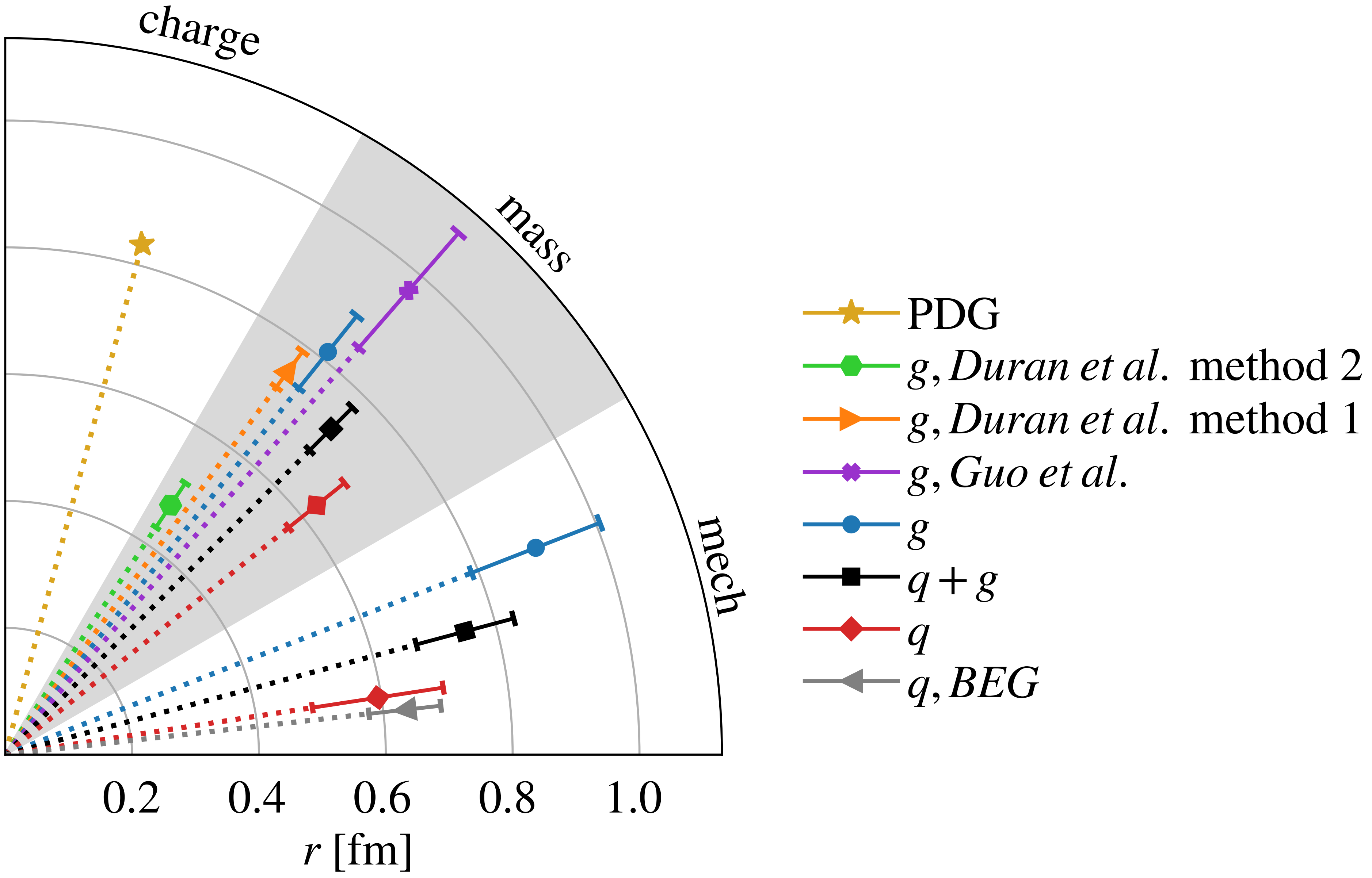
Densities & radii

$$\langle r_i^2 \rangle^{\text{mass}} = \frac{\int d^3\mathbf{r}\ r^2\ \epsilon_i(r)}{\int d^3\mathbf{r}\ \epsilon_i(r)}$$

$$\langle r_i^2 \rangle^{\text{mech}} = \frac{\int d^3\mathbf{r}\ r^2 F_i^{\parallel}(r)}{\int d^3\mathbf{r}\ F_i^{\parallel}(r)}$$



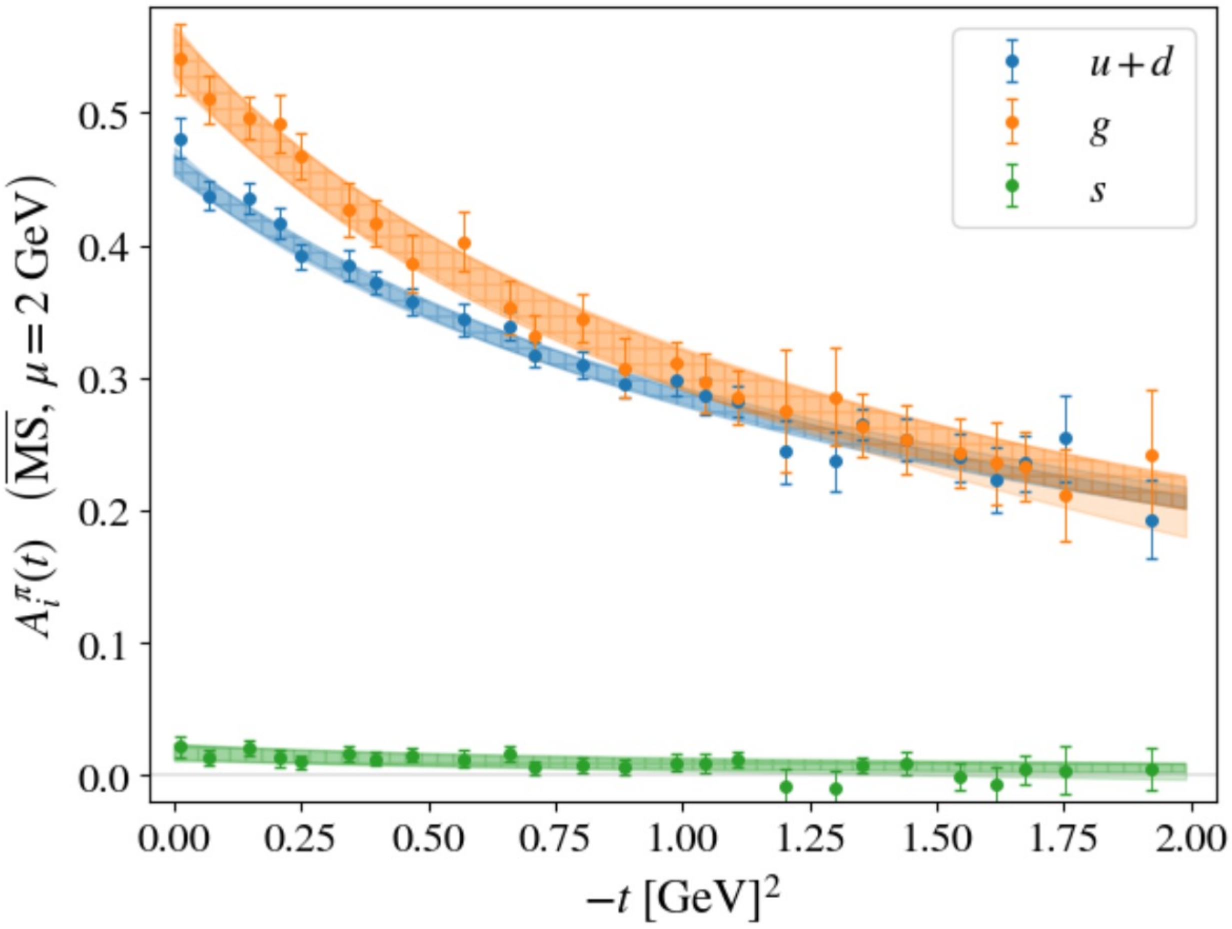
How big is a proton?

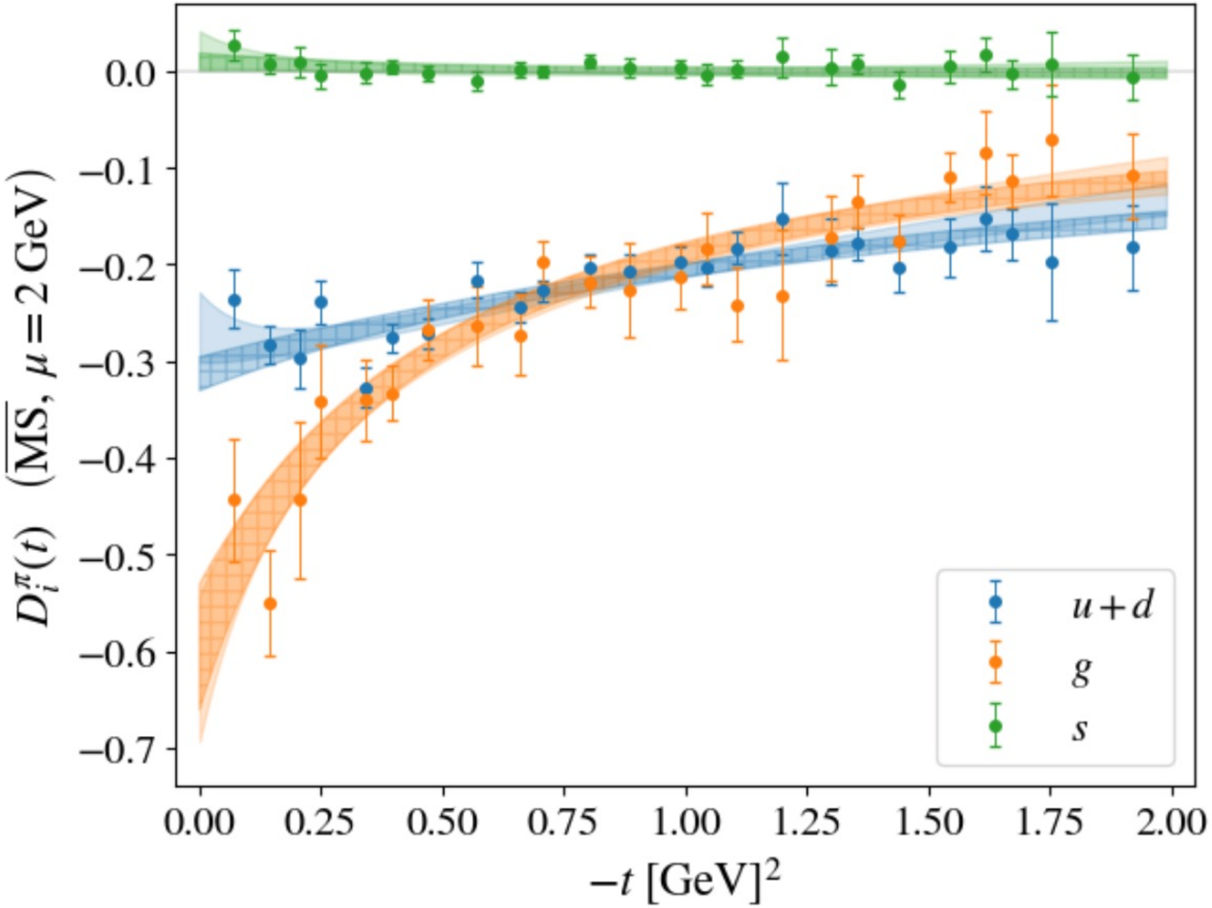


[Hackett Oare Pefkou Shanahan 2307.11707]

Pion GFFs (flavor decomp)

Hatched bands: monopole   Solid bands: z-expansion

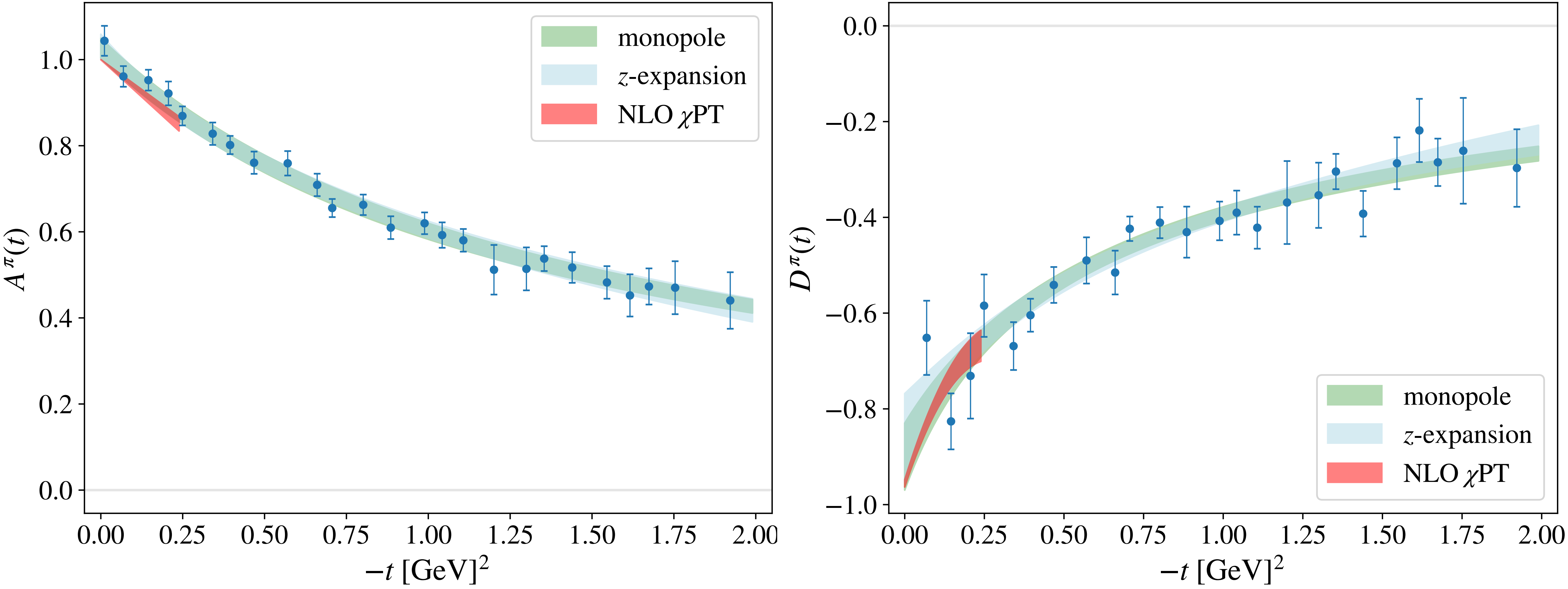




[Hackett Oare Pefkou Shanahan 2307.11707]

Pion GFFs (total)

Error on χPT estimate due to different estimates for LECs [Donaghue Leutwyler 1991]



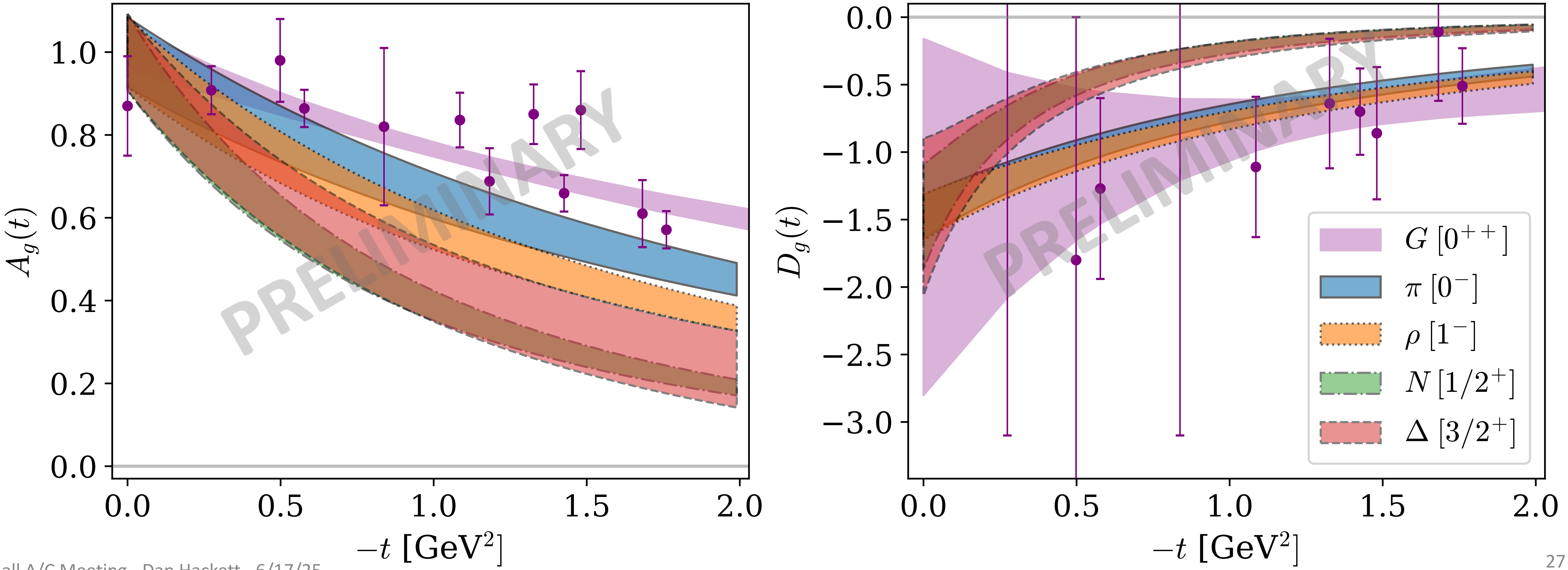
Scalar Glueball GFFs (in Yang-Mills)

[Abbott Hackett Pefkou Romero-Lopez Shanahan 2410.02706]

Glueball structure: new observables to discriminate among candidate observations?

Other hadrons from [Pefkou DH Shanahan 2107.10368]: $a \approx 0.11$ fm, $M_\pi \approx 450$ MeV

Normalized to match glueball GFFs at $A(0)$, $D(0)$



Conclusion

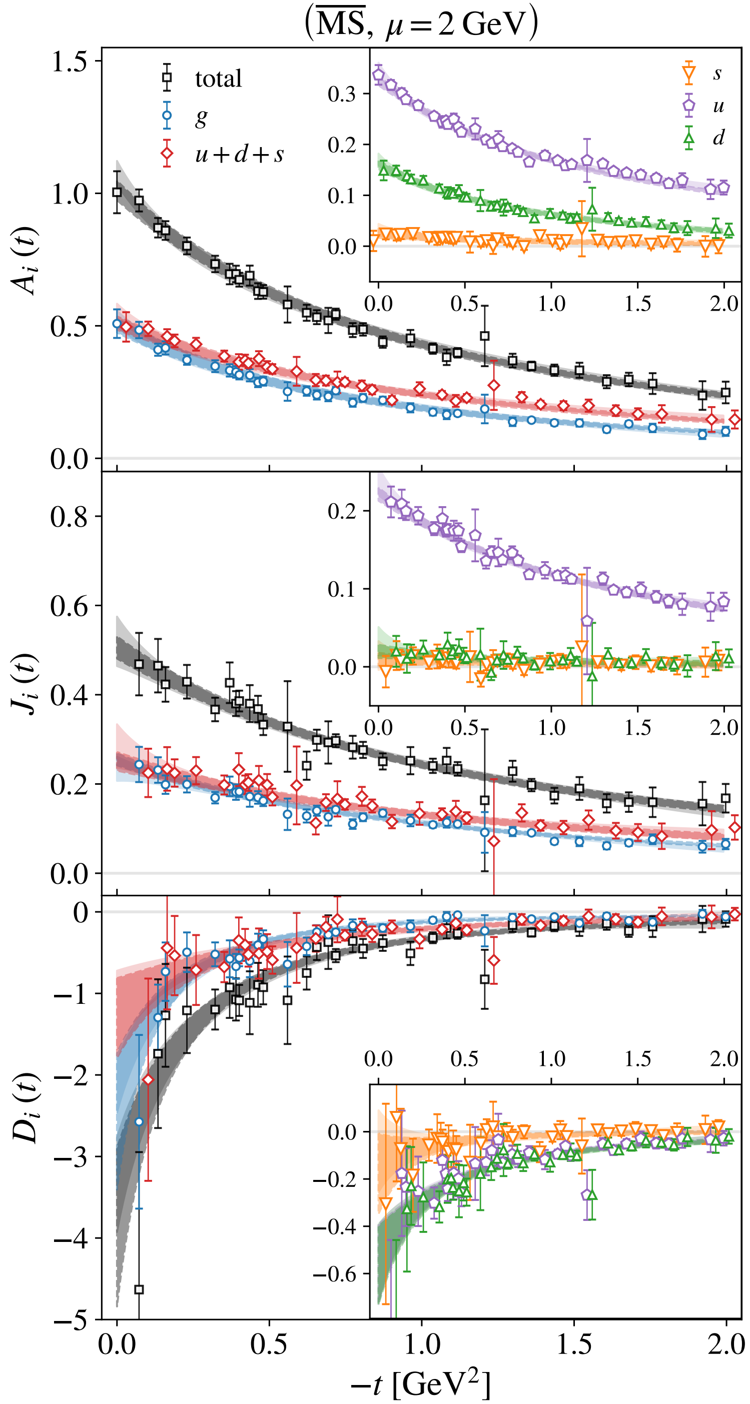
First lattice calculation of:

- complete flavor decomposition of nucleon GFFs
- *total* GFFs → *physical* (i.e. RGI) densities, radii
- $D(0)$

New first-principles descriptions of size and shape of nucleon

Towards a precision calculation, need:

- Multiple ensembles to take continuum & physical-mass limits
- Improved renormalization (GIRS? Flow? Sum rules?)
- Better methods to fully control excited state effects



Backup

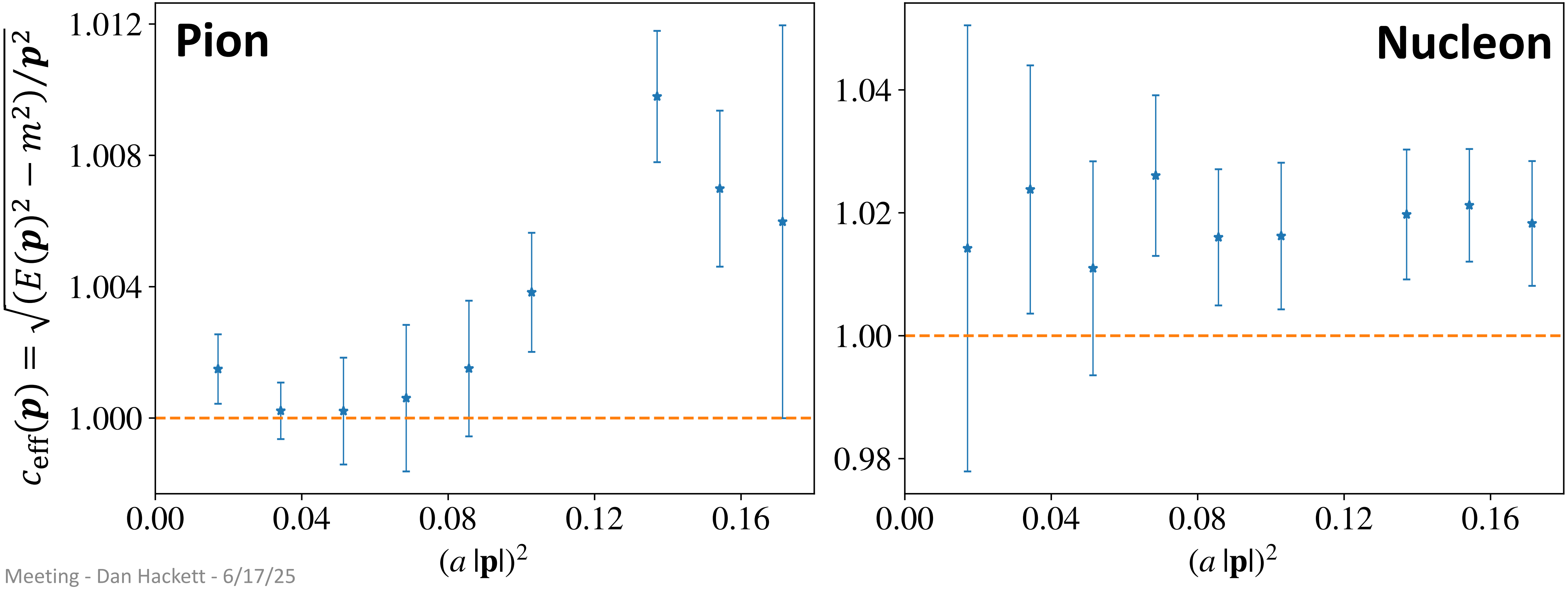
Two-point functions

Compute on 2511 configs, 1024 srcs/cfg (2x offset $4^3 \times 8$ grids)

Note: only one interpolating operator; both diagonal spin channels

Relativistic dispersion obeyed at ~ % level

→ Neglect errors in $aM_\pi = 0.0779$ and $aM_N = 0.4169$



Extracting bare matrix elements

1. Construct ratios

$$R(p,p';\tau,t_f) = \frac{C^{3\text{pt}}(p,p';t_f,\tau)}{C^{2\text{pt}}(p';t_f)} \sqrt{\frac{C^{2\text{pt}}(p;t_f-\tau)}{C^{2\text{pt}}(p';t_f-\tau)} \frac{C^{2\text{pt}}(p';\, t_f)}{C^{2\text{pt}}(p;t_f)} \frac{C^{2\text{pt}}(p';\, \tau)}{C^{2\text{pt}}(p;\tau)}}$$

$$= \#\, \boxed{\langle p'|T^b(\Delta)|p\rangle} + O\left(e^{-\Delta E\,\tau\,-\Delta E'(t_f-\tau)}\right)$$

2. Bin ratios together w/ same kinematic coeffs →

Number of distinct ratios:

conn q: 6982 → 3081

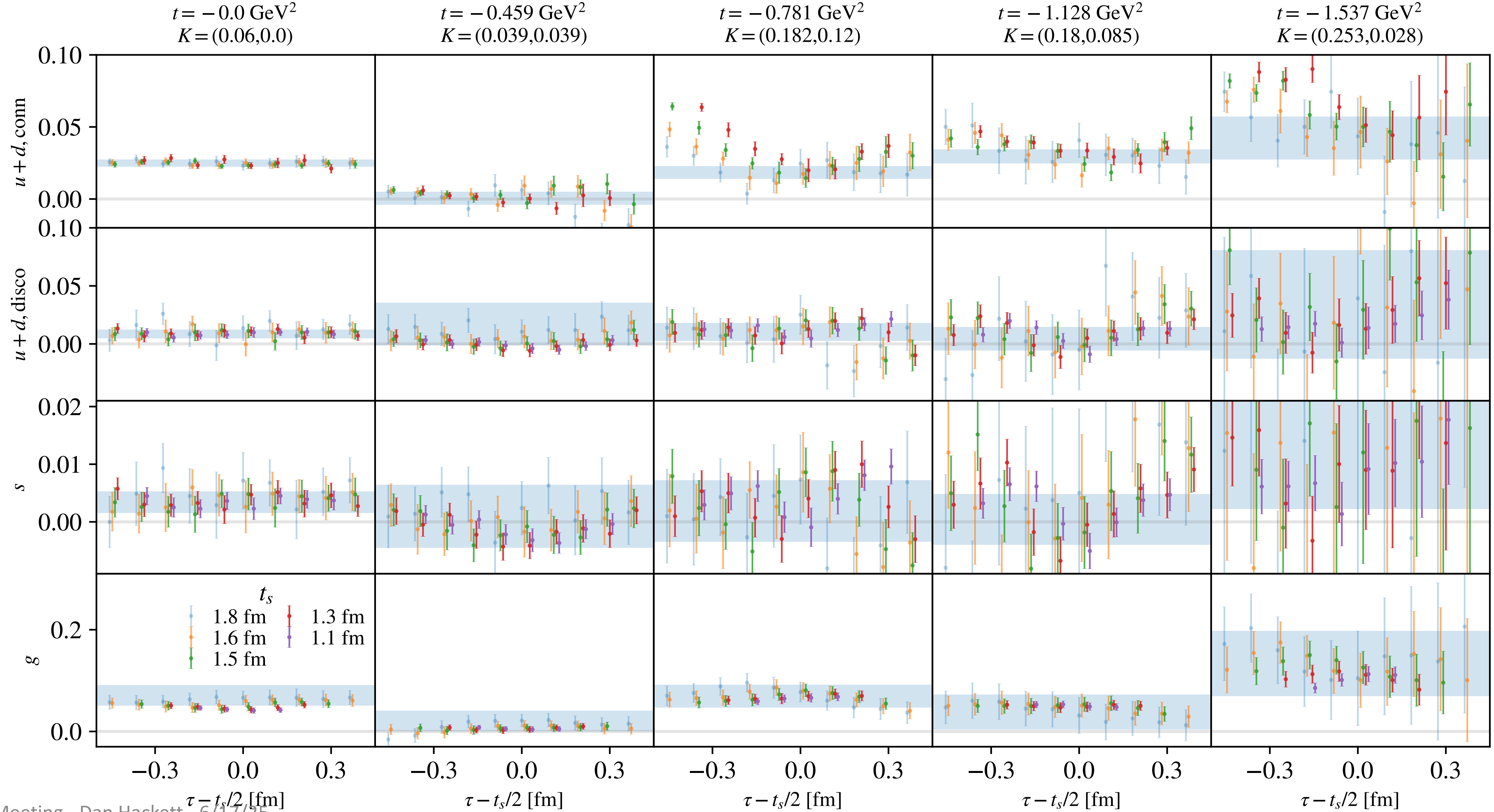
disc q/g: 1200296 → 11452

3. Fit using "summation method"

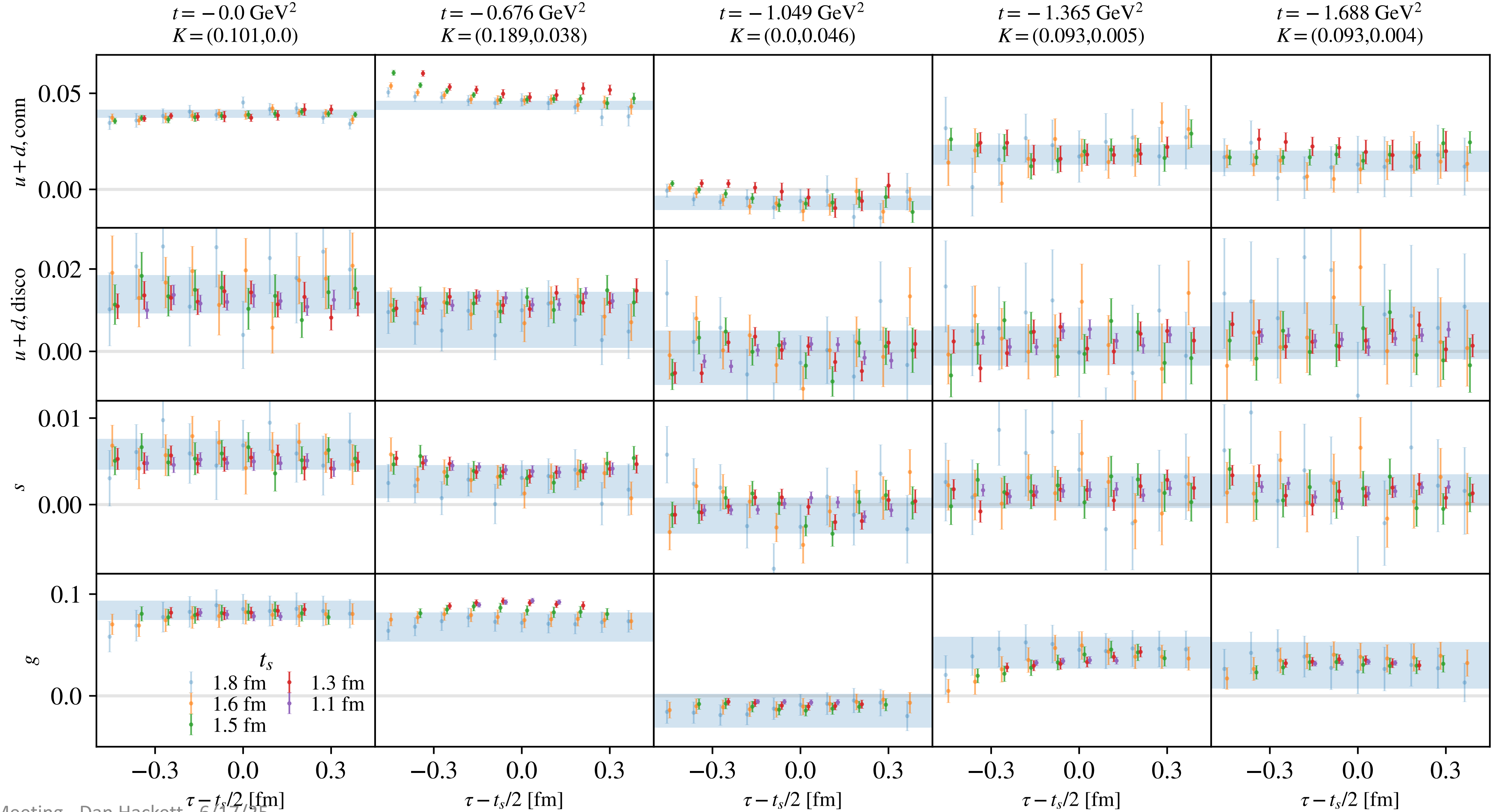
$$\Sigma(t_f) = \sum_{\tau=\tau_{\text{cut}}}^{t_f-\tau_{\text{cut}}} R(\tau,t_f) = (\text{const}) + \#\, \boxed{\langle p'|T^b(\Delta)|p\rangle}\, t_f + O\left(e^{-\delta E\, t_f}\right)$$

... w/ Bayesian model averaging over fit ranges, τ_{cut}

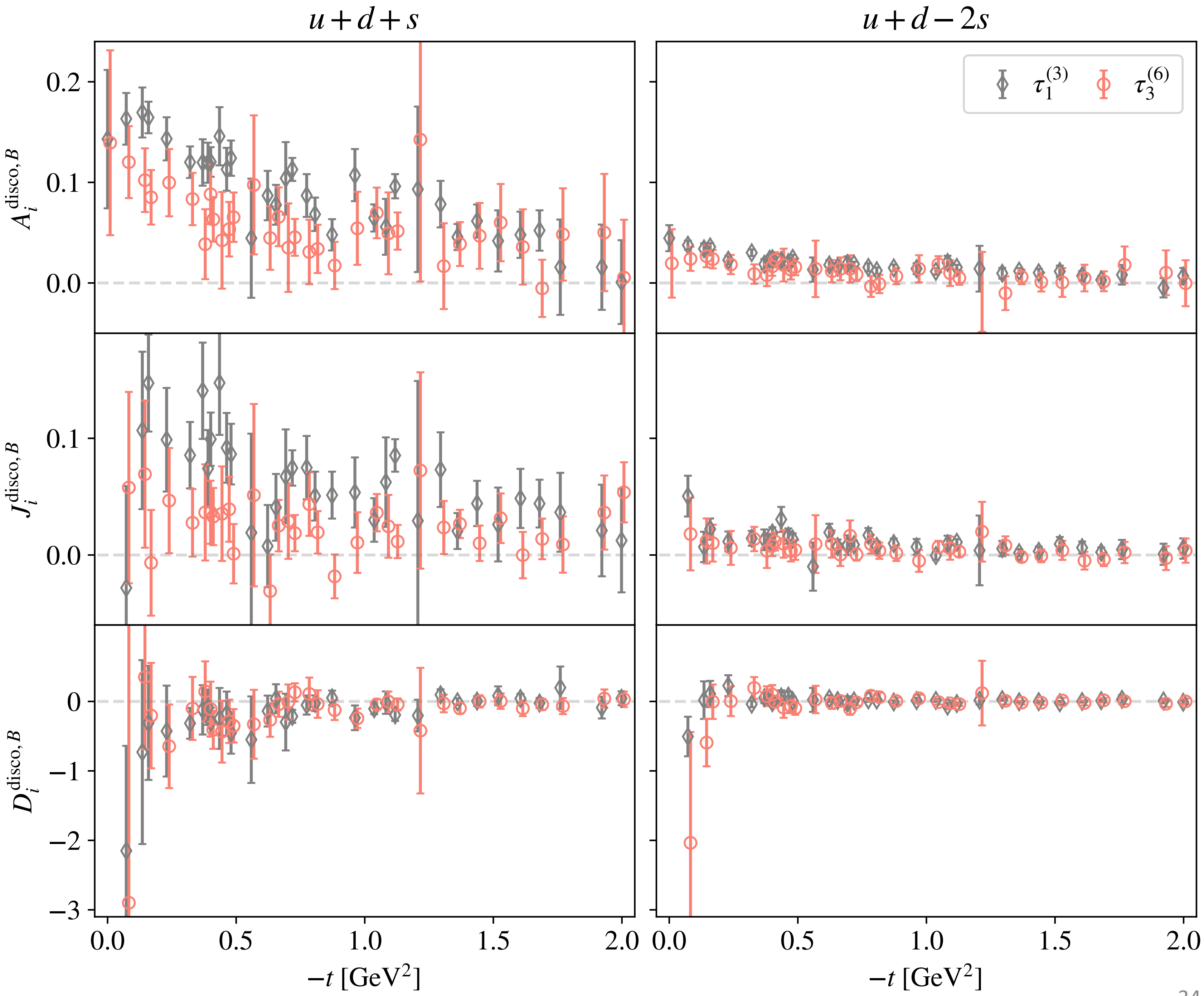
Example pion ratios: $\tau_1^{(3)}$



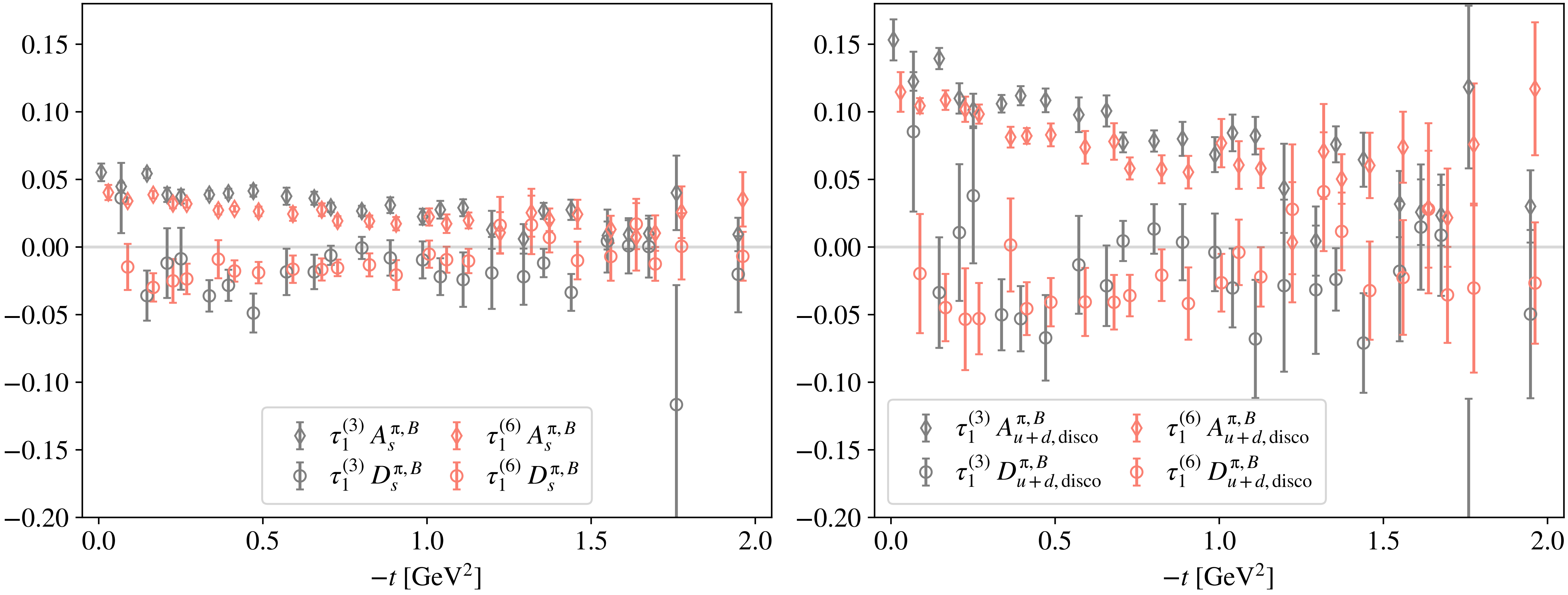
Example pion ratios: $\tau_3^{(6)}$



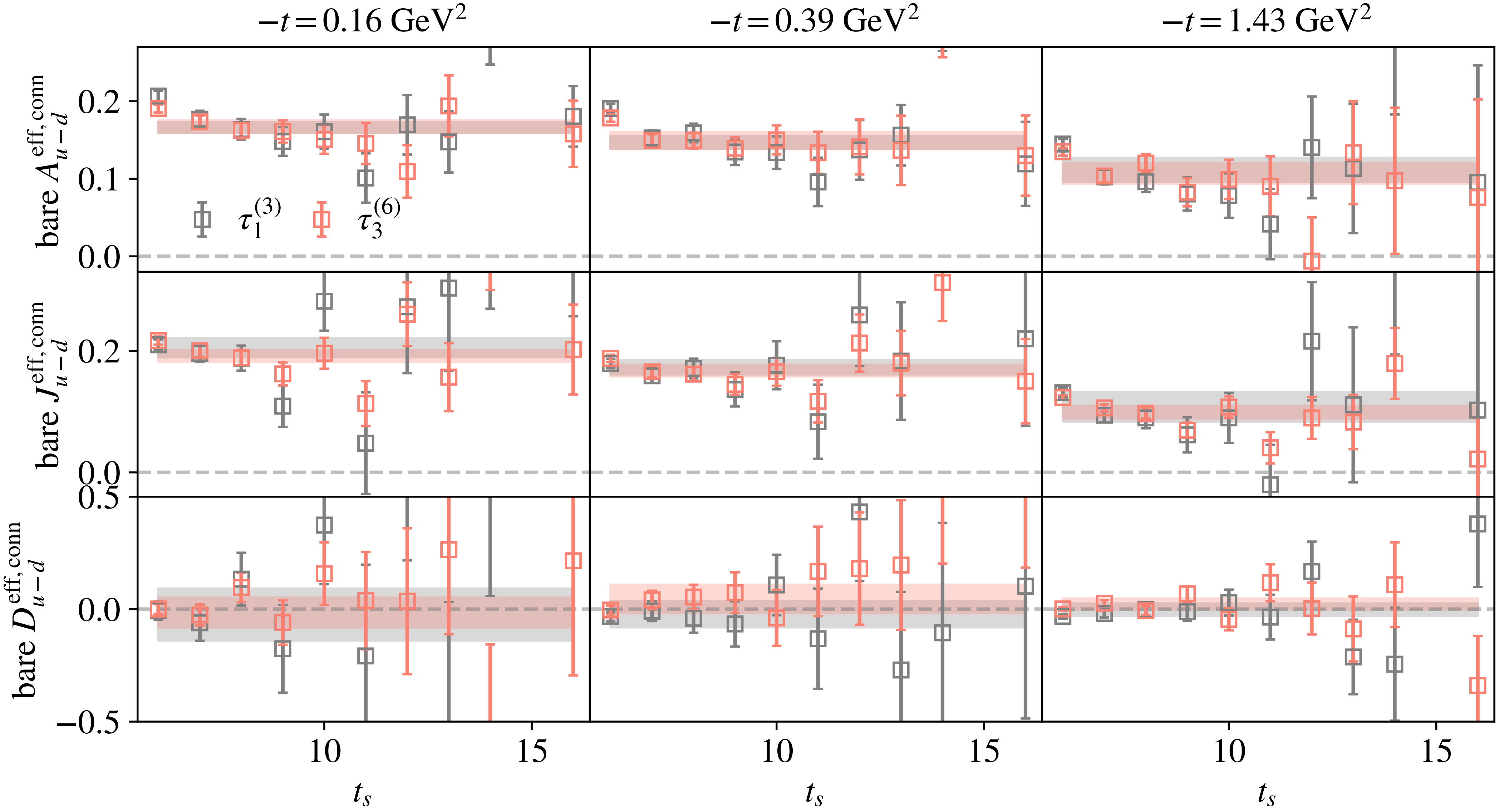
Nucleon: bare disconnected GFFs



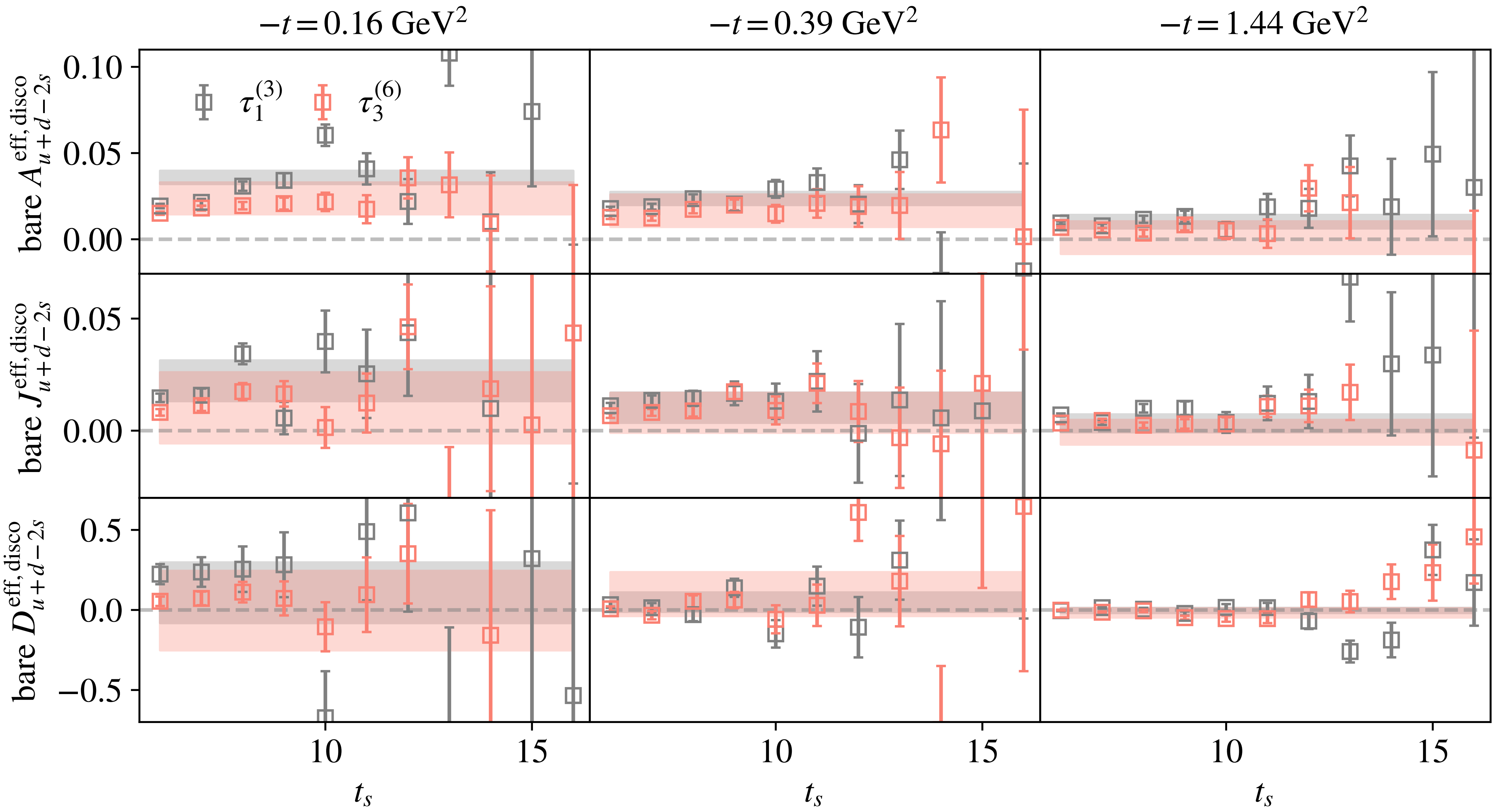
Pion: bare disconnected GFFs



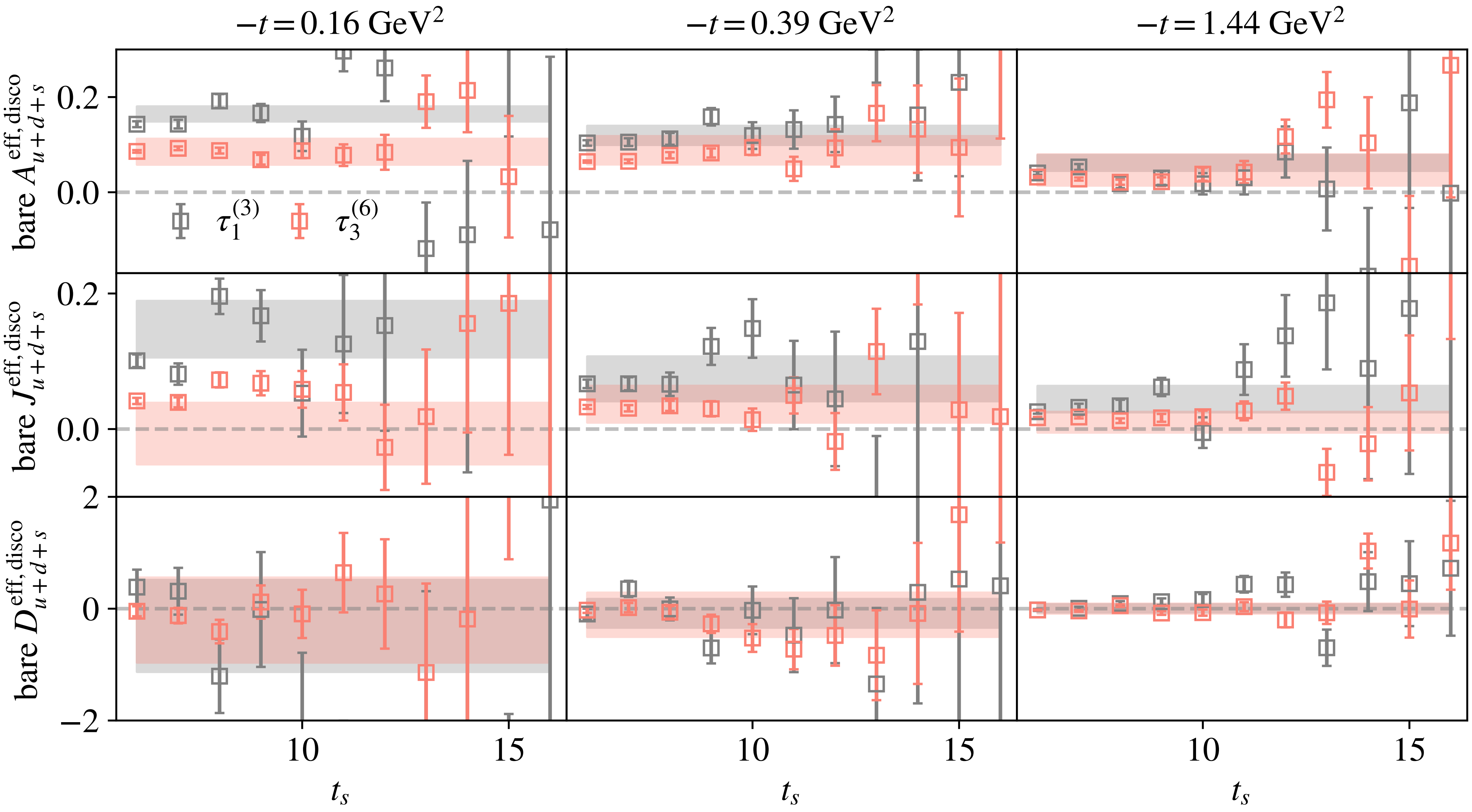
Nucleon: effective GFFs



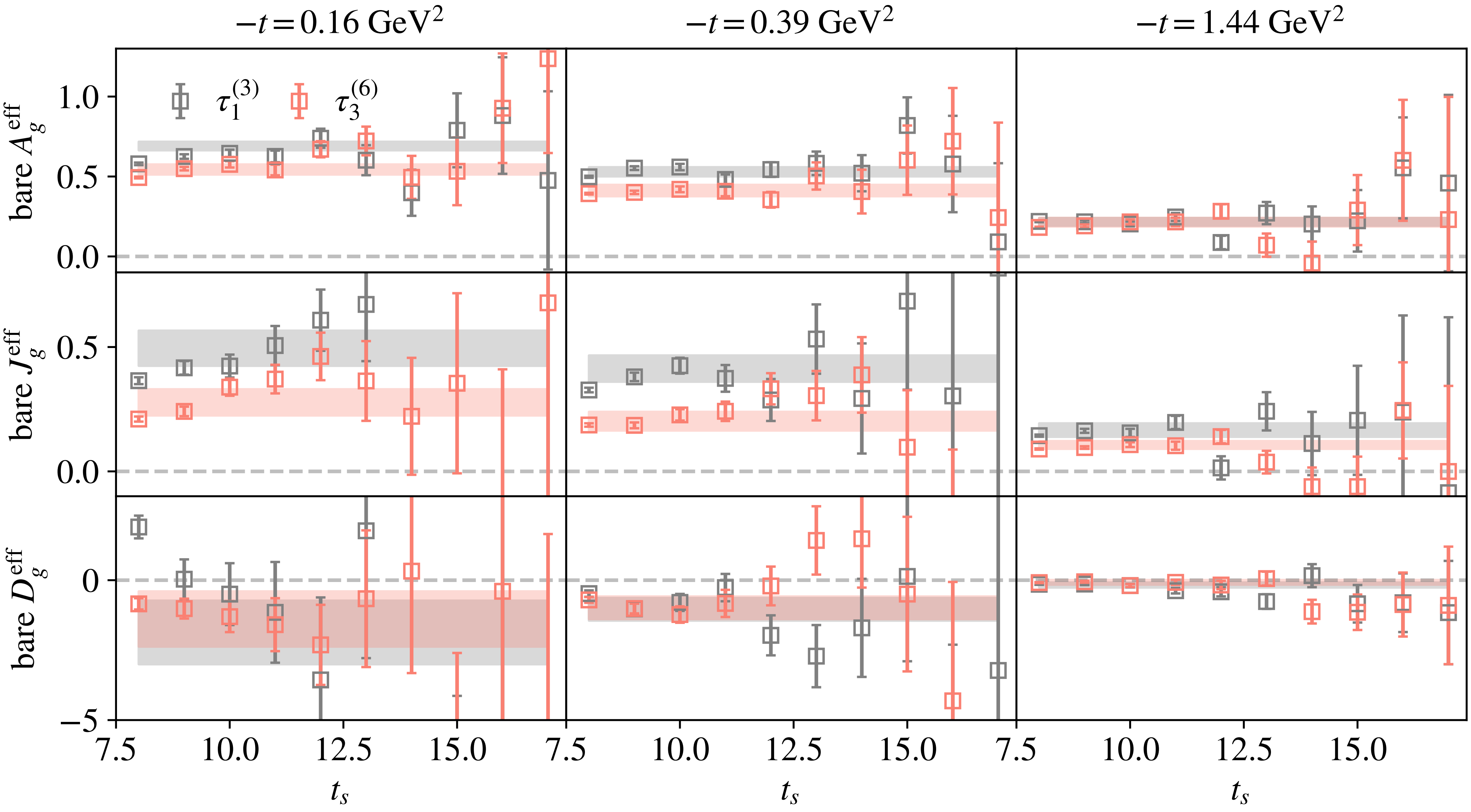
Nucleon: effective GFFs



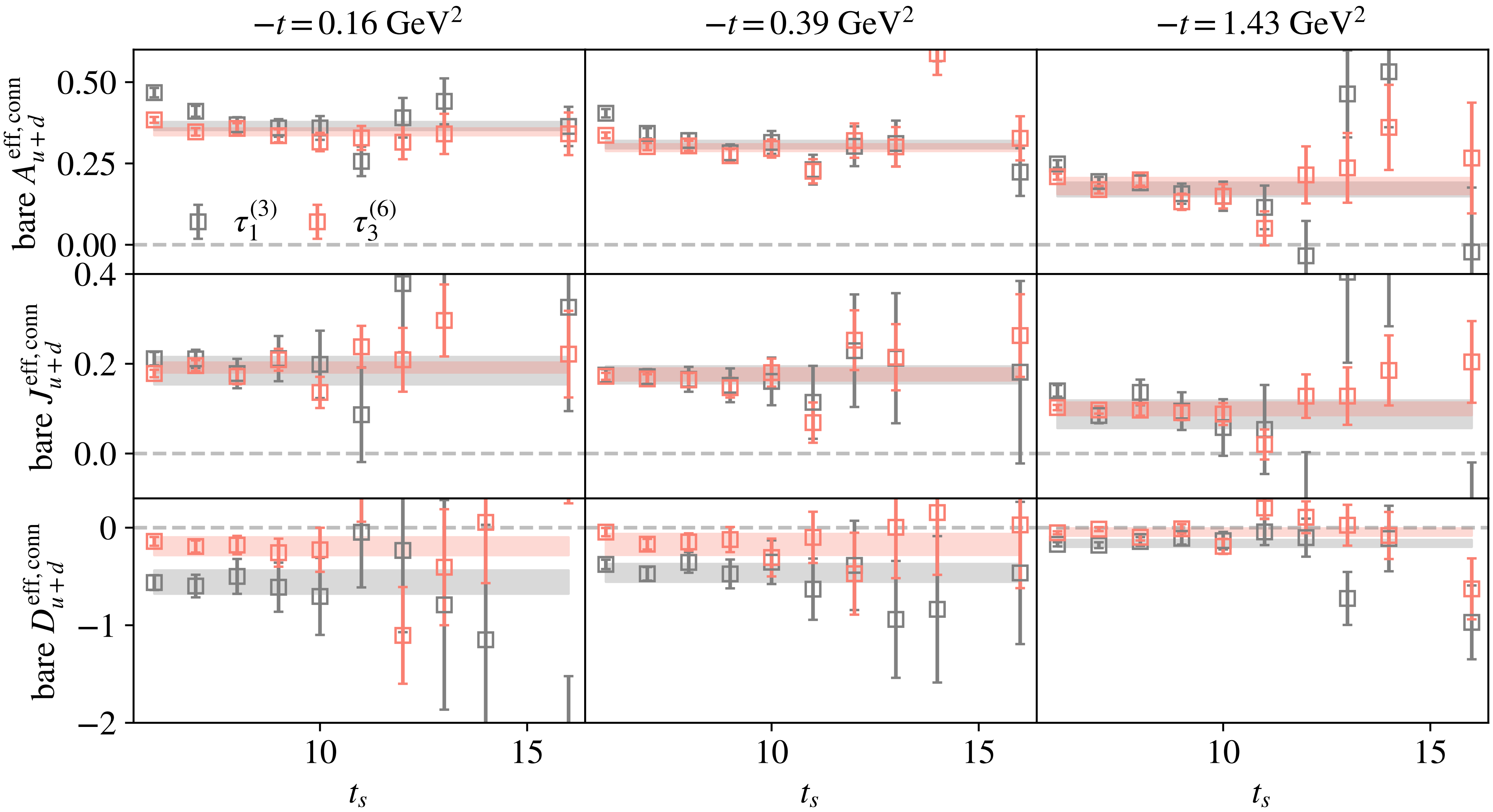
Nucleon: effective GFFs



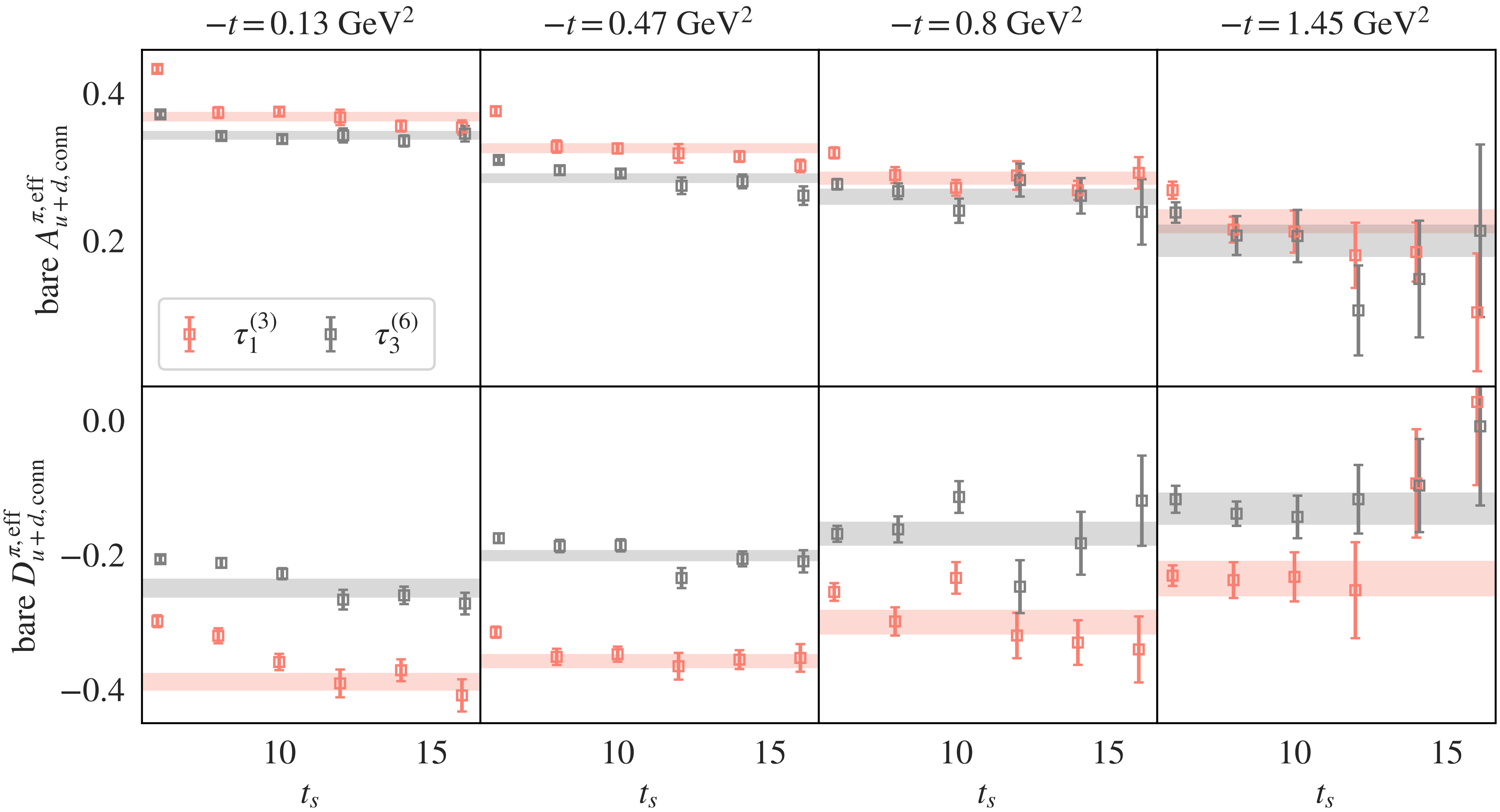
Nucleon: effective GFFs



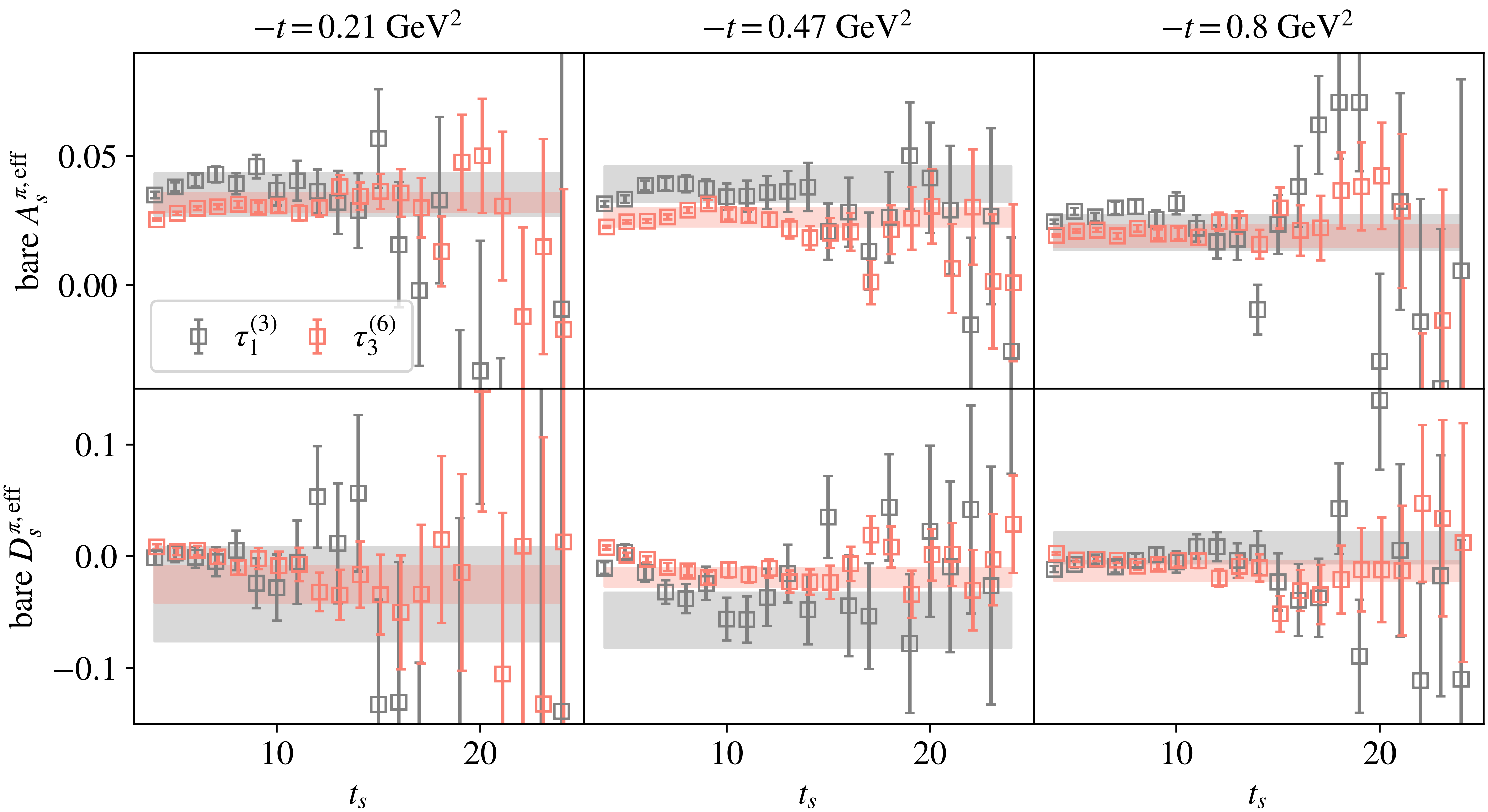
Nucleon: effective GFFs



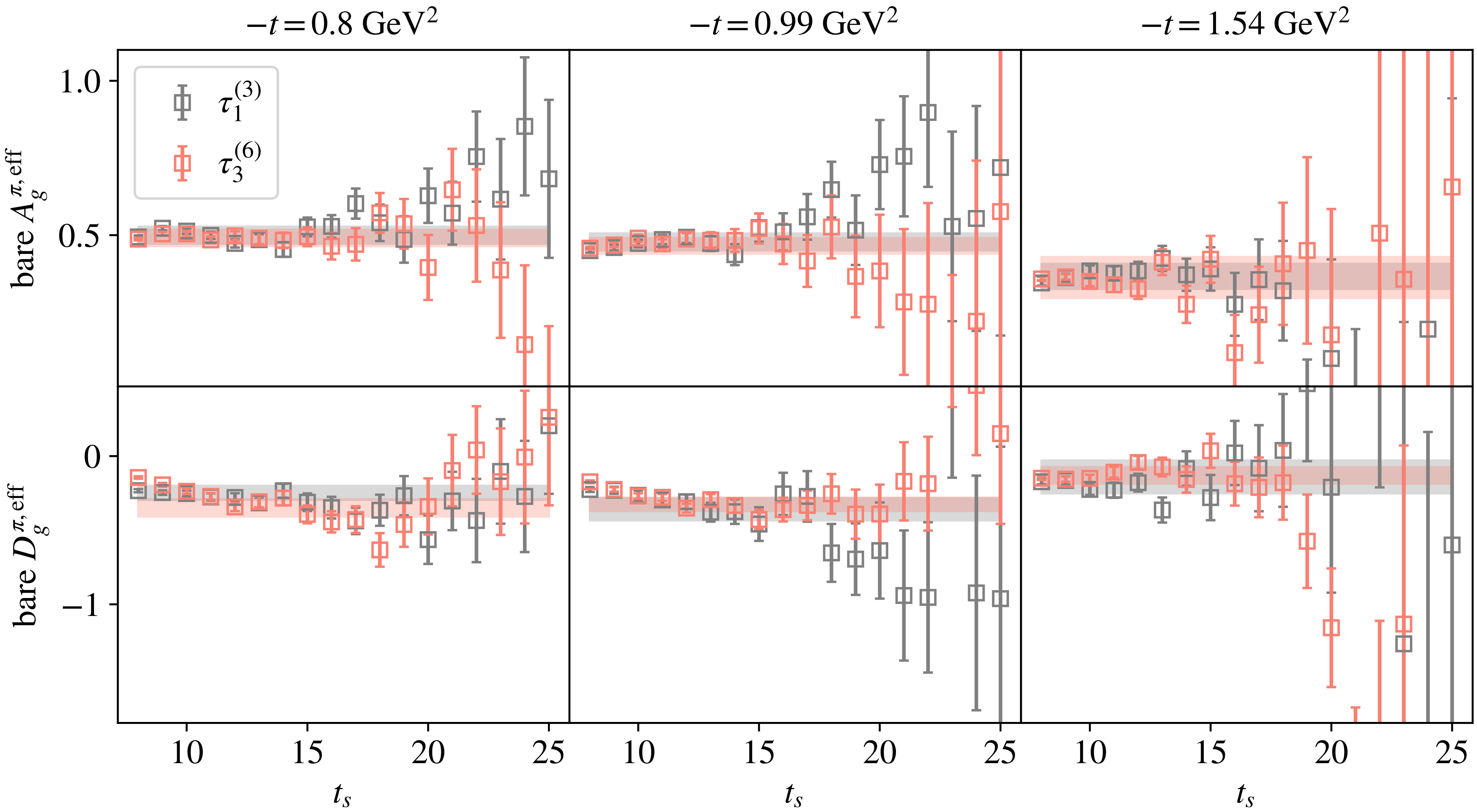
Pion: effective GFFs



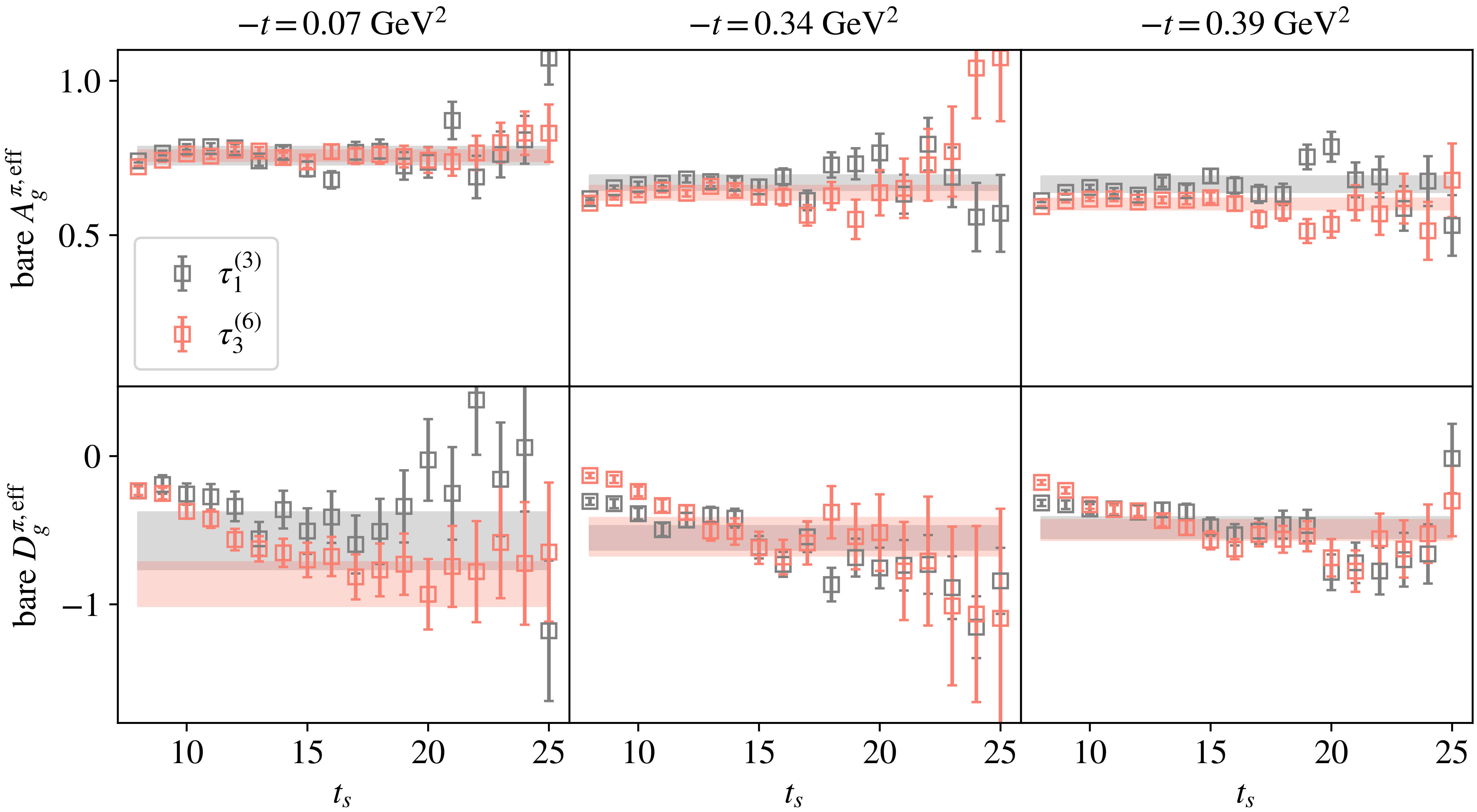
Pion: effective GFFs



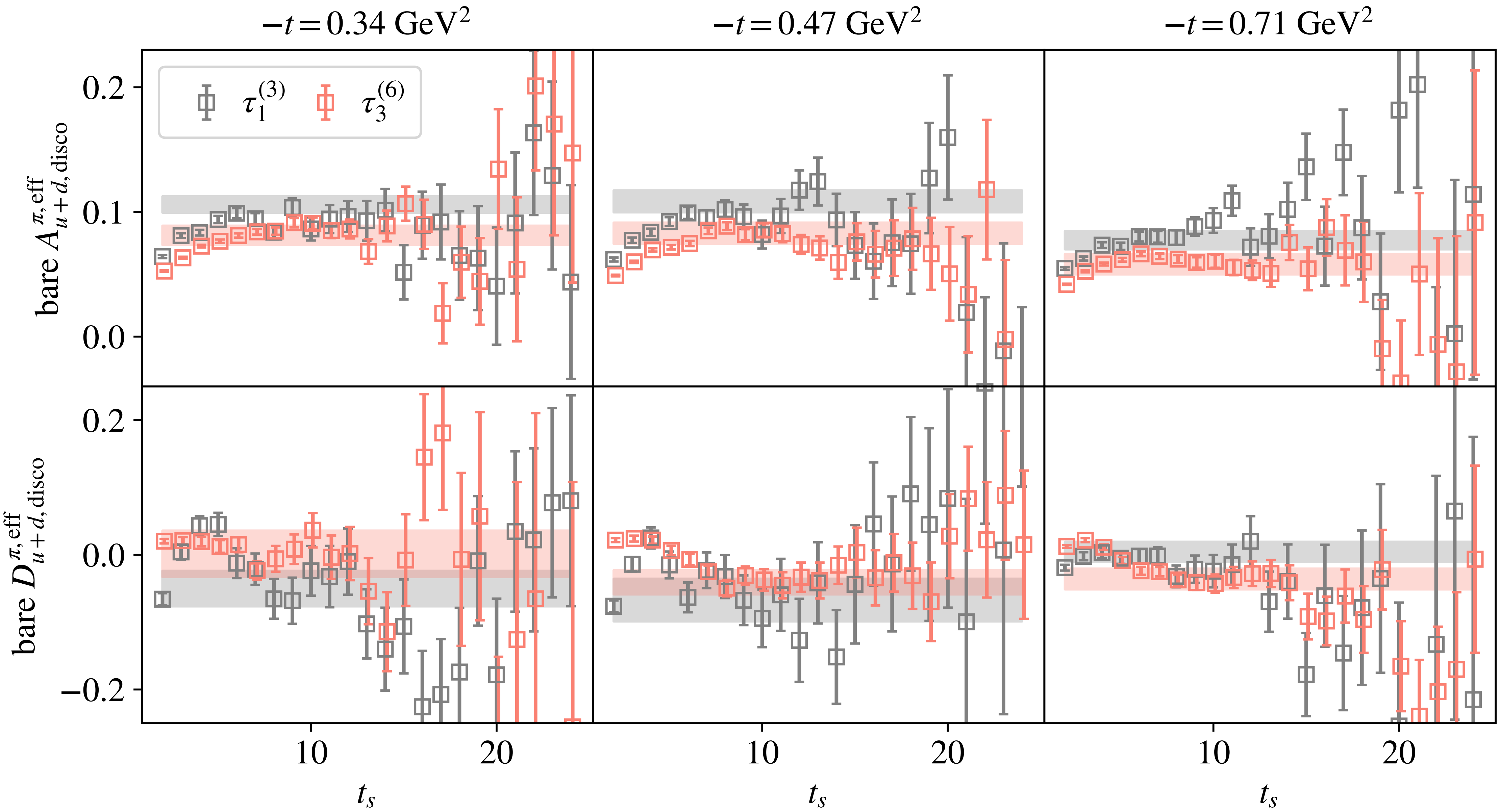
Pion: effective GFFs



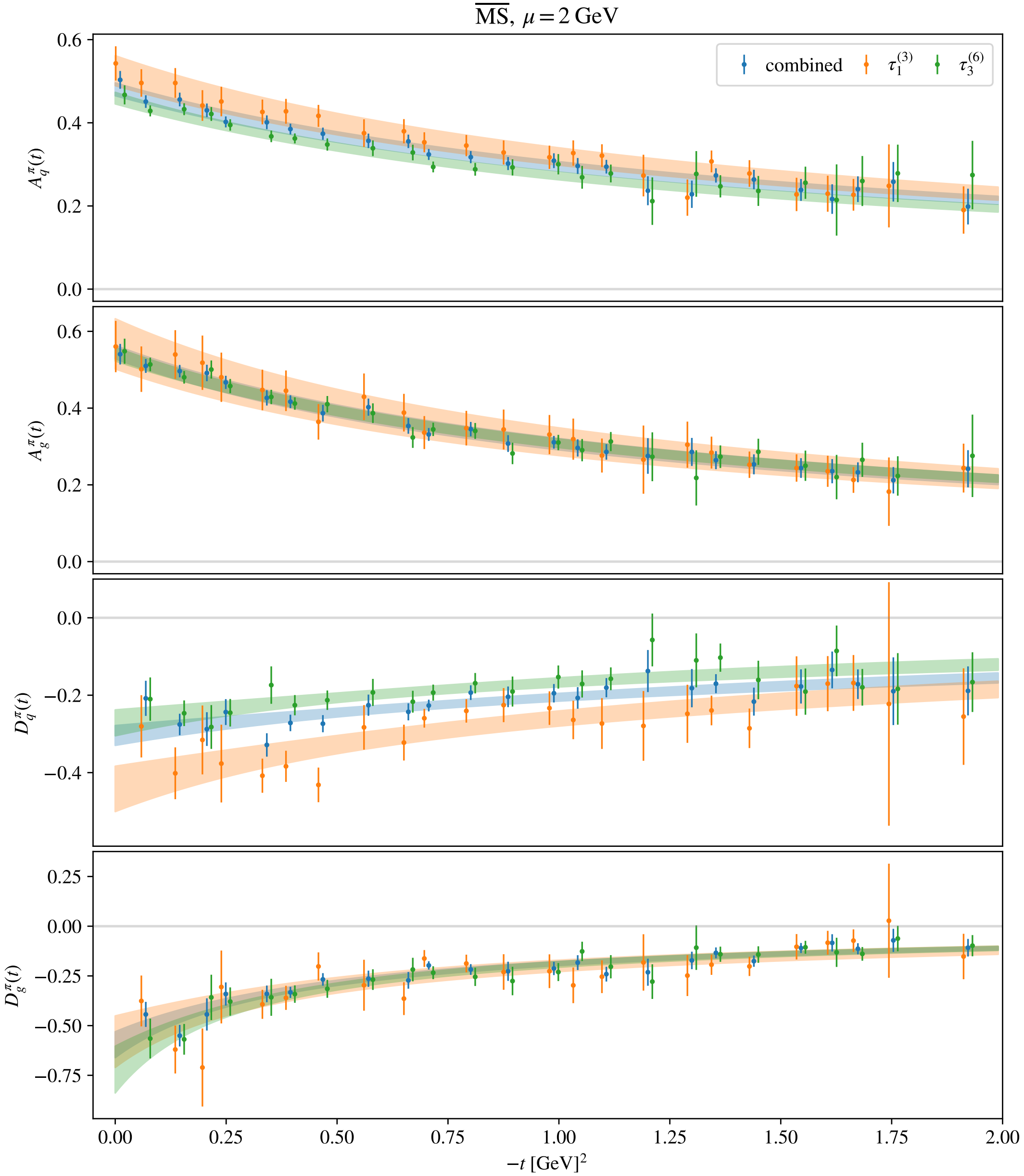
Pion: effective GFFs

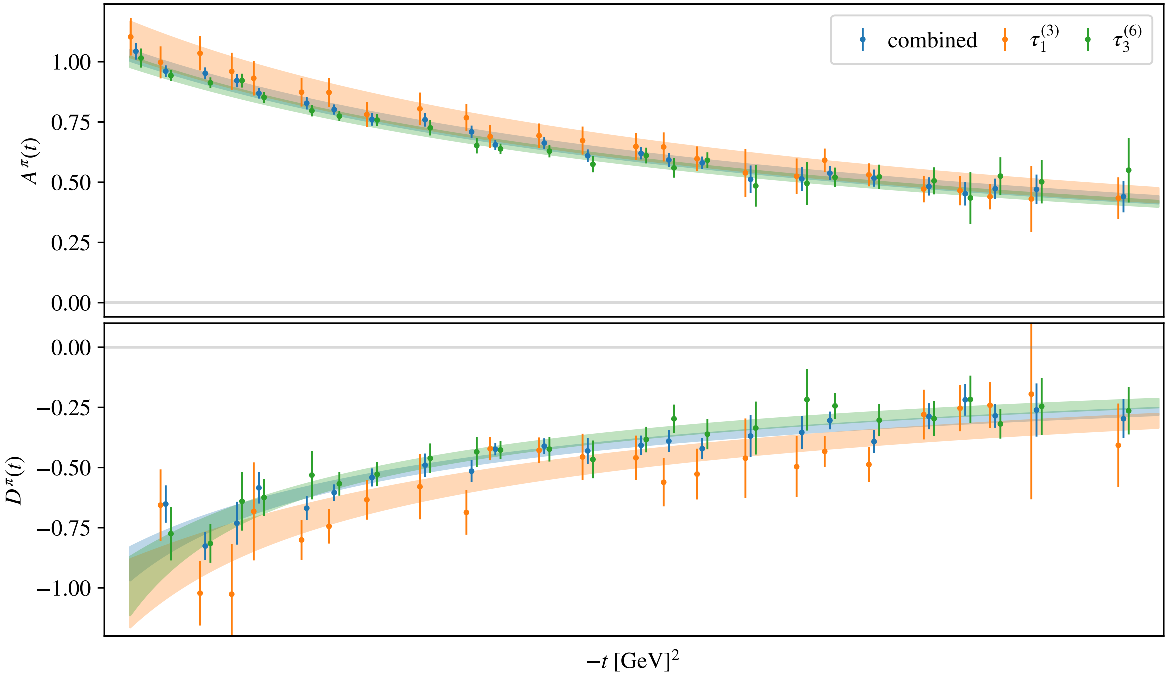


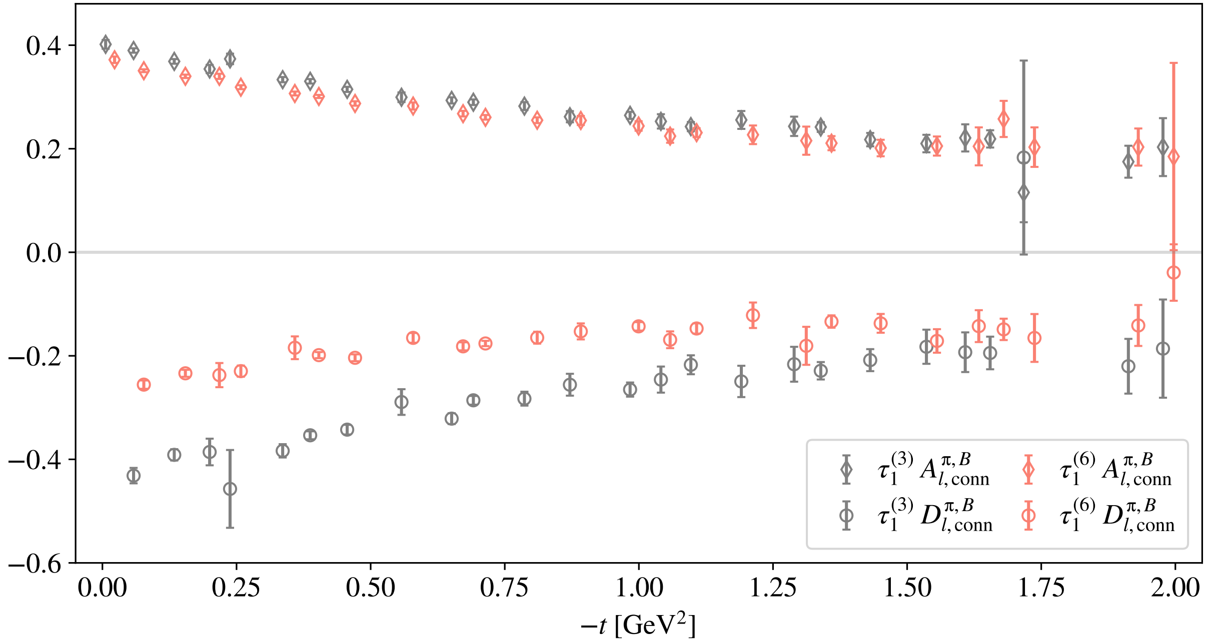
Pion: effective GFFs



Pion: split irreps







Nucleon: split irreps