Two-nucleon scattering by a central potential

Exercise 1

Consider two nucleons each of mass $m = 938$ MeV$/c^2$ interacting via a central potential given by

$$V(r) = \begin{cases} -V_0 & \text{for } r \leq R \\ 0 & \text{otherwise} \end{cases}$$

where V_0 is positive and $R = 1.45$ fm.

1. Phase Shift Computation:
   - Set $V_0 = 20\,\text{MeV}$.
   - Use the Numerov algorithm to compute the phase shifts $\delta_\ell(E)$ for $\ell = 0$ (S-wave), $\ell = 1$ (P-wave), and $\ell = 2$ (D-wave), as a function of the energy E in the range 0–200 MeV.
   - Plot the phase shifts δ_ℓ versus energy for each partial wave.
2. Radial Wave Functions:
   - Compute and plot the radial wave functions $u_\ell(r)$ for a few representative energies: $E = 1, 10, 100, 200\,\text{MeV}$, for $\ell = 0, 1, 2$.
3. Cross Sections:
   - Compute the partial cross sections

     $$\sigma_\ell(E) = \frac{4\pi}{k^2}(2\ell + 1)\sin^2\delta_\ell(E)$$

     and the total cross section

     $$\sigma_{\text{tot}}(E) = \sum_{\ell=0}^{2} \sigma_\ell(E)$$

     as a function of energy.
   - Plot the total and partial cross sections versus energy.
4. Validation via Analytical S-wave:
   - The S-wave phase shift δ_0 can be obtained analytically for a square well.
   - Use the analytic expression to validate your numerical code for $\ell = 0$. Compare the analytic and numeric phase shifts and cross sections.
5. Increased Potential Depth:
   - Repeat steps 1–4 with $V_0 = 60\,\text{MeV}$.

- Note that there is an ambiguity in the definition of the phase shifts. This will make your phase shift in S-wave discontinuous when approaching $\pi/2$. Make your phase shift continuous by adding the appropriate phase.

6. Repulsive Potential:

   - Repeat steps 1–4 with $V_0 = -60\,\mathrm{MeV}$.

Implementation

We seek the solution $u(r) = r\,R(r)$ of the radial equation:

$$-\frac{\hbar^2}{2\,\mu}u''(r) + V(r)\,u(r) + \frac{\hbar^2}{2\,\mu}\frac{\ell(\ell+1)}{r^2}\,u(r) = E\,u(r)\,, \tag{1}$$

where

$$E = \frac{\hbar^2\,k^2}{2\,\mu}\,, \quad \text{and} \quad \mu = \frac{m}{2}\,. \tag{2}$$

We define

$$v(r) = \frac{2\,\mu}{\hbar^2}\,V(r)\,, \tag{3}$$

and rewrite the equation above as

$$u''(r) + \left[k^2 - v(r) - \frac{\ell(\ell+1)}{r^2}\right]\,u(r) = 0\,,$$

or

$$u''(r) + K(r)u(r) = 0\,, \tag{4}$$

with

$$K(r) = \frac{2\,\mu}{\hbar^2}\,[E - V(r)] - \frac{\ell(\ell+1)}{r^2}\,. \tag{5}$$

As $r \to 0$, $u(r) \to r^{(\ell+1)}$. We use two grid points close to zero ($e.g.$, $r_1 = h$ and $r_2 = 2 \times h$, where h is the step in r), and the solution calculated in these points, that is $u_1 = u(r_1) = h^{\ell+1}$ and $u_2 = u(r_2) = (2 \times h)^{\ell+1}$, to start building the solution outwards using the Numerov's method.

The subsequent u_i's calculated at r_i's are obtained using the following algorithm

$$u_{i+1}\left(1 + \frac{h^2}{12}K_{i+1}\right) - u_i\left(2 - \frac{5\,h^2}{6}K_i\right) + u_{i-1}\left(1 + \frac{h^2}{12}K_{i-1}\right) + O(h^6) = 0\,.$$

You may use the notebook PhaseShiftExercise to implement and test your solution.

Exercise 2

Consider a reduced form of the Minnesota potential that consists of the sum of two Gaussians in the radial coordinate r:

$$v(r) = V_R(r) + V_t(r)\,, \tag{6}$$

where

$$V_R(r) = V_{0R}\, e^{-\kappa_R r}\,, \tag{7}$$
$$V_t(r) = -V_{0t}\, e^{-\kappa_t r}\,. \tag{8}$$

The parameters defining $v(r)$ are given in Table . The Minnesota potential is used in many-body calculations of nuclei.

V_{0R}	200.0 MeV	κ_R	1.487 fm^{-2}
V_{0t}	178.0 MeV	κ_t	0.639 fm^{-2}

Calculate the phase shifts and partial cross sections in S- D- and P-wave.

Exercise 3

The scattering of a nucleon from Oxigen-16 (mass number $A = 16$) can be modeled using a central Wood-Saxon interaction:

$$v(r)_{\rm WS} = -\,V_c\,\frac{1}{1 + e^{(\frac{r-R_c}{a})}} \tag{9}$$

where the parameters are:

$$V_c = -52.06 \text{ MeV}, \quad R_c = r_c\, A^{1/3},$$
$$r_c = 1.26 \text{ fm}, \quad a = 0.662 \text{ fm},$$

Calculate the phase shifts and partial cross sections in S- D- and P-wave. Note, you will have to use the appropriate reduced mass of the systems.