Spin lecture

I. INTRO, SPIN IN A PREFERRED FRAME (SLIDES, WILL BE ADDED HERE)

- What is spin (examples, audience, polarization, spin decomposition)
- Connection with rotations
- Density matrix (general, spin specific, reduced density matrix pure/mixed states, properties, diagonalization)
- entanglement entropy

Density matrix is a $2j+1$ by $2j+1$ Hermitian matrix. Can we decompose it on a basis of such matrices? Yes... We can use

1. The identity matrix
2. The 3 matrices of the rotation operators for spin j. These are Hermitian and traceless (SU(2)). This is a vector S^i for SU(2).
3. We can build higher order tensors that transform like SU(2) irreps ($L = 1, 2, \dots$) by taking symmetrized and traceless products of $S^i S^j, S^i S^j S^k, \dots$. We cannot use antisymmetrized products (SU(2) algebra). This generates orthogonal subspaces...multipole decomposition. This procedure has to stop (we run out of independent Hermitian matrices, the limit is $L = 2j$), which can be put on mathematical ground by using the Cayley-Hamilton theorem. In the end we have
$$\rho = \frac{1}{2j+1}\left(\mathbb{1} + s^i S^i + t^{ij}\mathcal{S}[S^i S^j] + \dots\right), \tag{1}$$
where the **real** coefficients (lower case) multiplying the matrices (upper case) share the same symmetry in indices (symmetric and traceless), which can be used to determine the number of independent polarization parameters. The s^i form a polarization vector, the t^{ij} a polarization tensor. Note that for $L > 1$ normalization coefficients of the different terms are conventional. For spin 1/2 this reduces to the well-known
$$\rho = \frac{1}{2}(1 + \boldsymbol{s}\cdot\sigma) \tag{2}$$
4. This multipole decomposition can also be understood from spin coupling:
$$j \otimes j = 0 \oplus 1 \oplus \dots \oplus 2j. \tag{3}$$
5. Note that the polarization coefficients can be obtained as the expectation values of the operators associated with the matrices they multiply:
$$\langle \hat{S}^i \rangle = \mathrm{Tr}[\rho S^i] = s^i \tag{4}$$
$$\langle \mathcal{S}[\hat{S}^i \hat{S}^j] \rangle = \mathrm{Tr}[\rho \mathcal{S}[S^i S^j]] \propto t^{ij} \tag{5}$$
$$\tag{6}$$

[Exercise:] How many independent parameters does the tensor polarization need? Use arguments based on 3 or 4 above.

Fun facts: the spin-1 $\lambda = 0$ pure state has $\boldsymbol{s} = 0$. Each spin-1 pure state has tensor polarization!

II. RELATIVITY, BOOSTS

Things get more involved once we include relativity and boosts, the possibility to change reference frames. The rotation group gets enlarged to the Lorentz group with commutation relations

$$\begin{aligned}
[\mathbb{J}_l, \mathbb{J}_m] &= \mathrm{i}\epsilon_{lmn}\mathbb{J}_n, \\
[\mathbb{J}_l, \mathbb{K}_m] &= \mathrm{i}\epsilon_{lmn}\mathbb{K}_n, \\
[\mathbb{K}_l, \mathbb{K}_m] &= -\mathrm{i}\epsilon_{lmn}\mathbb{J}_n.
\end{aligned} \tag{7}$$

We see that Boosts $\mathbb{K}_m$ transform as a vector under rotations, but more interestingly that the combination of boosts results in a *rotation*! Given the link between spin and rotations this already hints at boosts being able to cause spin rotation effects.

First, we discuss how things transform and we can get unitary representations of the Lorentz group for single particle states. This means we will detail what exactly we mean by something like $|\boldsymbol{p}, j\lambda\rangle, |\boldsymbol{p}', j\lambda\rangle$, and how we can connect them by boosts. In the following Λ denotes a general Lorentz transformation (can be boosts, rotation or a combination) and L_p is what is called the *standard boost*, and the discussion is specific for massive particles. We start in the rest frame of the massive particle, which has the reference momentum $\mathring{p}^\mu \equiv (m, \boldsymbol{0})$. We then consider representations for the *little group*, which is the subset of Lorentz transformations that keep the rest frame momentum invariant. These are the rotations of course, so we get SU(2) representations (spin) $|\mathring{p}, \lambda\rangle$. Next we define states in a moving frame by the following *definition*:

$$|p, j\lambda\rangle \equiv U[L_p]\, |\mathring{p}, j\lambda\rangle\,. \tag{8}$$

We will detail what $L_{\boldsymbol{p}}$ is precisely in a minute (for now the only thing that is important is that $L_p\, \mathring{p} = p$). Before we do so, we consider the effect of a general boost on these states

$$U(\Lambda)|p, j\lambda\rangle = U(\Lambda)U[L_p]\; |\mathring{p}, j\lambda\rangle = U(L_{\Lambda p})U^{-1}(L_{\Lambda p})U(\Lambda)U[L_p]\; |\mathring{p}, j\lambda\rangle = U(L_{\Lambda p})U(L^{-1}_{\Lambda p}\Lambda L_p)\; |\mathring{p}, j\lambda\rangle \tag{9}$$

Here we introduced $U(L_{\Lambda p})$ and its inverse to relate our states to the $|\Lambda_p, j\lambda$ states. If we consider the combination of Lorentz transformations $L^{-1}_{\Lambda p}\Lambda L_p$, we see that it leaves the rest frame momentum $\mathring{p}$ *invariant*, which means it can only be a rotation! This is the so-called **Wigner rotation** $R_w[\Lambda, p, L]$. Note that besides the actual Lorentz transformation Λ, it **also** depends on the momentum p and the choice of standard boost. However, this allows us to write

$$U(\Lambda)|p, j\lambda\rangle = U(L_{\Lambda p})|\mathring{p}, j\lambda'\rangle\, \mathcal{D}_{\lambda'\lambda}(R_w[\Lambda, p, L]) = |\Lambda p, j\lambda\rangle \mathcal{D}_{\lambda'\lambda}(R_w[\Lambda, p, L]), \tag{10}$$

which furnishes a unitary but infinitely dimensional representation of the Lorentz group.

The conclusion is that boosts, in general, cause spin rotations. Standard boosts acting on rest frame states by definition do not.

Short aside: by combining creation/annihilation operators with the $\mathcal{D}(L_p)$ part of the Wigner rotation one gets objects that transform with $\mathcal{D}(\Lambda)$, and we get **finite**-dimensional (no more p dependence) but **non-unitary** representations of the Lorentz group, which forms the basis of the field construction.

Before we continue, what are the standard boosts?

In principle, anything that gets you to p is possible, but there are 3 common choices:

1. Canonical L_c: pure boost in the direction of the momentum.
2. Helicity L_h: boost in the z-direction to $|\boldsymbol{p}|$ followed by a rotation to the final direction (θ, ϕ).
3. Light-front helicity L_f: boost in the z-direction to $|\boldsymbol{p}|$ followed by transverse light-front boosts (mixing p^- and p^1, p^2) to the final momentum.

They each have specific advantages:

1. Canonical: the Wigner rotation of a rotation r is that rotation

$$R_w([r, p, L_c]) = r. \tag{11}$$

   This is convenient when coupling angular momenta: all states/particles if represented by canonical states transform with the same Wigner rotation under a rotation, which allows for the use of CG coefficients as is usual in NRQM.

2. Helicity: the Wigner rotation of a rotation r is just a phase, so there is no spin rotation in that case. Hence helicity is conserved under rotations...
3. Light-front helicity: the light-front boosts form a closed subalgebra (any combination of light-front boosts does not result in a rotation). As a consequence the Wigner rotation of a light-front boost is the identity.

$$R_w([\Lambda_{LF}, p, L_f]) = \mathbb{1}. \tag{12}$$

So we can light-front boost all we want, this does not induce additional spin rotations for light-front helicity states.

How are states defined with different standard boosts defined? Let's consider two (general) different standard boosts L_α, L_β, we consider

$$|p, j\lambda\rangle_\alpha = U(L_{\alpha p})|\overset{\circ}{p}, j\lambda\rangle = U(L_{\alpha,p})U^{-1}(L_{\beta,p})\,|p, j\lambda\rangle_\beta = U(L_{\alpha,p}L^{-1}_{\beta,p})|p, j\lambda\rangle_\beta \equiv |p, j\lambda'\rangle\mathcal{D}(R_M[p, L_\alpha, L_\beta]), \tag{13}$$

where the last step is possible given that the combination of the two standard boosts is a rotation (leaves the rest frame momentum invariant) and the resulting rotation is called the **Melosh rotation**, which is again **momentum dependent**. Thanks to Melosh rotations we can always relate two different sets of spin-states that differ by the choice of standard boost. If we consider a state $|p, j\lambda\rangle$, depending on which choice of standard boost we pick, we end up with a different basis of spin states in its rest frame, where all those bases are related by their respective Melosh rotations. [A particle has an infinite number of rest frames, all related by rotations].

An example is the appearance of the Melosh rotations in the light-front deuteron wave function. They appear in the transition from light-front to canonical spin states, where the latter are the ones that are used in the angular momentum coupling of the two nucleons to the deuteron $J = 1$ state.

What does this all mean for the transformation of the density matrix when we change between frames? Let's consider

$$\sum_{\lambda\lambda'} \rho_{\lambda\lambda'}\langle p, j\lambda'|\hat{O}|p, j\lambda\rangle, \tag{14}$$

where $\hat{O}$ is some Lorentz invariant operator (the following argument can be generalized to cross sections etc.). We write

$$\begin{aligned}
\sum_{\lambda\lambda'} \rho(p)_{\lambda\lambda'}\langle p, j\lambda'|\hat{O}|p, j\lambda\rangle &= \sum_{\lambda\lambda'} \rho(p)_{\lambda\lambda'}\langle p, j\lambda'|U^{-1}(\Lambda)\,U(\Lambda)\hat{O}U^{-1}(\Lambda)\,U(\Lambda)|p, j\lambda\rangle && (15)\\
&= \sum_{\lambda\lambda'} \rho(p)_{\lambda\lambda'}\langle p, j\lambda'|U^{-1}(\Lambda)\,\hat{O}\,U(\Lambda)|p, j\lambda\rangle && (16)\\
&= \sum_{\lambda\lambda',\alpha\alpha'} \rho(p)_{\lambda\lambda'}\mathcal{D}^*_{\alpha'\lambda'}(R_w[\Lambda, p, L])\langle\Lambda p, j\alpha'|\hat{O}|\Lambda p, j\alpha\rangle\mathcal{D}_{\alpha\lambda}R_w[\Lambda, p, L], && (17)\\
&= \sum_{\alpha\alpha'} \left[\mathcal{D}^\dagger(R_w[\Lambda, p, L])\rho(p)\mathcal{D}(R_w[\Lambda, p, L])\right]_{\alpha\alpha'}\langle\Lambda p, j\alpha'|\hat{O}|\Lambda p, j\alpha\rangle && (18)\\
&= \sum_{\alpha\alpha'} \rho_{\alpha\alpha'}(\Lambda p)\langle\Lambda p, j\alpha'|\hat{O}|\Lambda p, j\alpha\rangle, && (19)
\end{aligned}$$

and we see that the density matrix transforms with the (momentum-dependent) Wigner rotations! This also means that the density matrix in the frame reached by the standard boost is identical to that in the rest frame (for that particular choice of standard boost).

Fun fact: the Von Neumann entropy is not a Lorentz invariant unless the states are momentum plane waves.

One drawback so far: density matrix defined using rotational invariant objects, suffers from Wigner rotations. We like to use covariant/invariant objects and so are our final answers in cross section calculations etc. Can we engineer something that bridges these things...yes! Attach to the density matrix the objects which also undergo (the opposite) wigner rotations, which are the wave functions (spinors, four vectors, tensors) of the particle under consideration.

[no detailed derivations, stating results here]

For instance for spin 1/2, we consider

$$\rho = \sum_{\lambda\lambda'} \rho_{\lambda\lambda'}u(p, \lambda)\bar{u}(p, \lambda') \tag{20}$$

which is now a covariant matrix in spinor indices. The standard properties of the density matrix now translate into

1. $\mathrm{Tr}\,\rho = 2m$
2. The Hermiticity requirement becomes $\gamma^0 \rho^\dagger \gamma^0 = \rho$
3. It obeys the wave function constraints (= Dirac equation here) $(p^\mu \gamma_\mu - m)\rho = \rho(p^\mu \gamma_\mu - m) = 0$. Note that condition 1 with one of the identities here implies the other.

[**Exercise:**] Show that by starting from a general expansion in the Dirac matrix basis $(1, \gamma_5, \gamma^\mu, \gamma^\mu\gamma_5, \sigma^{\mu\nu}\gamma_5)$ the standard expression can be obtained

$$\rho(p) = \sum_{\lambda\lambda'} \rho_{\lambda\lambda'} u(p,\lambda)\bar{u}(p,\lambda') = (p^\mu \gamma_\mu + m)\left(\frac{1 + (s\cdot\gamma)\gamma_5}{2}\right), \tag{21}$$

where $(p.s) = 0$ also follows from imposing the requirements. Hint: the third requirement can be imposed by writing $\rho = (p^\mu \gamma_\mu + m)(\ldots)$ where the dots can be expanded on the whole Dirac basis.

In this last equation, s^μ is called the polarization four vector and transforms as a (pseudo) four vector. It is the covariant generalization of the 3D spin vector introduced earlier. By matching expressions in the rest frame, one actually finds that in the rest frame $\overset{\circ}{s}{}^\mu = (0, \boldsymbol{s})$. For different types of spinors (standard boosts) s^μ can be found by boosting with the standard boost from the rest frame: $s^\mu = (L_p)^\mu{}_\nu \overset{\circ}{s}{}^\nu$.

Note that s^μ is a property of the density matrix, not individual states. This doesn't mean people do not abuse notation. Often found expressions in for instance TMD literature use

$$\langle p, S|\hat{O}|p, S\rangle = \text{expression}(S^\mu)\,, \tag{22}$$

(there's an implicit density matrix understood here, representing a general mixed state characterized by S^μ).

Similarly for spin-1 where four vector wave functions $\epsilon^\mu(\lambda)$ (with $(p\epsilon) = 0$ as constraint) we can introduce (and decompose)

$$\rho^{\mu\nu} = \sum_{\lambda\lambda'} \rho_{\lambda\lambda'} \epsilon^\mu(p,\lambda)\epsilon^{*\nu}(p,\lambda') = \frac{1}{3}\left(-g^{\mu\nu} + \frac{p^\mu p^\nu}{p^2}\right) + \frac{i}{2m}\epsilon^{\mu\nu\rho\sigma} p_\rho s_\sigma - t^{\mu\nu}, \tag{23}$$

where now again $s^\mu, t^{\mu\nu}$ are covariant versions of the 3D polarization vector and tensor, and reduce to them in the rest frame (and completely orthogonal to p^μ. Note that one shouldn't confuse the four vector s^μ (property of the density matrix), with the spin wave function ϵ^μ. Both s^μ and $t^{\mu\nu}$ can be written as quadratic functions of ϵ^μ.

III. OBSERVABLES: CASE STUDY SPIN-1

- covariant polarization parameters
- cross section expressions
- asymmetries

In this part we want to focus how all the previous aspects turn up in experimental observables, and how spin can be used to isolate certain structures. We take the spin-1 deuteron as an example target, as it's exhibits more complications/features compared to the spin-1/2 case.

We start from the diagonalized (3 by 3) density matrix, which has 3 eigenvalues (2 independent, as they sum to 1). From these eigenvalues the so-called degrees of vector and tensor polarization can be defined.

$$\mathcal{P} = n_+ - n_- \qquad\qquad -1 < \mathcal{P} < 1, \tag{24}$$

$$\mathcal{Q} = n_+ + n_- - 2n_0 \qquad\qquad -2 < \mathcal{Q} < 1. \tag{25}$$

In this diagonalized frame, we have $\mathcal{P} = s^z$, $\mathcal{Q} \propto t^{zz}$, which shows that they control the vector, resp. tensor polarization of the deuteron. Other notations for $\mathcal{Q} : t^{20}, A_{zz}$ (up to proportionality).

[**Exercise:**] Rewrite the diagonalized 3 by 3 density matrix using only $\mathcal{P}, \mathcal{Q}$. Which values of $\mathcal{P}, \mathcal{Q}$ do pure states $\lambda = \pm 1, 0$ have? Check that for these pure states the statement at the end of the first section is valid (each pure state has tensor polarization)

Any cross section is linear in the target density matrix, which means that it will be linear in $\{1, \mathcal{P}, \mathcal{Q}\}$. This leads to expressions like (for electron scattering)

$$d\sigma = d\sigma_{\text{unpol}} \left(1 + hA^e + \mathcal{P}A^V + h\mathcal{P}A^{eV} + \frac{\mathcal{Q}}{2}A^T + \frac{h\mathcal{Q}}{2}A^{eT} \right), \tag{26}$$

where $h = \pm 1/2$ is the electron helicity, and the different A^i define asymmetries. These can be separated in single and double spin asymmetries. For only electron polarization, one has the beam-spin asymmetry. The vector and tensor polarization generate single and double vector and tensor polarized asymmetries.

[**Exercise:**] Using the results of the previous exercise (values of $\mathcal{P}, \mathcal{Q}$ for the pure states), find which combination of the deuteron pure states and electron helicities isolate the different asymmetries (and unpolarized cross section). What are the limits for the tensor asymmetry introduced this way?

One thing Eq. (26) misses is information of the geometry. How does the orientation of the density matrix (polarization axis) compare to the kinematics of the reaction. In a way, Eq. (26) oversimplifies things (in notation). All asymmetries written in the equation carry additional dependence on this geometry, or equivalently the orientation of the polarization axis; only the strengths (eigenvalue related $\mathcal{P}, \mathcal{Q}$ are separated out.

However, we can get the full geometry from the covariant $s^\mu, t^{\mu\nu}$ in combination with the four vectors of particles in the reaction: p_D, q, p_h. We can construct a basis with the kinematical vectors that corresponds to the x, y, z axes in the collinear rest frame where the photon is along the negative z-axis. The expressions are completely invariant:

$$S_L = \frac{(sq)}{(p_D q)} \frac{m}{\sqrt{1+\gamma^2}}, \quad S_T \cos\phi_S = \ldots \tag{27}$$

They have a physical interpretation in the aforementioned rest frame (compare to the invariant $p^2 = m^2$ which has the interpretation as mass in the rest frame). It is these invariants that appear in cross section expressions for more involved processes (polarization, detected final state particles).

Advantage: we're using completely covariant objects and constructing invariants from them. It does not matter in which frame we do this!

How does it work in practice:

1. use a polarimetry reaction to determine invariants built from covariant polarization vector/tensor etc. [Subtlety, since our L/T separation is relative to q, this is event per event dependent, so an additional rotation might be needed to make this determination more uniform?]

2. this gives complete information to reconstruct the full covariant density matrix in any frame.

3. From there we can use the polarization information for any other reaction.

Does the form of standard boost (dynamics/form) play in a role in all of this? No. It is the covariant density matrix that is the object of interest, which is independent of all of this (measurement determines this). Different forms will lead to different $\rho_{\lambda\lambda'}$ density matrices (as the wave functions are different), which are density matrices in **different** rest frames (different standard boosts). These rest frames are then in general different from the one where the q-vector is along the z-axis. The additional rotation makes all density matrices agree again (wonderful !). We already have these values immediately from the decomposition with invariants (S_L, S_T etc.).