Exercise 1: Non relativistic reduction of the one-body current operator

The covariant one-body current operator is:

$$j^\mu = \bar{u}(\mathbf{p}'s') \left[e\, e_N \gamma^\mu + i \frac{\kappa_N}{2\, m_N} \sigma^{\mu\nu} q_\nu \right] u(\mathbf{p}s) \, , \tag{1}$$

where m_N denotes the nucleon mass, and e_N and μ_N are defined as follows:

$$e_N = (1 + \tau_z)/2 \; , \qquad \kappa_N = (\kappa_S + \kappa_V \tau_z)/2 \; , \qquad \mu_N = e_N + \kappa_N \; , \tag{2}$$

where κ_S and κ_V are the isoscalar and isovector combinations of the anomalous magnetic moments of the proton and neutron, ($\kappa_S = -0.12$ n.m. and $\kappa_V = 3.706$ n.m.).

The spinor are given by:

$$u(\mathbf{p}s) = N \begin{pmatrix} 1 \\ \frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{E+m_N} \end{pmatrix} \chi_s \, , \tag{3}$$

where N is the normalization factor. Use the normalization convention $u^\dagger(\mathbf{p}s)u(\mathbf{p}s) = 1$ to determine N. (Sol. $N = \sqrt{(E + m_N)/2\,E}$).

Show that the limit of $\frac{|\mathbf{p}_i|}{m_N} \to 0$, the charge and current operators read:

$$j^0 \to \rho = e\, e_N \tag{4}$$

$$j^i \to \mathbf{j} = \frac{e}{2\, m_N} \Big[e_N \, (\mathbf{p} + \mathbf{p}') + i\, \mu_N \, \boldsymbol{\sigma} \times \mathbf{q} \Big] \, . \tag{5}$$

You may find these relations useful:

$$\sqrt{\frac{E + m_N}{2\, m_N}} \to 1 - \frac{\mathbf{p}^2}{8\, m_N^2} \, , \tag{6}$$

$$\frac{1}{E + m_N} \to \frac{1}{2\, m_N} \left(1 - \frac{\mathbf{p}^2}{4\, m_N^2} \right) \, , \tag{7}$$

$$q^0 = E' - E \to \frac{1}{2\, m_N} \left(\mathbf{p}'^{\mathbf{2}} - \mathbf{p}^2 \right) \, , \tag{8}$$

$$\boldsymbol{\sigma} \cdot \mathbf{A} \boldsymbol{\sigma} \cdot \mathbf{B} = \mathbf{A} \cdot \mathbf{B} + i \boldsymbol{\sigma} \cdot \mathbf{A} \times \mathbf{B} \, . \tag{9}$$

Exercise 2: OPEP in time ordered perturbation theory

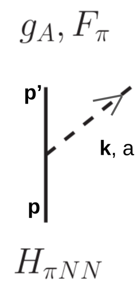

Figure 1: Vertex induced by the πNN interaction Hamiltonian.

The pion-nucleon interaction Hamiltonian is given by

$$H_{\pi NN} \quad = \quad \frac{g_A}{F_\pi} \int \mathrm{d}\mathbf{x}\, N^\dagger(\mathbf{x})\; [\boldsymbol{\sigma} \cdot \nabla \pi_a(\mathbf{x})]\; \tau_a\, N(\mathbf{x}) \;, \tag{10}$$

where σ_a and τ_a are the spin and isospin Pauli matrices. The expressions of the isospin triplet of pion fields, $\pi_a(\mathbf{x})$ is defined as

$$\pi_a(\mathbf{x}) = \sum_{\mathbf{p}} \frac{1}{\sqrt{2\,\omega_p}} \left[c_{\mathbf{p},a}\, \mathrm{e}^{i\mathbf{p}\cdot\mathbf{x}} + \text{h.c.} \right] \;, \tag{11}$$

where $a = x, y, z$, denotes the Cartesian component in isospin space, $c_{\mathbf{p},a}$ and $c^\dagger_{\mathbf{p},a}$ annihilate and create pions with momentum $\mathbf{p}$. They satisfy the following commutation relations:

$$[c_{\mathbf{p},a},\, c^\dagger_{\mathbf{p}',a'}] = \delta_{\mathbf{p}\,\mathbf{p}'}\, \delta_{a\,a'} \;. \tag{12}$$

The energy of the pion is given by $\omega_p \equiv (p^2 + m_\pi^2)^{1/2}$, where the pion mass $m_\pi \sim 138$ MeV is averaged over its charge states.

The charged pion field $\pi_\pm$ is expressed in terms of Cartesian components as

$$\pi_\pm(\mathbf{x}) = \frac{1}{\sqrt{2}} \left[\pi_x(\mathbf{x}) \mp i\pi_y(\mathbf{x}) \right] \;, \tag{13}$$

and π_+ (π_-) annihilates a positively (negatively) charged pion, or creates a negatively (positively) charged pion.

The nucleon field $N(\mathbf{x})$ is taken in the non-relativistic limit as

$$N(\mathbf{x}) \quad = \quad \sum_{\mathbf{p},\sigma\tau} b_{\mathbf{p},\sigma\tau}\, \mathrm{e}^{i\mathbf{p}\cdot\mathbf{x}} \chi_{\sigma\tau} \;, \tag{14}$$

where $b_{\mathbf{p},\sigma\tau}$ is the annihilation operator for a nucleon with momentum $\mathbf{p}$, and spin and isospin specified by the quantum numbers σ and τ, respectively. The short-hand notation $\chi_{\sigma\tau}$ is introduced to denote the spin-isospin state $\chi_\sigma\,\eta_\tau$. The b's and $b^\dagger$'s satisfy the standard anticommutation relations, appropriate for fermionic fields, *i.e.*

$$\left[b_{\mathbf{p},\sigma\tau},\, b^\dagger_{\mathbf{p}',\sigma'\tau'}\right]_+ = \delta_{\mathbf{p}\mathbf{p}'}\delta_{\sigma\sigma'}\delta_{\tau\tau'}\,, \tag{15}$$

where $[\ldots,\ldots]_+$ denotes the anticommutator.

Consider the vertex illustrated in the figure with momenta and isopsin cartesian component as indicated in the figure. The initial and final states are then

$$|\mathbf{p}\rangle = b^\dagger_{\mathbf{p}}|\,0\,\rangle \tag{16}$$

$$|\mathbf{p}';\mathbf{k},a\rangle = b^\dagger_{\mathbf{p}'}\,c^\dagger_{\mathbf{k},a}|\,0\,\rangle\,, \tag{17}$$

where the spin and isospin state were omitted for convince.

Show that the vertex illustrated in Fig. 2 is given by

$$V_{\pi NN} = \langle\mathbf{p}';\mathbf{k},a|H_{\pi NN}|\mathbf{p}\rangle = -i\frac{g_A}{F_\pi}\frac{\boldsymbol{\sigma}\cdot\mathbf{k}}{\sqrt{2\,\omega_k}}\,\tau_a\,, \tag{18}$$

where the delta function conserving the total momentum at the vertex has been dropped.

In time ordered perturbation theory the one-pion exchange amplitude is given by the sum of two contributions represented by the time ordered diagrams illustrated in Fig. 3

$$T_{\rm OPE} = \langle\mathbf{p}_2'|H_{\pi NN}|\mathbf{p}_2;\mathbf{k},a\rangle\,\frac{1}{E_i-E_{Ia}}\langle\mathbf{p}_1';\mathbf{k},a|H_{\pi NN}|\mathbf{p}_1\rangle \tag{19}$$

$$+\ \langle\mathbf{p}_1'|H_{\pi NN}|\mathbf{p}_1;-\mathbf{k},a\rangle\,\frac{1}{E_i-E_{Ib}}\langle\mathbf{p}_2';-\mathbf{k},a|H_{\pi NN}|\mathbf{p}_2\rangle\,. \tag{20}$$

In the equation above, E_i is the non-relativistic energy of the two nucleon in the initial state, and the energy E_{Ia} and E_{Ib} are the energies of the intermediate states

$$E_i = E_1+E_2 = 2m_N+\frac{\mathbf{p}_1^2}{2\,m_N}+\frac{\mathbf{p}_2^2}{2\,m_N}\,,$$

$$E_{Ia} = E_1'+E_2+\omega_k = 2m_N+\frac{\mathbf{p}_1'^2}{2m_N}+\frac{\mathbf{p}_2^2}{2m_N}+\omega_k\,, \tag{21}$$

$$E_{Ib} = E_1+E_2'+\omega_k = 2m_N+\frac{\mathbf{p}_1}{2m_N}+\frac{\mathbf{p}_2'^2}{2m_N}+\omega_k\,, \tag{22}$$

where m_N denotes the nucleon mass. The OPEP is in fact calculated in the static limit approximation, *i.e.* $m_N\to\infty$, so that the equations above become

$$\left.\frac{1}{(E_i-E_I)}\right|_{\rm static} = \frac{1}{-\omega_k}\,. \tag{23}$$

Show that the OPEP in momentum space reads:

$$v_\pi(\mathbf{k}) = -\frac{g_A^2}{F_\pi^2}\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2\,\frac{\mathbf{k}\cdot\boldsymbol{\sigma}_1\,\mathbf{k}\cdot\,\boldsymbol{\sigma}_2}{\omega_k^2}\,. \tag{24}$$

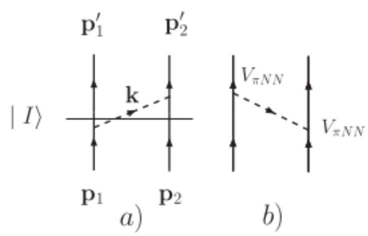

Figure 2: Time-ordered diagrams contributing to the OPEP.