Lectures note for the summer school: "Light-ion physics in the EIC era"

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1 Solving the two body Schrödinger equation

Let us consider for now a system of two bosons of spin 0 of mass m_1 and m_2. Their positions are described by the vectors $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$. The Hamiltonian operator reads

$$\hat{H} = -\frac{\nabla_1^2}{2m_1} - \frac{\nabla_2^2}{2m_2} + \hat{V}(\boldsymbol{r}_1, \boldsymbol{r}_2)\,. \tag{1}$$

Let us also suppose that our potential depends only on the relative position of the particles only i.e.

$$\hat{V}(\boldsymbol{r}_1, \boldsymbol{r}_2) = \hat{V}(\boldsymbol{r}_1 - \boldsymbol{r}_2)\,. \tag{2}$$

To solve this problem we introduce the relative coordinate

$$\boldsymbol{r} = \boldsymbol{r}_1 - \boldsymbol{r}_2\,, \tag{3}$$

and the center of mass coordinate

$$\boldsymbol{R} = \frac{m_1\boldsymbol{r}_1 + m_2\boldsymbol{r}_2}{m_1 + m_2}\,. \tag{4}$$

Using these definitions the position coordinates of the two particles can be rewritten as

$$\boldsymbol{r}_1 = \boldsymbol{R} + \frac{\mu}{m_1}\boldsymbol{r}\,, \qquad \boldsymbol{r}_2 = \boldsymbol{R} - \frac{\mu}{m_2}\boldsymbol{r}\,, \tag{5}$$

where $\mu = \frac{m_1 m_2}{m_1+m_2}$ is the reduced mass. Using these coordinates we can rewrite the Laplacians as

$$\boldsymbol{\nabla}_1 = \frac{\mu}{m_2}\boldsymbol{\nabla}_{\boldsymbol{R}} + \boldsymbol{\nabla}_{\boldsymbol{r}}\,, \qquad \boldsymbol{\nabla}_2 = \frac{\mu}{m_1}\boldsymbol{\nabla}_{\boldsymbol{R}} - \boldsymbol{\nabla}_{\boldsymbol{r}}\,, \tag{6}$$

from which it is easy to rewrite the kinetic energy operator

$$T = -\frac{\nabla_1^2}{2m_1} - \frac{\nabla_2^2}{2m_2} = -\frac{\nabla_{\boldsymbol{R}}^2}{2(m_1+m_2)} - \frac{\nabla_{\boldsymbol{r}}^2}{2\mu}\,. \tag{7}$$

Notice that now the motion of the center of mass is now completely decoupled. Let us now go back to our time-independent Schrödinger equation

$$\hat{H}\Psi(\boldsymbol{R}, \boldsymbol{r}) = E\Psi(\boldsymbol{R}, \boldsymbol{r})\,. \tag{8}$$

We can use as educated guess for our solution

$$\Psi(\boldsymbol{R}, \boldsymbol{r}) = \psi_R(\boldsymbol{R})\psi_r(\boldsymbol{r})\,, \tag{9}$$

and dividing Eq. (8) by the wave function we obtain

$$-\frac{1}{\psi_R(\boldsymbol{R})}\frac{\nabla_{\boldsymbol{R}}^2\psi_R(\boldsymbol{R})}{2(m_1+m_2)} + \left[-\frac{1}{\psi_r(\boldsymbol{r})}\frac{\nabla_{\boldsymbol{r}}^2\psi_r(\boldsymbol{r})}{2\mu} + \hat{V}(\boldsymbol{r})\right] = E\,. \tag{10}$$

Since the two terms separately depend on $\boldsymbol{R}$ and $\boldsymbol{r}$ and they are equal to a number it means they are both equal to a number. This means that we can separate the two equations as

$$-\frac{\nabla^2_{\boldsymbol{R}}\psi_R(\boldsymbol{R})}{2(m_1+m_2)} = E_R\psi_R(\boldsymbol{R})\,, \tag{11}$$

and

$$-\frac{\nabla^2_{\boldsymbol{r}}}{2\mu}\psi_r(\boldsymbol{r}) + \hat{V}(\boldsymbol{r})\psi_r(\boldsymbol{r}) = E_r\psi_r(\boldsymbol{r})\,. \tag{12}$$

The first equation is the free motion of the center of mass. The part we are going to focus our interest on is the solution of the relative motion equation. In the rest of the notes we will drob the subscript r when we will discuss the wave function of the relative motion.

2 The Deuteron wave function

The deuteron is the simplest nucleus formed by a proton and a neutron. Experimentally it has been found that:

- its binding energy is $E_d = -2.22457$ MeV;
- its total angular momentum and parity is $J^\pi = 1^+$;
- its magnetic moment is $\mu = 0.8574\ \mu$N;
- its quadrupole moment id $Q = 0.2859\ e\ \text{fm}^2$.

The nucleons are identical for the nuclear interaction therefore we can exploit the isospin symmetry to construct the wave function. We will consider the breaking of the isospin symmetry only at the level of the kinetic energy operator using as $m_1 = 938.272$ MeV the mass of the proton and $m_2 = 939.565$ MeV from which $\mu = 469.46$ MeV. The wave function will be then a function of the isospin (t_i), the spin (s_i) and the relative position of the two particles

$$\psi_d(\boldsymbol{r}, s_1, s_2, t_1, t_2)\,, \tag{13}$$

in more detail we can write it as a tensor product of

$$\psi_d(\boldsymbol{r}, s_1, s_2, t_1, t_2) \propto [\psi_r(\boldsymbol{r}) \otimes \psi_S(s_1, s_2)]_J \otimes \psi_T(t_1, t_2)\,. \tag{14}$$

Since the nucleon are fermions we want the nuclear wave function to be completely anti-symmetric under exchange of the two particles and that the total angular momentum is $J^\pi = 1^+$.

For the spin and isospin part of the wave function we want to construct states with well defined total spin and total isospin. We can do this using the Clebch-Gordan coefficients starting from the single spin (isospin) state

$$|SS_z\rangle = \chi_{SS_z} = [s_1 s_2]_{SS_z} = \sum_{s_{1z}, s_{2z}} \langle 1/2\, s_{1z}\, 1/2\, s_{2z}|S\, S_z\rangle |1/2\, s_{1z}\rangle |1/2\, s_{2z}\rangle \tag{15}$$

and analogously

$$|TT_z\rangle = \chi_{TT_z} = [t_1 t_2]_{TT_z} = \sum_{t_{1z}, t_{2z}} \langle 1/2\, t_{1z}\, 1/2\, t_{2z}|T\, T_z\rangle |1/2\, t_{1z}\rangle |1/2\, t_{2z}\rangle\,, \tag{16}$$

where we defined three different notation different notation to define the same state that we will use in the notes.

The spatial part of the wave function can be written as

$$\psi_{L,M}(\boldsymbol{r}) = f_L(r) Y_{L,M}(\hat{\boldsymbol{r}}) \tag{17}$$

where $Y_{\ell,m}$ are the standard spherical harmonics. The first thing we want to do is to construct the correct total angular momentum. To do so we can combine the angular momentum and spin part writing the spatial-spin wave function as

$$\psi_{JJ_z}(\boldsymbol{r}) = \sum_L f_L(r) \left[Y_L(\hat{\boldsymbol{r}})\chi_S\right]_{JJ_z}\,, \tag{18}$$

where

$$\left[Y_L(\hat{\boldsymbol{r}})\chi_S\right]_{JJ_z} = \sum_M \langle L\, M\, , S\, S_z | J\, J_z\rangle Y_{L,M}(\hat{\boldsymbol{r}})\chi_{SS_z}\,. \tag{19}$$

We can add now the isospin to our wave function to obtain the full wave function

$$\psi_{JJ_z}(\boldsymbol{r}, T, T_z) = \sum_L f_L(r)\left[Y_L(\hat{\boldsymbol{r}})\chi_S\right]_{JJ_z}\chi_{TT_z}\,. \tag{20}$$

Notice how from Eq. (13) we passed from the quantum numbers of the single particle s_1, s_2, t_1, t_2 and $\hat{\boldsymbol{r}}$ to the total isospin, total spin, orbital angilar momentum and total angular momentum L, S, J, J_z, T, T_z maintaining the same number of degrees of freedom.

We can now determine the quantum numbers. We want the parity to be positive so we want

$$\psi_{JJ_z}(\boldsymbol{r}, T, T_z) = \psi_{JJ_z}(-\boldsymbol{r}, T, T_z)\,. \tag{21}$$

The spin isospin structure does not change under parity as well as $f_L(r)$ that depends only on the magnitude of $\boldsymbol{r}$. The only piece in the wave function that change are the spherical harmonics

$$Y_{L,M}(\hat{\boldsymbol{r}}) = (-)^L Y_{L,M}(\hat{\boldsymbol{r}})\,. \tag{22}$$

Therefore in order to satisfy Eq. (21) we need L to be even. The possible values for the total spin are $S = 1$ and $S = 0$. However to have $J = 1$ with $S = 0$ would imply that $L = 1$ that does not satisfy the parity condition. The only possibility is that $S = 1$ which imply that $L = 0$ and $L = 2$ only are possible. The only possible values for the total isospin are $T = 0$ and $T = 1$ as for the spin. To determine it we need now to impose the anti-symmetry of the wave function exchanging the particle 1 and 2 i.e.

$$\psi^{12}_{JJ_z}(\boldsymbol{r}, T, T_z) = -\psi^{21}_{JJ_z}(\boldsymbol{r}, T, T_z)\,. \tag{23}$$

By using the Clebch-Gordan coefficients and spherical harmonic properties it is easy to see that

$$[t_2 t_1]_{TT_z} = (-1)^{T+1}[t_1 t_2]_{TT_z}\,, \tag{24}$$

$$[s_2 s_1]_{SS_z} = (-1)^{S+1}[s_1 s_2]_{SS_z}\,, \tag{25}$$

$$Y_{LM}(-\hat{\boldsymbol{r}}) = (-)^L Y_{LM}(\hat{\boldsymbol{r}})\,, \tag{26}$$

and so the wave function under the exchange of the particle 1 and 2 reads then

$$\psi^{21}_{JJ_z}(\boldsymbol{r}, T, T_z) = \sum_{L=0,2}(-)^{L+S+T} f_L(r)\left[Y_L(\hat{\boldsymbol{r}})\chi_S\right]_{JJ_z}\chi_{TT_z}\,, \tag{27}$$

which imposing Eq. (23) implies $T = 0$. The final wave function reads then

$$\psi_{1J_z}(\boldsymbol{r}, T = 0, T_z = 0) = \sum_{L=0,2} f_L(r)\left[Y_L(\hat{\boldsymbol{r}})\chi_{S=1}\right]_{1J_z}\chi_{T=0T_z=0}\,, \tag{28}$$

where the only term to be determined are the radial functions $f_L(r)$ that we will construct using the Schrödinger equation. Before discussing this, however, we will return back to our Schrödinger equation.

3 The radial Schrödinger equation

We are now focusing on the Schrödinger equation describing the relative motion of the to particles given in Eq. (12). The nuclear interaction that we are going to use is given by

$$\hat{V} = V_{CT}(r)\left(\frac{3+\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2}{4}\right)\left(\frac{1-\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2}{4}\right)\delta_{L,0} + (V^{(1)}_{OPE}(r)\hat{S}_{12} + V^{(2)}_{OPE}(r)\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)(\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)\,, \tag{29}$$

where $\hat{S}_{12}$ is the tensor operator

$$\hat{S}_{12} = 3(\boldsymbol{\sigma}_1\cdot\hat{\boldsymbol{r}})(\boldsymbol{\sigma}_2\cdot\hat{\boldsymbol{r}}) - \boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2 \tag{30}$$

with $\sigma_i(\tau_i)$ the Pauli matrices acting on the spin (isospin). Modern nuclear theory shows that the contact potential V_{CT} is needed only in the $^3S_1$ partial wave [vK20]. Its radial component is given by

$$V_{CT}(r) = \frac{C_S}{\pi^{3/2} R_S^3} e^{-(r/R_S)^2}\,, \tag{31}$$

where $R_S = 0.7$ fm and $C_S = -1.2565$ fm^2 is a constant fixed to reproduce the binding energy of the deuteron. R_S plays the role of a cut-off. The radial components of the one pion exchange (OPE) potential are given by

$$V^{(1)}_{OPE}(r) = \frac{g_A^2}{12\pi}\frac{m_\pi^3}{F_\pi^2}\frac{e^{-m_\pi r}}{m_\pi r}\left(1 + \frac{3}{m_\pi r} + \frac{3}{(m_\pi r)^2}\right) C_{R_L}(r)\,, \tag{32}$$

$$V^{(2)}_{OPE}(r) = \frac{g_A^2}{12\pi}\frac{m_\pi^3}{F_\pi^2}\frac{e^{-m_\pi r}}{m_\pi r} C_{R_L}(r)\,, \tag{33}$$

where

$$C_{R_L}(r) = 1 - \frac{1}{(r/R_L)^6\, e^{(r-R_L)/a_L} + 1}, \tag{34}$$

is the regularizing function with $R_L = 1.0$ fm and $a_L = R_L/2$. The other constants are the nucleon axial coupling $g_A = 1.29$, the pion decay constant $F_\pi = 184.80$ MeV and the mean mass of the pion $m_\pi = 138.04$ MeV. The regularization as well as the cut-off R_L and R_S chosen for this example are arbitrary. Other choices can be valid as well. Here we made this to show you an application close as possible to the present state of art (see Ref. [PGS+15]).

Now that we have the explicit form of the potential what we want is to derive the differential equations for the functions $f_L(r)$. To do so we can compute the matrix elements

$$\langle \left[Y_{L'}\chi_{S'}\right]\chi_{T'}|\hat{T} + \hat{V}|\psi_d\rangle = \langle \left[Y_{L'}\chi_{S'}\right]\chi_{T'}|E|\psi_d\rangle\,. \tag{35}$$

For the right hand side, since E is a number we have

$$\langle \left[Y_{L'}\chi_{S'}\right]\chi_{T'}|E|\psi_d\rangle = E f_{L'}(r) \tag{36}$$

The kinetic energy operator can be written in spherical coordinates as

$$\hat{T} = -\frac{1}{2\mu}\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{\hat{L}^2}{r^2}\right) \tag{37}$$

where $\hat{L}$ is the standard angular momentum operator. Since the spherical harmonics are the eigenfunction of $\hat{L}^2$ and no spin and isospin operator appears, the kinetic energy matrix element in Eq. (35) reduces to

$$\langle \left[Y_{L'}\chi_{S'}\right]\chi_{T'}|\hat{T}|\psi_d\rangle = -\frac{1}{2\mu}\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f_{L'}(r)}{\partial r}\right) + \frac{L'(L'+1)}{r^2} f_{L'}(r)\right)\,. \tag{38}$$

The remaining part is to compute the matrix elements of the nuclear interaction i.e.

$$\langle \left[Y_{L'}\chi_{S'}\right]\chi_{T'}|\hat{V}|\psi_d\rangle\,, \tag{39}$$

that with a bit of algebra reduces to compute the following matrix elemnts

$$\chi^\dagger_{T'}(\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)\chi_{T'}\,, \tag{40}$$

$$\chi^\dagger_{S'}(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)\chi_{S'}\,, \tag{41}$$

$$\int d\hat{r}\,\left[Y_{L'}(\hat{r})\chi_{S'}\right]^\dagger_{JJ_z}\hat{S}_{12}\left[Y_L(\hat{r})\chi_S\right]_{JJ_z}\,. \tag{42}$$

To do so we need to introduce the Wigner-j coefficients.

3.1 Some usefull definitions

From now on we will use the following notation for coupling angular momentum

$$\Big[T_K \Psi_L\Big]_{JJ_z} = \sum_{\kappa M} (K\kappa, LM|JJ_z) T_{K\kappa}\Psi_{LM} \ , \tag{43}$$

where $T_{K\kappa}$ is a generic spherical tensor operator of rank K and component z given by κ, while Ψ_{LM} rappresents a state of angular momentum L, M. $(K\kappa, LM|JJ_z)$ is a Clebsh-Gordan coefficient.

The Wigner-3j coefficient are defined as

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \equiv \frac{(-1)^{j_1-j_2-m_3}}{\sqrt{2j_3+1}} \langle j_1 m_1 j_2 m_2 \mid j_3 \, (-m_3)\rangle \, . \tag{44}$$

and represent an alternative to the Clebch-Gordan coefficient.

The spherical harmonics can be coupled and rewritten in term of the Wigner-3j coefficients by making use of the contraction rule

$$\Big[Y_{\ell_1} Y_{\ell_2}\Big]_{LM} = \frac{B^{\ell}_{\ell_1\ell_2}}{\sqrt{4\pi}} Y_{LM}(\hat{r}) \ , \qquad B^{L}_{\ell_1\ell_2} = \hat{\ell}_1\hat{\ell}_2(-)^{\ell_1+\ell_2} \begin{pmatrix} \ell_1 & \ell_2 & L \\ 0 & 0 & 0 \end{pmatrix}, \tag{45}$$

where $\hat{\ell} = \sqrt{2\ell+1}$.

The Wigner-6j symbol is used for expressing the coupling of three angular momenta

$$|(j_1 j_2)_{j_{12}} j_3 JM\rangle = \sum_{j_{23}} T^{j_1 j_2 j_3}_{j_{12} j_{23} J} |j_1 (j_2 j_3)_{j_{23}} JM\rangle \tag{46}$$

where

$$T^{j_1 j_2 j_3}_{j_{12} j_{23} J} = (-)^{j_1+j_2+j_3+J} \hat{j}_{12}\hat{j}_{23} \begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & J & j_{23} \end{Bmatrix} \ . \tag{47}$$

The same is true for the Wigner-9j coefficients that are use to couple four angular momenta

$$|(j_1 j_2)_{j_{12}} (j_3 j_4)_{j_{34}} JM\rangle \quad = \quad \sum_{j_{13} j_{24}} N^{j_1 j_2 j_3 j_4}_{j_{12} j_{34} j_{13} j_{23} J} |(j_1 j_3)_{j_{13}} (j_2 j_4)_{j_{24}} JM\rangle \tag{48}$$

where

$$N^{j_1 j_2 j_3 j_4}_{j_{12} j_{34} j_{13} j_{23} J} = \hat{j}_{13}\hat{j}_{24}\hat{j}_{12}\hat{j}_{34} \begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j_4 & j_{34} \\ j_{13} & j_{24} & J \end{Bmatrix} \ . \tag{49}$$

These coefficients can be computed using standard computational tools as *Mathematica* [Inc]. For most of the programming languages there are libraries that contains function in which the Wigner-j coefficients are computed. For whom is interested in a more detailed reading I suggest Ref. [Edm55].

3.2 Pure spin and isospin operators

We can now pass to compute some useful quantity for the spin and isospin. Let us start with

$$[\boldsymbol{\sigma}|1/2\rangle]_{1/2 S_z} = \sum_{\sigma_z s_z} \langle 1\, \sigma_z 1/2\, s_z | 1/2, S_z\rangle \sigma_{1\sigma_z} |1/2\, s_z\rangle \, . \tag{50}$$

Note that σ_{σ_z} are the spherical components of the Pauli matrices i.e.

$$\sigma_{10} = \sigma_z \, , \qquad \sigma_{1-1} = \frac{1}{\sqrt{2}} (\sigma_x - i\sigma_y) \ , \qquad \sigma_{1+1} = -\frac{1}{\sqrt{2}} (\sigma_x + i\sigma_y) \ . \tag{51}$$

Let us supposed that $S_z = 1/2$ then we have

$$\begin{aligned} [\boldsymbol{\sigma}|1/2\rangle]_{1/2,1/2} &= \langle 1\,0\,, 1/2\, 1/2 | 1/2\, 1/2\rangle \sigma_{10} |1/2\, 1/2\rangle + \langle 1\,1\,, 1/2\ -1/2 | 1/2\, 1/2\rangle \sigma_{11} |1/2\ -1/2\rangle \\ &= -\frac{1}{\sqrt{3}} |1/2\, 1/2\rangle + \sqrt{\frac{2}{3}} (-\sqrt{2}) |1/2\, 1/2\rangle = -\sqrt{3} |1/2\, 1/2\rangle \, , \end{aligned} \tag{52}$$

and analogously for $S_z = -1/2$. Therefore

$$[\boldsymbol{\sigma}|1/2\rangle]_{1/2S_z} = -\sqrt{3}|1/2\, S_z\rangle\,. \tag{53}$$

The operator $\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2$ can be rewritten in term of the spherical component of the Pauli matrices as

$$\begin{aligned}\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 &= -\sigma^1_{1+1}\sigma^2_{1-1} - \sigma^1_{1-1}\sigma^2_{1+1} + \sigma^1_{10}\sigma^2_{10} \\ &= -\sqrt{3}\langle 1\,+1\,,1\,-1|0,0\rangle\sigma^1_{1+1}\sigma^2_{1-1} - \sqrt{3}\langle 1\,-1\,,1\,+1|0,0\rangle\sigma^1_{1-1}\sigma^2_{1+1} - \sqrt{3}\langle 1\,0\,,1\,0|0,0\rangle\sigma^1_{10}\sigma^2_{10} \\ &= -\sqrt{3}\left[\sigma^1_1\sigma^2_1\right]_{00}\,.\end{aligned} \tag{54}$$

Similar thing happens for the operator $\boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2$.

We can now compute explicitly the effect of this operator on the state χ_S,

$$(\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2)\chi_{SS_z} = -\sqrt{3}\left[\sigma^1_1\sigma^2_1\right]_{00}\left[|s_1\rangle\,|s_2\rangle\right]_{SS_z} = -\sqrt{3}\langle 0\,0\,,S\,S_z|S\,S_z\rangle\left[\left[\sigma^1_1\sigma^2_1\right]_{00}\left[|s_1\rangle\,|s_2\rangle\right]_S\right]_{SS_z}\,, \tag{55}$$

that can be rewritten using the Wigner-9j coefficient as

$$(\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2)\chi_{SS_z} = -\sqrt{3}\sum_{\Sigma_1\Sigma_2}\hat{0}\hat{S}\hat{\Sigma}_1\hat{\Sigma}_2\begin{Bmatrix}1 & 1 & 0\\ 1/2 & 1/2 & S\\ \Sigma_1 & \Sigma_2 & S\end{Bmatrix}\left[\left[\sigma^1_1|s_1\rangle\right]_{\Sigma_1}\left[\sigma^2_1|s_2\rangle\right]_{\Sigma_2}\right]_{SS_z} \tag{56}$$

Using then Eq. (53) we can easily write

$$(\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2)\chi_{SS_z} = -6\sqrt{3}\hat{S}\begin{Bmatrix}1 & 1 & 0\\ 1/2 & 1/2 & S\\ 1/2 & 1/2 & S\end{Bmatrix}\left[|s_1\rangle|s_2\rangle\right]_{SS_z}\,, \tag{57}$$

From which it is straightforward to obtain

$$\chi^\dagger_{S'S'_z}(\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2)\chi_{SS_z} = -6\sqrt{3}\hat{S}\begin{Bmatrix}1 & 1 & 0\\ 1/2 & 1/2 & S\\ 1/2 & 1/2 & S\end{Bmatrix}\delta_{SS'}\delta_{S_zS'_z}\,, \tag{58}$$

and analogously for the isospin.

3.3 Tensor operator

Before computing the tensor operator we need again to derive some useful relations. The first one is to express $\boldsymbol{\sigma}\cdot\hat{\boldsymbol{r}}$ in term of the spherical harmonics. Using the fact that we can express

$$\frac{z}{r} = \sqrt{\frac{4\pi}{3}}Y_{10}, \qquad \frac{y}{r} = \sqrt{\frac{4\pi}{3}}i\frac{Y_{1-1}+Y_{1+1}}{\sqrt{2}}, \qquad \frac{x}{r} = \sqrt{\frac{4\pi}{3}}\frac{Y_{1-1}-Y_{1+1}}{\sqrt{2}}. \tag{59}$$

and Eq. (51) we can rewrite

$$\begin{aligned}\boldsymbol{\sigma}\cdot\hat{\boldsymbol{r}} &= \sigma_x\frac{x}{r} + \sigma_x\frac{z}{r} + \sigma_z\frac{z}{r} = \sqrt{\frac{4\pi}{3}}\left(-\sigma_{1+1}Y_{1-1} - \sigma_{1-1}Y_{1+1} + \sigma_{10}Y_{10}\right) \\ &= -\sqrt{12\pi}\left[\sigma_1 Y_1\right]_{00}\end{aligned} \tag{60}$$

where in the last step we used the same trick as in Eq. (54). We can now rewrite the tensor operator as

$$\hat{S}_{12} = 12\pi\left[\left(\sigma^1_1 Y_1(\hat{r})\right)_0\left(\sigma^2_1 Y_1(\hat{r})\right)_0\right]_{00} + \sqrt{3}\left[\sigma^1_1\sigma^2_1\right]_{00}\,. \tag{61}$$

Let us consider now the first term and rearrange it using the Wigner-9j coefficients

$$\left[\left(\sigma^1_1 Y_1(\hat{r})\right)_0\left(\sigma^2_1 Y_1(\hat{r})\right)_0\right]_{00} = \sum_{\ell,m}\hat{\ell}^2\begin{Bmatrix}1 & 1 & 0\\ 1 & 1 & 0\\ \ell & \ell & 0\end{Bmatrix}\langle\ell\,-m\,,\ell\,m|00\rangle\left[\sigma^1_1\sigma^2_1\right]_{\ell-m}\left(Y_1Y_1\right)_{\ell m} \tag{62}$$

that using Eq. (45) reduces to

$$\left[\left(\sigma_1^1 Y_1(\hat{r})\right)_0 \left(\sigma_1^2 Y_1(\hat{r})\right)_0\right]_{00} = \frac{3}{\sqrt{4\pi}} \sum_{\ell,m} \hat{\ell}^2 \left\{ \begin{array}{ccc} 1 & 1 & 0 \\ 1 & 1 & 0 \\ \ell & \ell & 0 \end{array} \right\} \left(\begin{array}{ccc} 1 & 1 & \ell \\ 0 & 0 & 0 \end{array} \right) \langle \ell\, -m\,, \ell\, m | 00 \rangle \left[\sigma_1^1 \sigma_1^2\right]_{\ell -m} Y_{\ell m}$$
$$= \frac{3}{\sqrt{4\pi}} \sum_{\ell,m} \hat{\ell}^2 \left\{ \begin{array}{ccc} 1 & 1 & 0 \\ 1 & 1 & 0 \\ \ell & \ell & 0 \end{array} \right\} \left(\begin{array}{ccc} 1 & 1 & \ell \\ 0 & 0 & 0 \end{array} \right) \left[\left[\sigma_1^1 \sigma_1^2\right]_\ell Y_\ell\right]_{00} . \tag{63}$$

The Wigner-3j coefficient that appear above is non zero only for $\ell = 0$ and 2. By inserting the values of the Wigner coefficients it is easy to show that

$$\left[\left(\sigma_1^1 Y_1(\hat{r})\right)_0 \left(\sigma_1^2 Y_1(\hat{r})\right)_0\right]_{00} = -\frac{1}{4\pi\sqrt{3}} \left[\sigma_1^1 \sigma_1^2\right]_{00} + \frac{1}{\sqrt{6\pi}} \left[Y_2 \left[\sigma_1^1 \sigma_1^2\right]_2\right]_{00} \tag{64}$$

from which we have that the tensor operator take the form

$$\hat{S}_{12} = 2\sqrt{6\pi} \left[Y_2 \left[\sigma_1^1 \sigma_1^2\right]_2\right]_{00} . \tag{65}$$

Using this form of the tensor operator and the Wigner-j coefficients it is easy to derive the matrix element

$$\int d\hat{r} \left[Y_{L'} \chi_{S'}\right]^\dagger_{JJ_z} \hat{S}_{12} \left[Y_L \chi_S\right]_{JJ_z} = 30\sqrt{6} \hat{J} \hat{L} \hat{S} \hat{L}' \hat{S}' (-)^L \left\{ \begin{array}{ccc} 2 & 2 & 0 \\ L & S & J \\ L' & S' & J \end{array} \right\} \left\{ \begin{array}{ccc} 1 & 1 & 2 \\ 1/2 & 1/2 & S \\ 1/2 & 1/2 & S' \end{array} \right\} \left(\begin{array}{ccc} 2 & L & L' \\ 0 & 0 & 0 \end{array} \right) . \tag{66}$$

We leave this last proof to the reader as exercise to get practice with the use of the Wigner-j coefficients.

3.4 Final expression for the radial equations

We can now write the expression for the radial equations. Let us write the first potential matrix element as

$$\langle \left[Y_{L=0} \chi_{S=1}\right]_{1 J_z} \chi_{T=0} | \hat{V} | \psi_d \rangle = (V_{CT}(r) - 3V_{OPE}^{(2)}(r)) f_0(r) - 6\sqrt{2} V_{OPE}^{(1)}(r) f_2(r) , \tag{67}$$

$$\langle \left[Y_{L=2} \chi_{S=1}\right]_{1 J_z} \chi_{T=0} | \hat{V} | \psi_d \rangle = (6V_{OPE}^{(1)} - 3V_{OPE}^{(2)}(r)) f_2(r) - 6\sqrt{2} V_{OPE}^{(1)}(r) f_0(r) , \tag{68}$$

From which the two radial equation reads

$$-\frac{1}{2\mu} \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial f_0(r)}{\partial r} \right) + V_{00}(r) f_0(r) + V_{02}(r) f_2(r) = E f_0(r) , \tag{69}$$

$$-\frac{1}{2\mu} \left(\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial f_2(r)}{\partial r} \right) + \frac{6}{r^2} f_2(r) \right) + V_{20}(r) f_0(r) + V_{22}(r) f_2(r) = E f_2(r) , \tag{70}$$

$$\tag{71}$$

where

$$V_{00}(r) = V_{CT}(r) - 3V_{OPE}^{(2)}(r) , \tag{72}$$

$$V_{20}(r) = V_{02}(r) = -6\sqrt{2} V_{OPE}^{(1)}(r) , \tag{73}$$

$$V_{22}(r) = 6V_{OPE}^{(1)}(r) - 3V_{OPE}^{(2)}(r) . \tag{74}$$

The numerical coefficients that multiply the radial terms can be easily computed using Eqs. (58) and (66).

4 Solution of the radial equation

There are several ways to solve the equations above. Among the other it is worthy to mention the Numerov method [**?**] that is finite difference method as well as the R-matrix approach [**?**] that is particularly useful in coupled channel problems.

In this lecture we will transform our problem to solve the two coupled differential equation in a eigenvalue problem. We can do this using the Rayleigh-Ritz variational principle. We write our wave function as a linear expansion over a complete basis set

$$|\psi\rangle = \sum_i c_i |\phi_i\rangle \,. \tag{75}$$

The variational principle says that the best the wave function is the one that minimize the expectation value of the operator $\hat{H} - E$ respect to the variational parameters i.e.

$$\frac{\delta}{\delta c_k^\star} \sum_{i,j} c_i c_j^\star \langle \phi_j | \hat{H} - E | \phi_i \rangle = 0 \,. \tag{76}$$

Using the orthonormality of the basis we obtain

$$\sum_i c_i \langle \phi_j | \hat{H} | \phi_i \rangle = E c_j \,, \tag{77}$$

which represents an eigenvalue problem.

Using the same idea we expand our functions $f_L(r)$ on a complete basis set that span all the squared integrable functions as,

$$f_L(r) = \sum_l c_l^L g_l(r) \,. \tag{78}$$

Clearly any basis that is complete and span all the vector space is a possible choice and will return exactly the same result if we can include an infinite number of states. However, for all practical purposes we need to truncate our basis, therefore, we want to choose one that have the asymptotic behavior as close as possible to the exact radial wave function. We can notice that the nuclear potential goes to zero when $r \to \infty$. This implies that the asymptotic form for the solution of the radial equation is

$$f_L(r) \sim e^{-\beta r} \,. \tag{79}$$

Analogously we want the radial wave function to be finite when the two particle are close to each other i.e.

$$f_L(r) \xrightarrow{r \to 0} \text{finite} \,. \tag{80}$$

Moreover we would like to have it flexible enough to describe the radial wave function and possibly easy to manage when we compute the kinetic energy matrix elements. A optimal choice is the following

$$g_l(r) = N_l L_l^{(2)}(\gamma r) e^{-\frac{\gamma r}{2}} \,, \tag{81}$$

$$N_l = \gamma^{3/2} \sqrt{\frac{l!}{(l+2)!}} \,. \tag{82}$$

where $L_l^{(2)}(x)$ are the generalized Laguerre polynomials of order 2 and γ is a non variational parameter that is used to adjust the integration grids. A typical range for γ is $3.5 - 5.5$ fm^{-1}. With this form the radial basis is orthonormal, i.e

$$\int_0^\infty dr\, r^2 g_{l'}(r) g_l(r) = \delta_{l,l'} \,. \tag{83}$$

4.1 Matrix elements

Using the result given in Eq. (38), the kinetic energy matrix element can be written as

$$\langle \hat{T}^L \rangle_{l',l} = -\frac{\hbar^2}{2\mu} \int_0^\infty dr\, g_{l'}(r) \left[\frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \right) + L(L+1) \right] g_l(r) \,. \tag{84}$$

Performing the derivatives (that is a bit tedious) and being careful with the normalizations, the kinetic energy matrix elements can be written as

$$\langle \hat{T}^L \rangle_{l',l} = +\frac{\hbar^2\gamma^2}{2\mu}\left[(l+L(L+1))I^{(2)}_{l',l} + (l+1)\, I^{(1)}_{l',l} - \sqrt{l(l+2)}I^{(2)}_{l',l-1} - \frac{1}{4}\delta_{l,l'}\right], \tag{85}$$

where

$$I^{(1)}_{l',l} = \frac{N_{l'}N_l}{\gamma^3}\int_0^{+\infty} dx\, x e^{-x} L^{(2)}_{l'}(x) L^{(2)}_l(x)\,, \tag{86}$$

$$I^{(2)}_{l,l'} = \frac{N_{l'}N_l}{\gamma^3}\int_0^{+\infty} dx\, e^{-x} L^{(2)}_{l'}(x) L^{(2)}_l(x)\,. \tag{87}$$

'These can be easily computed in term of l and l' only as

$$I^{(1)}_{l',l} = \frac{1}{2}\sqrt{\frac{(l+1)(l+2)}{(l'+1)(l'+2)}} \qquad \text{for} \quad l \le l'\,, \tag{88}$$

$$I^{(2)}_{l',l} = \frac{1}{2}\sqrt{\frac{(l+1)(l+2)}{(l'+1)(l'+2)}}\left((l'+1) - \frac{l}{3}\right) \qquad \text{for} \quad l \le l'\,. \tag{89}$$

(Note: I don't have a good reference for these formulas and I was not able to prove them so far).

Analogously, we can write the matrix element for the nuclear interaction as

$$\langle \hat{V}^{L',L} \rangle_{l',l} = \frac{N_{l'}N_l}{\gamma^3}\int_0^{\infty} dx\, x^2 e^{-x} L^{(2)}_{l'}(x) V_{L',L}(x/\gamma) L^{(2)}_l(x)\,, \tag{90}$$

that can be computed using for example Gauss-Laguerre quadrature

$$\langle \hat{V}^{L',L} \rangle_{l',l} \simeq \frac{N_{l'}N_l}{\gamma^3}\sum_{i=1}^{N} \omega_i L^{(2)}_{l'}(x_i) V_{L',L}(x_i/\gamma) L^{(2)}_l(x_i)\,, \tag{91}$$

where ω_i and x_i are the Gauss-Laguerre weights and points respectively.

Using the matrix elements computed in this way we can construct the Hamiltonian matrix and solve the eigenvalue problem using the Lapack libraries.

5 Some observables

To check that our description of the deuteron is meaningful we can try to compute some observables and compare it to the experimental values. The mean value of the operators that I propose you to compute are

- The mean square radius

$$\hat{r} = \frac{1}{2}\sum_{i=1,2} r_i^2\,, \tag{92}$$

  where i is the index of the particle.

- The magnetic moment

$$\hat{\mu}_z = \sum_{i=1,2}\left(\frac{1+\tau_z^i}{2}\right)\hat{L}_z^i + \sum_{i=1,2}\left[\mu_p\left(\frac{1+\tau_z^i}{2}\right) + \mu_n\left(\frac{1-\tau_z^i}{2}\right)\right]\sigma_z^i \tag{93}$$

- The quadrupole moment

$$\hat{Q} = \sum_{i=1,2}(3z_i^2 - r_i^2)\left(\frac{1+\tau_z^i}{2}\right) \tag{94}$$

6 Computing operators using Monte Carlo

Now that we have our wave function we want to compute some operators. The operators that we want to compute can be pretty complex and performing the calculation as it has been done before involving the spin, isospin, and the angular part using the Wigner coefficients and then performing numerically the radial integrals can be tedious and inefficient in particular if we consider more complex wave-functions. However, in this case we approach the problem in a complete numerical fashion. We are going to perform the integration numerically using a Monte Carlo technic.

6.1 Monte Carlo Integration

Let us consider the integral

$$I = \int d^d x \, P(\boldsymbol{x}) f(\boldsymbol{x}) \,, \tag{95}$$

where $\boldsymbol{x} = (x_1, \ldots, x_2)$ and $P(\boldsymbol{x})$ is a function such that $P(\boldsymbol{x}) \geq 0$ for any $\boldsymbol{x}$ and

$$\int d^d x \, P(\boldsymbol{x}) = 1 \,. \tag{96}$$

$P(\boldsymbol{x})$ is then a probability density and we can write the integral as the mean value of f with respect to P,

$$I = \langle f \rangle_P \,. \tag{97}$$

This is guaranteed by the central limit theorem that says that the quantity

$$X = \frac{1}{N} \sum_{\boldsymbol{x}_i \in P(\boldsymbol{x}), i=1}^{N} f(\boldsymbol{x}_i) \,, \tag{98}$$

in the limit $N \to \infty$ the probability distribution of X is

$$p(X) \propto \exp\left(\frac{(X - \langle f \rangle_p)^2}{2(\langle f^2 \rangle_p - \langle f \rangle_p^2)/N} \right) \tag{99}$$

that is a Gaussian distribution centered at $\langle f \rangle_p$ with variance $\sigma^2 = (\langle f^2 \rangle_p - \langle f \rangle_p^2)/N$. So to compute an integral we just need to construct the probability distribution P and then generate a sample of N configurations that follows this distribution. How do we generate them? What can be used the M(RT)^2 algorithm. We are not going to prove any part of the algorithm but just detail it. We only mention that it makes use of the detailed balance principle. The practical implementation is the following:

- Given $\boldsymbol{x}$, a new trial point $\boldsymbol{x}^t$ is generated, for example as

  $$\boldsymbol{x}^t = \boldsymbol{x} + \Delta \boldsymbol{\xi} \,, \tag{100}$$

  where Δ is a fixed step and $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_N)$ with the ξ_i uniformly distributed between $-1/2$ and $1/2$

- The $\boldsymbol{x}^t$ is accepted with probability

  $$\min\left\{1, \frac{P(\boldsymbol{x}^t)}{P(\boldsymbol{x})}\right\} \,. \tag{101}$$

  So if $P(\boldsymbol{x}^t)/P(\boldsymbol{x}) \geq 1$ the $\boldsymbol{x}^t$ is always accepted. If $P(\boldsymbol{x}^t)/P(\boldsymbol{x}) < 1$ $\boldsymbol{x}^t$ is accepted with probability $P(\boldsymbol{x}^t)/P(\boldsymbol{x})$. To implement this is to select a random number $\eta \in [0, 1]$ and accept $\boldsymbol{x}^t$ if $P(\boldsymbol{x}^t)/P(\boldsymbol{x}) > \eta$.

- The selection on how big Δ should be is more heuristic. A reasonable choice is to choose Δ such that on average half of the moves are accepted and half of them rejected.

6.2 Matrix element of an operator

We can now compute the matrix element of an operator $\hat{O}$ for the deuteron wave function. We can write it as

$$\langle \hat{O} \rangle = \frac{\langle \psi | \hat{O} | \psi \rangle}{\langle \psi | \psi \rangle} = \frac{1}{\langle \psi | \psi \rangle} \int d\boldsymbol{r}_1 \, d\boldsymbol{r}_2 \psi(\boldsymbol{r}_1, \boldsymbol{r}_2)^\dagger \hat{O} \psi(\boldsymbol{r}_1, \boldsymbol{r}_2) \, . \tag{102}$$

To use the sampling we discuss before we need to construct our probability density in the integral. We can do it in the following way, can use Monte Carlo sampling to compute the integral. So we can rewrite the integral as a sum over the samplings $R = \{\boldsymbol{r}_1^i, \boldsymbol{r}_2^i\}$ where i is the

$$\langle \hat{O} \rangle = \frac{1}{\langle \psi | \psi \rangle} \int \psi_i^\dagger \hat{O} \psi_i = \int \frac{|\psi_i|^2}{\langle \psi | \psi \rangle} \frac{\psi_i^\dagger \hat{O} \psi_i}{|\psi_i|^2} = \int P(\boldsymbol{r}_1^i, \boldsymbol{r}_2^i) \frac{\psi_i^\dagger \hat{O} \psi_i}{|\psi_i|^2} = \sum_{R_i \in P} \frac{\psi_i^\dagger \hat{O} \psi_i}{|\psi_i|^2} \, . \tag{103}$$

where $R_i = \{\boldsymbol{r}_1^i, \boldsymbol{r}_2^i\}$ is the sample. Clearly what we need now to do is to understand how to compute $\psi_i^\dagger \hat{O} \psi_i$.

The operator that we are dealing are typically spin matrices acting on a single particle σ_i^x, σ_i^y, σ_i^z, the isospin matrices acting on a single particle τ_z^i, and the derivative operators $\frac{\partial}{\partial x^i}$, $\frac{\partial}{\partial y^i}$, $\frac{\partial}{\partial z^i}$.

For the spin and isospin operator we know how they act on the single particle therefore we need to express our wave function in term of the spin and isospin of the single particles. To do so we can project our wave function as

$$\begin{aligned} \psi_{s_1 s_2, t_1 t_2}(\boldsymbol{r}_1, \boldsymbol{r}_2) &= \langle s_1, s_2; t_1, t_2; \boldsymbol{r}_1, \boldsymbol{r}_2 | \psi_{1, J_z} \rangle \\ &= \textstyle\sum_{L=0,2} f_L(r) \sum_{M=-L}^{L} \langle L, M \, , 1, S_z | 1, J_z \rangle \langle 1/2, s_1 \, , 1/2, s_2 | 1 S_z \rangle Y_{L,M}(\hat{\boldsymbol{r}}) \langle 1/2, t_1 \, , 1/2, t_2 | 00 \rangle \, . \end{aligned} \tag{104}$$

The representation of the deuteron wave function can be then organized as a matrix in which the columns are defined by the isospin states $|t_1, t_2\rangle$ and the row by the spin states $|s_1, s_2\rangle$ and the matrix elements given in Eq. (104). In the specific case of the deuteron the wave function like looks like

$$\psi_{1 J_z}(\boldsymbol{r}_1, \boldsymbol{r}_2) = \begin{bmatrix} \psi_{+1+1,+1-1} & \psi_{+1+1,-1+1} \\ \psi_{+1-1,+1-1} & \psi_{+1-1,-1+1} \\ \psi_{-1+1,+1-1} & \psi_{+1+1,-1+1} \\ \psi_{-1-1,+1-1} & \psi_{-1-1,-1+1} \end{bmatrix} . \tag{105}$$

Let us now take the operator σ_x^1 and apply to our wave function $\sigma_x^1 \psi_{1 J_z}(\boldsymbol{r}_1, \boldsymbol{r}_2)$. What σ_x^1 does is to invert the components with spin up in the first particle to the one in spin down and vice versa. It is easy then to see that

$$\sigma_x^1 \psi_{1 J_z}(\boldsymbol{r}_1, \boldsymbol{r}_2) = \begin{bmatrix} \psi_{-1+1,+1-1} & \psi_{+1+1,-1+1} \\ \psi_{-1-1,1-1} & \psi_{-1-1,-11} \\ \psi_{+1+1,+1-1} & \psi_{+1+1,-1+1} \\ \psi_{+1-1,+1-1} & \psi_{+1-1,-1+1} \end{bmatrix} . \tag{106}$$

Practically in this representation it exchange the top two rows with the bottom two. Similarly for σ_y^1 but in this case we need to add a phase to the matrix elements i.e.

$$\sigma_y^1 \psi_{1 J_z}(\boldsymbol{r}_1, \boldsymbol{r}_2) = i \begin{bmatrix} \psi_{-1+1,+1-1} & \psi_{+1+1,-1+1} \\ \psi_{-1-1,1-1} & \psi_{-1-1,-11} \\ -\psi_{+1+1,+1-1} & -\psi_{+1+1,-1+1} \\ -\psi_{+1-1,+1-1} & -\psi_{+1-1,-1+1} \end{bmatrix} . \tag{107}$$

For σ_z^1 we have then

$$\psi_{1 J_z}(\boldsymbol{r}_1, \boldsymbol{r}_2) = \begin{bmatrix} \psi_{+1+1,+1-1} & \psi_{+1+1,-1+1} \\ \psi_{+1-1,+1-1} & \psi_{+1-1,-1+1} \\ -\psi_{-1+1,+1-1} & -\psi_{+1+1,-1+1} \\ -\psi_{-1-1,+1-1} & -\psi_{-1-1,-1+1} \end{bmatrix} . \tag{108}$$

We let the reader work out the operators σ_x^2, σ_y^2 and σ_z^2 as well as the operators τ_z^1 and τ_z^2.

To compute the derivatives, we just use a numerical approach. For example

$$\frac{\partial}{\partial x^1}\psi_{1J_z}(\boldsymbol{r}_1,\boldsymbol{r}_2) = \frac{\psi_{1J_z}(x_1+h,y_1,z_1,x_2,y_2,z_2) - \psi_{1J_z}(x_1-h,y_1,z_1,x_2,y_2,z_2)}{2h}\,. \tag{109}$$

On the practical point what we need to do is to compute the wave function again in two new points generated slightly moving each of the coordinates of $+h$ and $-h$ and then subtract the two. A typical value of h is 0.001 fm. We can repeat this operation for each component of the position of the particles and have all the derivatives we need. The calculation of the derivative does not change the form of the matrix.

We have know all the ingredients needed to compute the operators.

7 Elastic form factor of nuclei

If we consider the interaction of an electromagnetic probe (i.e. an electron) with a nucleus we can assume in first approximation that this is mediated by a single photon. A common and widely used approximation known as impulse approximation assumes that the photon interacts only with a nucleons inside the nucleus. This interaction can be parametrized through the following operators:

- The charge operator given by

$$\rho(\omega,\boldsymbol{q}) = \sum_{i=1}^{A}\left[G_E^p(Q^2)\left(\frac{1+\tau_z^i}{2}\right) + G_E^n(Q^2)\left(\frac{1-\tau_z^i}{2}\right)\right]e^{i\boldsymbol{q}\cdot\boldsymbol{r}_i}\,. \tag{110}$$

- The current operator is given by

$$\begin{aligned}\boldsymbol{j}(\omega,\boldsymbol{q}) = &\frac{1}{2m}\sum_{i=1}^{A}\left[G_E^p(Q^2)\left(\frac{1+\tau_z^i}{2}\right) + G_E^n(Q^2)\left(\frac{1-\tau_z^i}{2}\right)\right]e^{i\boldsymbol{q}\cdot\boldsymbol{r}_i}(\boldsymbol{q}-2i\nabla_i)\\ &-\frac{i}{2m}\sum_{i=1}^{A}\left[G_M^p(Q^2)\left(\frac{1+\tau_z^i}{2}\right) + G_M^n(Q^2)\left(\frac{1-\tau_z^i}{2}\right)\right]e^{i\boldsymbol{q}\cdot\boldsymbol{r}_i}(\boldsymbol{q}\times\boldsymbol{\sigma}_i)\end{aligned}$$

where the functions G_E^i and G_M^i are the electric and magnetic form factors of the nucleons that in the code will be parametrized as in Ref. [Kel04] and m is the nucleon mass. Clearly, other terms in the currents appear too. Of particular interest are the one known as meson exchange currents or two-body currents that implies that the photon interacts with the mesons exchanged by the nucleons.

7.1 Cross section and form factors

The elastic scattering of an electron on a nucleus is a process in which the final state of the nucleus remain the same as the initial. The elastic scattering cross section then reads

$$\frac{d\sigma}{d\Omega} = 4\pi\sigma_M f_{\rm rec}^{-1}\left[\frac{Q^4}{q^4}F_L^2 + \left(\frac{Q^2}{2q^2} + \tan^2\theta_e/2\right)F_T^2\right],, \tag{111}$$

where σ_M is the Mott cross section, q and Q are the electron three- and four-momentum transfers, f_{rec} is the recoil correction $f_{rec} = 1 + (2E/m_A)\sin^2\theta_e/2$, E and θ_e are the electron initial energy and scattering angle in the laboratory, and m_A is the mass of the target nucleus. In the case of elastic scattering, the electron energy transfer is $\omega_{el} = \sqrt{q^2+m_A^2} - m_A$ The form factors F_L and F_T are expressed in terms of reduced matrix elements (RMEs) of charge (C_L), magnetic (M_L), and electric (E_L) multipole operators defined as

$$\begin{aligned}F_L^2(q) =& \frac{1}{2J_i+1}\sum_{L=0}^{\infty}|\langle J_f\|C_L(q)\|J_i\rangle|^2\\ F_T^2(q) =& \frac{1}{2J_i+1}\sum_{L=1}^{\infty}\left[|\langle J_f\|M_L(q)\|J_i\rangle|^2 + |\langle J_f\|E_L(q)\|J_i\rangle|^2\right]\end{aligned} \tag{112}$$

where in the elastic electron-scattering the E_L operator vanish because of time reversal invariance. Our goal for the lectures is to compute the longitudinal and transverse form factor. To do so we need to find the relations that connect the operator matrix elements with the multipole matrix element you see here. This would require another full lecture. Here we just give the final formulas for the current

$$\langle JJ;\mathbf{q}|j_y(\mathbf{q})|JJ\rangle = \sqrt{4\pi}\sum_{L\geq 1} i^{L+1}\frac{\langle J,J\,J,-J|L,0\rangle}{\sqrt{L(L+1)}}P_L^1(\cos\theta)\langle J||M_L(q)||J\rangle\,, \tag{113}$$

where $P_L^1(x)$ are the associated Legendre functions and θ is the angle between $\boldsymbol{q}$ and the z-axis defined by the polarization of the nucleus. For the charge we have

$$\langle JJ;\mathbf{q}|\rho(\mathbf{q})|JJ\rangle = \sqrt{4\pi}\sum_{L=0}^{\infty} i^{L}\langle J,J\,J,-J|L,0\rangle P_L(\cos\theta)\langle J|\,|C_L(q)|\,|J\rangle, \tag{114}$$

where $P_L(x)$ are the Legendre polynomials. As you can see the matrix elements are computed with the nucleus polarized along the z-axis with $J_z = J$.

For the deuteron we have $J = 1$ so we have

$$\langle 11;\mathbf{q}|j_y(\mathbf{q})|11\rangle = -\sqrt{\pi}P_1^1(\cos\theta)M_1(q)\,, \tag{115}$$

where we shorten the notation for the multipoles. Selecting $\boldsymbol{q} = q\hat{\boldsymbol{x}}$ we have $\theta = \pi/2$ and $P_1^1(0) = -1$ from which we have

$$M_1(q) = \frac{1}{\sqrt{\pi}}\langle 11;\mathbf{q}|j_y(q\hat{\boldsymbol{x}})|11\rangle,\,. \tag{116}$$

At the same time the charge operator matrix element reads

$$\langle 11;\mathbf{q}|\rho(\mathbf{q})|11\rangle = -\sqrt{\frac{4\pi}{3}}\left(P_0(\cos\theta)C_0(q) + \frac{1}{\sqrt{2}}P_2(\cos\theta)C_2(q)\right)\,. \tag{117}$$

To separate the two components we need then to chose two different angles for example we can chose $\boldsymbol{q} = q\hat{\boldsymbol{z}}$ and so $\cos\theta = 1$

$$\langle 11;\mathbf{q}|\rho(q\hat{\boldsymbol{z}})|11\rangle = -\sqrt{\frac{4\pi}{3}}\left(C_0(q) - \frac{1}{\sqrt{2}}C_2(q)\right)\,, \tag{118}$$

and $\boldsymbol{q} = q\hat{\boldsymbol{x}}$ and so $\cos\theta = \pi/2$ that gives

$$\langle 11;\mathbf{q}|\rho(q\hat{\boldsymbol{x}})|11\rangle = -\sqrt{\frac{4\pi}{3}}\left(C_0(q) + \frac{1}{2\sqrt{2}}C_2(q)\right)\,. \tag{119}$$

Inverting the formulas we obtain then

$$C_0(q) = \frac{1}{\sqrt{12\pi}}\left(\langle 11;\mathbf{q}|\rho(q\hat{\boldsymbol{z}})|11\rangle + 2\langle 11;\mathbf{q}|\rho(q\hat{\boldsymbol{x}})|11\rangle\right) \tag{120}$$

$$C_2(q) = \sqrt{\frac{2}{3\pi}}\left(\langle 11;\mathbf{q}|\rho(q\hat{\boldsymbol{x}})|11\rangle - \langle 11;\mathbf{q}|\rho(q\hat{\boldsymbol{z}})|11\rangle\right)\,. \tag{121}$$

Computing then the matrix elements using the Monte Carlo techniques we can then obtain the longitudinal and transverse form factors exploiting these formulas.

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