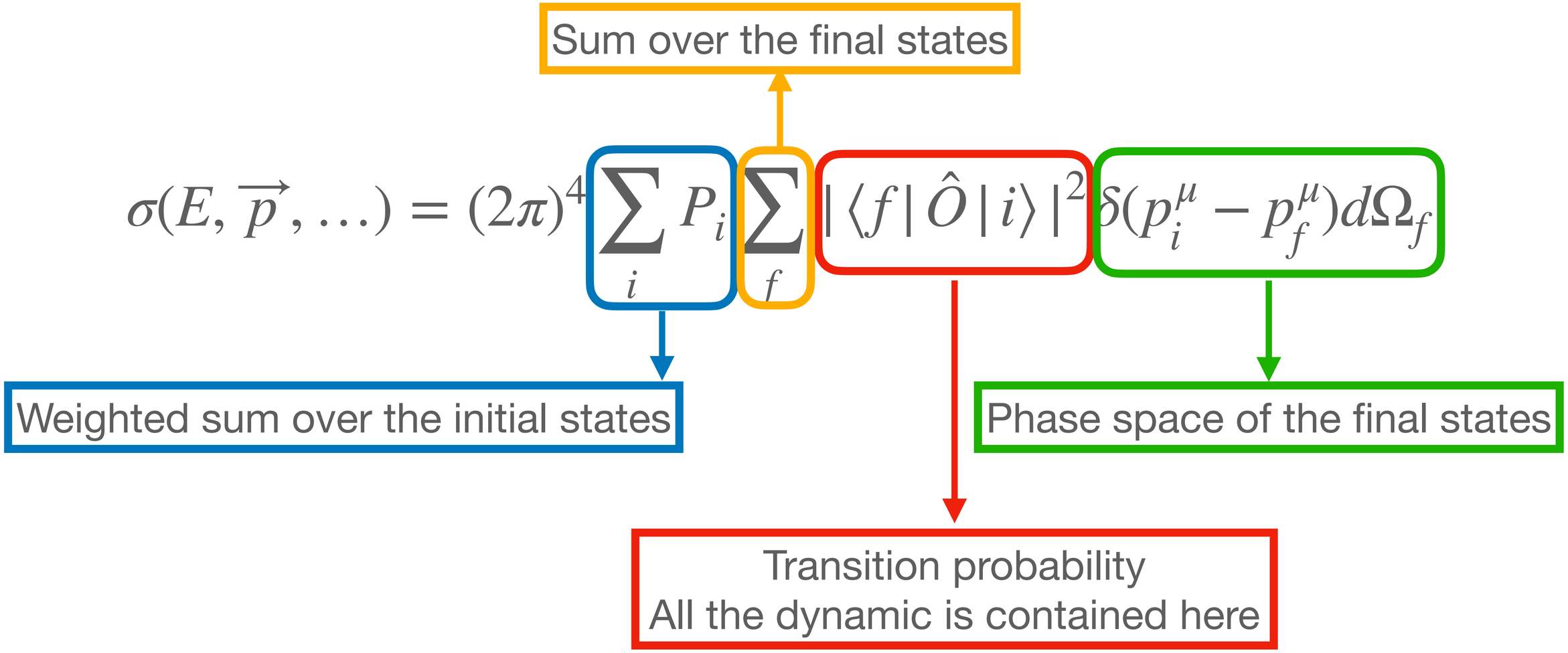
Few-body methods for light nuclei

Summer school “Light ion physics in the EIC era: from nuclear structure to high-energy physics” Florida International University, Miami, FL

Alex Gnech (agnech@odu.edu)

Cross sections

Fermi's Golden Rule (for weakly interacting processes)



Transition probability (matrix element)

when we deal with nuclei interacting with external fields

$$M_{fi} = \langle f | \hat{O} | i \rangle$$

- The operator is made of a “lepton” part and a nuclear part

$$\hat{O} = l_\mu \; j^\mu$$

- The nuclear and lepton waves appear in the initial and final state

$$|f,i\rangle = |\psi^N_{f,i}\rangle \otimes |\psi^L_{f,i}\rangle \qquad H^N |\psi^N_{f,i}\rangle = E_{f,i} |\psi^N_{f,i}\rangle$$

- M_{fi} can be factorized in a “lepton” and a nuclear matrix element

$$\langle f | \hat{O} | i \rangle = \langle \psi^L_f | l_\mu | \psi^L_i \rangle \langle \psi^N_f | j^\mu | \psi^N_i \rangle$$

Nuclear physics from first principles

Ab-initio approach

- How do we construct the Hamiltonian and the currents for nuclei? (Saori Pastore's lectures)

$$\hat{H}|\psi^N\rangle = E|\psi^N\rangle \qquad \hat{j}_\mu$$

- How do we compute the nuclear wave functions?

Outline of the lectures

- The deuteron wave function using a realistic potential
- Beyond A=2: the Hyperspherical Harmonic approach
- Short overview of few-body methods
- Electron-scattering processes (elastic scattering)

Nuclear interaction (OPE)

$$\hat{V} = V_{CT}(r)\left(\frac{3+\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2}{4}\right)\left(\frac{1-\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2}{4}\right)\delta_{L,0} + (V^{(1)}_{OPE}(r)\hat{S}_{12} + V^{(2)}_{OPE}(r)\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)(\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)\,,$$

$$\hat{S}_{12} = 3(\boldsymbol{\sigma}_1\cdot\hat{\boldsymbol{r}})(\boldsymbol{\sigma}_2\cdot\hat{\boldsymbol{r}}) - \boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2$$

$$V^{(1)}_{OPE}(r) = \frac{g_A^2}{12\pi}\frac{m_\pi^3}{F_\pi^2}\frac{e^{-m_\pi r}}{m_\pi r}\left(1+\frac{3}{m_\pi r}+\frac{3}{(m_\pi r)^2}\right)C_{R_L}(r)\,,$$

$$V^{(2)}_{OPE}(r) = \frac{g_A^2}{12\pi}\frac{m_\pi^3}{F_\pi^2}\frac{e^{-m_\pi r}}{m_\pi r}C_{R_L}(r)\,,$$

$$C_{R_{\rm L}}(r) = 1 - \frac{1}{(r/R_{\rm L})^6\, e^{(r-R_{\rm L})/a_{\rm L}}+1},$$

$$V_{CT}(r) = \frac{C_S}{\pi^{3/2}R_{\rm S}^3}e^{-(r/R_{\rm S})^2}$$

$$\chi^\dagger_{T'}(\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)\chi_{T'}\,,$$

$$\chi^\dagger_{S'}(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)\chi_{S'}\,,$$

$$\int d\hat{r}\,\left[Y_{L'}(\hat{r})\chi_{S'}\right]^\dagger_{JJ_z}\hat{S}_{12}\left[Y_L(\hat{r})\chi_S\right]_{JJ_z}\,.$$

Wigner-j coefficients

$$\hat{j} = \sqrt{2j+1}$$

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \equiv \frac{(-1)^{j_1-j_2-m_3}}{\sqrt{2j_3+1}} \langle j_1 m_1 j_2 m_2 \mid j_3 \, (-m_3) \rangle$$

Wigner-3j

$$|(j_1 j_2)_{j_{12}} j_3 J M\rangle = \sum_{j_{23}} T^{j_1 j_2 j_3}_{j_{12} j_{23} J} |j_1 (j_2 j_3)_{j_{23}} J M\rangle$$

Wigner-6j

$$T^{j_1 j_2 j_3}_{j_{12} j_{23} J} = (-)^{j_1+j_2+j_3+J} \hat{j}_{12} \hat{j}_{23} \begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & J & j_{23} \end{Bmatrix}$$

$$|(j_1 j_2)_{j_{12}} (j_3 j_4)_{j_{34}} J M\rangle = \sum_{j_{13} j_{24}} N^{j_1 j_2 j_3 j_4}_{j_{12} j_{34} j_{13} j_{23} J} |(j_1 j_3)_{j_{13}} (j_2 j_4)_{j_{24}} J M\rangle$$

Wigner-9j

$$N^{j_1 j_2 j_3 j_4}_{j_{12} j_{34} j_{13} j_{23} J} = \hat{j}_{13} \hat{j}_{24} \hat{j}_{12} \hat{j}_{34} \begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j_4 & j_{34} \\ j_{13} & j_{24} & J \end{Bmatrix} .$$

Pure spin operator

$$[\boldsymbol{\sigma}|1/2\rangle]_{1/2S_z} = -\sqrt{3}|1/2\, S_z\rangle$$

$$\begin{aligned}\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 &= -\sigma^1_{1+1}\sigma^2_{1-1} - \sigma^1_{1-1}\sigma^2_{1+1} + \sigma^1_{10}\sigma^2_{10} \\ &= -\sqrt{3}\langle 1\,+1\,,1\,-1|0,0\rangle\sigma^1_{1+1}\sigma^2_{1-1} - \sqrt{3}\langle 1\,-1\,,1\,+1|0,0\rangle\sigma^1_{1-1}\sigma^2_{1+1} - \sqrt{3}\langle 1\,0\,,1\,0|0,0\rangle\sigma^1_{10}\sigma^2_{10} \\ &= -\sqrt{3}\left[\sigma^1_1\sigma^2_1\right]_{00}\,.\end{aligned} \tag{54}$$

$$(\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2)\chi_{SS_z} = -\sqrt{3}\left[\sigma^1_1\sigma^2_1\right]_{00}\left[|s_1\rangle\,|s_2\rangle\right]_{SS_z} = -\sqrt{3}\langle 0\,0\,,S\,S_z|S\,S_z\rangle\,\left[\left[\sigma^1_1\sigma^2_1\right]_{00}\left[|s_1\rangle\,|s_2\rangle\right]_S\right]_{SS_z}\,,$$

$$(\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2)\chi_{SS_z} = -\sqrt{3}\sum_{\Sigma_1\Sigma_2}\hat{0}\hat{S}\hat{\Sigma}_1\hat{\Sigma}_2\begin{Bmatrix} 1 & 1 & 0 \\ 1/2 & 1/2 & S \\ \Sigma_1 & \Sigma_2 & S \end{Bmatrix}\left[\left[\sigma^1_1|s_1\rangle\right]_{\Sigma_1}\left[\sigma^2_1|s_2\rangle\right]_{\Sigma_2}\right]_{SS_z}$$

Pure spin operator

$$(\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2)\chi_{SS_z} = -6\sqrt{3}\hat{S} \left\{ \begin{array}{ccc} 1 & 1 & 0 \\ 1/2 & 1/2 & S \\ 1/2 & 1/2 & S \end{array} \right\} [|s_1\rangle|s_2\rangle]_{SS_z} \ ,$$

$$\chi^{\dagger}_{S'S'_z}(\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2)\chi_{SS_z} = -6\sqrt{3}\hat{S} \left\{ \begin{array}{ccc} 1 & 1 & 0 \\ 1/2 & 1/2 & S \\ 1/2 & 1/2 & S \end{array} \right\} \delta_{SS'}\delta_{S_zS'_z} \ ,$$

Tensor operator

$$\hat{S}_{12} = 3(\boldsymbol{\sigma}_1 \cdot \hat{\boldsymbol{r}})(\boldsymbol{\sigma}_2 \cdot \hat{\boldsymbol{r}}) - \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2$$

$$\begin{aligned}\boldsymbol{\sigma} \cdot \hat{\boldsymbol{r}} &= \sigma_x \frac{x}{r} + \sigma_x \frac{z}{r} + \sigma_z \frac{z}{r} = \sqrt{\frac{4\pi}{3}} \left(-\sigma_{1+1} Y_{1-1} - \sigma_{1-1} Y_{1+1} + \sigma_{10} Y_{10}\right) \\ &= -\sqrt{4\pi} [\sigma_1 Y_1]_{00}\end{aligned}$$

$$\hat{S}_{12} = 12\pi \left[\left(\sigma_1^1 Y_1(\hat{r})\right)_0 \left(\sigma_1^2 Y_1(\hat{r})\right)_0\right]_{00} + \sqrt{3} \left[\sigma_1^1 \sigma_1^2\right]_{00} .$$

$$\left[\left(\sigma_1^1 Y_1(\hat{r})\right)_0 \left(\sigma_1^2 Y_1(\hat{r})\right)_0\right]_{00} = \sum_{\ell,m} \hat{\ell}^2 \begin{Bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ \ell & \ell & 0 \end{Bmatrix} \langle \ell - m\, , \ell\, m | 00 \rangle \left[\sigma_1^1 \sigma_1^2\right]_{\ell - m} (Y_1 Y_1)_{\ell m}$$

$$\left[Y_{\ell_1} Y_{\ell_2}\right]_{LM} = \frac{B^{\ell}_{\ell_1 \ell_2}}{\sqrt{4\pi}} Y_{LM}(\hat{r}) \, , \qquad B^{L}_{\ell_1 \ell_2} = \hat{\ell}_1 \hat{\ell}_2 (-)^{\ell_1 + \ell_2} \begin{pmatrix} \ell_1 & \ell_2 & L \\ 0 & 0 & 0 \end{pmatrix},$$

Tensor operator

$$\hat{S}_{12} = 3(\boldsymbol{\sigma}_1 \cdot \hat{\boldsymbol{r}})(\boldsymbol{\sigma}_2 \cdot \hat{\boldsymbol{r}}) - \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2$$

$$\begin{aligned}\left[\left(\sigma_1^1 Y_1(\hat{r})\right)_0 \left(\sigma_1^2 Y_1(\hat{r})\right)_0\right]_{00} &= \frac{3}{\sqrt{4\pi}} \sum_{\ell,m} \hat{\ell}^2 \left\{ \begin{array}{ccc} 1 & 1 & 0 \\ 1 & 1 & 0 \\ \ell & \ell & 0 \end{array} \right\} \left(\begin{array}{ccc} 1 & 1 & \ell \\ 0 & 0 & 0 \end{array} \right) \langle \ell - m\, , \ell\, m | 00 \rangle \left[\sigma_1^1 \sigma_1^2\right]_{\ell - m} Y_{\ell m} \\ &= \frac{3}{\sqrt{4\pi}} \sum_{\ell,m} \hat{\ell}^2 \left\{ \begin{array}{ccc} 1 & 1 & 0 \\ 1 & 1 & 0 \\ \ell & \ell & 0 \end{array} \right\} \left(\begin{array}{ccc} 1 & 1 & \ell \\ 0 & 0 & 0 \end{array} \right) \left[\left[\sigma_1^1 \sigma_1^2\right]_\ell Y_\ell\right]_{00} .\end{aligned}$$

$$\left[\left(\sigma_1^1 Y_1(\hat{r})\right)_0 \left(\sigma_1^2 Y_1(\hat{r})\right)_0\right]_{00} = -\frac{1}{4\pi\sqrt{3}} \left[\sigma_1^1 \sigma_1^2\right]_{00} + \frac{1}{\sqrt{6\pi}} \left[Y_2 \left[\sigma_1^1 \sigma_1^2\right]_2\right]_{00}$$

$$\hat{S}_{12} = 2\sqrt{6\pi} \left[Y_2 \left[\sigma_1^1 \sigma_1^2\right]_2\right]_{00}$$

Exercise Ia (for home)

Prove

$$\hat{S}_{12} = 2\sqrt{6\pi}\left[Y_2\left[\sigma_1^1\sigma_1^2\right]_2\right]_{00}$$

$$\int d\hat{r}\left[Y_{L'}\chi_{S'}\right]^{\dagger}_{JJ_z}\hat{S}_{12}\left[Y_L\chi_S\right]_{JJ_z} = 30\sqrt{6}\hat{J}\hat{L}\hat{S}\hat{L}'\hat{S}'(-)^L\begin{Bmatrix} 2 & 2 & 0 \\ L & S & J \\ L' & S' & J \end{Bmatrix}\begin{Bmatrix} 1 & 1 & 2 \\ 1/2 & 1/2 & S \\ 1/2 & 1/2 & S' \end{Bmatrix}\begin{pmatrix} 2 & L & L' \\ 0 & 0 & 0 \end{pmatrix}$$

Exercise Ib

$$V_{00}(r) = V_{CT}(r) - 3V_{OPE}^{(2)}(r)\,,$$
$$V_{20}(r) = V_{02}(r) = -6\sqrt{2}V_{OPE}^{(1)}(r)\,,$$
$$V_{22}(r) = 6V_{OPE}^{(1)}(r) - 3V_{OPE}^{(2)}(r)\,.$$

https://drive.google.com/drive/folders/1LLUUTaHBT2IyHaZ-p-ISeRm7x_MHttyV?usp=sharing

Exercise II

Compute the deuteron wave function

- Kinetic Energy Matrix Element

$$\langle \hat{T}^L \rangle_{l',l} = +\frac{\hbar^2\gamma^2}{2\mu}\left[(l+L(L+1))I^{(2)}_{l',l} + (l+1)\,I^{(1)}_{l',l} - \sqrt{l(l+2)}I^{(2)}_{l',l-1} - \frac{1}{4}\delta_{l,l'}\right]$$

$$I^{(1)}_{l',l} = \frac{N_{l'}N_l}{\gamma^3}\int_0^{+\infty} dx\, x e^{-x} L^{(2)}_{l'}(x) L^{(2)}_{l}(x)\,,$$

$$I^{(2)}_{l,l'} = \frac{N_{l'}N_l}{\gamma^3}\int_0^{+\infty} dx\, e^{-x} L^{(2)}_{l'}(x) L^{(2)}_{l}(x)\,.$$

$$I^{(1)}_{l',l} = \frac{1}{2}\sqrt{\frac{(l+1)(l+2)}{(l'+1)(l'+2)}} \qquad \text{for} \quad l \le l'\,,$$

$$I^{(2)}_{l',l} = \frac{1}{2}\sqrt{\frac{(l+1)(l+2)}{(l'+1)(l'+2)}}\left((l'+1)-\frac{l}{3}\right) \qquad \text{for} \quad l \le l'\,.$$

- Potential Energy Matrix Element

$$\langle \hat{V}^{L',L} \rangle_{l',l} = \frac{N_{l'}N_l}{\gamma^3}\int_0^{\infty} dx\, x^2 e^{-x} L^{(2)}_{l'}(x) V_{L',L}(x/\gamma) L^{(2)}_{l}(x)\,,$$

$$\langle \hat{V}^{L',L} \rangle_{l',l} \simeq \frac{N_{l'}N_l}{\gamma^3}\sum_{i=1}^{N} \omega_i L^{(2)}_{l'}(x_i) V_{L',L}(x_i/\gamma) L^{(2)}_{l}(x_i)$$

Exercise III (if you have time)

Compute the matrix element of the following operators

- Charge mass radii $$\hat{r} = \frac{1}{2} \sum_{i=1,2} r_i^2 \,,$$
- Magnetic moment $$\hat{\mu}_z = \sum_{i=1,2} \left(\frac{1+\tau_z^i}{2} \right) \hat{L}_z^i + \sum_{i=1,2} \left[\mu_p \left(\frac{1+\tau_z^i}{2} \right) + \mu_n \left(\frac{1-\tau_z^i}{2} \right) \right] \sigma_z^i$$
- Quadrupole moment $$\hat{Q} = \sum_{i=1,2} (3z_i^2 - r_i^2) \left(\frac{1+\tau_z^i}{2} \right)$$

The Hyperspherical Harmonic method

A. Kievsky, *et al.*, J. Phys. G, **35**, 063101 (2008) and L.E. Marcucci, *et al.*, Front. Phys. **8**, 69 (2020)

Eigenvalue problem for bound state

- **Rayleigh-Ritz variational principle**

$$E = \min_{c} \frac{\langle \Psi(c) | \hat{H} | \Psi(c) \rangle}{\langle \Psi(c) | \Psi(c) \rangle}$$

- A possible choice for the variational wave function is

$$|\psi(c)\rangle = \sum_{i} c_i |f_i\rangle$$

Complete orthonormal basis → **Hyperspherical Harmonics**

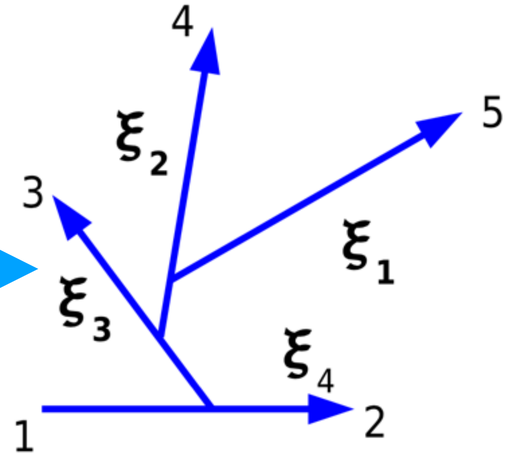
- **(Generalized) Eigenvalue problem**

$$\sum_{j} c_j \langle f_i | \hat{H} | f_j \rangle = E c_i$$

Hyperspherical coordinates

- **Kinetic energy in Jacobi vectors**

$$T = -\frac{\hbar^2}{2m}\sum_{i=1}^{A}\nabla^2_{\vec{r}_i} = T_{CM} - \frac{\hbar^2}{m}\sum_{i=1}^{A-1}\nabla^2_{\boxed{\vec{\xi}_i}}$$



- **Kinetic energy in hyperspherical coordinates**

$\vec{\xi}_i$
- Angular part of the Jacobi vec. $\hat{\xi}_i$
- Angles between Jacobi vec. ϕ_i
- Hyperradius $\rho^2 = \sum_i \xi_i^2$

$$T_0 = -\frac{\hbar^2}{m}\left(\frac{\partial^2}{\partial\rho^2} + \frac{3A-4}{\rho}\frac{\partial}{\partial\rho} - \frac{\boxed{\Lambda^2(\Omega)}}{\rho^2}\right)$$

$$\cos\phi_i = \frac{\xi_i}{\sqrt{\xi_1^2 + \xi_2^2 + \cdots + \xi_i^2}}$$

The Hyperspherical Harmonic basis

- Eigenstates of the $\Lambda^2(\Omega)$ operator

$$\Lambda^2(\Omega)\mathcal{Y}_{[K]}(\Omega) = K(K+3A-5)\mathcal{Y}_{[K]}(\Omega)$$

Eigenvalue

The parameter we use to control our expansion

- The trivial case (A=2)

$$\Lambda^2(\theta,\phi)\mathcal{Y}_{[K]}(\Omega) = K(K+1)\mathcal{Y}_{[K]}(\Omega)$$

$$L^2 Y_{L,M}(\theta,\phi) = L(L+1)Y_{L,M}(\theta,\phi)$$

Explicit expression

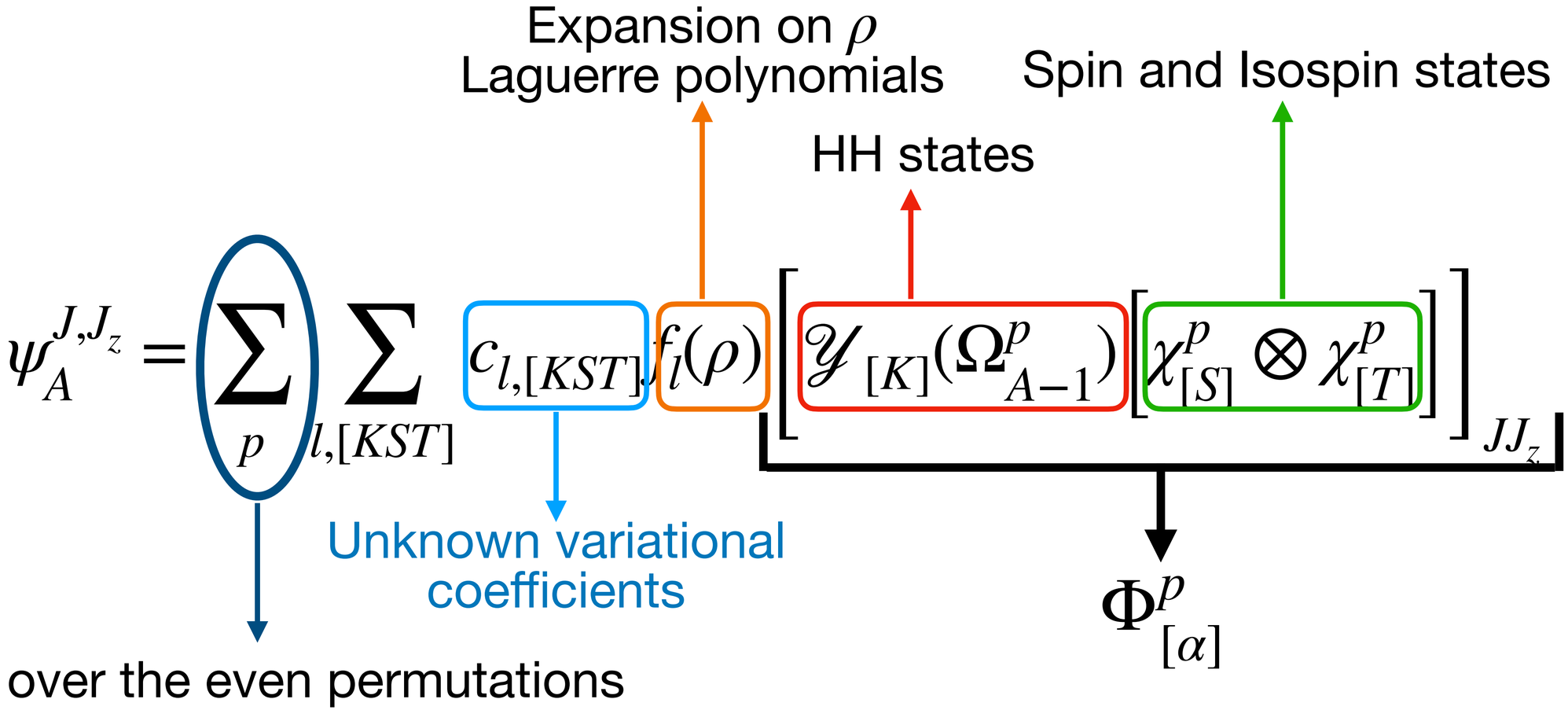
Hyperangular momentum operator

$$\Lambda_i^2(\Omega_i) = \frac{\partial^2}{\partial\varphi_i^2} + \left[3(i-2)\mathrm{cotan}\,\varphi_i + 2(\mathrm{cotan}\,\varphi_i - \tan\varphi_i)\right]\frac{\partial}{\partial\varphi_i} + \frac{{L_i}^2}{\cos^2\varphi_i} + \frac{\Lambda_{i-1}^2(\Omega_{i-1})}{\sin^2\varphi_i}\ ,$$

Spherical Harmonic for A=3

$$Y_{[K]}(\Omega_2) = \mathcal{N}_n^{\ell_2,\nu_2}(\cos\varphi_2)^{\ell_2}(\sin\varphi_2)^{\ell_1}Y_{\ell_1 m_1}(\hat{x}_1)Y_{\ell_2 m_2}(\hat{x}_2)\ P_n^{\ell_1+1/2,\ell_2+1/2}(\cos 2\varphi_2)$$

The HH wave function



This permits to antisymmetrize the wave function selecting specific quantum numbers

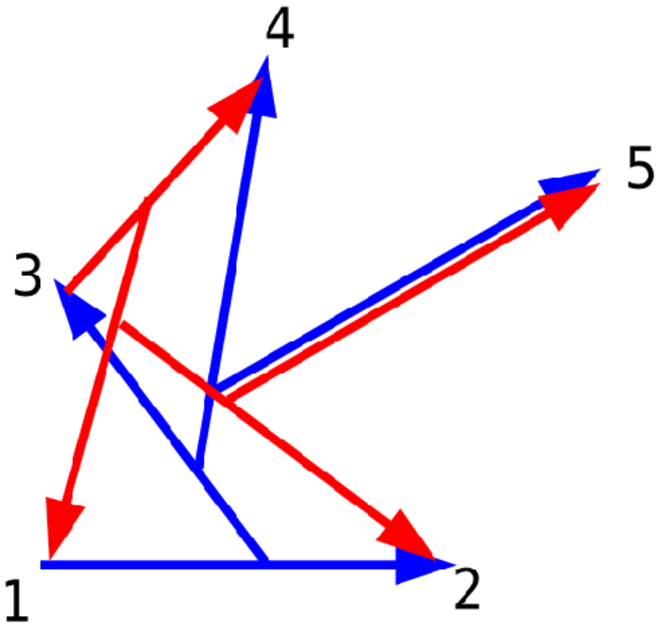
Construction of the basis I

- Geometric property of the HH

$$\mathcal{Y}_{[K]}(\Omega^p) = \sum_{[K'](K=K')} a^{p\to 1}_{[K],[K']} \mathcal{Y}_{[K']}(\Omega^1)$$

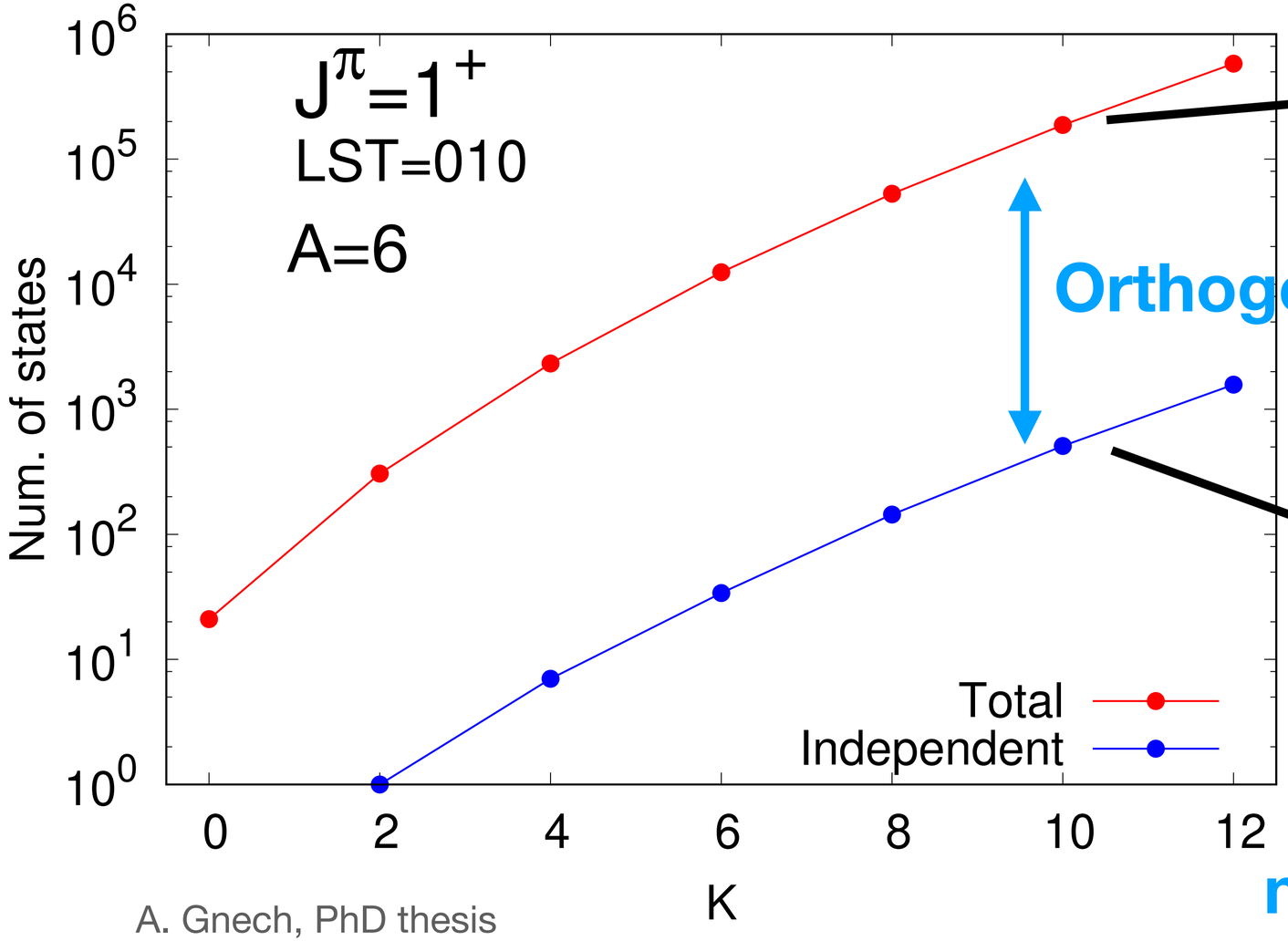
- Transform the sum over the permutations on sum of geometric coefficients

$$\sum_{\text{even } p} \Phi^p_{[\alpha]} = \sum_{\text{even } p} \sum_{[\alpha']} a^{p\to 1}_{[\alpha],[\alpha']} \Phi^1_{[\alpha']} = \sum_{[\alpha']} A_{[\alpha],[\alpha']} \Phi^1_{[\alpha']}$$



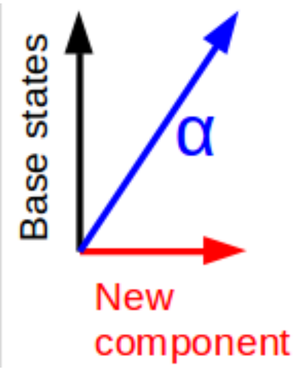
Knowing the coefficients is knowing the entire basis

Construction of the basis II



Orthogonalization

Total number of non orthogonal states



Total number of orthogonal states

The dimension of the final matrix to diagonalize is small

A. Gnech, PhD thesis

A. Gnech et al., Phys Rev. C **102**, 014001 (2020)

Matrix elements

- For any operator is possible to write it in the following way

$$\langle HH_{[\alpha]} | \sum_{i<j} \hat{O}_{ij} | HH_{\beta} \rangle = \frac{A(A-1)}{2} \langle HH_{[\alpha]} | \hat{O}_{12} | HH_{\beta} \rangle =$$

$$= \frac{A(A-1)}{2} \sum_{[\alpha'],[\beta']} A_{[\alpha],[\alpha']} A_{[\beta],[\beta']} \langle \Phi^1_{[\alpha']} | \hat{O}_{12} | \phi^1_{[\beta']} \rangle$$

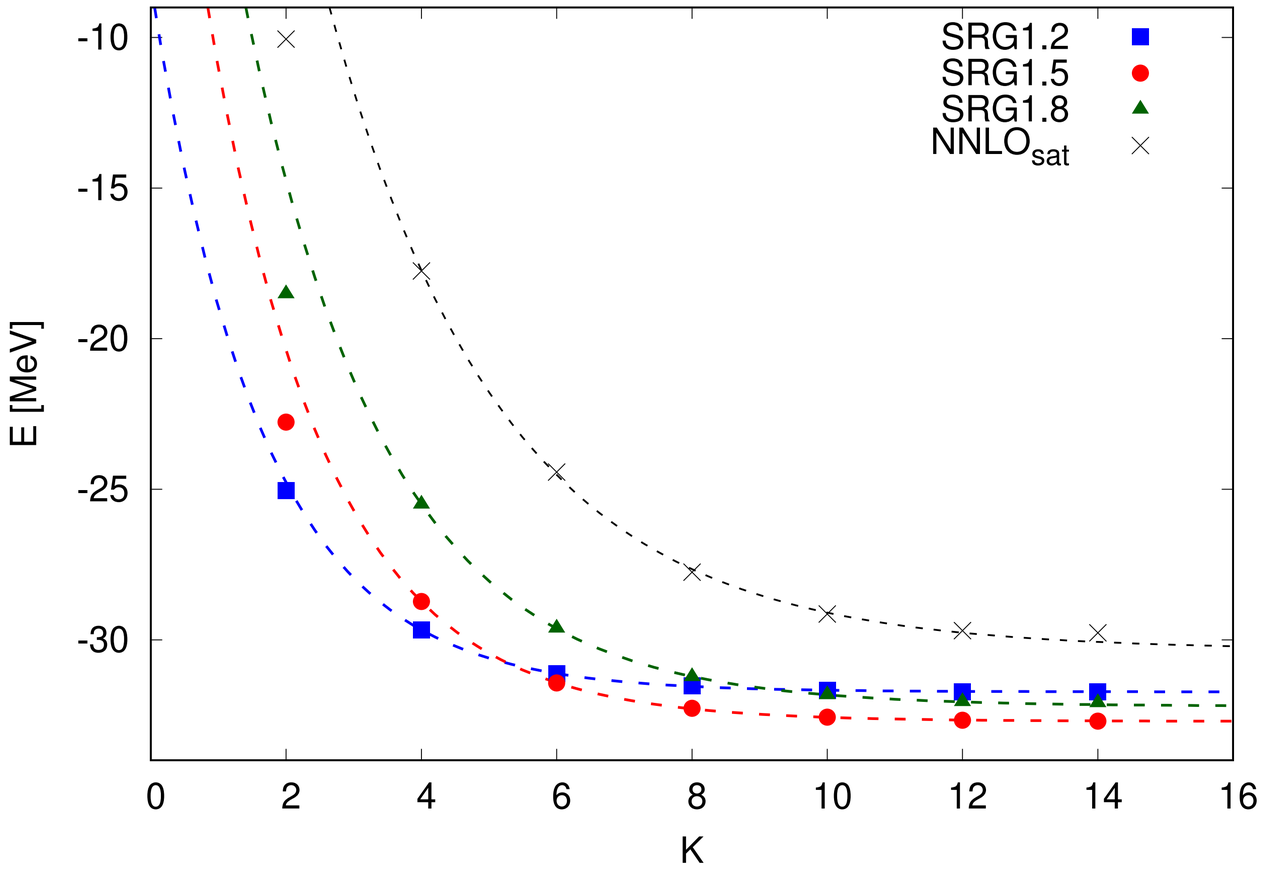
Only geometric (operator independent)
Computationally expensive

Depends on the operator
Implies few numerical integrations

A. Gnech, PhD thesis
A. Gnech et al., Phys Rev. C **102**, 014001 (2020)

Convergence of the HH basis

^{6}Li ground state



- Interaction N3LO500 [1] SRG evolved [2]
- No 3-body forces are considered
- Extrapolation is needed

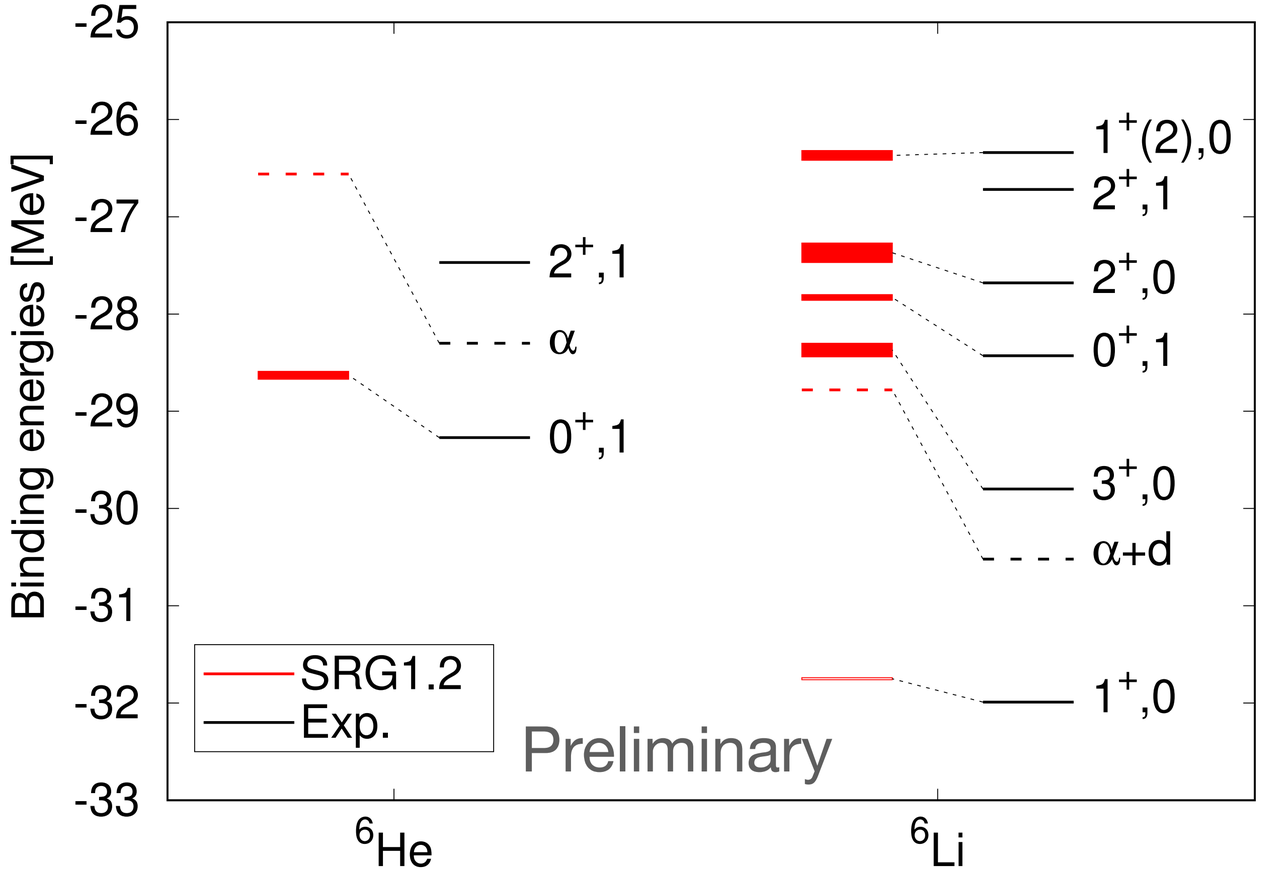
$$E(K) = E(\infty) + e^{-bK}$$

SRG 1.2	SRG 1.5	SRG 1.8	NNLO_sat	Exp.
-31.81(1)	-32.91(2)	-32.68(9)	-30.71(15)	-31.99

[1] D. R. Entem and R. Machleidt, Phys. Rev. C **68**, 041001(R) (2003)
[2] S. K. Bogner, R. J. Furnstahl, and R. J. Perry, Phys. Rev. C **75**, 061001(R) (2007)

The A=6 spectra

By fixing J it is possible to obtain also the excited states



^{6}Li

J^π, T	Exp.	SRG1.2
$1^+, 0$	-31.99	-31.78(1)
$\alpha + d$	-30.52	-28.78
$3^+, 0$	-29.80	-28.37(7)
$0^+, 1$	-28.43	-26.37(5)
$2^+, 0$	-27.86	-27.4(1)
$2^+, 1$	-26.72	–
$1^+_2, 0$	-26.34	-27.83(3)

^{6}He

J^π, T	Exp.	SRG1.2
$0^+, 1$	-29.27	-28.63(4)
α	-28.30	-26.56
$2^+, 1$	-27.47	–

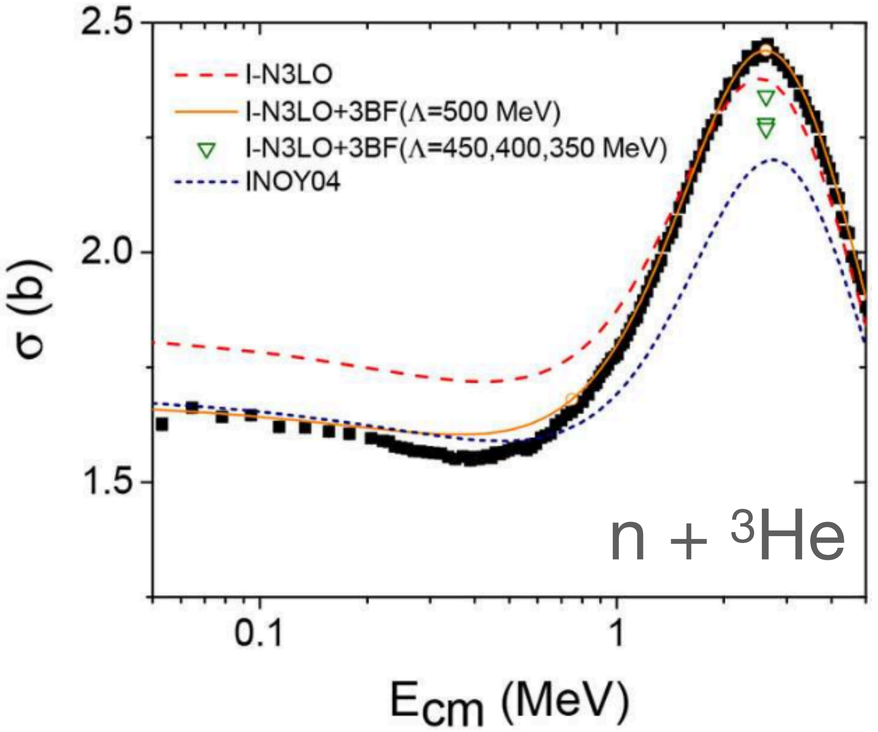
K=12

K=8

An overview on few-body methods in nuclear physics

Fadeev-Yakubovsky

- Jacobi coordinates + decomposition of wave function components.
- Both configuration (A=5) and momentum space (A=4).
- Exact asymptotic behavior.
- High precision approach.



$$\Psi = \sum_{i<j=1}^{N} \Phi_{ij} = \Phi_{12} + \Phi_{13} + \Phi_{23},$$

$$\Phi_{ij} = G_0(E) V_{ij} \Psi; \qquad G_0(E) = (E - H_0)^{-1},$$

$$(E - H_0 - V_{ij}) \Phi_{ij} = V_{ij} \sum_{kl \neq ij} \Phi_{kl},$$

[Lazauskas R and Carbonell J, Few-Body Syst 60 (2019) and Front. Phys. 7:251 (2020).]

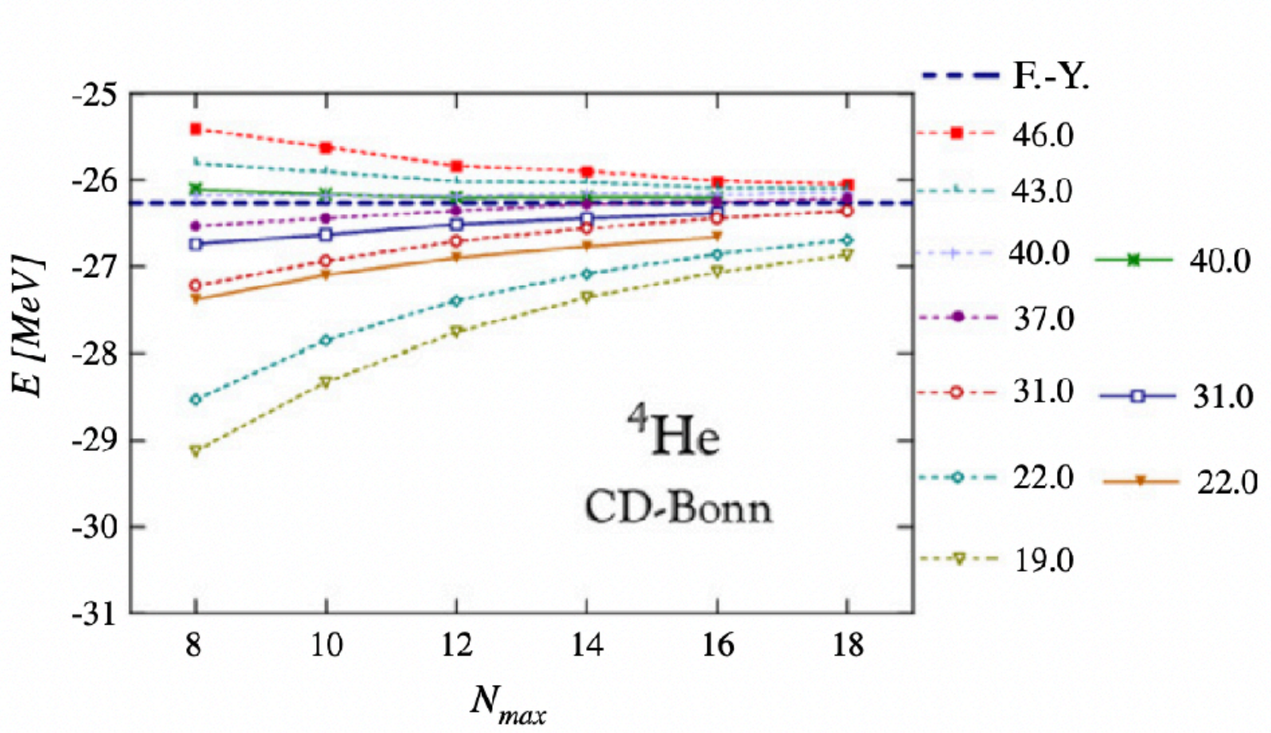
No-Core Shell model

- Complete harmonic oscillator basis

$$\Psi = \sum_{i}^{N_{max}} c_i \phi_i \qquad H_{OH}\phi_i = E_i \phi_i$$

- Variational solution for the bound state

- Scattering states using the NCSM with continuum



[B.R. Barrett, et al.,Progress in Particle and Nuclear Physics,Volume 69, (2013)]

Variational Monte Carlo

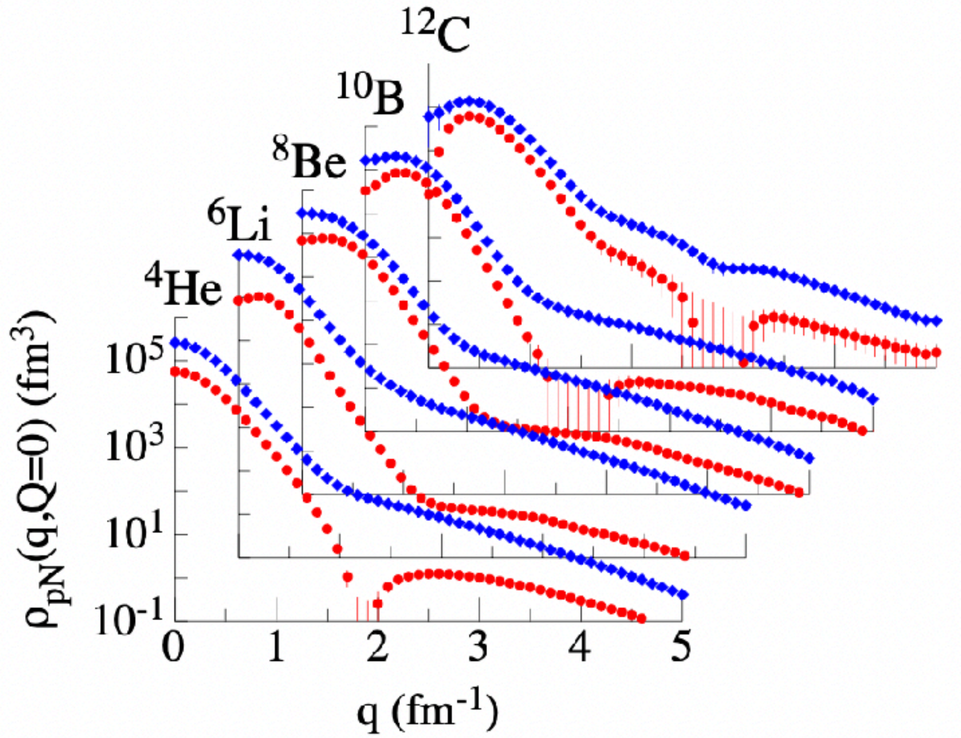
See Saori Pastore lectures

- Physically motivated trial wave function that includes explicit correlations

$$|\Psi_T\rangle \;=\; \mathcal{F}|\Phi\rangle.$$

- Optimization via energy optimization

$$E_V = \frac{\langle\Psi_T|H|\Psi_T\rangle}{\langle\Psi_T|\Psi_T\rangle} \geq E_0$$



VMC pn and pp pair momentum distributions

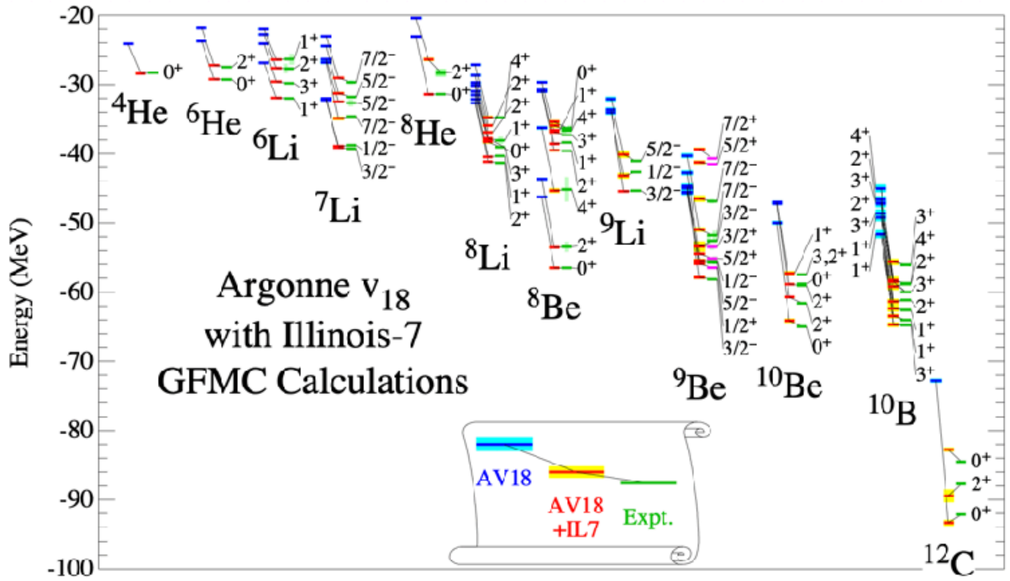
[J Carlson et al.,
Rev. Mod. Phys. 87, 1067 (2015)]

GFMC and AFDMC

Green Function MonteCarlo

Projection out of the ground state from a quantum state.
Full spin-isospin structure.

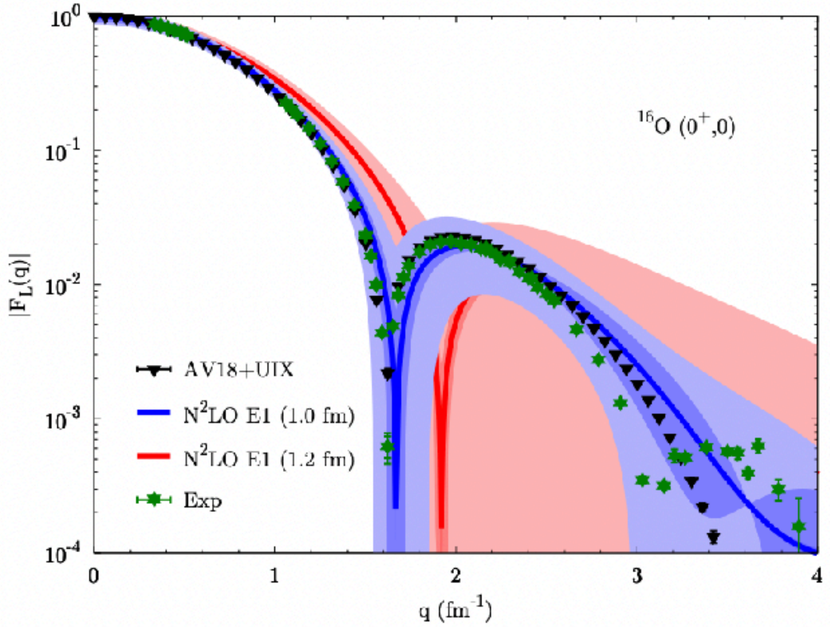
$$|\Psi_0\rangle \propto \lim_{\tau\to\infty} \exp[-(H-E_0)\tau]|\Psi_T\rangle$$



Auxiliary Field Diffusion Monte Carlo

Outer product of particle position and single nucleon spin-isospin spinor states

$$|\boldsymbol{R}\,S\rangle = |\boldsymbol{r}_1 s_1\rangle \otimes |\boldsymbol{r}_2 s_2\rangle \cdots \otimes |\boldsymbol{r}_n s_n\rangle$$



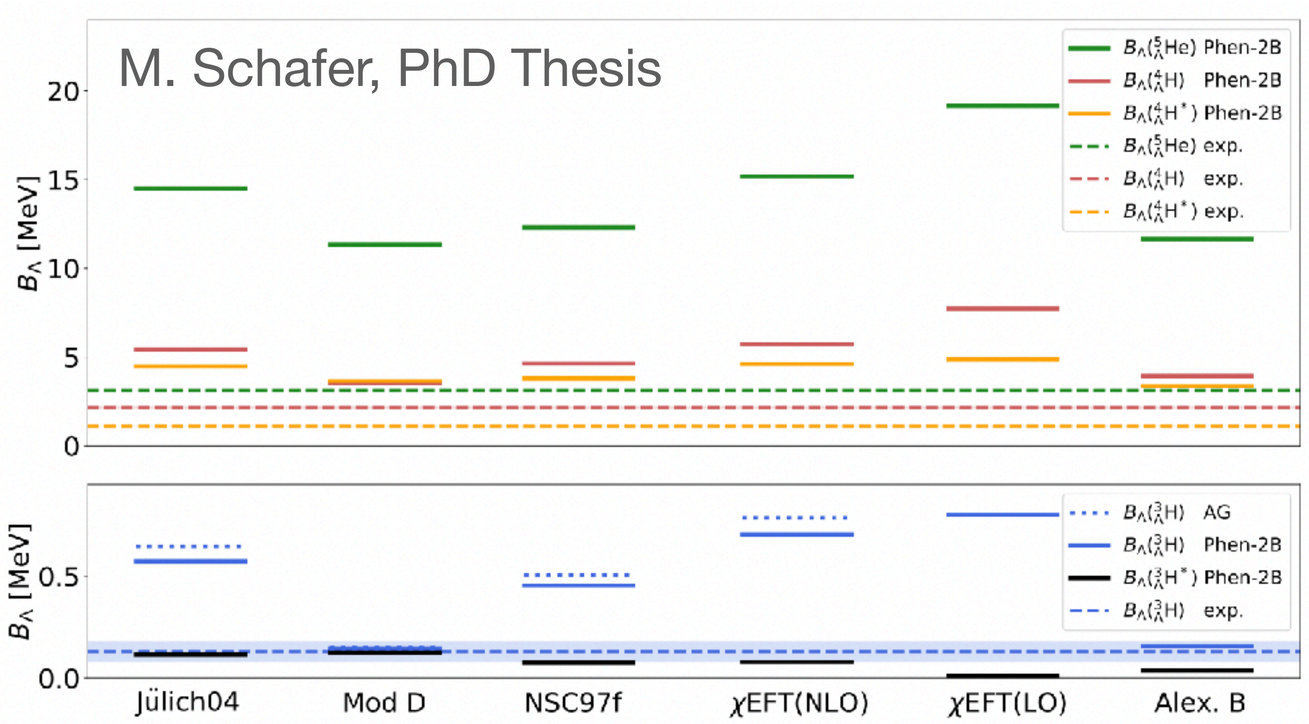
[J Carlson et al., Rev. Mod. Phys. 87, 1067 (2015)]

Stochastic Variational Method

- Use of linear combination of correlated Gaussian as basis expansion.
- The parameters in the matrix A are optimized using a stochastic approach.
- New element on the basis are added stochastically as well.

$$\psi_{SM_S}(\mathbf{x}, A) = \mathcal{A}\{G_A(\mathbf{x})\chi_{SM_S}\},$$

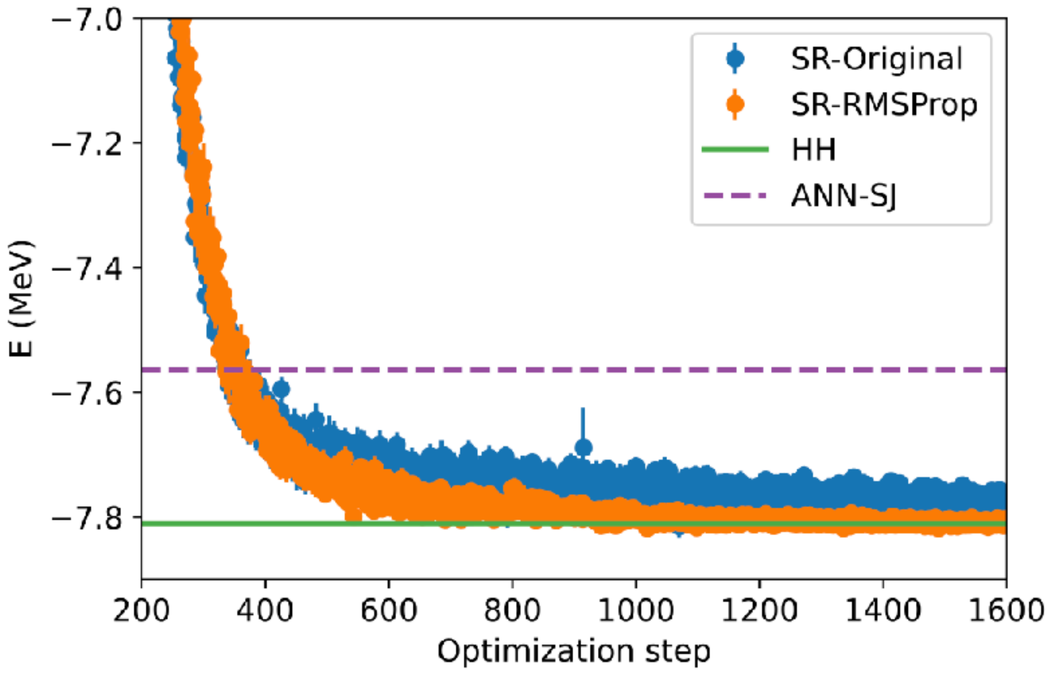
$$G_A(\mathbf{x}) = \mathrm{e}^{-\frac{1}{2}\mathbf{x}^T A\mathbf{x}},$$



[Suzuki, Y., Varga, K.: Stohastic Variational Approach to Quantum-Mechanical Few-Body Problems. Springer, Berlin (1998)]

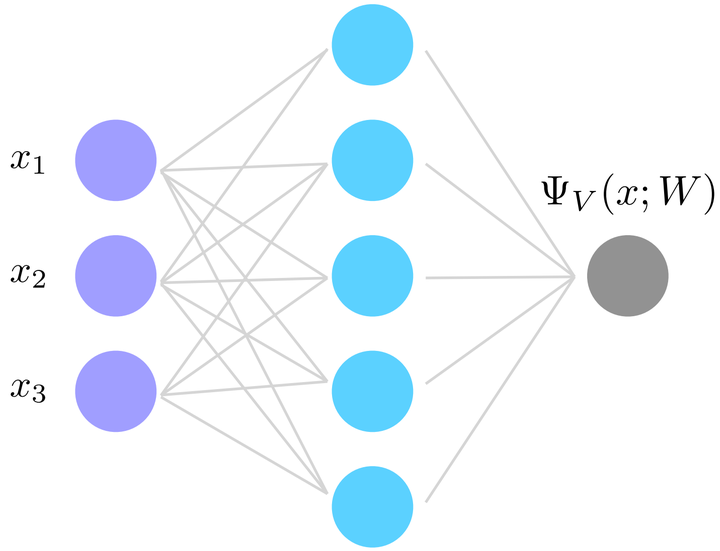
Neural Network Quantum States

See Alessandro Lovato lecture



$$E(W) = \frac{\langle \Psi_V(W) | H | \Psi_V(W) \rangle}{\langle \Psi_V(W) | \Psi_V(W) \rangle}$$

$$W_{new} = W_{old} + \delta W(\partial_W \Psi, \partial_W^2 \Psi, E(W))$$

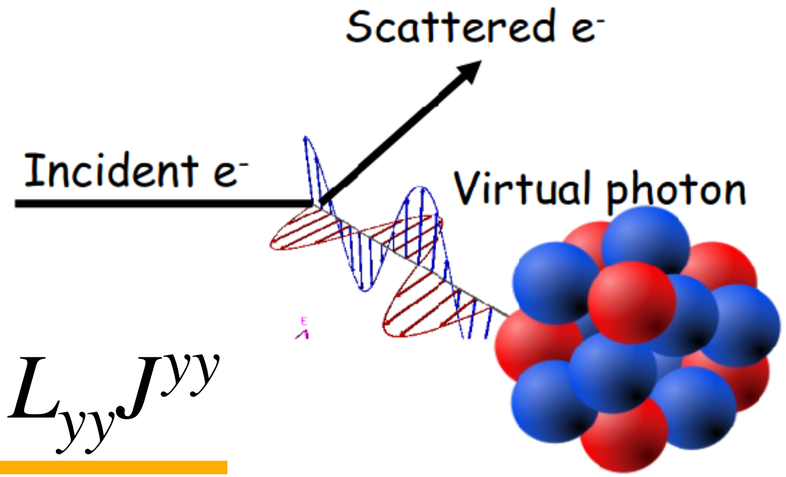


[A. Lovato, et al., Phys. Rev. Res. 4, 043178 (2022)]

Elastic electron-scattering on nuclei

Elastic scattering of electrons on nuclei

$$\langle f | \hat{O} | i \rangle = \langle \psi_f^L | l_\mu | \psi_i^L \rangle \langle \psi_f^N | j^\mu | \psi_i^N \rangle$$



$$|\langle f | \hat{O} | i \rangle|^2 = L_{\mu\nu} J^{\mu\nu} = L_{00} J^{00} + L_{xx} J^{xx} + L_{yy} J^{yy}$$

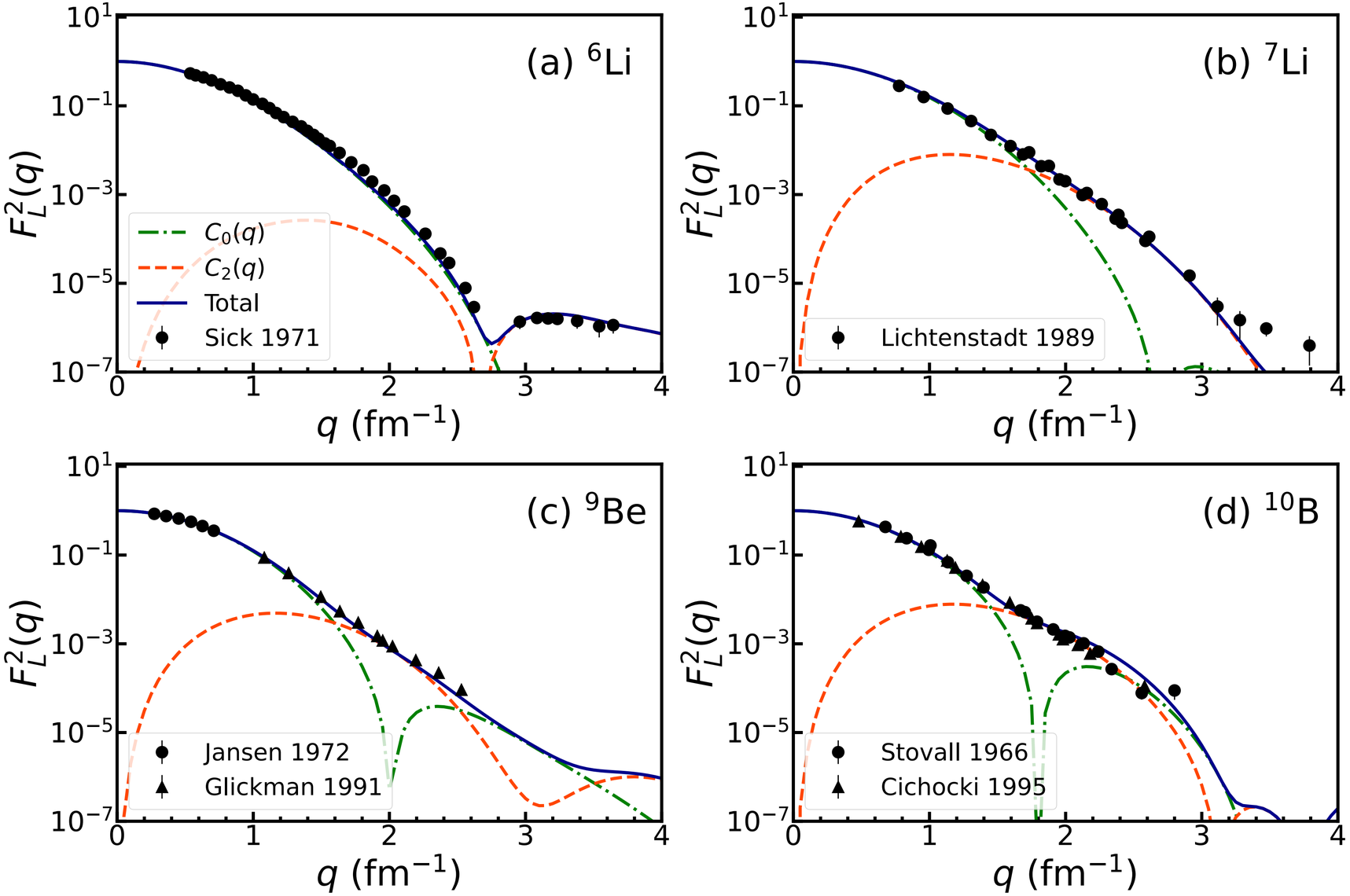
$$\frac{d\sigma}{d\Omega} = 4\pi\sigma_M f_{\text{rec}}^{-1} \left[\frac{Q^4}{q^4} F_L^2(q) + \left(\frac{Q^2}{2q^2} + \tan^2 \theta_e/2 \right) F_T^2(q) \right]$$

F_L longitudinal form factor
F_T transverse form factor

Mott's cross section (scattering of electrons from a point-like charge)

G.B. King, et al. PRC 110, 054325 (2024)

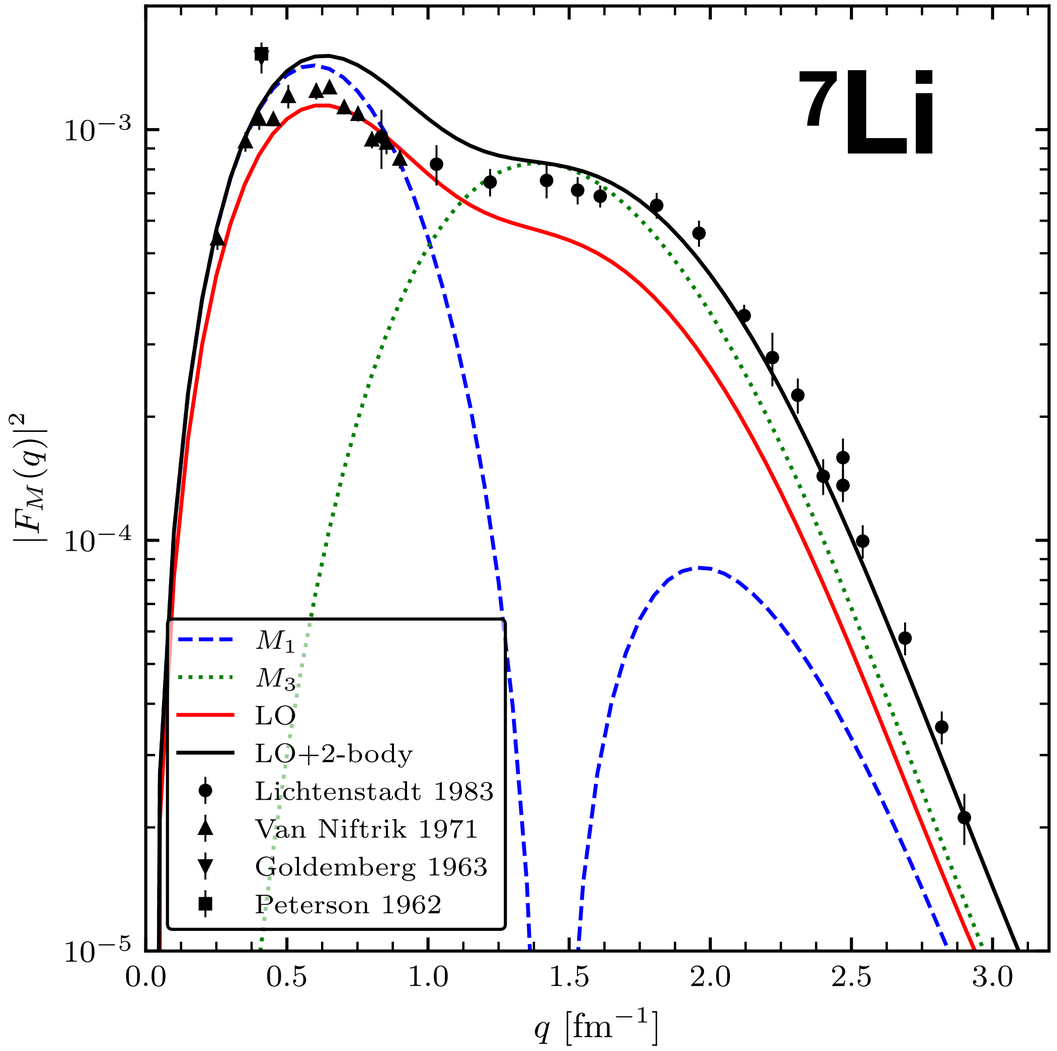
Charge form factor predictions

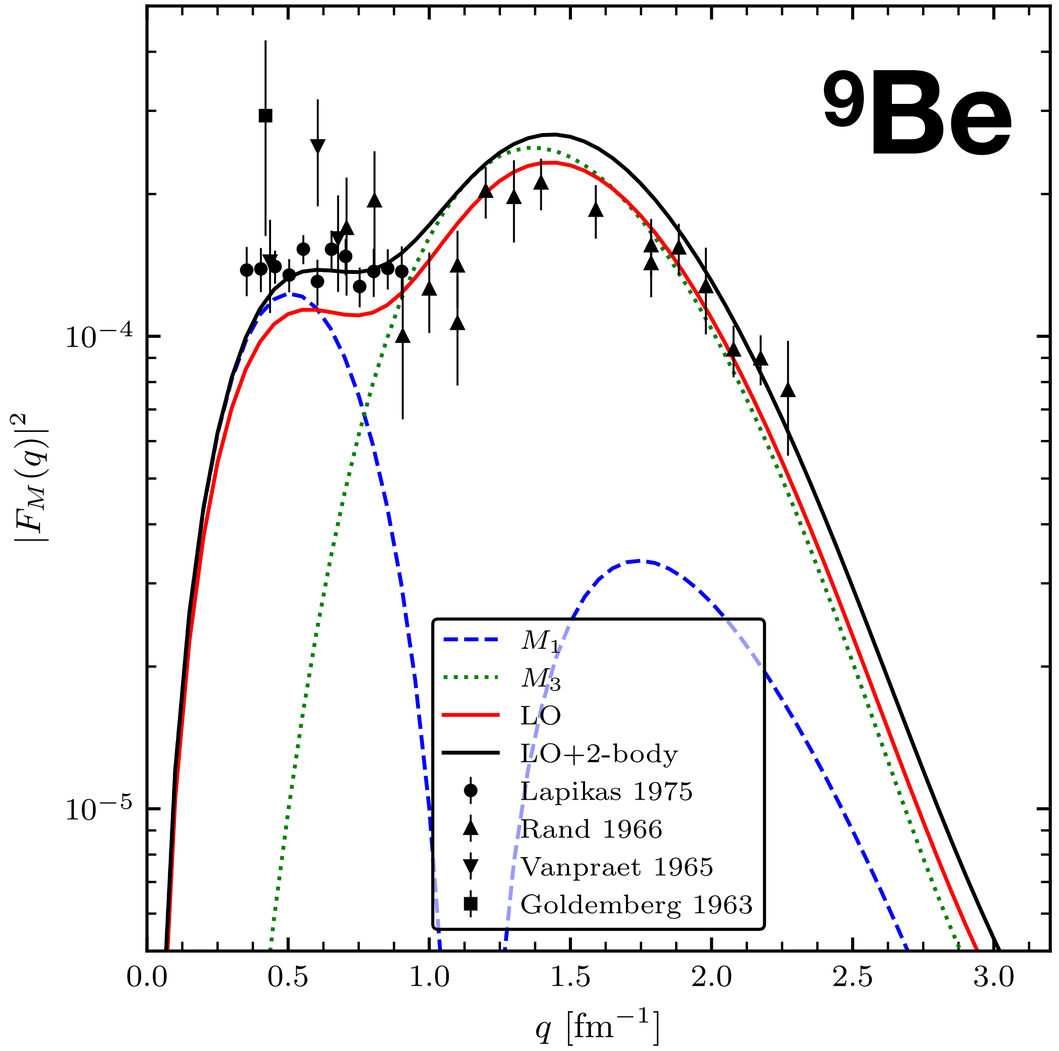


G. Chambers-Wall, et al. PRL 133, 142501 (2024)

Magnetic form factor predictions

Lithium-7 and Berilium-9 (isovector dominated)

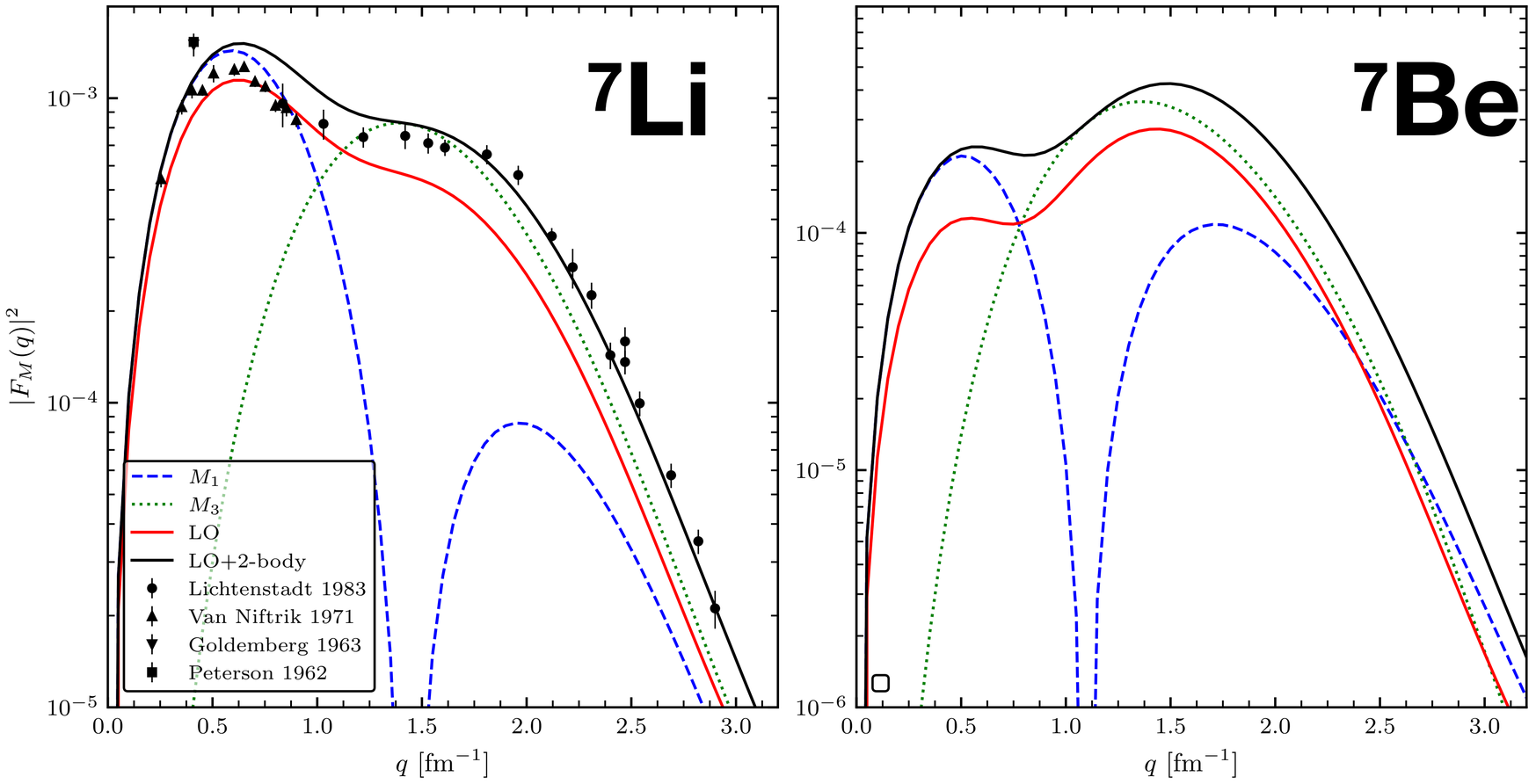




Mirror nuclei structure

- M_1 is enhanced respect to M_3 for nuclei with an unpaired neutron in the p-shell.

- We observed a similar behavior for the mirror systems ^{9}Li-^{9}C and ^{9}Be-^{9}B



Pure prediction (no previous literature) + no experimental confirmation

Mirror nuclei structure

The reason

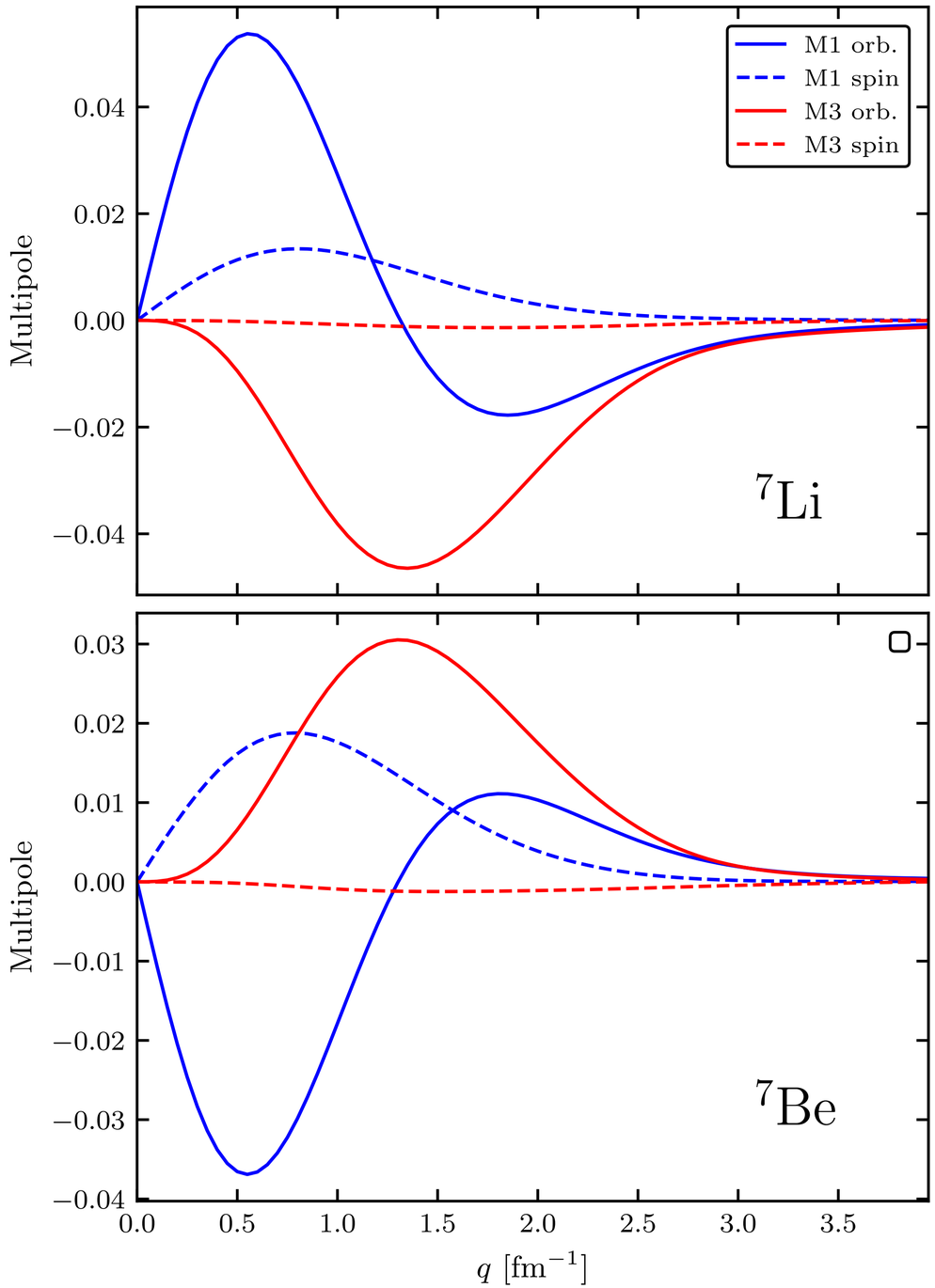
$$\mathbf{j}^{\mathrm{LO}}(\mathbf{q}) = \frac{\epsilon_i(q_\mu^2)}{2\,m}\,\left[\mathbf{p}_i\,,\,\mathrm{e}^{i\mathbf{q}\cdot\mathbf{r}_i}\right]_+ + i\,\frac{\mu_i(q_\mu^2)}{2\,m}\,\mathrm{e}^{i\mathbf{q}\cdot\mathbf{r}_i}\,\sigma_i\times\mathbf{q}\,,$$

Convection current

Magnetic current

No contribution to M_3

Change sign if there is an unpaired neutron/proton

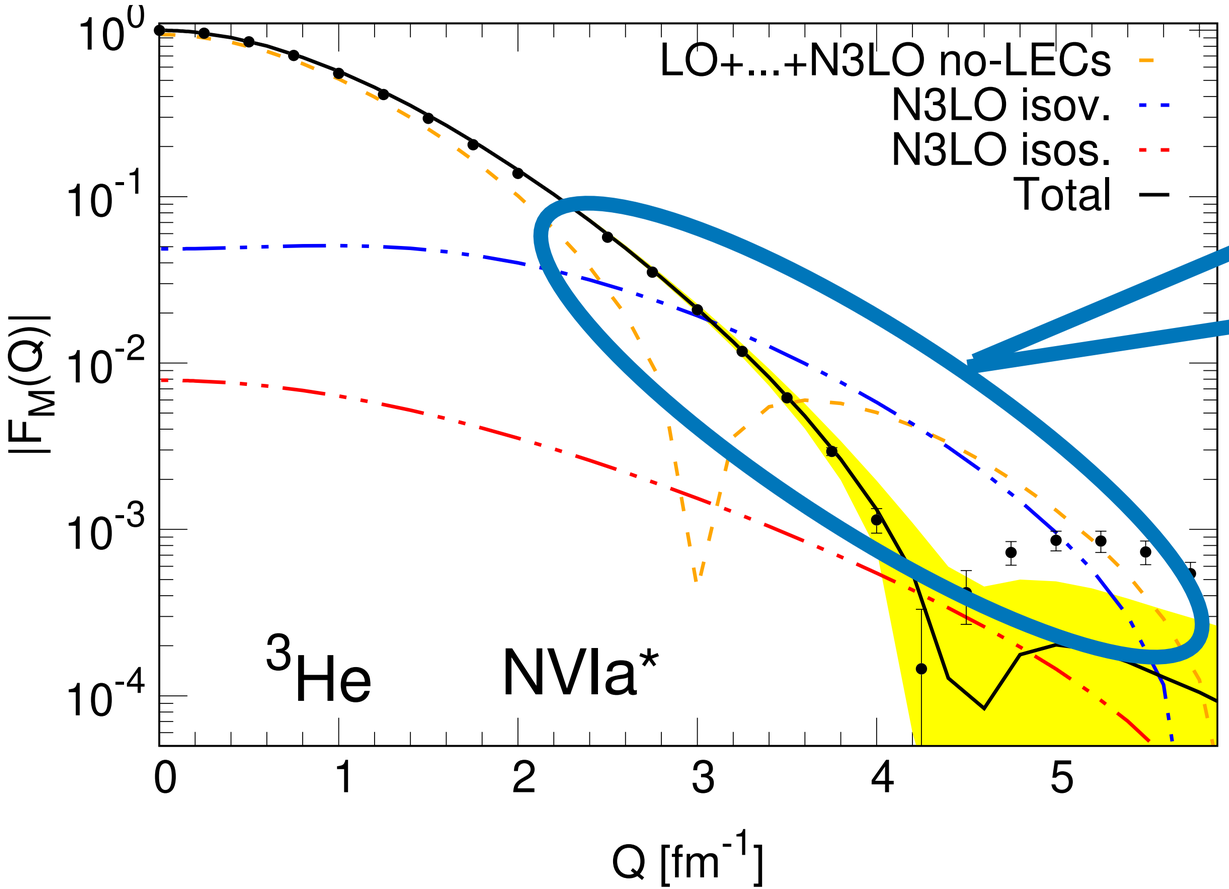


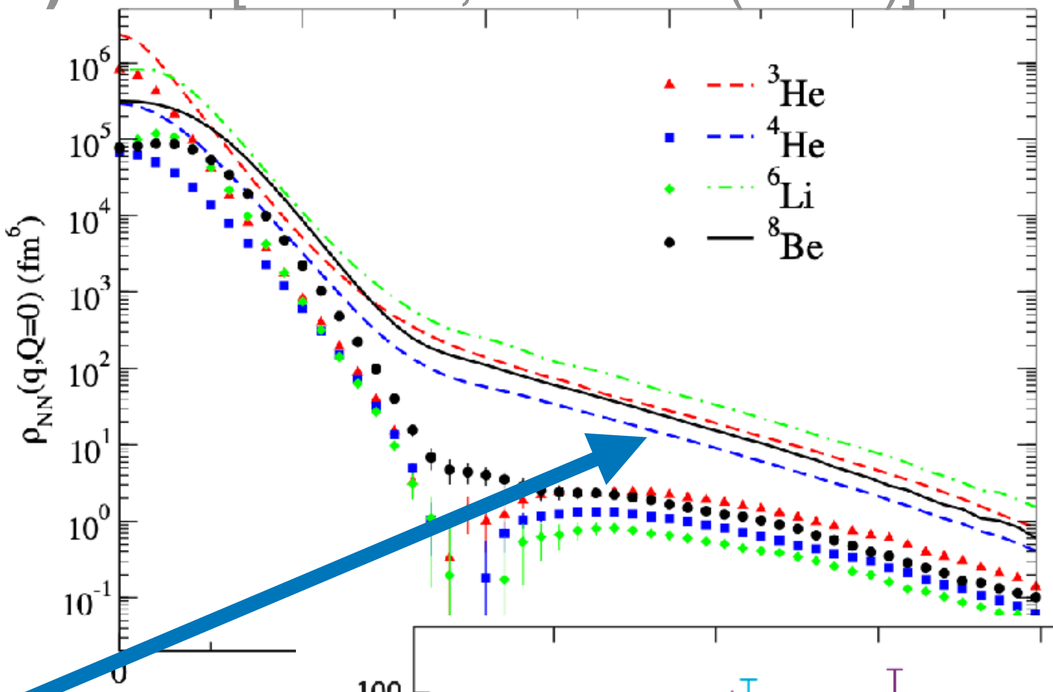
A.G. and R. Schiavilla, PRC 106, 04401 (2020)

Form factors and correlations

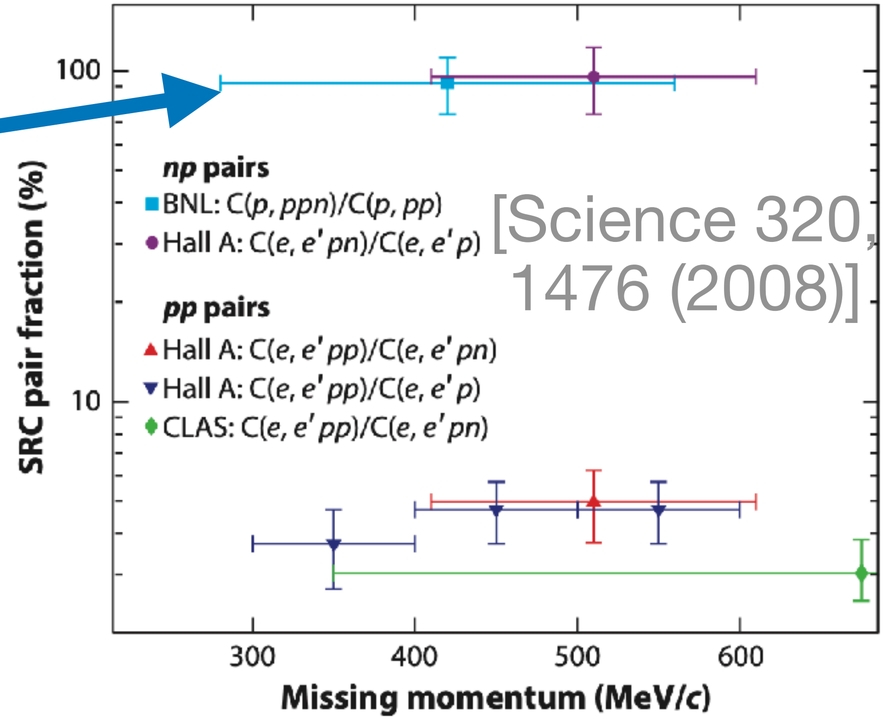
Isovector currents transform S/T=0/1 in S/T=1/0 pairs

np dominance





[PRL 98, 132501 (2007)]

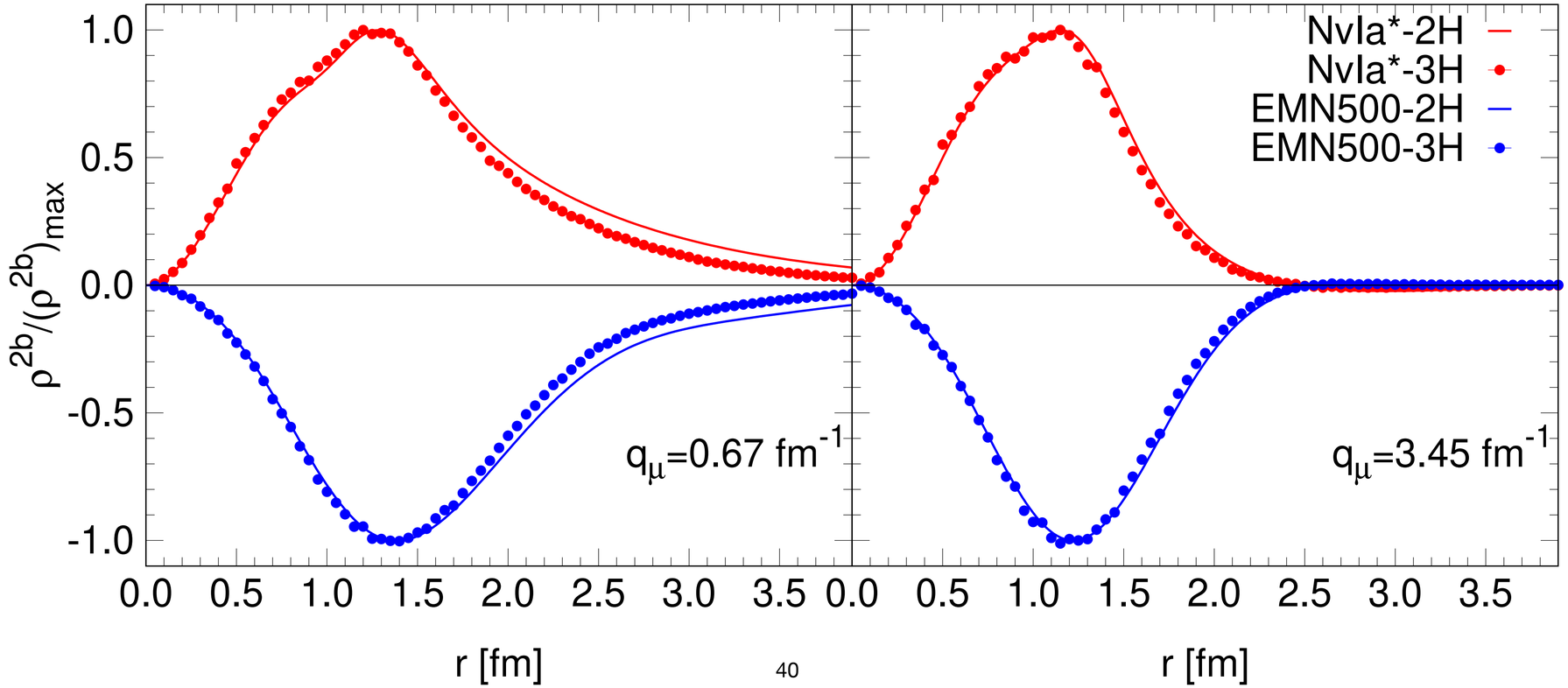


[Science 320, 1476 (2008)]

How correlations works in nuclei

Universal behavior of isovector transitions

Correlated np pairs ⟷ **Universal 2-body wave functions** ⟷ **Universal 2-body transition densities**



Elastic form factor

$$\frac{d\sigma}{d\Omega} = 4\pi\sigma_M f_{\text{rec}}^{-1}\left[\frac{Q^4}{q^4}F_L^2 + \left(\frac{Q^2}{2q^2} + \tan^2\theta_e/2\right)F_T^2\right],$$

$$F_L^2(q) = \frac{1}{2J_i+1}\sum_{L=0}^{\infty}|\langle J_f\|C_L(q)\|J_i\rangle|^2$$

$$F_T^2(q) = \frac{1}{2J_i+1}\sum_{L=1}^{\infty}\left[|\langle J_f\|M_L(q)\|J_i\rangle|^2 + |\langle J_f\|E_L(q)\|J_i\rangle|^2\right]$$

Reduced matrix element calculation

- Matrix element - reduced matrix element relation

$$\langle JJ;\mathbf{q}|\rho(\mathbf{q})|JJ\rangle = \sqrt{4\pi}\sum_{L=0}^{\infty} i^L \langle J,J\,J,-J|L,0\rangle P_L(\cos\theta)\langle J||C_L(q)||J\rangle,$$

- Only two independent multipoles (i.e two independent direction of q)

$$\begin{aligned} C_0(q) &= \frac{1}{\sqrt{12\pi}}\left(\langle 11;\mathbf{q}|\rho(q\hat{\boldsymbol{z}})|11\rangle + 2\langle 11;\mathbf{q}|\rho(q\hat{\boldsymbol{x}})|11\rangle\right) \\ C_2(q) &= \sqrt{\frac{2}{3\pi}}\left(\langle 11;\mathbf{q}|\rho(q\hat{\boldsymbol{x}})|11\rangle - \langle 11;\mathbf{q}|\rho(q\hat{\boldsymbol{z}})|11\rangle\right) . \end{aligned}$$

Matrix element of the charge operator

$$\langle \hat{O} \rangle = \frac{1}{\langle \psi | \psi \rangle} \int \psi_i^\dagger \hat{O} \psi_i = \int \frac{|\psi_i|^2}{\langle \psi | \psi \rangle} \frac{\psi_i^\dagger \hat{O} \psi_i}{|\psi_i|^2} = \int P(\boldsymbol{r}_1^i, \boldsymbol{r}_2^i) \frac{\psi_i^\dagger \hat{O} \psi_i}{|\psi_i|^2} = \sum_{R_i \in P} \frac{\psi_i^\dagger \hat{O} \psi_i}{|\psi_i|^2} .$$

$$\rho(\omega, \boldsymbol{q}) = \sum_{i=1}^{A} \left[G_E^p(Q^2) \left(\frac{1+\tau_z^i}{2} \right) + G_E^n(Q^2) \left(\frac{1-\tau_z^i}{2} \right) \right] e^{i\boldsymbol{q}\cdot\boldsymbol{r}_i} .$$

Exercise IV

Compute the charge form factor of the deuteron

- Write the wave function as spin-isospin amplitudes

$$\psi_{s_1 s_2, t_1 t_2}(\boldsymbol{r}_1, \boldsymbol{r}_2) = \langle s_1, s_2; t_1, t_2; \boldsymbol{r}_1, \boldsymbol{r}_2 | \psi_{1, J_z} \rangle$$
$$= \sum_{L=0,2} f_L(r) \sum_{M=-L}^{L} \langle L, M, 1, S_z | 1, J_z \rangle \langle 1/2, s_1, 1/2, s_2 | 1 S_z \rangle Y_{L,M}(\hat{\boldsymbol{r}}) \langle 1/2, t_1, 1/2, t_2 | 00 \rangle$$

$$\psi_{1 J_z}(\boldsymbol{r}_1, \boldsymbol{r}_2) = \begin{bmatrix} \psi_{+1+1,+1-1} & \psi_{+1+1,-1+1} \\ \psi_{+1-1,+1-1} & \psi_{+1-1,-1+1} \\ \psi_{-1+1,+1-1} & \psi_{+1+1,-1+1} \\ \psi_{-1-1,+1-1} & \psi_{-1-1,-1+1} \end{bmatrix}$$

- To compute the matrix element of the charge the only operator you have to apply is the operator τ_z^i