Rapidity factorization and rapidity evolution in QCD

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QCD evolution 2025
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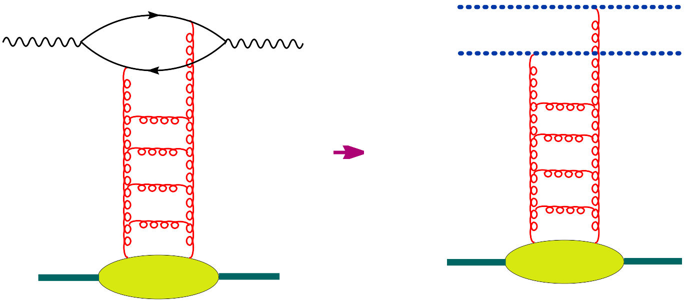
Outline

1. Part 1: High-energy scattering and rapidity evolution of color dipoles:
   - Rapidity factorization for small-x DIS
   - Shock-wave picture of high-energy scattering.
   - Non-linear rapidity evolution of color dipoles
   - Rapidity evolution of color dipoles at the NLO
2. Part 2: Rapidity-only TMD factorization and power corrections
   - TMD factorization for particle production in hadron collisions.
   - Leading-N_c power corrections for DY hadronic tensor.
   - Results for Z-boson production: angular asymmetries.
   - Estimates for unpolarized SIDIS at EIC.
3. Part 3: Rapidity factorization and rapidity evolution of TMDs:
   - Rapidity-only cutoff *vs* UV+rapidity regularization
   - Rapidity evolution of TMDs in the Sudakov region.
   - Argument of coupling constant by BLM.
   - Rapidity-only factorization at one loop.
4. Conclusions

DIS at high energy

- At high energies, particles move along straight lines ⇒ the amplitude of $\gamma^* N \to \gamma^* N$ scattering reduces to the matrix element of a two-Wilson-line operator (color dipole).



Space-time picture in the target (proton) frame.

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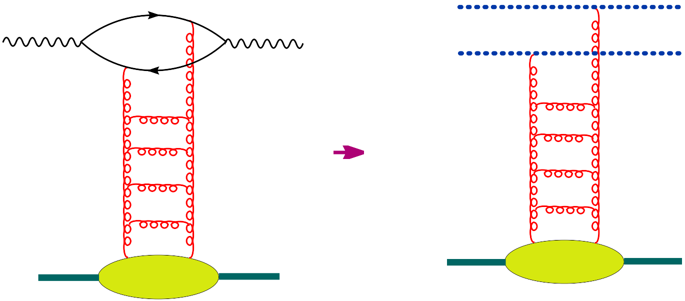


Space-time picture in the target (proton) frame.

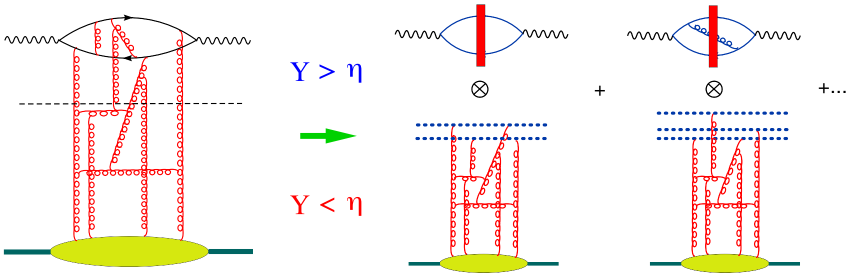
Energy dependence of amplitude is governed by the rapidity evolution of color dipoles

Spectator frame: propagation in the shock-wave background.

Each path is weighted with the gauge factor $Pe^{ig\int dx_\mu A^\mu}$. Quarks and gluons do not have time to deviate in the transverse space $\Rightarrow$ we can replace the gauge factor along the actual path with the one along the straight-line path.

[$x \to z$: free propagation]$\times$
$[U^{ab}(z_\perp)$ - instantaneous interaction with the $\eta < \eta_2$ shock wave$]\times$
[$z \to y$: free propagation]

Rapidity factorization for DIS at small x



η - rapidity factorization scale

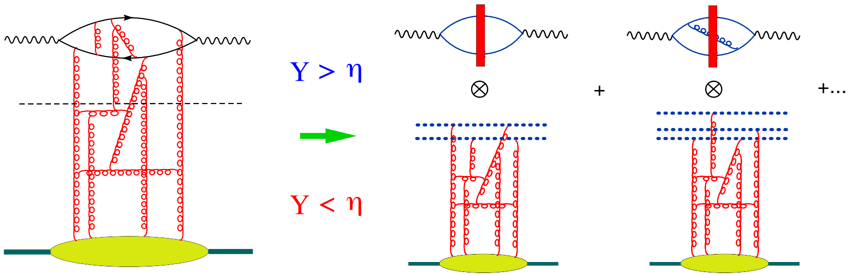
Rapidity Y > η - coefficient function ("impact factor")

Rapidity Y < η - matrix elements of (light-like) Wilson lines with rapidity divergence cut by η

$$U_x^\eta = \text{Pexp}\Big[ig\int_{-\infty}^{\infty} dx^+ A_+^\eta(x_+, x_\perp)\Big]$$

$$A_\mu^\eta(x) = \int \frac{d^4k}{(2\pi)^4}\theta(e^\eta - |\alpha_k|)e^{-ik\cdot x}A_\mu(k)$$

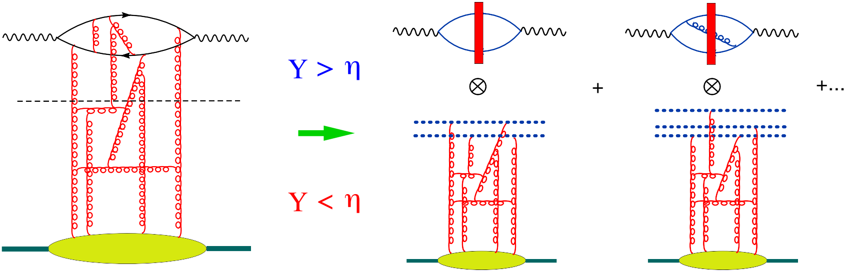
High-energy OPE in color dipoles



The high-energy operator expansion is

$$T\{\hat{j}_\mu(x)\hat{j}_\nu(y)\} = \int d^2z_1 d^2z_2\ I^{\rm LO}_{\mu\nu}(z_1,z_2,x,y){\rm Tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}$$
$$+\ \text{NLO contribution}$$

High-energy OPE in color dipoles



η - rapidity factorization scale

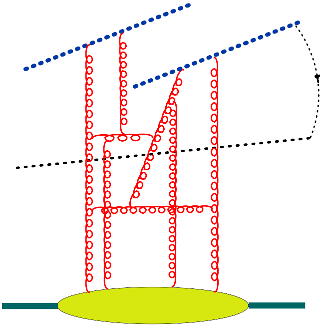
Evolution equation for color dipoles

$$\frac{d}{d\eta}\mathrm{tr}\{U_x^\eta U_y^{\dagger\eta}\} = \frac{\alpha_s}{2\pi^2}\int d^2z \frac{(x-y)^2}{(x-z)^2(y-z)^2}[\mathrm{tr}\{U_x^\eta U_y^{\dagger\eta}\}\mathrm{tr}\{U_x^\eta U_y^{\dagger\eta}\}$$
$$- N_c\mathrm{tr}\{U_x^\eta U_y^{\dagger\eta}\}] + \alpha_s K_{\mathrm{NLO}}\mathrm{tr}\{U_x^\eta U_y^{\dagger\eta}\} + O(\alpha_s^2)$$

(Linear part of K_{NLO} = $K_{\mathrm{NLO\ BFKL}}$)

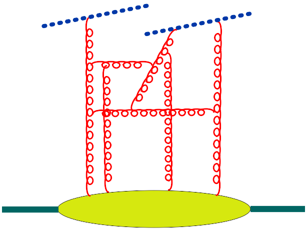
Evolution equation for color dipoles

To get the evolution equation, consider the dipole with the rapidies up to η_1 and integrate over the gluons with rapidities $\eta_1 > \eta > \eta_2$. This integral gives the kernel of the evolution equation (multiplied by the dipole(s) with rapidities up to η_2).



$\Rightarrow$

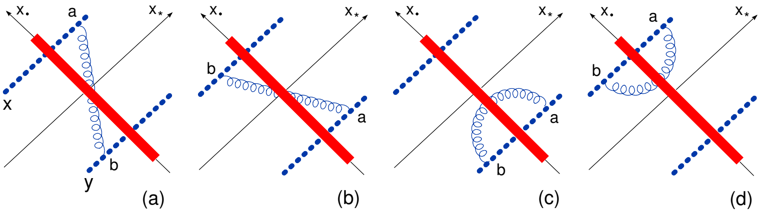
$$\alpha_s(\eta_1 - \eta_2)K_{\text{evol}} \otimes$$



Evolution equation in the leading order

$$\frac{d}{d\eta}\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\} = K_{\mathrm{LO}}\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\} + \ldots \quad \Rightarrow$$

$$\frac{d}{d\eta}\langle\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\}\rangle_{\mathrm{shockwave}} = \langle K_{\mathrm{LO}}\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\}\rangle_{\mathrm{shockwave}}$$



$$U^{ab}_z = \mathrm{Tr}\{t^a U_z t^b U^\dagger_z\} \quad \Rightarrow (U_x U^\dagger_y)^{\eta_1} \to (U_x U^\dagger_y)^{\eta_1} + \alpha_s(\eta_1-\eta_2)(U_x U^\dagger_z U_z U^\dagger_y)^{\eta_2}$$

$\Rightarrow$ Evolution equation is non-linear

Non linear evolution equation

$$\hat{\mathcal{U}}(x,y) \equiv 1 - \frac{1}{N_c}\mathrm{Tr}\{\hat{U}(x_\perp)\hat{U}^\dagger(y_\perp)\}$$

BK equation

$$\frac{d}{d\eta}\hat{\mathcal{U}}(x,y) \;=\; \frac{\alpha_s N_c}{2\pi^2}\int\frac{d^2z\,(x-y)^2}{(x-z)^2(y-z)^2}\Big\{\hat{\mathcal{U}}(x,z)+\hat{\mathcal{U}}(z,y)-\hat{\mathcal{U}}(x,y)-\hat{\mathcal{U}}(x,z)\hat{\mathcal{U}}(z,y)\Big\}$$

I. B. (1996), Yu. Kovchegov (1999)
Alternative approach: JIMWLK (1997-2000)

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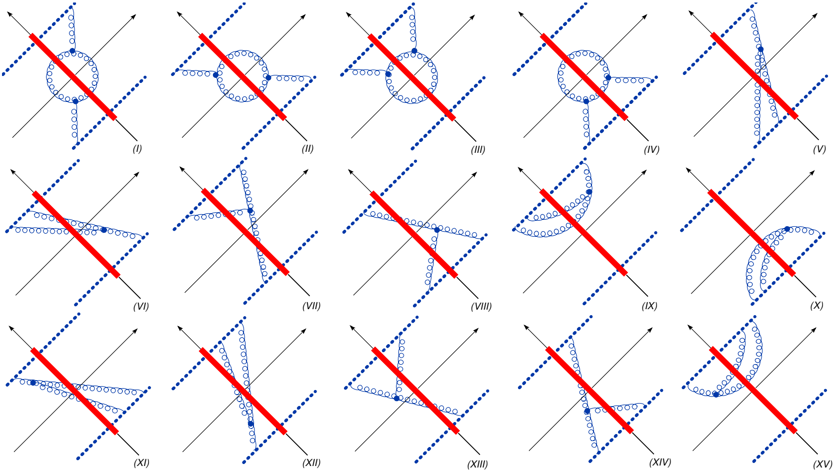
LLA for DIS in pQCD $\Rightarrow$ BFKL (LLA: $\alpha_s \ll 1, \alpha_s\eta \sim 1$)

LLA for DIS in sQCD $\Rightarrow$ BK eqn (LLA: $\alpha_s \ll 1, \alpha_s\eta \sim 1,\ \alpha_s A^{1/3} \sim 1$)

(s for semiclassical)

Sample diagrams for NLO BK kernel



NLO evolution of "composite dipoles" in QCD

I. B. and G. Chirilli

$$a\frac{d}{da}[\mathrm{tr}\{U_{z_1}U^\dagger_{z_2}\}]^{\rm comp}_a = \frac{\alpha_s}{2\pi^2}\int d^2z_3\,\Big([\mathrm{tr}\{U_{z_1}U^\dagger_{z_3}\}\mathrm{tr}\{U_{z_3}U^\dagger_{z_2}\} - N_c\mathrm{tr}\{U_{z_1}U^\dagger_{z_2}\}]^{\rm comp}_a$$

$$\times\frac{z_{12}^2}{z_{13}^2z_{23}^2}\Big[1+\frac{\alpha_sN_c}{4\pi}\big(b\ln z_{12}^2\mu^2 + b\frac{z_{13}^2-z_{23}^2}{z_{13}^2z_{23}^2}\ln\frac{z_{13}^2}{z_{23}^2}+\frac{67}{9}-\frac{\pi^2}{3}\big)\Big]$$

$$+\frac{\alpha_s}{4\pi^2}\int\frac{d^2z_4}{z_{34}^4}\Big\{\Big[-2+\frac{z_{14}^2z_{23}^2+z_{24}^2z_{13}^2-4z_{12}^2z_{34}^2}{2(z_{14}^2z_{23}^2-z_{24}^2z_{13}^2)}\ln\frac{z_{14}^2z_{23}^2}{z_{24}^2z_{13}^2}\Big]$$

$$\times[\mathrm{tr}\{U_{z_1}U^\dagger_{z_3}\}\mathrm{tr}\{U_{z_3}U^\dagger_{z_4}\}\{U_{z_4}U^\dagger_{z_2}\} - \mathrm{tr}\{U_{z_1}U^\dagger_{z_3}U_{z_4}U^\dagger_{z_2}U_{z_3}U^\dagger_{z_4}\} - (z_4\to z_3)]$$

$$+\frac{z_{12}^2z_{34}^2}{z_{13}^2z_{24}^2}\Big[2\ln\frac{z_{12}^2z_{34}^2}{z_{23}^2z_{14}^2}+\Big(1+\frac{z_{12}^2z_{34}^2}{z_{13}^2z_{24}^2-z_{23}^2z_{14}^2}\Big)\ln\frac{z_{13}^2z_{24}^2}{z_{23}^2z_{14}^2}\Big]$$

$$\times[\mathrm{tr}\{U_{z_1}U^\dagger_{z_3}\}\mathrm{tr}\{U_{z_3}U^\dagger_{z_4}\}\mathrm{tr}\{U_{z_4}U^\dagger_{z_2}\} - \mathrm{tr}\{U_{z_1}U^\dagger_{z_4}U_{z_3}U^\dagger_{z_2}U_{z_4}U^\dagger_{z_3}\} - (z_4\to z_3)]\Big\}$$

$$b=\frac{11}{3}N_c-\frac{2}{3}n_f$$

$K_{\rm NLO\ BK}$ = Running coupling part + Conformal "non-analytic" (in j) part
$+$ Conformal analytic ($\mathcal{N}=4$) part

Linearized $K_{\rm NLO\ BK}$ reproduces the known result for the forward NLO BFKL kernel.

Argument of coupling constant

$$\frac{d}{d\eta}\hat{\mathcal{U}}(z_1, z_2) =$$

$$\frac{\alpha_s(?_\perp)N_c}{2\pi^2}\int dz_3 \frac{z_{12}^2}{z_{13}^2 z_{23}^2}\Big\{\hat{\mathcal{U}}(z_1, z_3) + \hat{\mathcal{U}}(z_3, z_2) - \hat{\mathcal{U}}(z_1, z_2) - \hat{\mathcal{U}}(z_1, z_3)\hat{\mathcal{U}}(z_3, z_2)\Big\}$$

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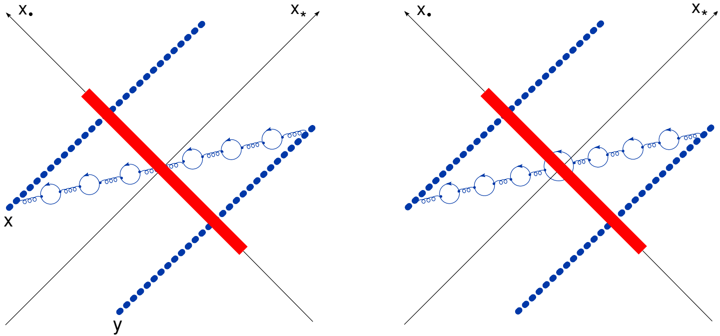
BLM or renormalon approach: summation of quark bubbles



$$-\tfrac{2}{3}n_f \to b = \tfrac{11}{3}N_c - \tfrac{2}{3}n_f$$

Argument of coupling constant (rcBK)

$$\frac{d}{d\eta}\mathrm{Tr}\{\hat{U}_{z_1}\hat{U}^\dagger_{z_2}\} = \frac{\alpha_s(z_{12}^2)}{2\pi^2}\int d^2z\,[\mathrm{Tr}\{\hat{U}_{z_1}\hat{U}^\dagger_{z_3}\}\mathrm{Tr}\{\hat{U}_{z_3}\hat{U}^\dagger_{z_2}\} - N_c\mathrm{Tr}\{\hat{U}_{z_1}\hat{U}^\dagger_{z_2}\}]$$
$$\times\Big[\frac{z_{12}^2}{z_{13}^2 z_{23}^2} + \frac{1}{z_{13}^2}\Big(\frac{\alpha_s(z_{13}^2)}{\alpha_s(z_{23}^2)} - 1\Big) + \frac{1}{z_{23}^2}\Big(\frac{\alpha_s(z_{23}^2)}{\alpha_s(z_{13}^2)} - 1\Big)\Big] + \ldots$$

I.B.; Yu. Kovchegov and H. Weigert (2006)

When the sizes of the dipoles are very different the kernel reduces to:

$$\frac{\alpha_s(z_{12}^2)}{2\pi^2}\frac{z_{12}^2}{z_{13}^2 z_{23}^2} \qquad |z_{12}| \ll |z_{13}|, |z_{23}|$$
$$\frac{\alpha_s(z_{13}^2)}{2\pi^2 z_{13}^2} \qquad |z_{13}| \ll |z_{12}|, |z_{23}|$$
$$\frac{\alpha_s(z_{23}^2)}{2\pi^2 z_{23}^2} \qquad |z_{23}| \ll |z_{12}|, |z_{13}|$$

$\Rightarrow$ the argument of the coupling constant is given by the size of the smallest dipole.

Part 2: Rapidity-only TMD factorization and power corrections

TMD factorization

TMD factorization formula for particle production in hadron-hadron scattering looks like

$$\frac{d\sigma}{d\eta d^2q_\perp} = \sum_{\text{flavors}} e_f^2 \int d^2k_\perp \mathcal{D}_{f/A}(x_A, k_\perp)\mathcal{D}_{f/B}(x_B, q_\perp - k_\perp)C(q, k_\perp)$$
$$+ \text{ power corrections } + \text{ “Y} - \text{terms}''$$

- $\mathcal{D}_{f/A}(x_A, k_\perp)$ is the TMD density of a parton f in hadron A with fraction of momentum x_A and transverse momentum $k_\perp$,
- $\mathcal{D}_{f/B}(x, q_\perp - k_\perp)$ is a similar quantity for hadron B,
- $C_i(q, k)$ are determined by the cross section $\sigma(f\!f \to \mu^+\mu^-)$ of production of DY pair of invariant mass q^2 in the scattering of two partons.

Examples: Drell-Yan process with Q being the mass of DY pair and Higgs production by gluon-gluon fusion

TMD approach is relevant when the transverse momentum $q_\perp \ll Q$

Long-term project: TMD factorization valid at small and moderate x

$$\frac{d\sigma}{d\eta d^2q_\perp} = \sum_{\text{flavors}} e_f^2 \int d^2k_\perp \mathcal{D}_{f/A}(x_A,k_\perp)\mathcal{D}_{f/B}(x_B,q_\perp-k_\perp)C(q,k_\perp)$$
$$+ \text{ power corrections } + \text{ “Y} - \text{terms''}$$

The quantities $\mathcal{D}_{f/A}(x_A,k_\perp)$, $\mathcal{D}_{f/B}(x_B,q_\perp-k_\perp)$, and $C(q,k_\perp)$ are defined with cutoffs. The dependence on the cutoffs cancels in their product order by order in α_s.

At moderate x_A, x_B: CSS/SCET approach. The TMDs $\mathcal{D}_{f/A}(x_A,k_\perp)$ are defined with a combination of UV and rapidity cutoffs.

At $x_A, x_B \ll 1$: k_T-factorization approach. The TMDs are defined with rapidity-only cutoffs.

It is impossible to extend CSS to small x. (Recently: LO BFKL from SCET)

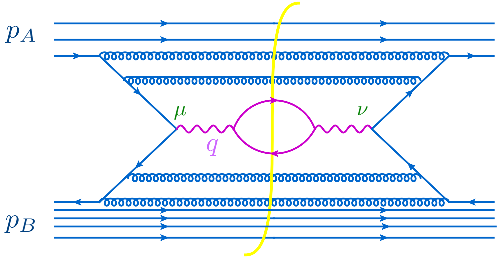
It *is* possible to study TMD factorization at moderate x using small-x methods (rapidity-only factorization etc.) (A. Tarasov, G. Chirilli, I.B, 2015-2023)

Example: full list of power corrections $\sim \frac{1}{Q^2}$ for DY hadronic tensor.
They are not obtained (yet?) by CSS or SCET

Classical example: DY hadronic tensor

DY cross section is given by the product of leptonic tensor and hadronic tensor. The hadronic tensor $W_{\mu\nu}$ is defined as

$$W_{\mu\nu}(p_A, p_B, q) \; = \; \frac{1}{(2\pi)^4} \int d^4x \; e^{-iqx} \langle p_A, p_B | J_\mu(x) J_\nu(0) | p_A, p_B \rangle$$



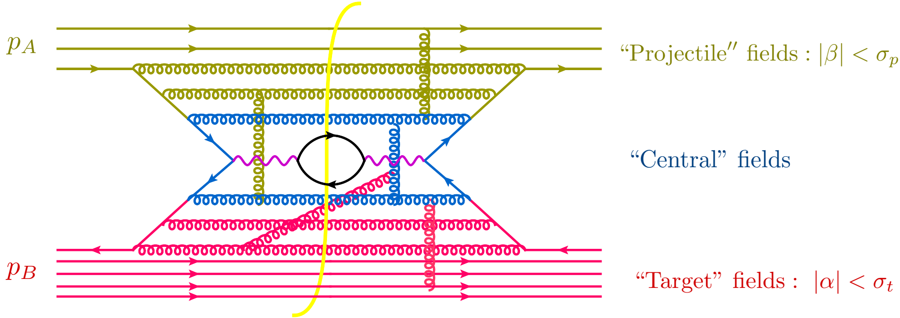
p_A, p_B = hadron momenta, q = the momentum of DY pair, and J_μ is the electromagnetic or Z-boson current.

There are four tensor structures W_T, W_L, W_Δ, $W_{\Delta\Delta}$

TMD factorization from rapidity factorization (A. Tarasov and I.B.)

Sudakov variables:

$$p = \alpha p_1 + \beta p_2 + p_\perp, \qquad p_1 \simeq p_A, \;\; p_2 \simeq p_B, \;\; p_1^2 = p_2^2 = 0$$



The result of the integration over "central" fields in the background of projectile and target fields is a series of TMD operators made from projectile (or target) fields multiplied by powers of $\frac{1}{Q^2} \Rightarrow$ power corrections

TMD representation for W_i

The hadronic tensor in the Sudakov region $q^2 \equiv Q^2 \gg q_\perp^2$ can be studied by TMD factorization. For example, functions W_T and $W_{\Delta\Delta}$ can be represented as

$$W_i = \sum_{\text{flavors}} e_f^2 \int d^2k_\perp \mathcal{D}^{(i)}_{f/A}(x_A, k_\perp)\mathcal{D}^{(i)}_{f/B}(x_B, q_\perp - k_\perp)C_i(q, k_\perp)$$
$$+ \text{ power corrections } + \text{ Y} - \text{terms}$$

There is, however, a problem with this equation for the functions W_L and W_Δ.

W_T and $W_{\Delta\Delta}$ are determined by leading-twist quark TMDs, but W_Δ and W_L start from terms $\sim \frac{q_\perp}{Q}$ and $\sim \frac{q_\perp^2}{Q^2}$ determined by quark-quark-gluon TMDs.

The power corrections $\sim \frac{q_\perp}{Q}$ were found more than two decades ago but there was no calculation of power corrections $\sim \frac{q_\perp^2}{Q^2}$ until recently.

Goal: TMD factorization formula

TMD factorization formula structure :

$$\langle p'_A, p'_B|J(x_1)J(x_2)|p_A,p_B\rangle = \sum_{\text{TMD operators}} \int dz_1^- dz_2^- dw_1^+ dw_2^+ \mathfrak{C}_i(x_1,x_2;z_i^-,w_i^+;\sigma_p,\sigma_t)$$
$$\times \langle p'_A|\hat{\mathcal{O}}_i^{\sigma_p}(z_2^-,x_{2_\perp};z_1^-,x_{1_\perp})|p_B\rangle\langle p'_B|\hat{\mathcal{O}}_i^{\sigma_t}(z_2^+,x_{2_\perp};z_1^+,x_{1_\perp})|p'_B\rangle$$

$q_\perp^2 \ll Q^2 \Rightarrow$ no dynamics in the transverse space (to be demonstrated below)

$\hat{\mathcal{O}}_i^{\sigma_p}$ - “projectile” TMD operators with $\beta < \sigma_p$ cutoff, e.g

$$\mathcal{O}(z_{1-},z_{1_\perp},z_{2-},z_{2_\perp}) \equiv \bar\psi(z_{1-},z_{1_\perp})[z_{1-},-\infty]_{z_{1_\perp}}\Gamma[-\infty,z_{2+}]_{z_2}\psi(z_{2+},z_{2_\perp})$$

$\hat{\mathcal{O}}_i^{\sigma_p}$ - “target” TMD operators with $\alpha < \sigma_t$ cutoff, e.g

$$\mathcal{O}(z_{1+},z_{1_\perp},z_{2+},z_{2_\perp}) \equiv \bar\psi(z_{1+},z_{1_\perp})[z_{1+},-\infty]_{z_{1_\perp}}\Gamma[-\infty,z_{2+}]_{z_{2_\perp}}\psi(z_{2+},z_{2_\perp}).$$

Standard notation for straight-line gauge link

$$[x,y] \equiv \mathrm{P}e^{ig\int_0^1 du\ (x-y)^\mu A_\mu(ux+(1-u)y)} - \text{gauge link}$$

Convenient notations

$$[x_+,y_+]_{z_\perp} \equiv [x_+,0_-,z_\perp;y_+,0_-,z_\perp], \quad [x_-,y_-]_{z_\perp} \equiv [x_-,0_+,z_\perp;y_-,0_+,z_\perp]$$

Means: "double operator expansion"

Intermediate step: double operator expansion

$$\hat{J}(x_1)\hat{J}(x_2) = \sum_{I,J} \int dz_1^- dz_2^- dw_1^+ dw_2^+ \mathfrak{C}_{IJ}(x_1, x_2; z_i^-, w_i^+; \sigma_p, \sigma_t)$$
$$\times \hat{\mathcal{O}}_I^{\sigma_p}(z_2^-, x_{2_\perp}; z_1^-, x_{1_\perp}) \hat{\mathcal{O}}_J^{\sigma_t}(z_2^+, x_{2_\perp}; z_1^+, x_{1_\perp})$$

To find relevant operators and coefficients, it is convenient to consider "matrix" elements of the l.h.s. and r.h.s. in suitable background field

Suitable field $\mathbb{A}$: solution of classical YM equations with boundary condition that at the remote past the field is a sum of projectile and target fields

$$\langle \hat{J}(x_1)\hat{J}(x_2)\rangle_{\mathbb{A}} = \sum_{I,J} \int dz_1^- dz_2^- dw_1^+ dw_2^+ \mathfrak{C}_{IJ}(x_1, x_2; z_i^-, w_i^+; \sigma_p, \sigma_t)$$
$$\times \langle \hat{\mathcal{O}}_I^{\sigma_p}(z_2^-, x_{2_\perp}; z_1^-, x_{1_\perp}) \hat{\mathcal{O}}_J^{\sigma_t}(z_2^+, x_{2_\perp}; z_1^+, x_{1_\perp})\rangle_{\mathbb{A}}$$

Method of solution:

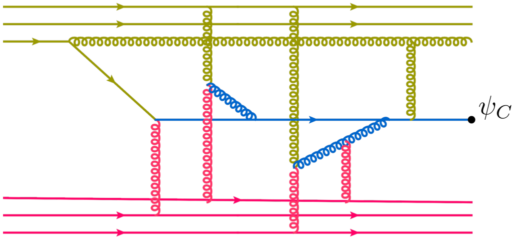
- Start with $\Psi_{\text{trial}} = \psi_A + \psi_B$ and $\mathbb{A}_{\text{trial}} = \bar{A}_\mu + \bar{B}_\mu$ in the gauge $A^+ = 0$, $A^- = 0$
- Correct by computing Feynman diagrams (with retarded propagators) with sources $(\not{P} + m)(\psi_A + \psi_B)$ and $J_\nu = D^\mu F^{\mu\nu}(A + B)$

ψ_C in the tree approximation

It is convenient to choose projectile/target fields as

Projectile fields: $\beta = 0 \Rightarrow A(x^-, x_\perp),\ \psi_A(x^-, x_\perp)$

Target fields: $\alpha = 0 \Rightarrow B(x^+, x_\perp),\ \psi_B(x^-, x_\perp)$



Classical background fields: Ψ , $\mathbb{A}_\mu$

ψ_C = sum of tree diagrams in external A, ψ_A and B, ψ_B fields with sources

$J_\psi = (\not{P} + m)(\psi_A + \psi_B)$, $\qquad J_\nu = D^\mu F^{\mu\nu}(A + B)$

Classical fields in the leading order in $p_\perp^2/p_\parallel^2 \sim q_\perp^2/Q^2$

The solution of such YM equations in general case is yet unsolved problem (goes under the name "glasma" $\Leftrightarrow$ scattering of two "color glass condensates").

Fortunately, for our case of particle production with $\frac{q_\perp}{Q} \ll 1$ we can use this small parameter and construct the approximate solution.

At the tree level transverse momenta are $\sim q_\perp^2$ and longitudinal are $\sim Q^2 \Rightarrow$

$$\Psi, \mathbb{A} \;=\; \text{series in } \frac{q_\perp}{Q}: \qquad \Psi = \psi^{(0)} + \psi^{(1)} + \ldots, \qquad \mathbb{A} = A^{(0)} + A^{(1)} + \ldots$$

NB: After the expansion

$$\frac{1}{p^2 + i\epsilon p_0} \;=\; \frac{1}{p_\parallel^2 - p_\perp^2 + i\epsilon p_0} \;=\; \frac{1}{p_\parallel^2} - \frac{1}{p_\parallel^2 + i\epsilon p_0} p_\perp^2 \frac{1}{p_\parallel^2 + i\epsilon p_0} \;+\ldots$$

the dynamics in transverse space is trivial.

Fields are either at the point $x_\perp$ or at the point $0_\perp \Rightarrow$ TMDs

Leading-N_c power corrections

Power corrections are $\sim$ leading twist $\times\Big(\frac{q_\perp}{Q}$ or $\frac{q_\perp^2}{Q^2}\Big)\times\big(1+\frac{1}{N_c}+\frac{1}{N_c^2}\big)$.

NB: almost all $\bar{q}Gq$ TMDs not suppressed by $\frac{1}{N_c}$ are determined by the $\bar{q}q$ TMDs due to QCD equations of motion

Leading twist (for the projectile nucleon): $\varrho\equiv\sqrt{s/2}$

$$\frac{1}{8\pi^3 s}\int dx^- d^2x_\perp\ e^{-i\alpha\varrho x^- + i(k,x)_\perp}\ \langle N|\hat{\bar{\psi}}(x^-,x_\perp)\ \not{p}_2\hat{\psi}(0)|N\rangle\ =\ f_1(\alpha,k_\perp^2)$$

Power correction:

$$\frac{1}{8\pi^3 s}\int dx^- dx_\perp\ e^{-i\alpha\varrho x^- + i(k,x)_\perp}$$
$$\times\ \langle N|\hat{\bar{\psi}}(x^-,x_\perp)\hat{\not{A}}(x^-,x_\perp)\not{p}_2\gamma_i\hat{\psi}(0)|N\rangle$$
$$=\ k_i f_1(\alpha,k_\perp) - \alpha k_i\big[f_\perp(\alpha,k_\perp) + ig^\perp(\alpha,k_\perp)\big],$$

(Mulders & Tangerman, 1996)

Application: angular coefficients of Z-boson production

In CMS and ATLAS experiments $s = 8$ TeV, $Q = 80 - 100$ GeV and $Q_\perp$ varies from 0 to 120 GeV.

Our analysis is valid at $Q_\perp = 10 - 30$ GeV and $Y \simeq 0$ ($x_A \sim x_B \sim 0.1$) so that power corrections are small but sizable.

Angular distribution of DY leptons in the Collins-Soper frame ($c_\phi \equiv \cos\phi$, $s_\phi \equiv \sin\phi$ etc.)

$$\frac{d\sigma}{dQ^2 dy d\Omega_l} = \frac{3}{16\pi}\frac{d\sigma}{dQ^2 dy}\Big[(1+c_\theta^2) + \frac{A_0}{2}(1-3c_\theta^2) + A_1 s_{2\theta}c_\phi + \frac{A_2}{2}s_\theta^2 c_{2\phi} + A_3 s_\theta c_\phi + A_4 c_\theta + A_5 s_\theta^2 s_{2\phi} + A_6 s_{2\theta}s_\phi + A_7 s_\theta s_\phi\Big]$$

Back-of-the envelope estimation: take only f_1 contribution at large N_c, use "factorization hypothesis" for TMD $f_1(x,k_\perp) \simeq f(x)g(k_\perp)$ and calculate integrals over $k_\perp$ in the leading log approximation using $f_1(x,k_\perp^2) \simeq \frac{f(x)}{k_\perp^2}$

Comparison of A_0 with LHC results

Logarithmic estimate of A_0 (m_z - Z-boson mass, m - proton mass)

$$A_0 \;=\; \frac{Q_\perp^2}{m_z^2}\,\frac{1+2\frac{\ln m_z^2/Q_\perp^2}{\ln Q_\perp^2/m^2}}{1+\frac{Q_\perp^2}{m_z^2}\,\frac{\ln m_z^2/Q_\perp^2}{\ln Q_\perp^2/m^2}} \qquad (*)$$

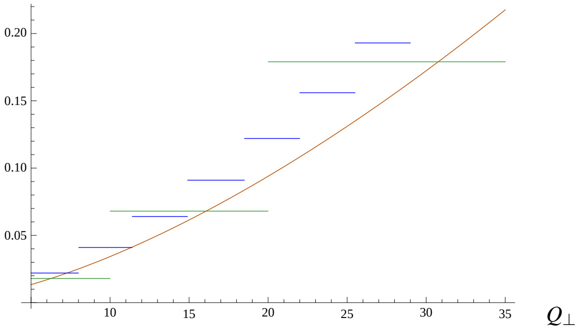


Figure: Comparison of prediction $(*)$ with lines depicting angular coefficient A_0 in bins of $Q_\perp$ and $Y < 1$ from CMS (arXiv:1504.03512) and ATLAS (arXiv1606.00689)

Comparison of A_2 with LHC results

Logarithmic estimate of A_2

$$A_2 = \frac{Q_\perp^2}{m_z^2}\frac{1}{1+\frac{Q_\perp^2}{m_z^2}\frac{\ln m_z^2/Q_\perp^2}{\ln Q_\perp^2/m^2}} \qquad (**)$$

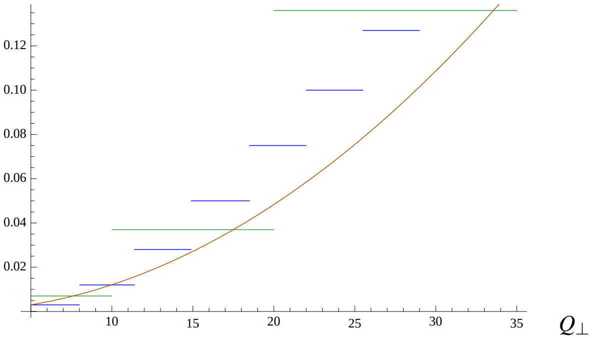


Figure: Comparison of prediction $(**)$ with lines depicting angular coefficient A_2 in bins of $Q_\perp$ and $Y < 1$ from CMS (arXiv:1504.03512) and ATLAS (arXiv1606.00689)

Result of estimation

Result of the estimation: A_0 and A_2 in agreement with data, rest of $A_i = 0$.
Experimentally, at $q_\perp \leq \frac{1}{4} m_z$ other A_i are an order of magnitude smaller that $A_0 \sim A_2$
$\Rightarrow$ it looks like f_1 is numerically the most important TMD for unpolarized DY cross sections.

Application: unpolarized SIDIS

Result for power corrections in SIDIS: $\mathrm{PC}^{\mathrm{SIDIS}}(q) = \mathrm{PC}^{\mathrm{DY}}(-q)$ with replacements $f_1(-\alpha_q, -q_\perp - k_\perp) \to \bar{D}_1\Big(z = \frac{1}{\alpha_q}, q_\perp + k_\perp\Big)$,
$h_1^\perp(-\alpha_q, -q_\perp - k_\perp) \to -\bar{H}_1^\perp\Big(z = \frac{1}{\alpha_q}, q_\perp + k_\perp\Big)$ *etc.*

Hopefully, our analysis is valid for EIC kinematics at $Q \geq 10$ GeV and $Q_\perp \sim 3$GeV so that power corrections are small but sizable.

The unpolarized cross section is parametrized by four functions $F_{UU,T}, F_{UU,L}, F_{UU}^{\cos\phi_h}, F_{UU,T}^{\cos 2\phi_h}$

Estimation: similarly to the DY case, take only f_1 and D_1 contribution at large N_c,

$$
\begin{aligned}
F_{UU,T} &= xz\int dk_\perp\Big(1 - \frac{2(q,k)_\perp}{Q^2}\Big)\Phi(q,k_\perp)\\
F_{UU,L} &= x\int dk_\perp \frac{4k_\perp^2}{Q^2}\Phi(q,k_\perp)\\
F_{UU}^{\cos\phi_h} &= x\int dk_\perp \frac{2(q,k)_\perp}{Qq_\perp}\Phi(q,k_\perp)\\
F_{UU}^{\cos 2\phi_h} &= -x\int dk_\perp \frac{2(q,k)_\perp}{Q^2}\Phi(q,k_\perp) \qquad \Rightarrow \quad F_{UU}^{\cos 2\phi_h} = -\frac{q_\perp}{Q}F_{UU}^{\cos\phi_h}
\end{aligned}
$$

$$
\Phi(q,k_\perp) \equiv D_1(z, q_\perp + k_\perp)f_1(x,k_\perp) + \bar{D}_1(z, q_\perp + k_\perp)\bar{f}_1(x,k_\perp)
$$

Back-of-the-envelope estimation

To understand the magnitude of power corrections, define

$$R_{UU,T} = \frac{F_{UU,T}}{F^{\rm l.t.}_{UU,T}} - 1, \quad R_{UU,L} = \frac{F_{UU,L}}{F^{\rm l.t.}_{UU,T}}, \quad R_{UU}^{\cos\phi_h} = \frac{F_{UU}^{\cos\phi_h}}{F^{\rm l.t.}_{UU,T}}, \quad R_{UU}^{\cos 2\phi_h} = \frac{F_{UU}^{\cos 2\phi_h}}{F^{\rm l.t.}_{UU,T}}$$

Back-of-the envelope estimation: similarly to the DY case, take only f_1 and D_1 contribution at large N_c, use “factorization hypothesis” for TMD PDFs and FFs $\phi(x, k_\perp) \simeq \phi(x)\psi(k_\perp)$ and calculate integrals over $k_\perp$ in the leading log approximation using $f_1(x, k_\perp^2) \simeq \frac{f(x)}{k_\perp^2}$ and $D_1(z, k_\perp^2) \simeq \frac{D(z)}{k_\perp^2}$

Result:

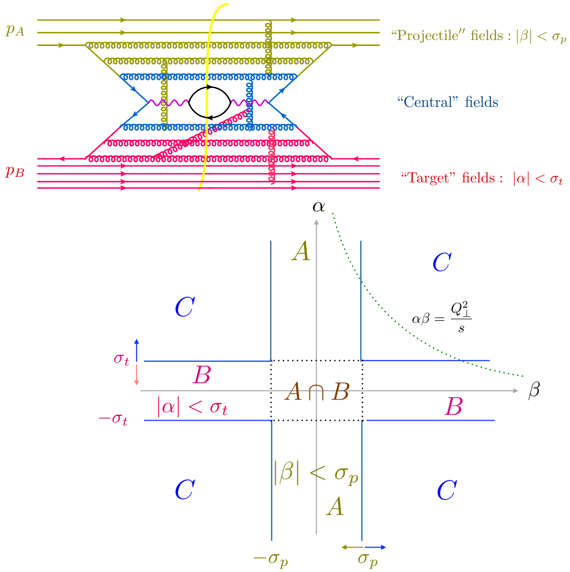
$$R_{UU,T} = R_{UU}^{\cos 2\phi_h} = \frac{q_\perp^2}{Q^2}, \quad R_{UU}^{\cos\phi_h} = -\frac{q_\perp}{Q}$$

$$R_{UU,L} = 2\frac{q_\perp^2}{Q^2} + \frac{\int dk_\perp 4(k, q+k)_\perp \Phi(q,k)}{Q^2 \int dk_\perp \Phi(q,k)} \simeq 2\frac{q_\perp^2}{Q^2}\frac{\ln\frac{Q^2}{m^2}}{\ln\frac{q_\perp^2}{m^2}}$$

This estimate does not depend on z and x.

Part 3: Rapidity factorization and rapidity evolution of TMDs

Rapidity-only cutoffs and matching of logs



Matching: $\ln \sigma_p$ in the projectile TMDs and $\ln \sigma_t$ in the target TMDs should cancel with $\ln \sigma_p$ and $\ln \sigma_t$ in the coefficient functions.

$A \cap B$, $k_\perp \sim m_\perp$: Glauber gluons
$A \cap B$, $k_\perp \ll m_\perp$: soft gluons

$A \cap B$ gluons $\equiv$ soft/Glauber (sG) gluons cancel out

Rapidity-only cutoff

Typical diagram in the background
field $\Psi(\beta_B, p_{B_\perp}) \;=\; \varrho\int dz^+ dz_\perp\ \Psi(z^+, z_\perp) e^{i\varrho\beta_B z^+ - i(p_B, z)_\perp}$

$$\langle [x^+, -\infty]_x \Gamma\psi(y^+, y_\perp)\rangle_\Psi \;=\; g^2 c_F \int \text{đ}\beta_B \text{đ}p_{B_\perp} e^{-ip_B y}\Gamma\Psi(\beta_B, p_{B_\perp})$$

$$\times \int_0^\infty \text{đ}\alpha \int \frac{\text{đ}p_\perp}{p_\perp^2} \frac{\beta_B s e^{-i\frac{p_\perp^2}{\alpha s}\varrho\Delta^+ + i(p,\Delta)}}{\alpha\beta_B s + (p-p_B)_\perp^2 + i\epsilon} \qquad \leftarrow \text{ divergent as } \alpha \to \infty$$

$$\langle [x^+, -\infty]_x \Gamma\psi(y^+, y_\perp, -\delta^-)\rangle_\Psi \;=\; g^2 c_F \int \text{đ}\beta_B \text{đ}p_{B_\perp} e^{-ip_B y}\Gamma\Psi(\beta_B, p_{B_\perp}) \qquad \sigma \equiv \frac{1}{\varrho\delta^-}$$

$$\times \int_0^\infty \text{đ}\alpha \int \frac{\text{đ}p_\perp}{p_\perp^2} \frac{\beta_B s e^{-i\frac{p_\perp^2}{\alpha s}\varrho\Delta^+ + i(p,x-y)_\perp}}{\alpha\beta_B s + (p-p_B)_\perp^2 + i\epsilon} e^{-i\frac{\alpha}{\sigma}} \qquad \leftarrow \text{ convergent as } \alpha \to \infty$$

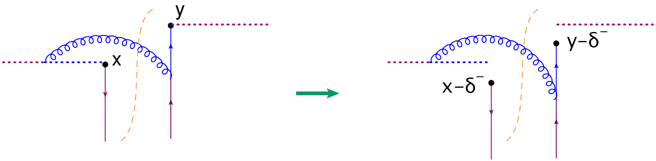


Figure: Point-splitting visualization of “smooth” rapidity-only cutoff.

Rapidity-only cutoff *vs* UV+rapidity regularization

Typical divergent integral ($\varepsilon = \frac{d}{2} - 2$, $\bar{d}^n p \equiv \frac{d^n p}{(2\pi)^n}$)

$$-i\mu^{-2\varepsilon}\int \bar{d}\alpha \bar{d}\beta \bar{d}p_\perp \frac{1}{\beta - i\epsilon}\frac{1}{\alpha\beta s - p_\perp^2 + i\epsilon}\frac{s(\beta-\beta_B)}{\alpha(\beta-\beta_B)s - p_\perp^2 + i\epsilon}\big(1 - e^{i(p,x)_\perp}\big)$$

$$= \mu^{-2\varepsilon}\int \frac{\bar{d}p_\perp}{p_\perp^2}\big(1 - e^{i(p,x)_\perp}\big)\int_0^{\beta_B}\frac{\bar{d}\beta}{\beta_B}\frac{\beta_B - \beta}{\beta - i\epsilon} = -\frac{1}{8\pi^2}\frac{\Gamma(\varepsilon)}{(x_\perp^2\mu^2)^\varepsilon}\int_0^{\beta_B}\frac{d\beta}{\beta_B}\frac{\beta_B-\beta}{\beta - i\epsilon}$$

Regularization with $A^-(z^+) \to A^-(z^+)e^{\pm\delta z^+}$

$$-\frac{1}{8\pi^2}\frac{\Gamma(\varepsilon)}{(x_\perp^2\mu^2)^\varepsilon}\int_0^{\beta_B}\frac{d\beta}{\beta_B}\frac{\beta_B-\beta}{\beta - i\delta} \simeq \frac{1}{8\pi^2}\Big(-\frac{1}{\varepsilon} + \ln\mu^2\frac{x_\perp^2}{4} + \gamma_E\Big)\Big(\ln\frac{\beta_B}{-i\delta} - 1\Big)$$

Rapidity-only cutoff

$$-i\int \bar{d}\alpha \bar{d}\beta \bar{d}p_\perp \frac{1}{\beta - i\epsilon}\frac{e^{-i\frac{\alpha}{\sigma}}}{\alpha\beta s - p_\perp^2 + i\epsilon}\frac{s(\beta-\beta_B)}{\alpha(\beta-\beta_B)s - p_\perp^2 + i\epsilon}\big(1 - e^{i(p,x)_\perp}\big)$$

$$= \int \frac{\bar{d}p_\perp}{p_\perp^2}\big(1 - e^{i(p,x)_\perp}\big)\int_0^{\infty}\bar{d}\alpha \frac{\beta_B s}{\alpha\beta_B s + p_\perp^2}e^{-i\frac{\alpha}{\sigma}} = \frac{1}{16\pi^2}\ln^2\Big(-i\beta_B\sigma s\frac{x_\perp^2}{4}e^{\gamma_E}\Big)$$

Rapidity evolution of TMDs

Quark TMD operator

$$\mathcal{O}(z_{1+}, z_{1_\perp}, z_{2+}, z_{2_\perp}) \equiv \bar{\psi}(z_{1+}, z_{1_\perp})[z_{1+}, -\infty]_{z_1}\Gamma[-\infty, z_{2+}]_{z_2}\psi(z_{2+}, z_{2_\perp})$$

Sudakov regime: $Q^2 \gg Q_\perp^2 \Leftrightarrow z_{12+}z_{12-} \ll z_{12_\perp}^2$

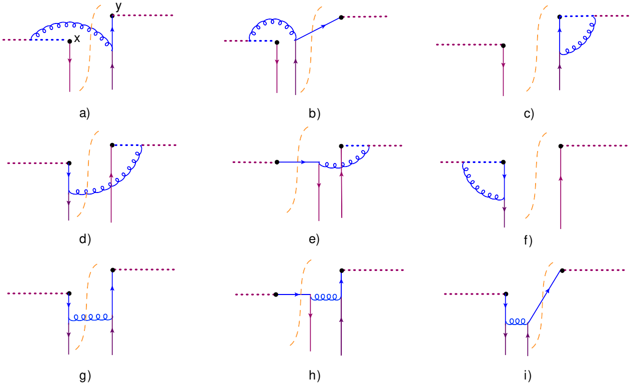


Figure: Diagrams for leading-order rapidity evolution of quark TMD in the Sudakov regime.

Conformally invariant evolution at the LO (G.A. Chirilli & I.B.,2019)

Evolution equation ($\lambda \equiv \sigma(x-y)_\perp^2 \frac{s}{4}$)

$$\lambda \frac{d}{d\lambda}\mathcal{O}(x_+, y_+;\lambda)$$
$$= \Big[\int_{x_+}^\infty dx'_+ \frac{1}{x'_+ - y_+} e^{i\frac{\lambda\sqrt{2/s}}{x'_+ - y_+}} \mathcal{O}(x_+, y_+;\lambda) - \int_{y_+}^\infty dy'_+ \frac{\mathcal{O}(x_+, y_+;\lambda) - \mathcal{O}(x_+t, y'_+;\lambda)}{y'_+ - y_+}$$
$$+ \int_{y_+}^\infty dy'_+ \frac{1}{y'_+ - x_+} e^{i\frac{\lambda\sqrt{2/s}}{y'_+ - x_+}} \mathcal{O}(x_+, y_+;\lambda) - \int_{x_+}^\infty dx'_+ \frac{\mathcal{O}(x_+, y_+;\lambda) - \mathcal{O}(x'_+, y_+;\lambda)}{x'_+ - x_+}\Big]$$

If we use rapidity cutoff at $\sigma = \frac{8\varsigma}{|x-y|_\perp \sqrt{s}} \quad \Leftrightarrow \lambda = \varsigma|x-y|\sqrt{s}$,
the solution

$$\mathcal{O}(x_+, y_+;\sigma) = e^{-\frac{\tilde\alpha_s}{2}\left(\ln^2 \frac{2(x-y)_\perp^2 \varsigma^2}{x_+ y_+} - \ln^2 \frac{2(x-y)_\perp^2 \varsigma_0^2}{x_+ y_+}\right)} e^{4\tilde\alpha_s \psi(1)\ln\frac{\varsigma}{\varsigma_0}} \int dx'_+ dy'_+\ \mathcal{O}(x'_+, y'_+;\sigma_0)$$
$$\times (x_+ y_+)^{-\tilde\alpha_s \ln\frac{\varsigma}{\varsigma_0}} \left[\frac{i\Gamma\left(1-\tilde\alpha_s \ln\frac{\varsigma}{\varsigma_0}\right)}{(x_+ - x'_+ + i\epsilon)^{1-\tilde\alpha_s \ln\frac{\varsigma}{\varsigma_0}}} - \frac{i\Gamma\left(1-\tilde\alpha_s \ln\frac{\varsigma}{\varsigma_0}\right)}{(x_+ - x'_+ - i\epsilon)^{1-\tilde\alpha_s \ln\frac{\varsigma}{\varsigma_0}}}\right]$$
$$\times \left[\frac{i\Gamma\left(1-\tilde\alpha_s \ln\frac{\varsigma}{\varsigma_0}\right)}{(y_+ - y'_+ + i\epsilon)^{1-\tilde\alpha_s \ln\frac{\varsigma}{\varsigma_0}}} - \frac{i\Gamma\left(1-\tilde\alpha_s \ln\frac{\varsigma}{\varsigma_0}\right)}{(y_+ - y'_+ - i\epsilon)^{1-\tilde\alpha_s \ln\frac{\varsigma}{\varsigma_0}}}\right]$$

is obviously invariant under the inversion $x_+ \to \frac{x_+}{x_\perp^2}$, $y_+ \to \frac{y_+}{y_\perp^2}$.

Argument of coupling constant by BLM/renormalon method

A problem with leading-order rapidity evolution: what is the argument of coupling constant?

In CSS approach - no problem, argument is defined by renormgroup

With rapidity-only evolution (BFKL, BK and the like) - argument of α_s may be obtained from the NLO calculations. BLM approach: calculate the small part of the NLO result, namely quark loop contribution to gluon propagator, and promote $-\frac{2}{3}n_f$ to the full $b = \frac{11}{3}N_c - \frac{2}{3}n_f$.

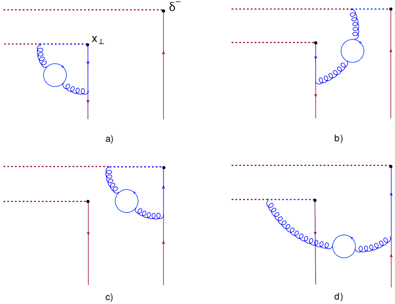
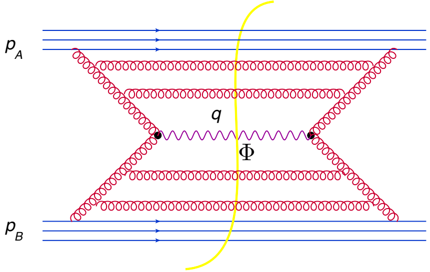


Figure: Quark loop correction to quark TMD evolution

Result: BLM optimal scale is logarithmically halfway between transverse momentum ($b_\perp^{-1/2}$) and energy ($\sigma\beta_B s$) of TMD both for quarks and gluons

Coefficient function for TMD factorization at one loop

Particle production by gluon fusion



$$s \gtrsim Q^2 \quad \gg \quad Q_\perp^2 \gtrsim m^2$$
$$q^2 \equiv Q^2 = M_\Phi^2, \quad Q_\perp^2 \equiv q_\perp^2$$

Goal: one-loop TMD factorization formula for hadronic tensor.

Result of calculations:

$$W(p_A, p_B; q) \;=\; \int db_\perp\; e^{i(q,b)_\perp} \mathcal{D}_{g/A}(x_A, b_\perp; \sigma_a) \mathcal{D}_{g/B}(x_B, b_\perp; \sigma_b)$$
$$\times\; \exp\Big\{\frac{\alpha_s N_c}{2\pi}\Big[\ln^2 \frac{b_\perp^2 s \sigma_p \sigma_t}{4} - 2\big(\ln\frac{x_A}{\sigma_t} + \gamma\big)\big(\ln\frac{x_B}{\sigma_p} + \gamma\big) + \frac{\pi^2}{2}\Big]\Big\}$$
$$+ \text{ NLO terms} \sim O\big(\alpha_s^2\big) \;+\; \text{power corrections}$$

One-loop coefficient function

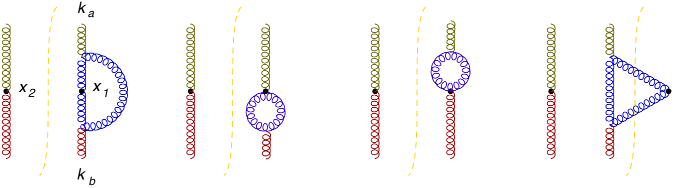
Calculation of coefficient function $\mathfrak{C}_1$ in the background field $\mathbb{A} = \bar{A} + \bar{B} + \bar{C}$

$$\int dz_2^- dz_{2_\perp} dz_1^- dz_{1_\perp} dw_1^+ dw_{1_\perp} dw_2^+ dw_{2_\perp} \frac{\alpha_s N_c}{2\pi} \mathfrak{C}_1(x_1, x_2; z_i^-, z_{i_\perp}, w_i^+, w_{i_\perp}; \sigma_p, \sigma_t)$$
$$\times F^{-i,a}(z_2^+, z_{2_\perp}) F^{-j,a}(z_1^+, z_{1_\perp}) F^{+i,a}(z_2^-, z_{2_\perp}) F^{+j,a}(z_1^-, z_{1_\perp})$$
$$= \frac{N_c^2 - 1}{16} g^4 \langle \tilde{F}^a_{\mu\nu} \tilde{F}^{a\mu\nu}(x_2) F^b_{\lambda\rho} F^{b\lambda\rho}(x_1) \rangle_{\mathbb{A} = \bar{A} + \bar{B}}$$
$$- \langle \hat{\mathcal{O}}^{ij,\sigma_p}(x_2^-, x_{2_\perp}; x_1^-, x_{1_\perp}) \hat{\mathcal{O}}^{ij;\sigma_t}(x_2^+, x_{2_\perp}; x_1^+, x_{1_\perp}) \rangle_{\mathbb{A} = \bar{A} + \bar{B}}$$

(for the purpose of calculating leading-twist coefficient function the “correction field” C can be neglected: $\mathbb{A} = \bar{A} + \bar{B}$)

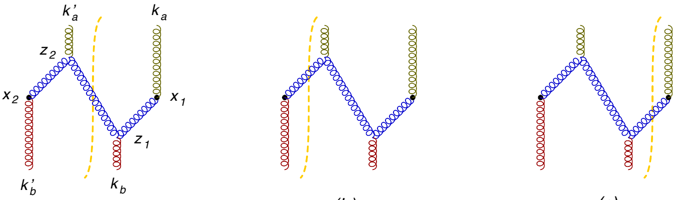
Diagrams for $\langle \tilde{F}^a_{\mu\nu}\tilde{F}^{a\mu\nu}(x_2)F^b_{\lambda\rho}F^{b\lambda\rho}(x_1)\rangle_{\mathbb{A}}$ in background fields

"Virtual" diagrams



(a) (b) (c) (d)

"Real" diagrams

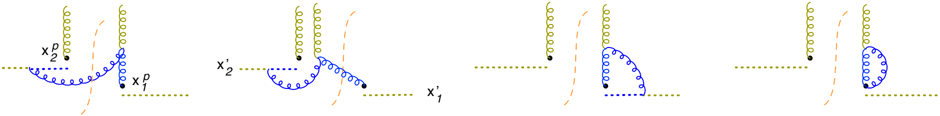


(a) (b) (c)

Diagrams for subtracted TMD matrix elements

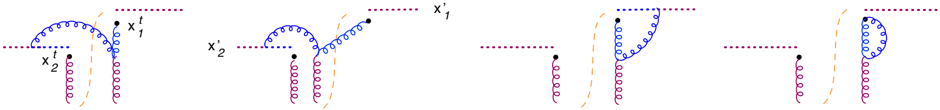
“Projectile” TMD matrix elements.

The $e^{-i\frac{\beta}{\sigma_p}}$ regularization is depicted by point splitting: positions of $F's$ are separated from the beginnings of gauge links. (Violations of gauge invariance are power corrections).



(a) (b) (c) (d)

“Target” TMD matrix elements. The $e^{-i\frac{\alpha}{\sigma_t}}$ regularization is depicted by point splitting.



(a) (b) (c) (d)

Result for the coefficient function

$$\frac{1}{16}(N_c^2-1)\langle p_A',p_B'|g^2F^a_{\mu\nu}F^{a\mu\nu}(x_2)g^2F^b_{\lambda\rho}F^{b\lambda\rho}(x_1)|p_A,p_B\rangle$$
$$= \langle p_A'|\hat{\mathcal{O}}^{\sigma_p}_{ij}(x_2^-,x_{2_\perp};x_1^-,x_{1_\perp})|p_A\rangle\langle p_B'|\hat{\mathcal{O}}^{ij;\sigma_t}(x_2^+,x_{2_\perp};x_1^+,x_{1_\perp})|p_B\rangle$$
$$+\int dz_1^-dz_2^-dw_1^+dw_2^+\frac{\alpha_sN_c}{2\pi}\mathfrak{C}_1(x_1,x_2;z_i^-,w_i^+;\sigma_p,\sigma_t)$$
$$\times\langle p_A'|\hat{\mathcal{O}}^{\sigma_p}_{ij}(z_2^-,x_{2_\perp};z_1^-,x_{1_\perp})|p_A\rangle\langle p_B'|\hat{\mathcal{O}}^{ij;\sigma_t}(z_2^+,x_{2_\perp};z_1^+,x_{1_\perp})|p_B\rangle$$

$$\mathfrak{C}_1(\alpha_a',\alpha_a,\beta_b',\beta_b;x_1,x_2;\sigma_p,\sigma_t)$$
$$=\ln^2\frac{x_{12_\perp}^2 s\sigma_p\sigma_t}{4}-\ln\frac{(-i\alpha_a')e^{\gamma_E}}{\sigma_t}\ln\frac{(-i\beta_b')e^{\gamma_E}}{\sigma_p}-\ln\frac{(-i\alpha_a)e^{\gamma_E}}{\sigma_t}\ln\frac{(-i\beta_b)e^{\gamma_E}}{\sigma_p}+\pi^2$$

The solution of TMD evolution equations compatible with this first-order result is

$$\mathfrak{C}(x_{1_\perp},x_{2_\perp};\alpha_a',\alpha_a,\beta_b',\beta_b;\sigma_p,\sigma_t)\;=\;e^{\frac{\alpha_sN_c}{2\pi}\mathfrak{C}_1(x_{12_\perp},\alpha_a',\alpha_a,\beta_b',\beta_b;\sigma_p,\sigma_t)}$$

$\Rightarrow$ hadronic tensor is

$$W(\alpha_a',\alpha_a,\beta_b',\beta_b,x_{1_\perp},x_{2_\perp})\;=\;\int d\!\!\!{}^-\alpha_a' d\!\!\!{}^-\alpha_a d\!\!\!{}^-\beta_b' d\!\!\!{}^-\beta_b\; e^{\frac{\alpha_sN_c}{2\pi}\mathfrak{C}_1(x_{12_\perp},\alpha_a',\alpha_a,\beta_b',\beta_b;\sigma_p,\sigma_t)}$$
$$\times\langle p_A'|\hat{\mathcal{O}}^{\sigma_p}_{ij}(\alpha_a',\alpha_a,x_{2_\perp},x_{1_\perp})|p_A\rangle\langle p_B'|\hat{\mathcal{O}}^{ij;\sigma_t}(\beta_b',\beta_b,x_{2_\perp},x_{1_\perp})|p_B\rangle\;+\ldots$$

Forward case ($\equiv$ particle production by gluon fusion)

Recall $\alpha_q \equiv x_A$, $\beta_q \equiv x_B$.

$$W(p_A,p_B;q) \;=\; \int db_\perp\; e^{i(q,b)_\perp} W(p_A,p_B;\alpha_q,\beta_q,b_\perp),$$

$$W(p_A,p_B;\alpha_q,\beta_q,b_\perp) \;=\; \frac{\pi^2}{2} Q^2 \mathcal{G}_{ij}^{\sigma_p}(\alpha_q,b_\perp;p_A)\,\mathcal{G}^{ij;\sigma_t}(\beta_q,b_\perp;p_B)$$
$$\times\;\exp\Big\{\frac{\alpha_s N_c}{2\pi}\Big[\ln^2\frac{b_\perp^2 s\sigma_p\sigma_t}{4} - 2\big(\ln\frac{\alpha_q}{\sigma_t}+\gamma\big)\big(\ln\frac{\beta_q}{\sigma_p}+\gamma\big) + \frac{\pi^2}{2}\Big]\Big\}$$
$$+\;\text{NLO terms} \sim O\big(\alpha_s^2\big) \;+\; \text{power corrections} \qquad (1)$$

where $\mathcal{G}_{ij}^{\sigma_p}$, $\mathcal{G}_{ij}^{\sigma_t}$ are gluon TMDs:

$$\langle p_A|\hat{\mathcal{O}}_{ij}^{\sigma_p}(z^-,0^-,b_\perp)|p_A\rangle \;=\; -g^2\varrho^2\int_0^1 du\; u\,\mathcal{G}_{ij}^{\sigma_p}(u,b_\perp)\cos u\varrho z^-,$$

$$\langle p_B|\hat{\mathcal{O}}_{ij}^{\sigma_t}(z^-,0^-,b_\perp)|p_B\rangle \;=\; -g^2\varrho^2\int_0^1 du\; u\,\mathcal{G}_{ij}^{\sigma_t}(u,b_\perp)\cos u\varrho z^-.$$

Matching of coefficient function and TMDs

The r.h.s. of this evolution formula (1) does not depend on cutoffs σ_p and σ_t as long as $\sigma_p \geq \tilde{\sigma}_p = \frac{4b_\perp^{-2}}{x_A s}$ and $\sigma_t \geq \tilde{\sigma}_t \equiv \frac{4b_\perp^{-2}}{x_B s}$. Thus, the result of double-log Sudakov evolution reads

$$\begin{aligned} W(p_A,p_B;x_A,x_B,b_\perp) &= \frac{\pi^2}{2} Q^2 \mathcal{G}_{ij}^{\tilde{\sigma}_p}(x_A,b_\perp;p_A)\mathcal{G}^{ij;\tilde{\sigma}_t}(x_B,b_\perp;p_B) \\ &\times \exp\Big\{-\frac{\alpha_s N_c}{2\pi}\Big[\Big(\ln\frac{Q^2 b_\perp^2}{4}+2\gamma\Big)^2 - 2\gamma^2 - \frac{\pi^2}{2}\Big]\Big\} + O(\alpha_s^2) \text{ terms } + \text{ power corrections} \end{aligned}$$

This result is universal for moderate x and small-x hadronic tensor. The difference lies in the continuation of the evolution beyond Sudakov region.

Double-log Sudakov evolution should stop at $x_B \tilde{\sigma}_p s \simeq b_\perp^{-2}$. After that:

- If $x_B \sim 1$ - DGLAP-type evolution from $\tilde{\sigma}_t = \frac{b_\perp^{-2}}{x_B s}$ to $\sigma_{\text{fin}} = \frac{m_N^2}{s}$: summation of $\big(\alpha_s \ln\frac{b_\perp^{-2}}{m_N^2}\big)^n$
- If $x_B \ll 1$ - BFKL-type evolution from $\tilde{\sigma}_t = \frac{b_\perp^{-2}}{x_B s}$ to $\sigma_{\text{fin}} = \frac{b_\perp^{-2}}{s}$: summation of $\big(\alpha_s \ln x_B\big)^n$

Conclusions

1. The rapidity-only factorization is the most convenient tool for high-energy QCD.
   - Current status of the BFKL/BK evolution: NLO evolution and NLO impact factors. (For $\mathcal{N}=4$ SYM - NNLO)
   - Impact factors for various processes from shock-wave approach: LO and NLO
   - "Hybrid factorization" - phenomenological mix of DGLAP and BK evolutions
2. Rapidity-only TMD factorization works:
   - Full list of $\frac{1}{Q^2}$ power corrections for DY and SIDIS.
   - Back-of-the-envelope estimates of power corrections seems to agree with exp. data.
   - Rapidity-only evolution with BLM prescription for running coupling gives the same universal formula for Sudakov double logs at both small and moderate x for both quark and gluon TMDs.
   - Rapidity factorization at the one-loop level gives Sudakov-type double logs for both small and intermediate x_B

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Thank you for attention!