LaMET's Asymptotic Analysis

Speaker: Yushan Su, University of Maryland
on behalf of the authors:
Jiunn-Wei Chen, Xiang Gao, Jinchen He, Jun Hua, Xiangdong Ji,
Andreas Schaefer, Yushan Su, Wei Wang, Yi-Bo Yang, Jian-Hui Zhang,
Qi-An Zhang, Rui Zhang, and Yong Zhao
of 2505.14619

Outline

- LaMET as a forward problem formalism
- Large z asymptotic analysis
- Data quality for asymptotic analysis

Lattice QCD Calculation on PDFs

Moments from local operators
$\langle x^N \rangle = \int dx\, x^N f(x)$

- Textbook knowledge: Peskin&Schroeder 1995
- Early calculations: LHPC 2010, RBC/UKQCD 2010
- Recent results: RQCD 2018, χQCD 2018, NME 2021, ETMC 2022, Mainz 2024

Short distance factorization from non-local operators

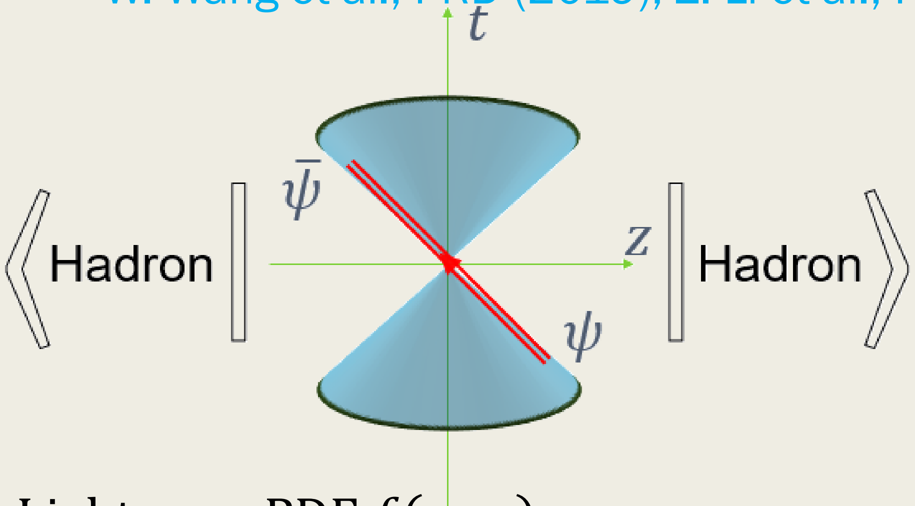
- Moments
  - Idea proposed: V. M. Braun&D. Muller 2008
  - Calculation just started: JLab 2017, BNL/ANL 2020
- Short distance correlation (SDF) and x-dependence modeling
  - Recent calculations: JLab 2017, 2019, 2021

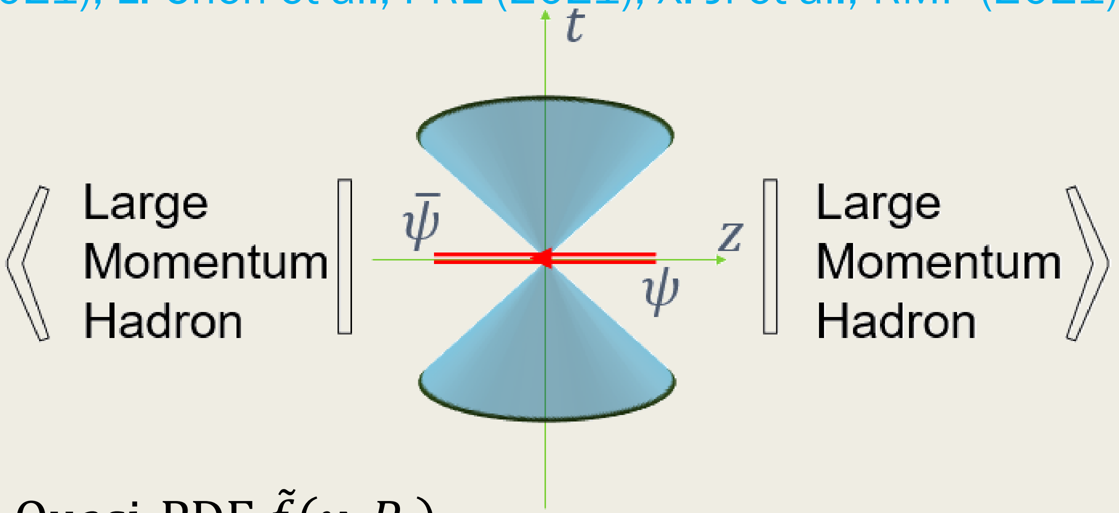
x-dependence from large momentum effective theory (LaMET)

- Idea proposed: Ji 2013, Ji 2014
- Major collaborations: BNL/ANL, LPC, ETMC, MSU

Large Momentum Effective Theory (LaMET)

Ji, PRL 110 (2013); Ji, SCPMA 57 (2014); X. Xiong et al., PRD (2014); T. Izubuchi et al., PRD (2018); W. Wang et al., PRD (2019); Z. Li et al., PRL (2021); L. Chen et al., PRL (2021); X. Ji et al., RMP (2021)





Light cone PDF $f(x,\mu) =$
$\int \frac{d\xi^-}{4\pi} e^{-i\,\xi^- P^+ x} \langle P | \bar{\psi}(\xi^-) U(\xi^-,0) \gamma^+ \psi(0) | P \rangle$

Quasi-PDF $\tilde{f}(y,P_z) =$
$\int \frac{dz}{4\pi} e^{i\,z\,P_z\,y} \langle P_z | \bar{\psi}(z) U(z,0) \gamma^z \psi(0) | P_z \rangle$, where $P_z \gg \Lambda_{\mathrm{QCD}}$

- The equivalence of collinear modes between f and $\tilde{f}$ is shown in Libby Sterman analysis in X. Ji et al., RMP (2021), arXiv:2004.03543
- The UV difference is absorbed into the matching coefficient C

$$f(x,\mu) = \int \frac{dy}{|y|} C\left(\frac{x}{y}, \frac{\mu}{|y|P^z}\right) \tilde{f}(y,P^z) + O\left[\frac{\Lambda_{\mathrm{QCD}}^2}{x^2 P_z^2}, \frac{\Lambda_{\mathrm{QCD}}^2}{(1-x)^2 P_z^2}\right]$$

LaMET analysis is a forward process with controlled precision

- Lattice data generation and hybrid renormalization (Coor)

$$\tilde{h}(z, P_z, a) = \langle P_z | \bar{\psi}(z) U(z, 0) \gamma^z \psi(0) | P_z \rangle$$

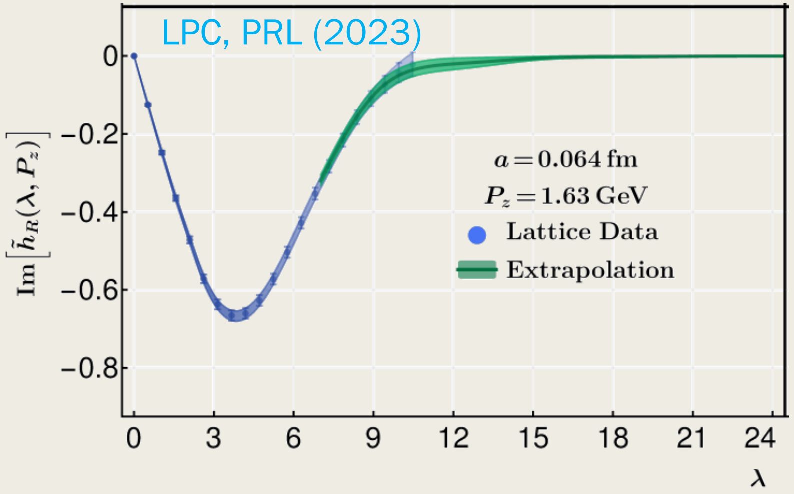
$$\tilde{h}^R(z, P_z) = \frac{\tilde{h}(z, P_z, a)}{\tilde{h}(z, 0, a)} \theta(z_s - z) + \frac{\tilde{h}(z, P_z, a)}{Z_R(z, a, \mu)} \frac{Z_R(z_s, a, \mu)}{\tilde{h}(z_s, 0, a)} \theta(z - z_s)$$

- Asymptotic extrapolation and Fourier transformation (Coor to Mom)

$$\tilde{h}^{R\prime}(z, P_z) = \begin{cases} \tilde{h}^R(z, P_z), z < z_L \\ \left[\frac{c_1}{(i\lambda)^{d_1}} + e^{-i\lambda} \frac{c_2}{(-i\lambda)^{d_2}}\right] e^{-m_{\text{eff}} z}, z > z_L \end{cases}$$

$$\tilde{f}(y, P^z) = \int_{-\infty}^{+\infty} \frac{d\lambda}{2\pi} e^{i\lambda y} \tilde{h}^{R\prime}(\lambda/P_z, P_z)$$

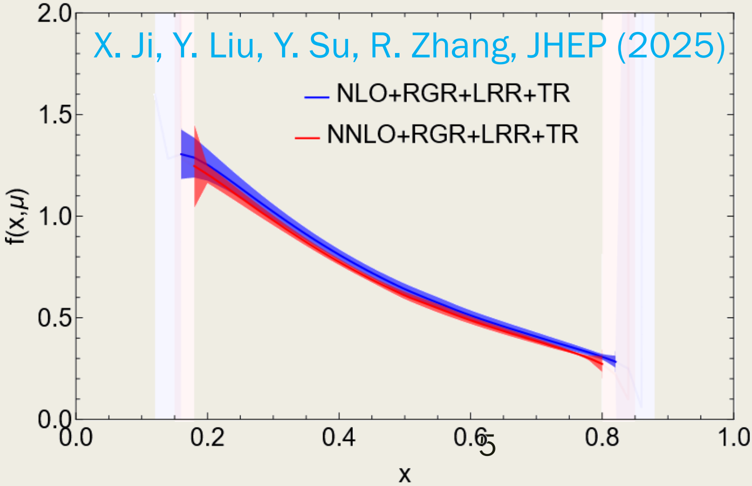
The errors from asymptotic extrapolation to FT are bounded by the exponential decay factor.



- Perturbative matching (Mom)

$$f(x, \mu) = \int \frac{dy}{|y|} C\left(\frac{x}{y}, \frac{\mu}{|y| P^z}\right) \tilde{f}(y, P^z) + O\left[\frac{\Lambda_{\text{QCD}}^2}{x^2 P_z^2}, \frac{\Lambda_{\text{QCD}}^2}{(1-x)^2 P_z^2}\right]$$

where the precision in the moderate$-x$ range is under controlled.



Lattice data generation and hybrid renormalization

- Generate lattice QCD matrix elements

$$\tilde{h}(z, P_z, a) = \langle P_z | \bar{\psi}(z) U(z, 0) \gamma^z \psi(0) | P_z \rangle$$

- Hybrid renormalization Ji:2020brr

  Motivation: cancel UV divergence without introducing uncontrolled non-perturbative effects at large distance

  Method:

$$\tilde{h}^R(z, P_z) = \frac{\tilde{h}(z, P_z, a)}{\tilde{h}(z, 0, a)} \theta(z_s - z) + \frac{\tilde{h}(z, P_z, a)}{Z_R(z, a, \mu)} \frac{Z_R(z_s, a, \mu)}{\tilde{h}(z_s, 0, a)} \theta(z - z_s)$$

  short distance ($z < z_s$): ratio scheme using zero momentum matrix element $\tilde{h}(z, 0, a)$

  Long distance ($z > z_s$): $\overline{\mathrm{MS}}$ scheme using the renormalization factor $Z_R(z, a, \mu)$

- The renormalization factor $Z_R(z, a, \mu)$, converting lattice to $\overline{\mathrm{MS}}$ scheme, is extracted from zero momentum matrix element $\tilde{h}(z, 0, a)$, by fitting the a dependence and comparing with $\overline{\mathrm{MS}}$ scheme perturbation theory. This process is called self-renormalization.

  LatticePartonLPC:2021gpi

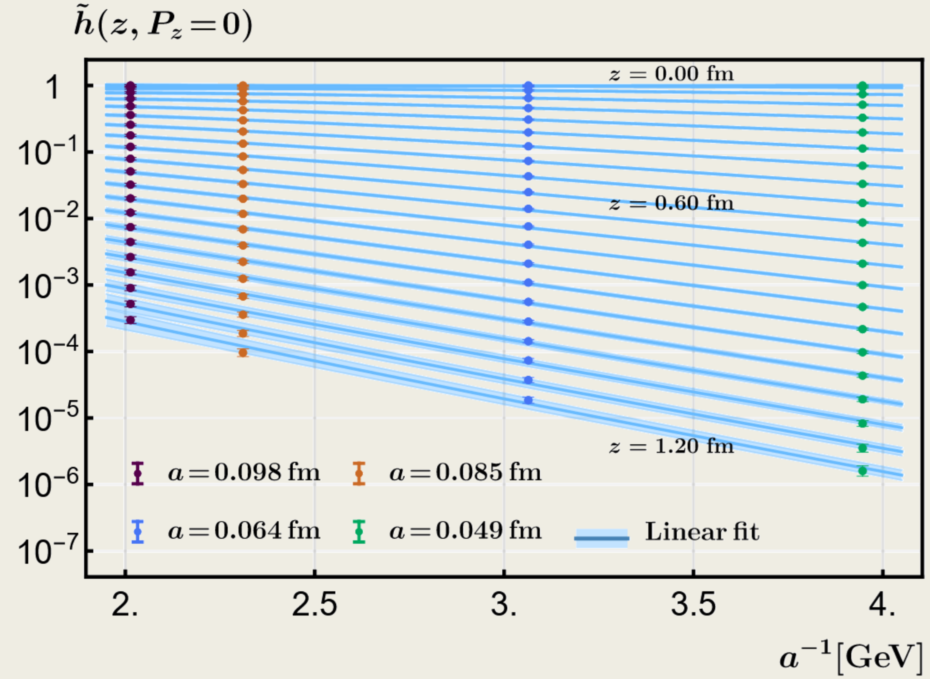
Self-renormalization to extract $Z_R(z,a,\mu)$ – a substep of hybrid renormalization

LatticePartonLPC:2021gpi

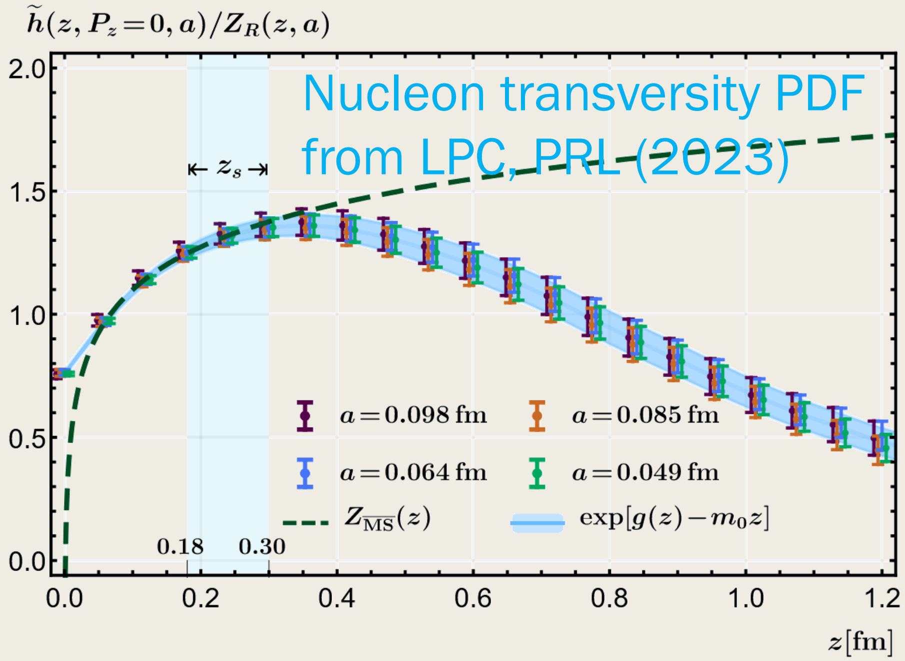
- The parametrization of $Z_R(z,a,\mu)$ motivated by perturbation theory

$$Z_R(z,a,\mu) = \mathrm{Exp}\left[\frac{kz}{a\ln(a\Lambda_{\mathrm{QCD}})} + \frac{3C_F}{11-\frac{2N_f}{3}}\ln\left[\frac{\ln\left[\frac{1}{(a\Lambda_{\mathrm{QCD}})}\right]}{\ln\left[\frac{\mu}{\Lambda_{\mathrm{QCD}}}\right]}\right] + \ln\left[1+\frac{d}{\ln(a\Lambda_{\mathrm{QCD}})}\right] + m_0 z + f(z)a^2\right]$$

- Fit the UV divergence (a-dependence) from zero momentum matrix element



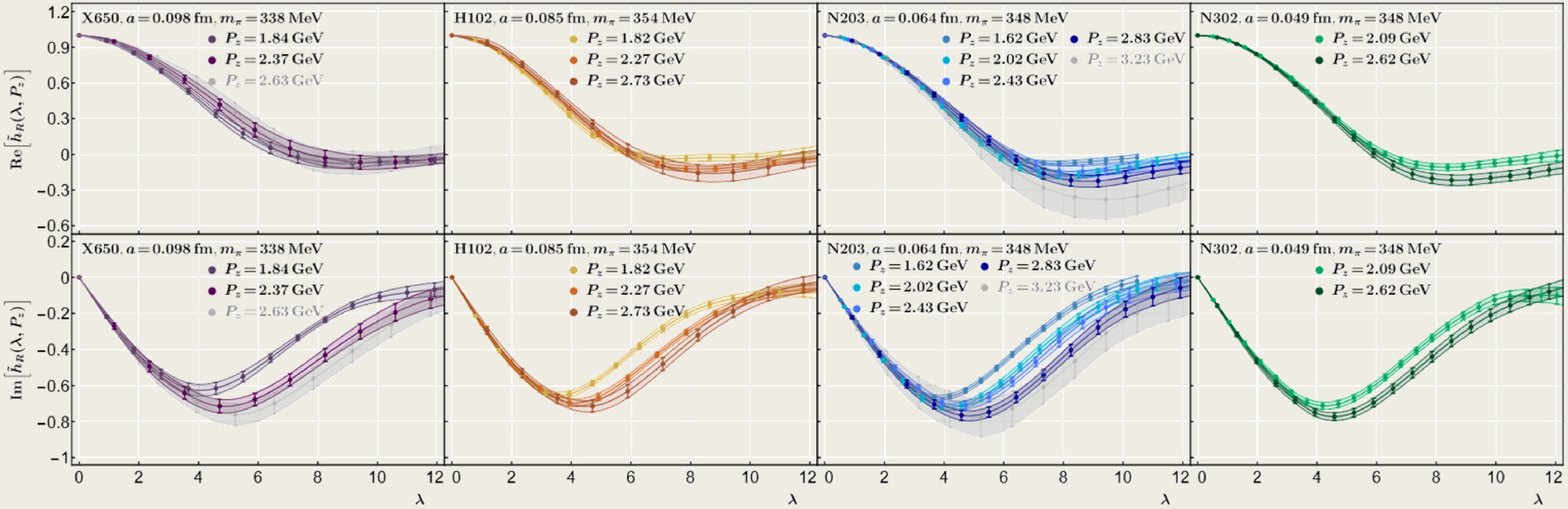
- Fit the scheme conversion factor, such as m_0, by comparing data and $\overline{\mathrm{MS}}$ scheme perturbation theory



The hybrid renormalized matrix elements

- The coordinate space matrix elements

Nucleon transversity PDF from LPC, PRL (2023)



- Convert to momentum space using FT?

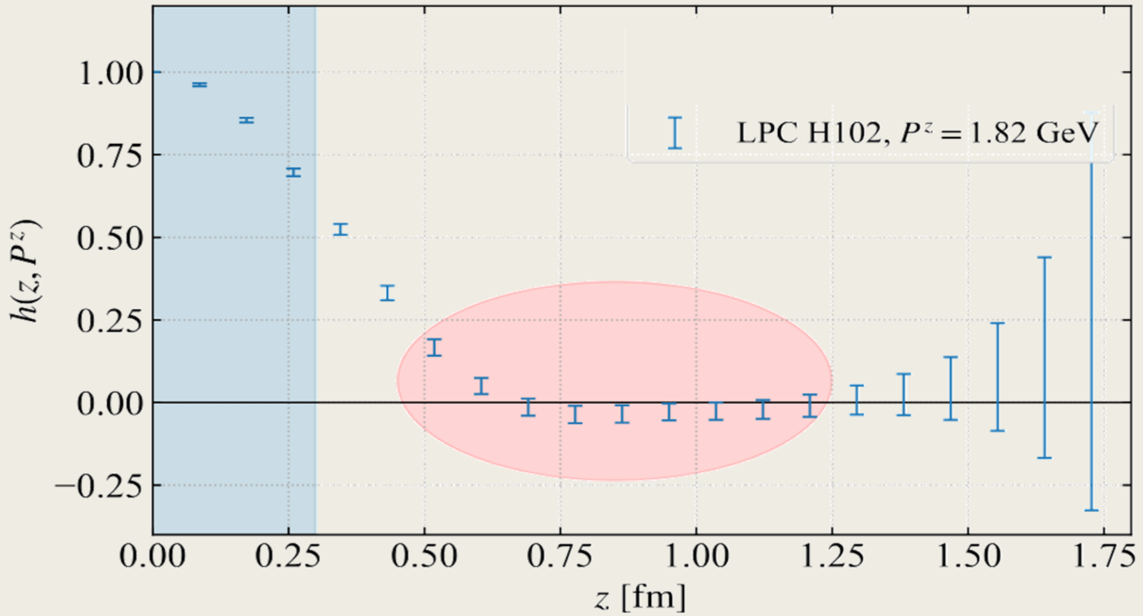
$$\tilde{f}(y,P^z) = \int_{-\infty}^{+\infty} \frac{d\lambda}{2\pi} e^{i\lambda y} \tilde{h}^R(\lambda/P_z, P_z)$$

An issue of FT

- FT needs correlator in the **infinite** range ($\lambda = zP^z$)

$$\tilde{f}(y, P^z) = \int_{-\infty}^{+\infty} \frac{d\lambda}{2\pi} e^{i\lambda y} \tilde{h}^R(\lambda/P_z, P_z)$$

- Lattice data are available up to a **finite** distance, such as $z \sim 1\mathrm{fm}$, due to bad signal to noise ratio at larger distance.

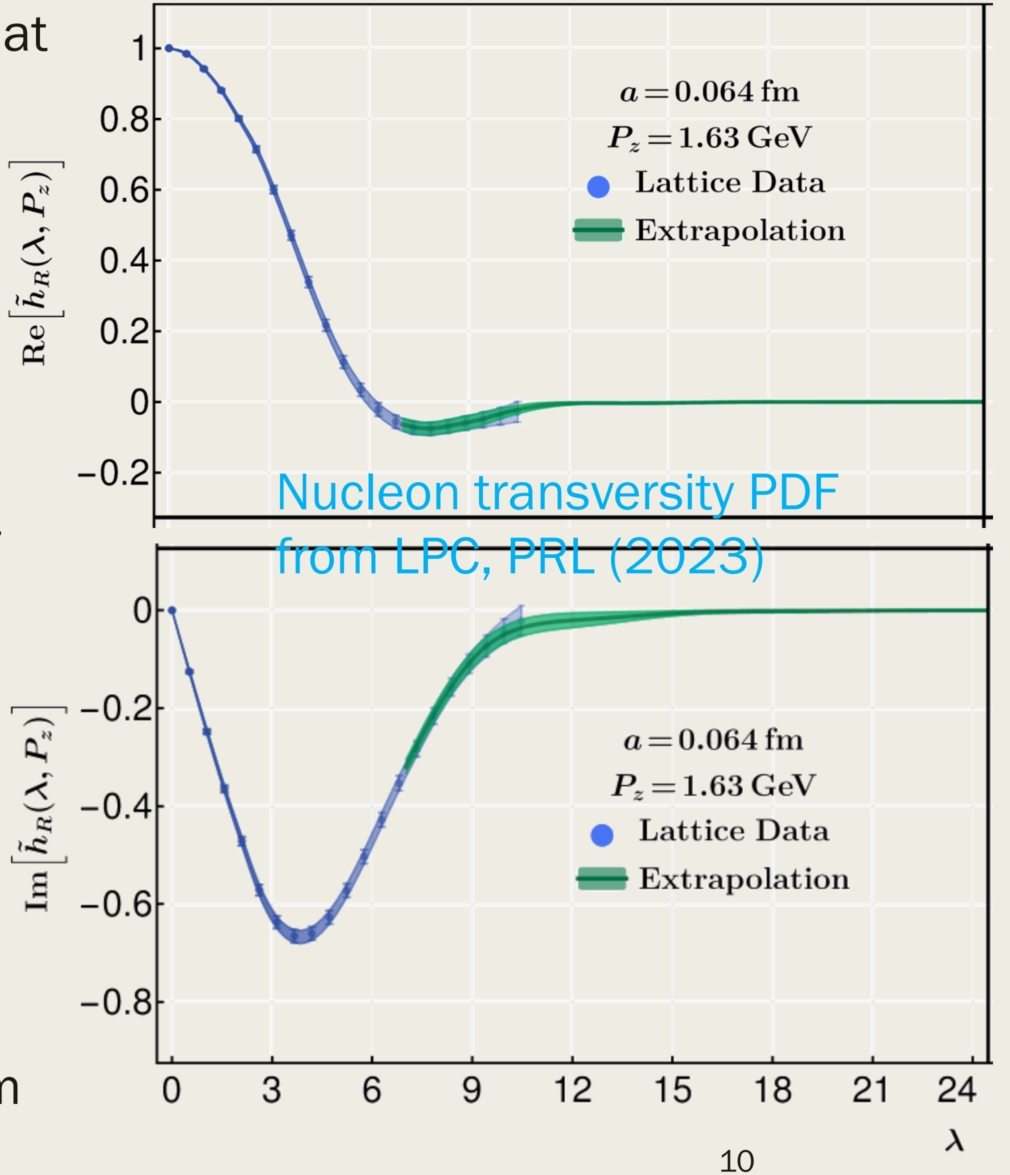


Nucleon transversity PDF from LPC, PRL (2023)

- We hope to resolve this issue by extrapolating data to infinite range based on physical guidance.

Large distance asymptotic analysis

- The basic idea: Euclidean correlator decays exponentially at large distance.
- The strategy: extrapolate data to infinite range based on exponential decay.
  - Fit data at large distance with asymptotic form
    $$\left[\frac{c_1}{(i\lambda)^{d_1}} + e^{-i\lambda}\frac{c_2}{(-i\lambda)^{d_2}}\right] e^{-m_{\text{eff}} z}$$
  - Combine data and asymptotic form to obtain the full-range correlator
    $$\tilde{h}^{R\prime}(z, P_z) = \begin{cases} \tilde{h}^R(z, P_z), z < z_L \\ \left[\frac{c_1}{(i\lambda)^{d_1}} + e^{-i\lambda}\frac{c_2}{(-i\lambda)^{d_2}}\right] e^{-m_{\text{eff}} z}, z > z_L \end{cases}$$
  - Perform the FT
    $$\tilde{f}(y, P^z) = \int \frac{d\lambda}{2\pi} e^{i\lambda y} \tilde{h}^{R\prime}(\lambda/P_z, P_z)$$
- The errors originated from asymptotic fitting to momentum space distribution are bounded by the exponential decay.



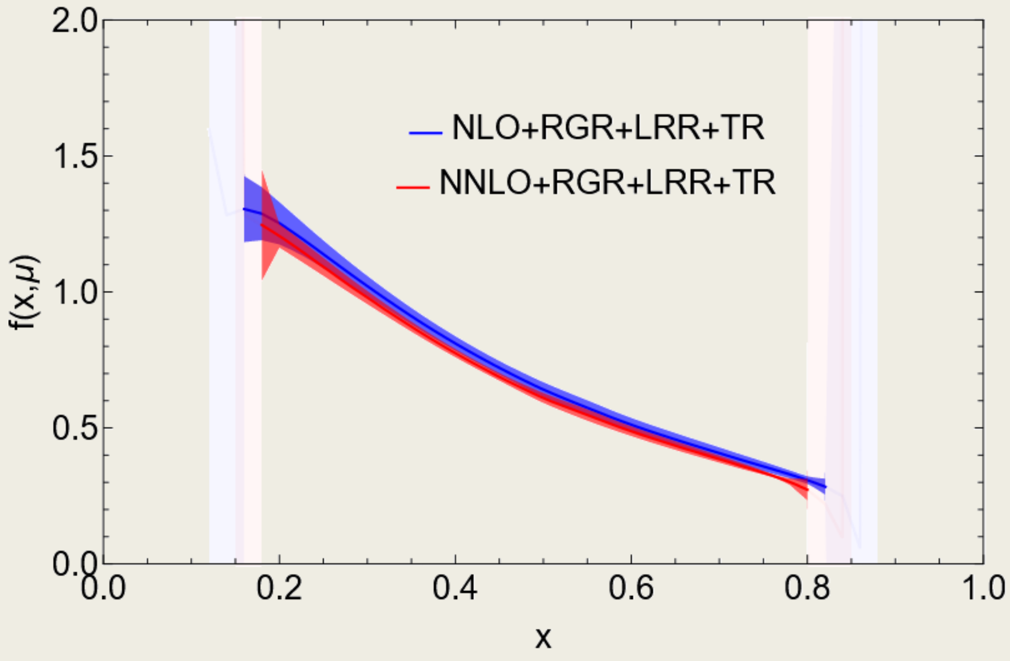
Nucleon transversity PDF from LPC, PRL (2023)

Perturbative matching

- Matching: convert quasi to light-cone

$$f(x,\mu)=\int\frac{dy}{|y|}C\left(\frac{x}{y},\frac{\mu}{|y|P^z}\right)\tilde{f}(y,P^z)+O\left[\frac{\Lambda_{\mathrm{QCD}}^2}{x^2P_z^2},\frac{\Lambda_{\mathrm{QCD}}^2}{(1-x)^2P_z^2}\right]$$

- Improved perturbative matching kernel $C\left(\frac{x}{y},\frac{\mu}{|y|P^z}\right)$
  - DGLAP log resummation (RGR): $\sim\alpha^n\ln^n(4x^2P_z^2/\mu^2)$ Su:2022fiu
  - Leading renormalon resummation (LRR): $\sim\alpha^{n+1}\beta_0^n n!$ Zhang:2023bxs
  - Threshold log resummation (TR): $\sim\alpha^n\ln^{2n}(4(1-x)^2P_z^2/\mu^2)$ Ji:2023pba, Liu:2023onm, Ji:2024hit
- Controlled precision in the moderate-x range, e.g. $0.2\lesssim x\lesssim 0.8$ for $P_z=2\mathrm{GeV}$.
- The reliable x-region could be extended with larger momentum in the future.



Pion valence PDF from
X. Ji, Y. Liu, Y. Su, R. Zhang, JHEP (2025)

Large z asymptotic analysis under LaMET

- Theoretical arguments for the exponential decay
- Exponential decay in lattice data
- Controlled precision of asymptotic analysis

Theoretical arguments for the exponential decay

- The quasi-PDF matrix element is two heavy-light quark vertices separated in z direction

$$\tilde{h}(z, P_z) = \langle P_z | \bar{\psi}(z) Q(z) \gamma^z Q(0) \psi(0) | P_z \rangle$$

- Insert a complete basis as the intermediate particles

$$\tilde{h}(z, P_z) = e^{-i\, z\, P^z} \sum_n \int \frac{d^3\vec{k}}{(2\pi)^3 2\sqrt{m_n^2 + \vec{k}^2}} e^{i\, z\, k^z} \langle P_z | \bar{\psi} Q | n_{\vec{k}} \rangle \langle n_{\vec{k}} | \gamma^z \bar{Q} \psi | P_z \rangle$$

- QCD confinement requires the branch cuts or singularity poles of the integrand have finite imaginary parts (mass gaps), which are the origins of exponential decay. For example, push the contour of k^z upward around the branch cut of $\sqrt{m_n^2 + \vec{k}^2}$ and redefine $k^z = i\, K^z$ gives one of the exponential decaying contributions,

$$\tilde{h}(z, P_z) = e^{-i\, z\, P^z} \sum_n \int \frac{d^2 k_\perp}{(2\pi)^2} \int_{\sqrt{m_n^2 + k_\perp^2}}^{+\infty} \frac{dK^z}{2\pi \sqrt{K_z^2 - (m_n^2 + k_\perp^2)}} e^{-z\, K^z} \langle P_z | \bar{\psi} Q | n_{\vec{k}} \rangle \langle n_{\vec{k}} | \gamma^z \bar{Q} \psi | P_z \rangle + \cdots$$

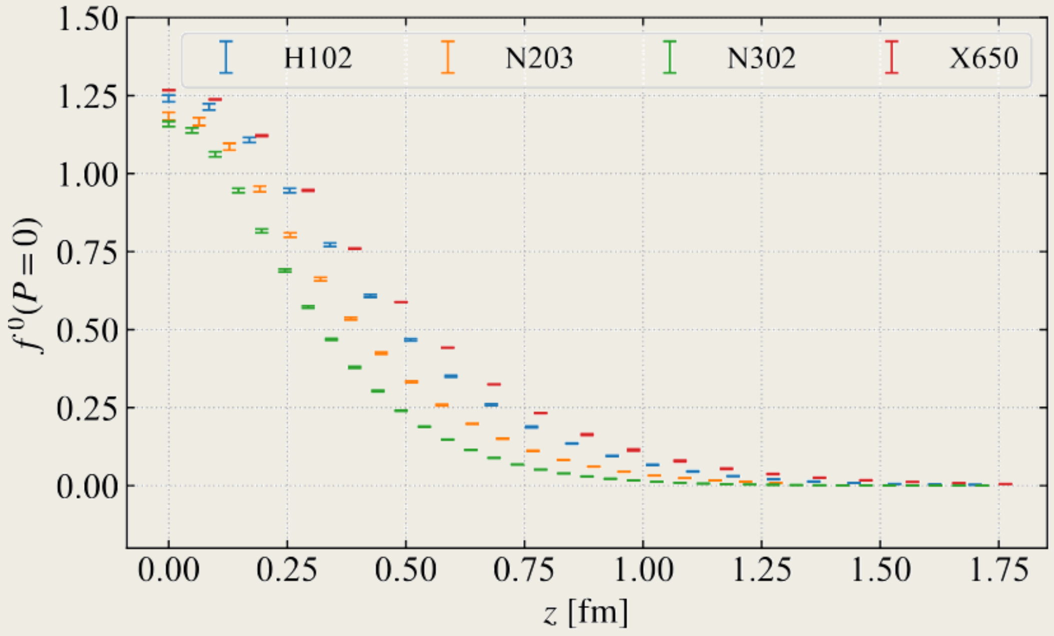
- In the large z limit, the dominant contribution comes from $K^z \sim \sqrt{m_n^2 + k_\perp^2}$, $k_\perp \sim 0$, $n = \mathrm{G}$ (ground state),

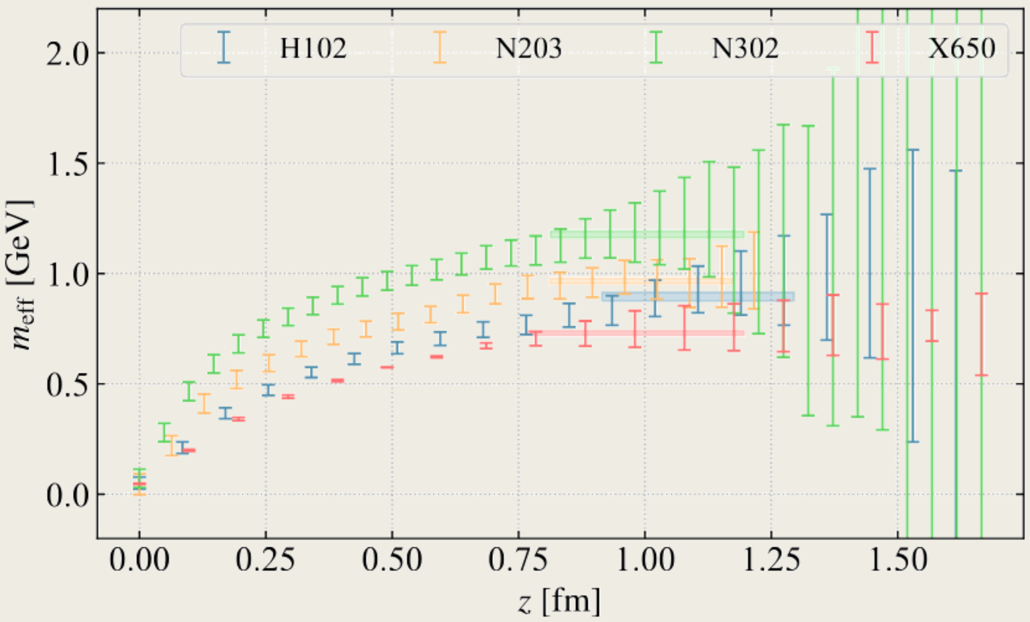
$$\tilde{h}(z \to \infty, P_z) \sim e^{-z\, m_G}$$

  which contains an exponential decay factor regarding the “ground state energy” of a hadron with a heavy quark.

Exponential decay in lattice data

- The transversity PDF data generated by LPC LPC, PRL (2023)





- Left panel: bare matrix element $f^0(z)$
- Right panel: "effective mass in z direction" of bare matrix element $\mathrm{m_{eff}}[z] = \frac{1}{a}\ln\frac{f^0(z)}{f^0(z+a)}$

  The plateau indicates the data hit the exponential decay region of the ground state.

Controlled precision of asymptotic analysis

- Recall the asymptotic analysis procedure:
  Fit the asymptotic form in the exponential decay region;
  Combine data and asymptotic form to obtain the full range correlator $\tilde{h}^{R\prime}(z, P_z)$;
  Perform the Fourier transformation to get the momentum space distribution.

- The error from asymptotic fitting could lead to the error of momentum space distribution, which is bounded due to the exponential decay: ANL/BNL, PRL (2022)

$$\delta\tilde{f}(x, P^z) < \frac{4N_x|h(z, P; \lambda_L)|_{\max}}{\pi x}$$

  where $|h(z, P; \lambda_L)|_{\max}$ is the maximum value of $\tilde{h}^{R\prime}(z, P_z)$ for $\lambda_L < \lambda < \infty$
  N_x is an integer at which the contribution from $\lambda > \lambda_L + N_x 2\pi/x$ is negligible

- e.g. for $P^z = 2\mathrm{GeV}$, $m_{\mathrm{eff}} = 0.2\mathrm{GeV}$, the exponential suppression after N_x periods is $\exp\left(-\frac{N_x 2\pi}{x}\frac{m_{\mathrm{eff}}}{P^z}\right) = \exp\left(-0.63\frac{N_x}{x}\right)$.

  For $x = 0.5$, $N_x \sim O(1)$ is a reasonable estimate. For example, we may take $N_x = 1$.

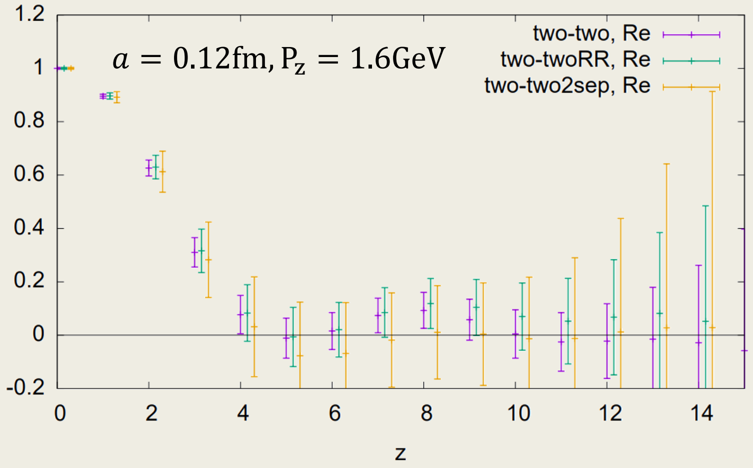
  If $|h(z, P; \lambda_L)|_{\max} = 0.05$, the error is bounded as $\delta\tilde{f}(x, P^z) < 0.13$.

- It likely overestimates the true uncertainty, as it assumes that $\tilde{h}^{R\prime}(z, P_z)$ can vary arbitrarily between 0 and its maximum value $|h(z, P; \lambda_L)|_{\max}$ at each z.

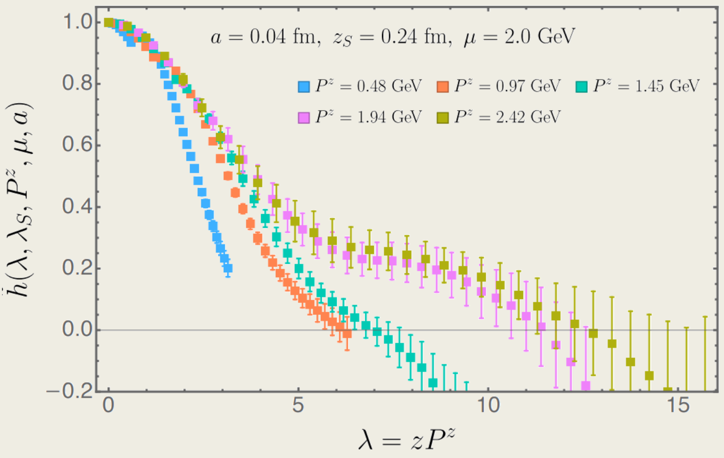
- The error $\delta\tilde{f}$ is governed by signal and noise, not the signal-to-noise ratio!

Data quality at large distance is improving

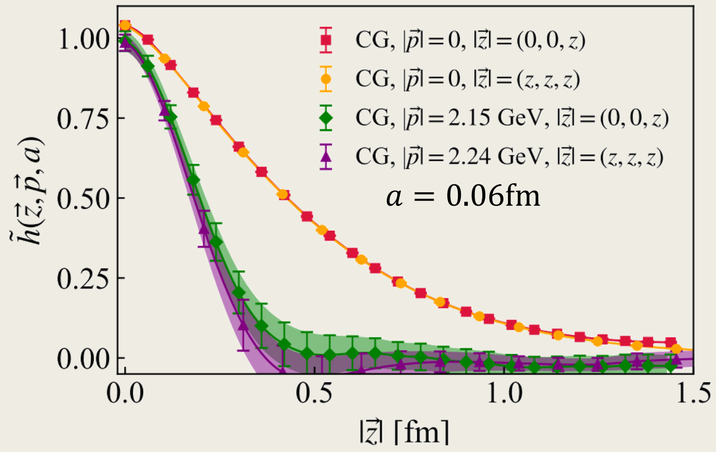
- e.g. Pion valence collinear PDF @ $m_\pi \sim 300\text{MeV}$



LP3, PRD (2019)



ANL/BNL, PRL (2022)

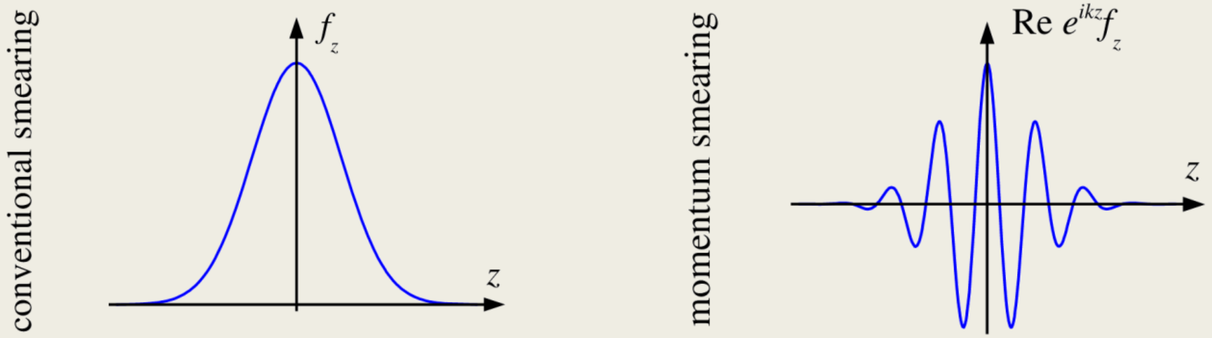


X. Gao, W. Liu, Y. Zhao, PRD (2024)

- Multiple reasons for the improvement
  - Computer resources and techniques
  - Novel physics-motivated ideas: momentum smearing source, Coulomb gauge method, kinematically-enhanced hadron interpolation operator…
  - …

Momentum smearing source

- Basic idea: introduce a **phase** to the gaussian smearing source to enhance the overlap with a **large momentum** hadron



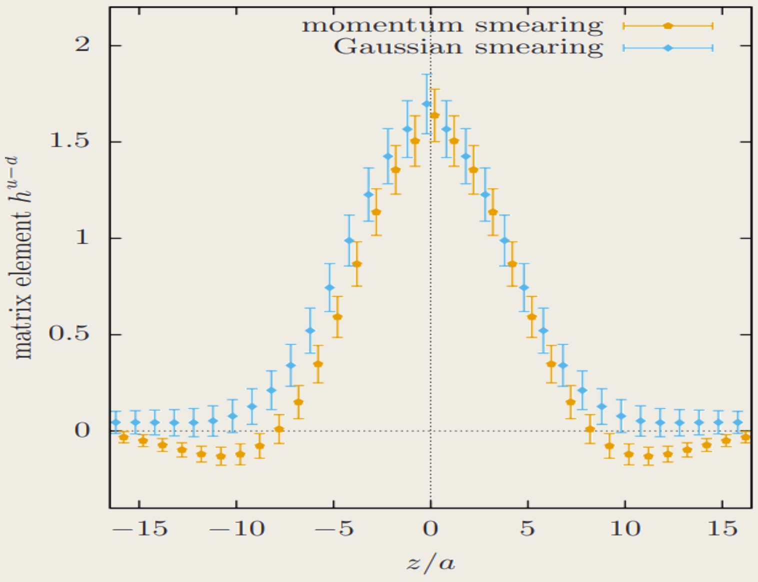
G. Bali, B. Lang, B. Musch, and A. Schäfer, PRD (2016)

- Effects:

Gaussian smearing source with 30000 measurements

is comparable with

momentum smearing source with 150 measurements



ETMC, PRD (2017)

Coulomb gauge method

- Basic arguments:

Gauge link in quasi-PDF ⇒ linear divergence ⟹ suppress signal-to-noise ratio at large distance

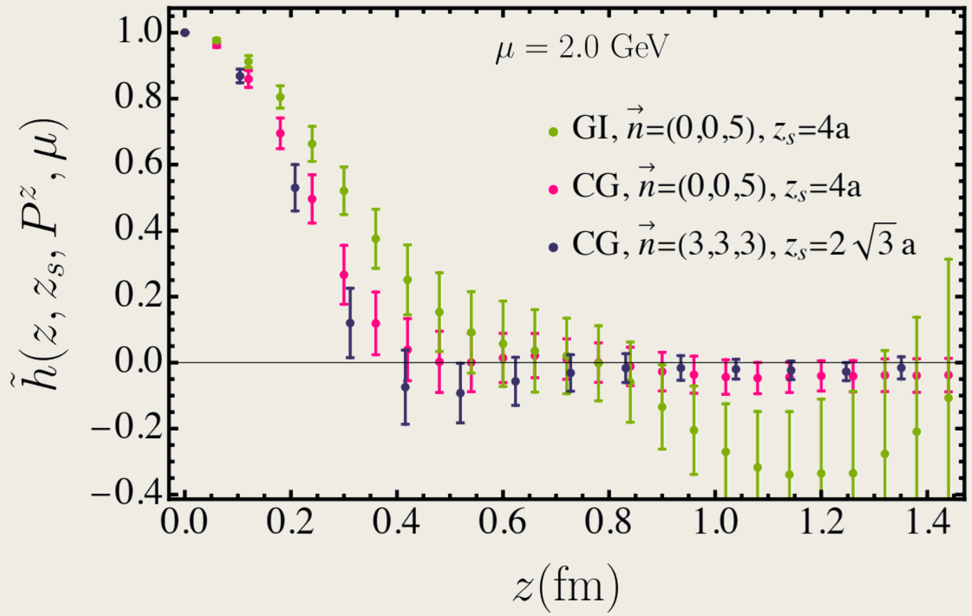
Replace gauge link with Coulomb gauge dressing factors ⇒ no linear divergence ⇒ higher precision

$$\bar{\psi}(z)U(z,0)\gamma^z\psi(0) \rightarrow \bar{\psi}(z)U_C^\dagger(z)\gamma^z U_C(0)\psi(0)$$

Y. Zhao, PRL (2024)

- Effects:

In the coulomb gauge method, the data at large distance become less noisy.



X. Gao, W. Liu, Y. Zhao, PRD (2024)

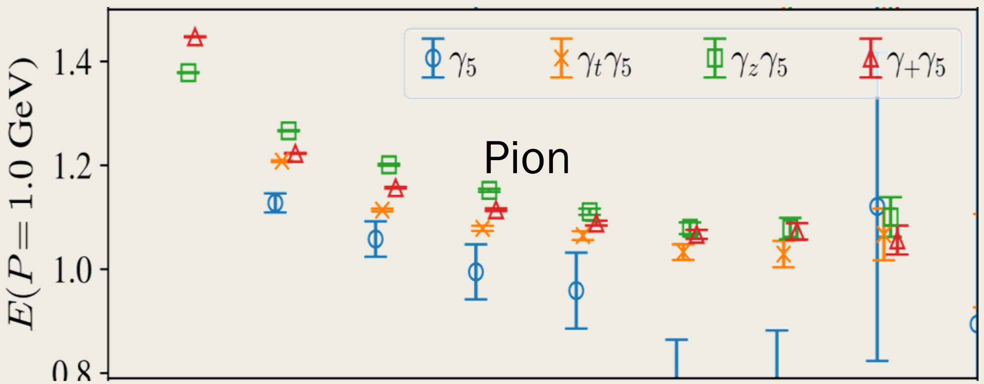
Kinematically-enhanced hadron interpolation operator

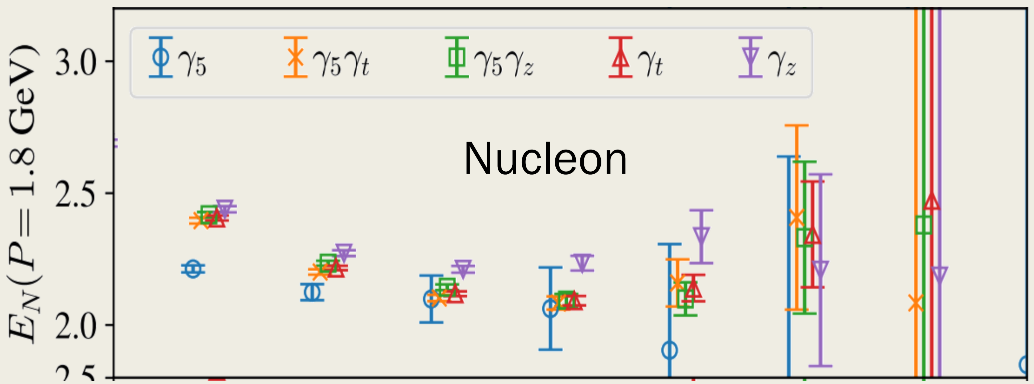
- Basic idea: modify the hadron interpolation operator to mimic the light-front operator

  For nucleon: $\epsilon_{abc}(d_a^T C\gamma_5 u_b)\mathcal{P}_+ u_c \to \epsilon_{abc}\left(d_a^T C\gamma_5\gamma_\mu u_b\right)\mathcal{P}_+ u_c$

  For pion: $\bar{u}\gamma_5 d \to \bar{u}\gamma_5\gamma_\mu d$

- The signal-to-noise ratio of 2pt is enhanced by the factor $O(P_z^2/M^2)$
- Numerical tests on the effective mass

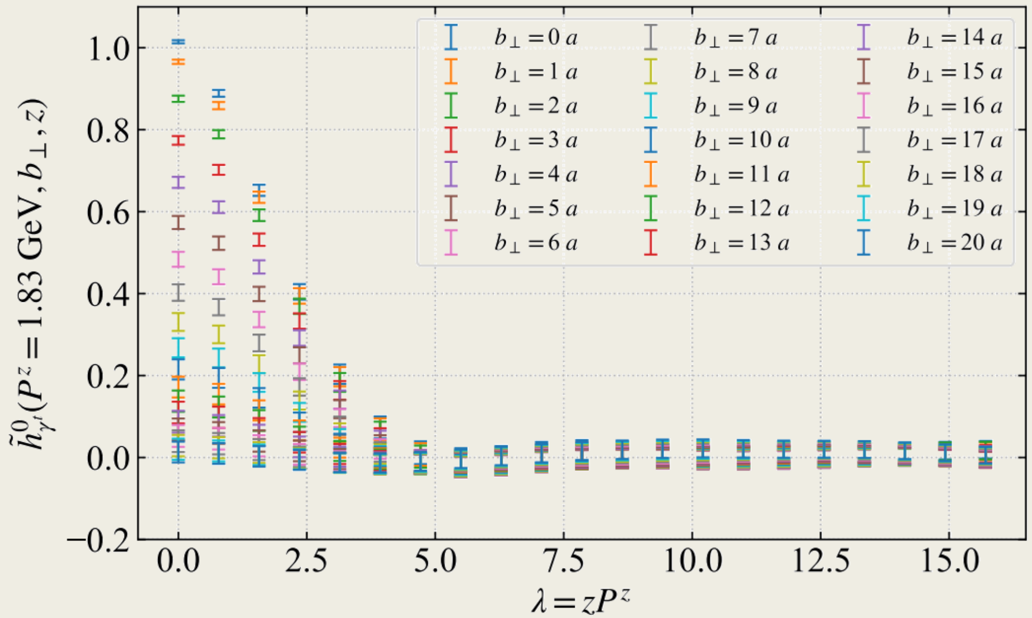


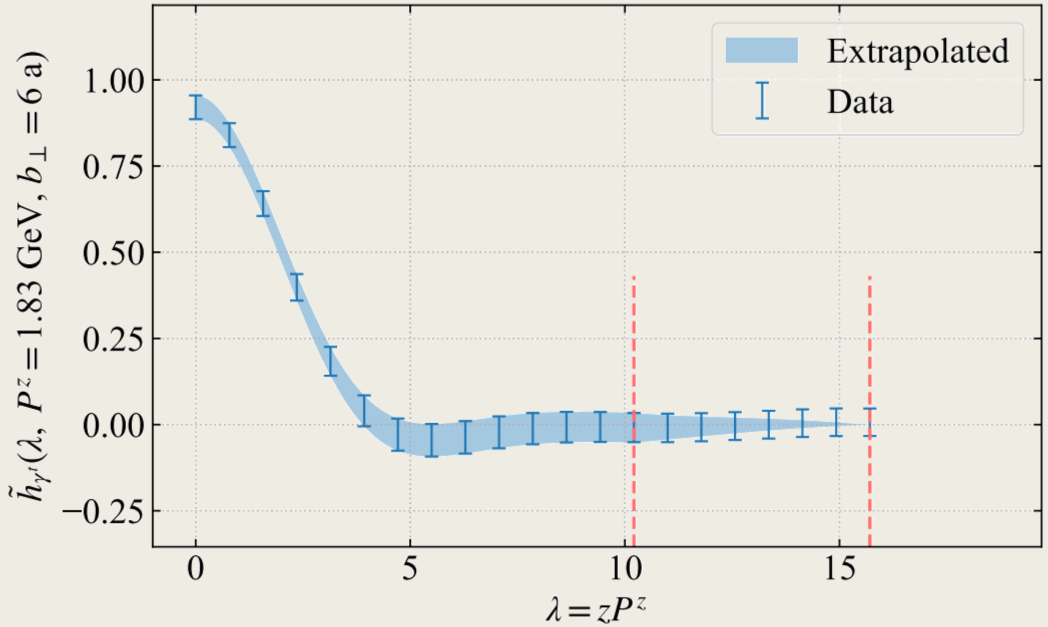


R. Zhang, A. Grebe, D. Hackett, M. Wagman, Y. Zhao, 2501.00729

State of art calculations

- Pion valence quasi TMDPDF under momentum smearing source and Coulomb gauge method





D. Bollweg, X. Gao, J. He, S. Mukherjee, Y. Zhao, 2504.04625

- The large-distance data show exponential decay and tend to zero quickly, while the errors do not increase.
- The precision of asymptotic extrapolation and Fourier transformation is well controlled.
- Haven't used the kinematically-enhanced hadron interpolation operator yet!
- Therefore, there are good quality data for asymptotic analysis under LaMET. The situation could be even better in the near future.

Conclusions

- LaMET is a forward problem by construction;
- The large z asymptotic analysis has a controlled precision, due to the exponential decay behavior;
- There are good-quality data for asymptotic analysis under LaMET. The situation could be even better in the near future.

Appendix

LaMET analysis is a forward process with controlled precision

- Lattice data generation and hybrid renormalization

$$\tilde{h}(z,P_z,a) = \langle P_z|\bar{\psi}(z)U(z,0)\gamma^z\psi(0)|P_z\rangle$$

$$\tilde{h}^R(z,P_z) = \frac{\tilde{h}(z,P_z,a)}{\tilde{h}(z,0,a)}\theta(z_s-z) + \frac{\tilde{h}(z,P_z,a)}{Z_R(z,a,\mu)}\frac{Z_R(z_s,a,\mu)}{\tilde{h}(z_s,0,a)}\theta(z-z_s)$$

- Asymptotic extrapolation and Fourier transformation ($\lambda = zP_z$)

$$\tilde{h}^{R\prime}(z,P_z) = \begin{cases} \tilde{h}^R(z,P_z), z < z_L \\ \left[\frac{c_1}{(i\lambda)^{d_1}} + e^{-i\lambda}\frac{c_2}{(-i\lambda)^{d_2}}\right] e^{-m_{\rm eff}z}, z > z_L \end{cases}$$

$$\tilde{f}(y,P^z) = \int \frac{d\lambda}{2\pi} e^{i\lambda y}\tilde{h}^{R\prime}(\lambda/P_z,P_z)$$

Current situation: Lattice data are available up to z_L ($0.7\sim1.0$ fm), in the exponential decay region. The data for $z > z_L$ are noisy and cannot be directly used.

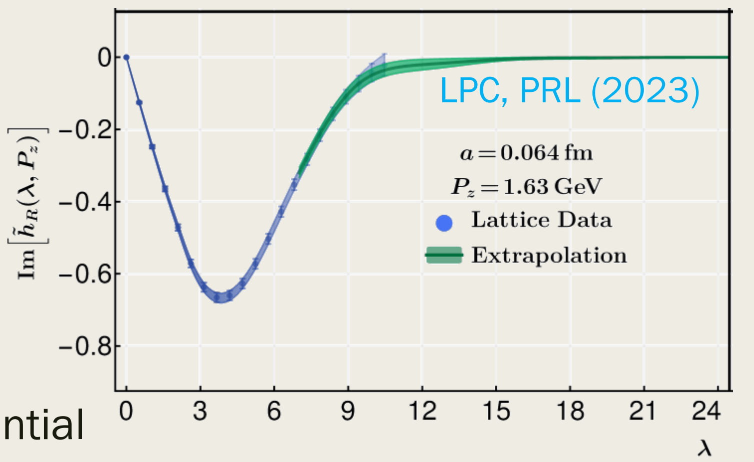
Our strategy (large z asymptotic analysis):

Fit the parameters $c_1, c_2, d_1, d_2, m_{\rm eff}$ of the asymptotic form using the large-z data.

Obtain the full range correlator $\tilde{h}^{R\prime}$ by combining data and asymptotic form.

Perform the Fourier transformation on $\tilde{h}^{R\prime}$.

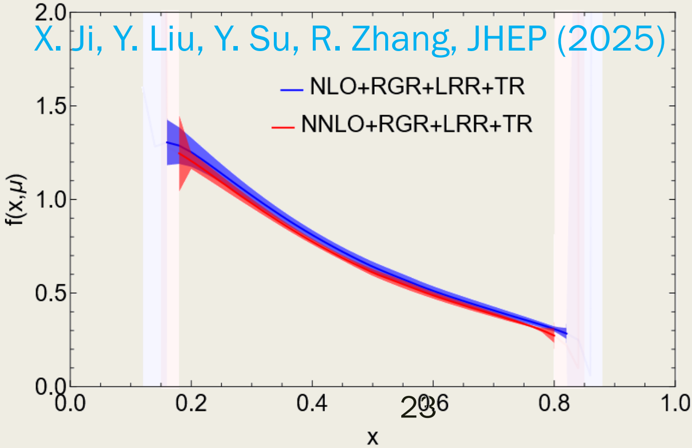
The errors are bounded by the exponential decay factor.



- Perturbative matching

$$f(x,\mu) = \int \frac{dy}{|y|} C\left(\frac{x}{y},\frac{\mu}{|y|P^z}\right)\tilde{f}(y,P^z) + O\left[\frac{\Lambda^2_{\rm QCD}}{x^2P_z^2},\frac{\Lambda^2_{\rm QCD}}{(1-x)^2P_z^2}\right]$$

where the precision in the moderate$-x$ range is under controlled.

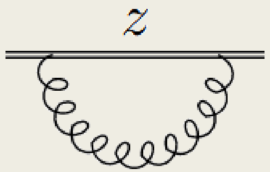


Data generation and hybrid renormalization

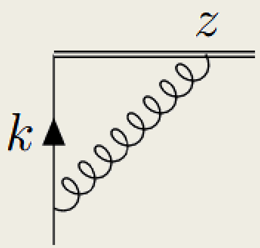
- Generating the non-local quark bilinear correlator in a large momentum hadron on lattice QCD

$$\tilde{h}(z, P_z, a) = \langle P_z | \bar{\psi}(z) U(z, 0) \gamma^z \psi(0) | P_z \rangle$$

- The UV divergences are regularized by the finite lattice spacing a



$$\sim \mathrm{Exp}\left[-\frac{m_{-1}(a)}{a} z\right]$$



$$\sim \ln[a]$$

- Hybrid renormalization ($z < z_s$ ratio scheme, $z > z_s$ $\overline{\mathrm{MS}}$ scheme)

$$\tilde{h}^R(z, P_z) = \frac{\tilde{h}(z, P_z, a)}{\tilde{h}(z, 0, a)} \theta(z_s - z) + \frac{\tilde{h}(z, P_z, a)}{Z_R(z, a, \mu)} \frac{Z_R(z_s, a, \mu)}{\tilde{h}(z_s, 0, a)} \theta(z - z_s)$$

which does not introduce uncontrolled non-perturbative effects at large distance

Pseudo-PDF method

- Lattice data generation and renormalization

$$\tilde{h}(z, P_z, a) = \langle P_z | \bar{\psi}(z) U(z,0) \gamma^z \psi(0) | P_z \rangle$$

$$\tilde{h}^R(z, P_z) = \frac{\tilde{h}(z, P_z, a)}{\tilde{h}(z, 0, a)}$$

- Perturbative matching A. Radyushkin, PRD(2017)

$$\tilde{h}^R(z, P_z) = \int_0^1 d\alpha \mathcal{C}(\alpha, z^2\mu^2) h_{\mathrm{LC}}(\alpha\lambda, \mu) + \mathcal{O}(z^2 \Lambda_{\mathrm{QCD}}^2)$$

- Fourier transformation

$$f(x, \mu) \equiv \int \frac{d\lambda}{2\pi} e^{-ix\lambda} h_{\mathrm{LC}}(\lambda, \mu)$$

  - Idealized situation: data hit asymptotic region.

    $h_{\mathrm{LC}}(\lambda \to \infty, \mu) \sim \frac{1}{\lambda^\alpha}$ is a polynomial decay much slower than the exponential decay in LaMET case.

    The errors from asymptotic analysis should be much larger than LaMET.
  - Practical calculation: $P_z \sim 2\mathrm{GeV}$, $z \lesssim 0.3\mathrm{fm}$, $\lambda \lesssim 3$, far less than the $\lambda \to \infty$ asymptotic region.

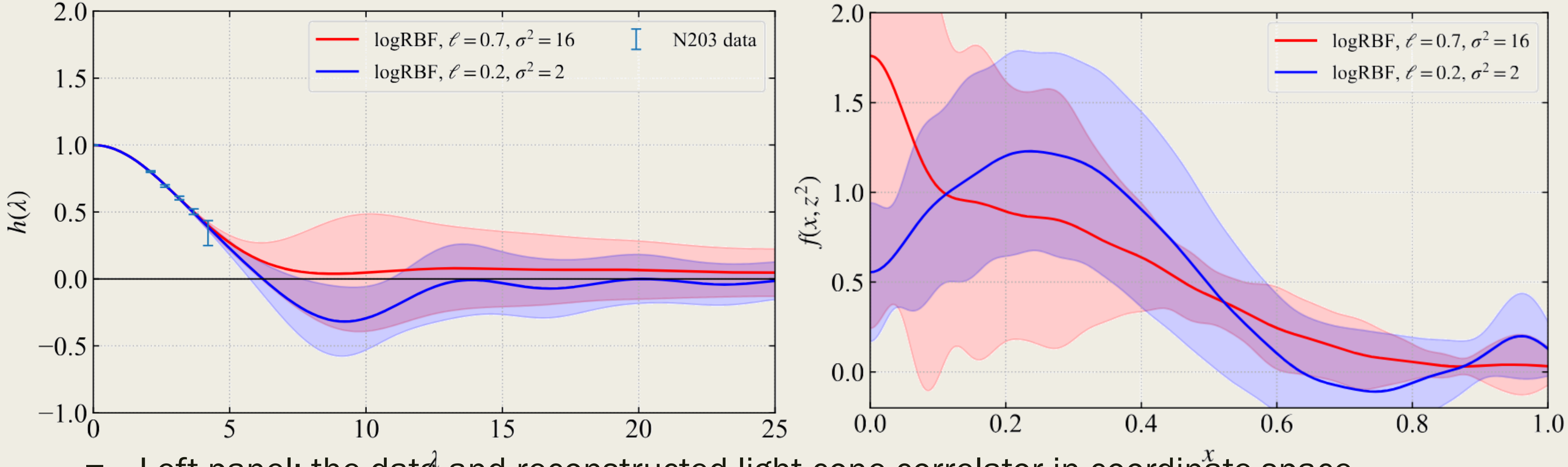
    One faces an inverse problem to obtain the momentum space PDF.

An example of inverse problem in Pseudo-PDF

LPC, PRL (2023)

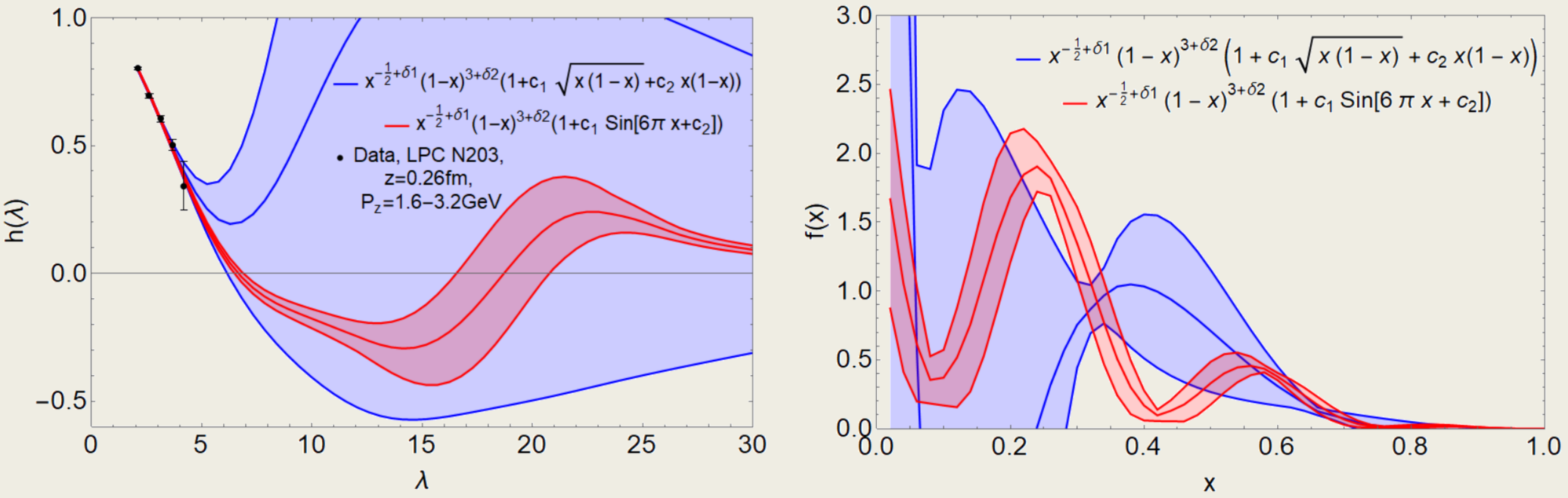
- Reconstructing PDFs from short-distance correlators of nucleon transversity PDF evaluated at $z = 0.26$ fm with proton momenta $P_z = 1.6, 2.0, 2.4, 2.8, 3.2$ GeV based on logRBF kernels with different hyperparameters

JLab, PRD 111 (2025) 3, 034515

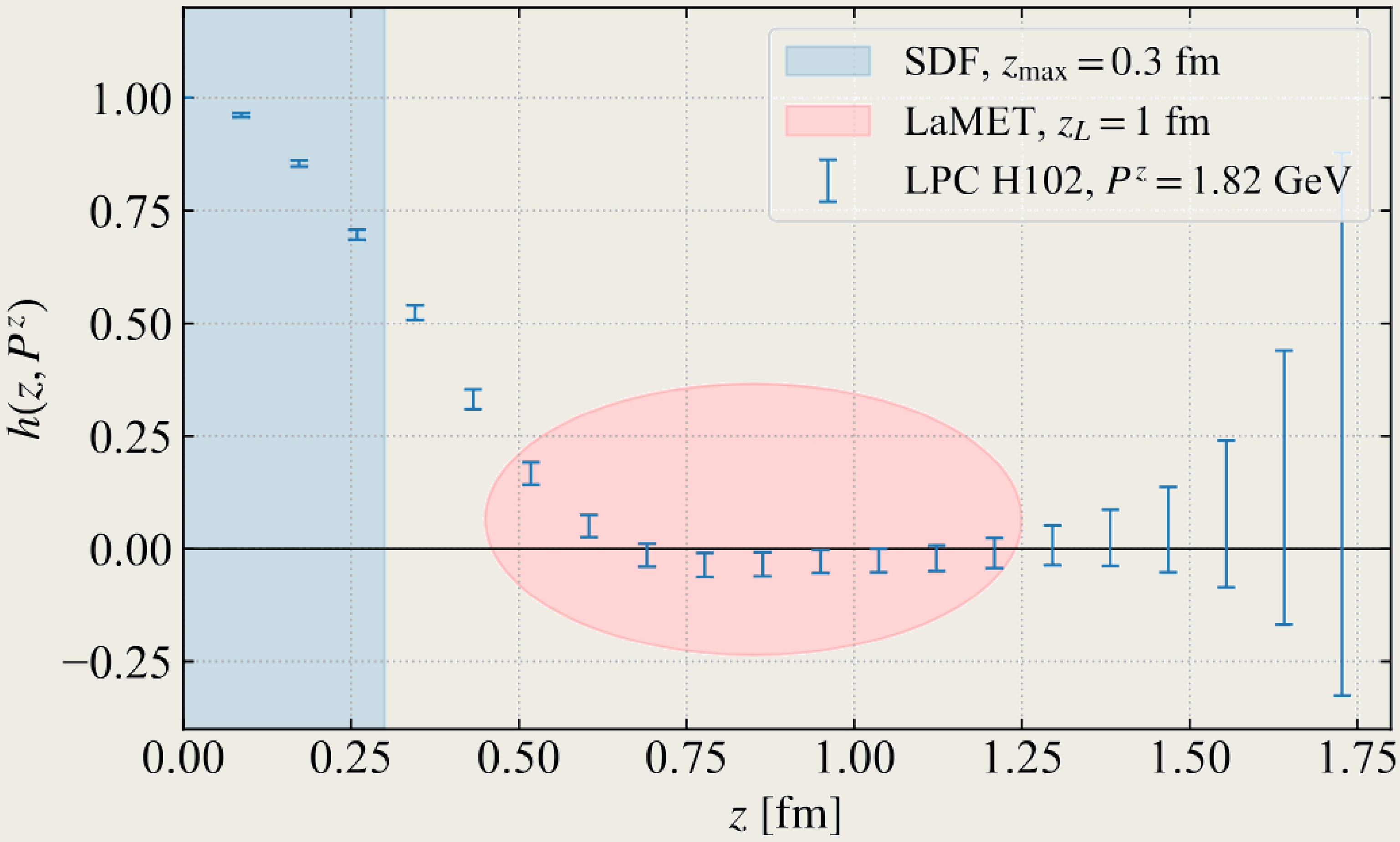


- Left panel: the data and reconstructed light-cone correlator in coordinate space.
- Right panel: the reconstructed light-cone PDF in momentum space
- Large uncertainties in the PDF reconstruction

An example of inverse problem in Pseudo-PDF

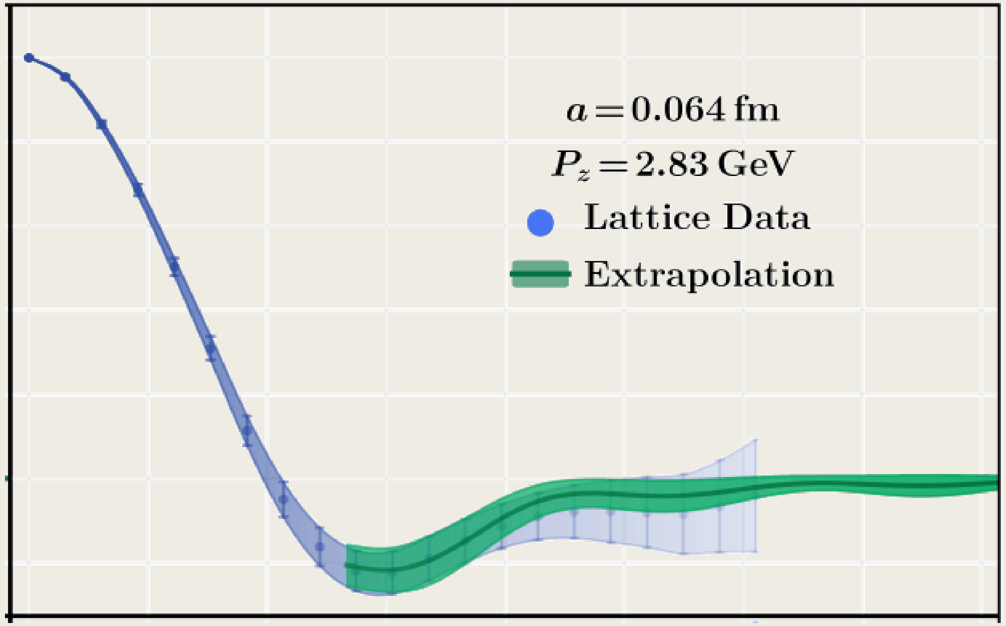


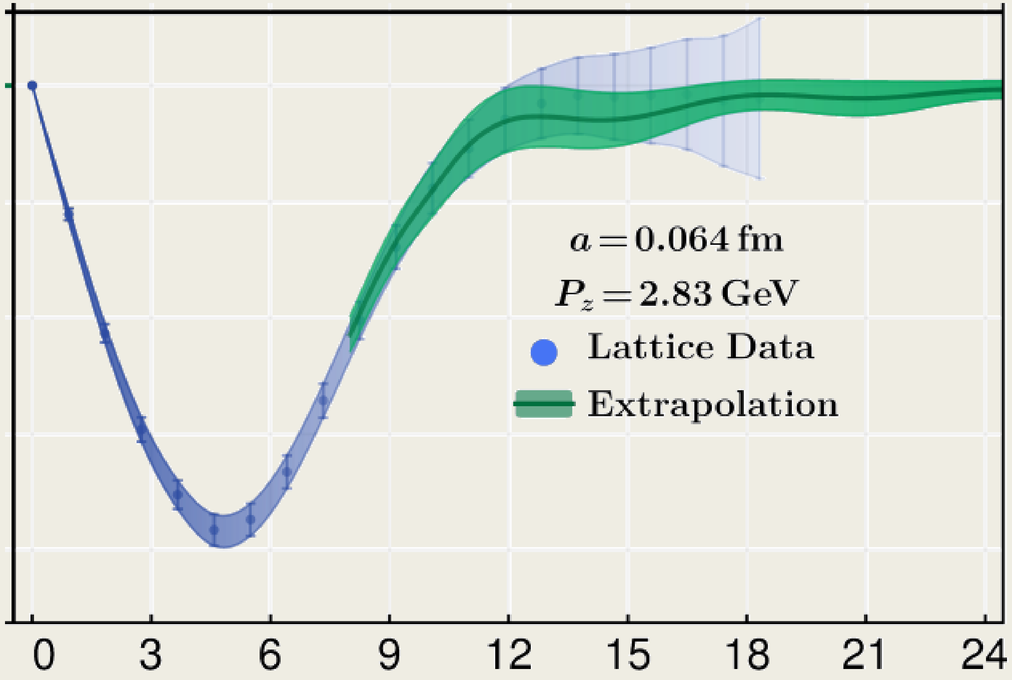
LPC transversity data not in Coulomb gauge



LPC, PRL (2023)

Extrapolation for LPC transversity data not in Coulomb gauge





LPC, PRL (2023)

Physical origin of the effective mass in the asymptotic decay

- The spectral decomposition of quasi-PDF correlation at large distance

$$\tilde{h}(z \to \infty, P_z) \sim e^{-z\, m_G}$$

- The physical origin of m_G:

$$m_G(a) = \delta m(a, \tau) + E_{\mathrm{bin}}(\tau)$$

  $\delta m(a, \tau) \sim \frac{1}{a}$ is the linear divergence of heavy quark self energy

  $E_{\mathrm{bin}}(\tau)$ is the remaining part of the energy

- Renormalon ambiguity

  $\delta m(a, \tau)$ contains the renormalon ambiguity at $O(\Lambda_{\mathrm{QCD}})$ and is dependent on the renormalon regularization scheme τ. Since $m_G(a)$ is not ambiguous, $E_{\mathrm{bin}}(\tau)$ should be τ dependent, cancelling the scheme τ dependence of $\delta m(a, \tau)$.

- Mass renormalization

  After renormalization of linear divergence, the exponential decay is related to $E_{\mathrm{bin}}(\tau)$. One can always choose a scheme τ so that $E_{\mathrm{bin}}(\tau)$ is positive and at $O(\Lambda_{\mathrm{QCD}})$. And error of asymptotic analysis is under control.

Controlled precision of asymptotic analysis

- Recall the asymptotic analysis procedure:

  Fit the asymptotic form in the exponential decay region;

  Combine data and asymptotic form to obtain the full range correlator $\tilde{h}^{H\prime}(z, P_z)$;

  Perform the Fourier transformation to get the momentum space distribution

- The error from asymptotic fitting could lead to the error of FT, which is constrained by the exponential decay:

  ANL/BNL, PRL (2022)

  - Suppose there are two extrapolations $\tilde{h}_1(\lambda)$ and $\tilde{h}_2(\lambda)$, their difference is $\delta\tilde{h}(\lambda) = \tilde{h}_1(\lambda) - \tilde{h}_2(\lambda)$, which should satisfy $\delta\tilde{h}(\lambda_L) = \delta\tilde{h}(\infty) = 0$. ($\lambda_L$ is the starting point to use exponential decay)
  - The error of FT is $\delta\tilde{f}(x, P^z) = \int_{\lambda_L}^{+\infty} \frac{d\lambda}{\pi} \cos(x\,\lambda)\,\delta\tilde{h}(\lambda)$
  - Since the integrand decays exponentially, it becomes negligible in a few (denoted as N_x) periods. Therefore, the error is approximated as $\delta\tilde{f}(x, P^z) \approx \int_{\lambda_L}^{\lambda_L + N_x 2\pi/x} \frac{d\lambda}{\pi} \cos(x\,\lambda)\,\delta\tilde{h}(\lambda)$
  - The error bound can by derived

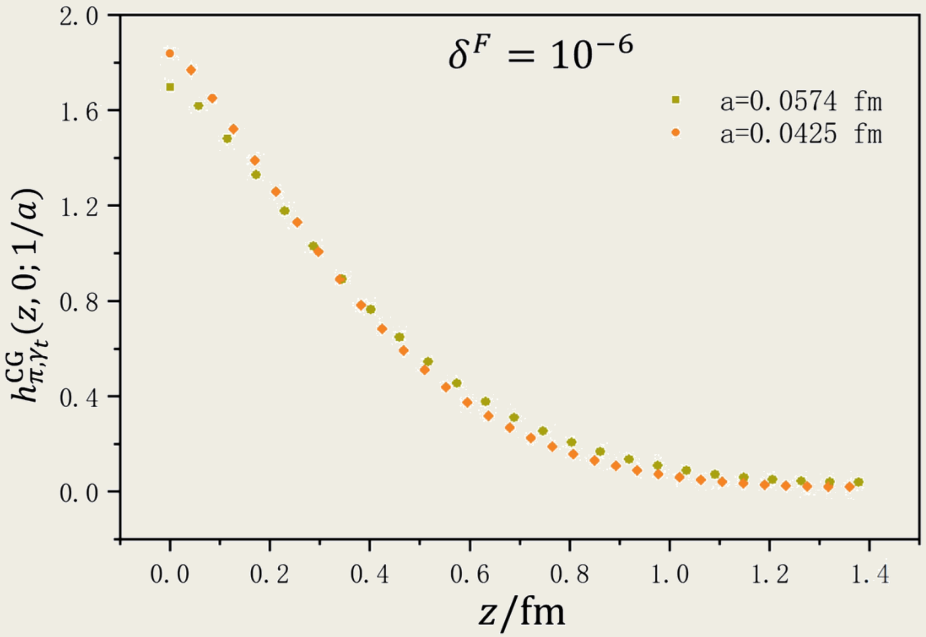
$$\delta\tilde{f}(x, P^z) \le \int_{\lambda_L}^{\lambda_L + \frac{N_x 2\pi}{x}} \frac{d\lambda}{\pi} |\cos(x\,\lambda)| \left|\delta\tilde{h}(\lambda)\right|_{\max}$$

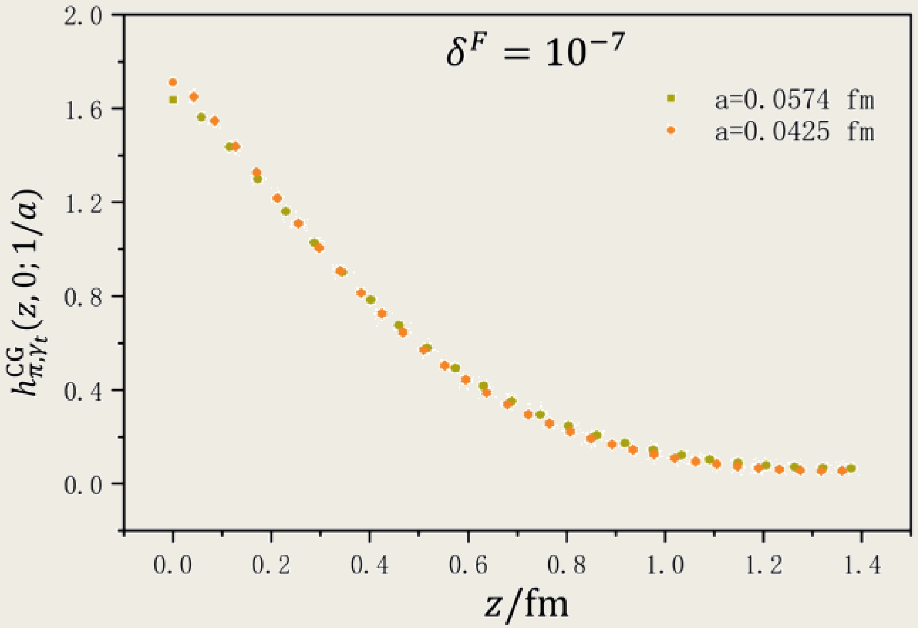
$$\le |h(z, P; \lambda_L)|_{\max} 4N_x \int_{\lambda_L}^{\lambda_L + \frac{\pi}{2x}} \frac{d\lambda}{\pi} |\cos(x\,\lambda)| = \frac{4N_x |h(z,P;\lambda_L)|_{\max}}{\pi\, x},$$

where $|h(z, P; \lambda_L)|_{\max}$ is the maximum value of $\tilde{h}^{H\prime}(z, P_z)$ for $\lambda_L < \lambda < \infty$.

Coulomb gauge PDF

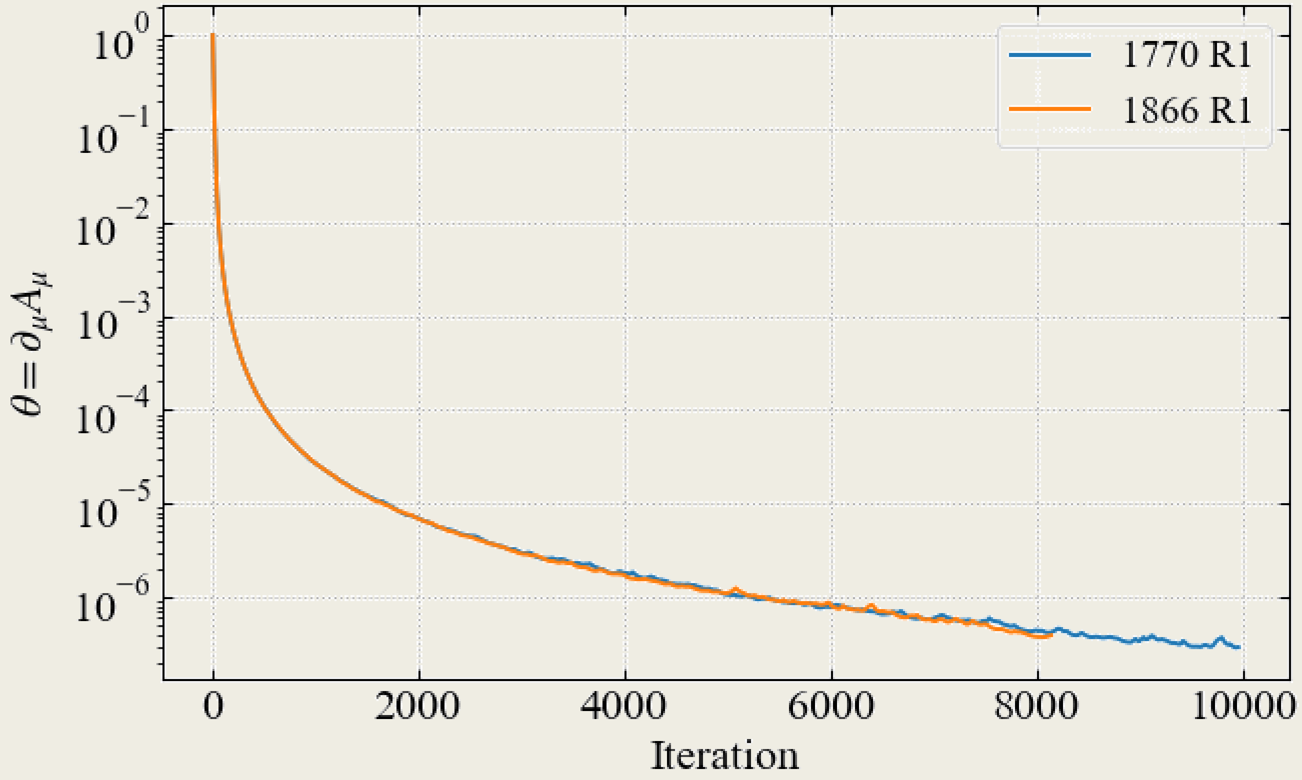
- In LPC, PRD.110.074505, the Coulomb gauge quasi-PDF matrix elements become consistent when the gauge fixing precision is $\delta^F = 10^{-7}$





- In D. Bollweg, X. Gao, J. He, S. Mukherjee, Y. Zhao, 2504.04625, they use $\delta^F = 10^{-8}$.

Coulomb gauge PDF



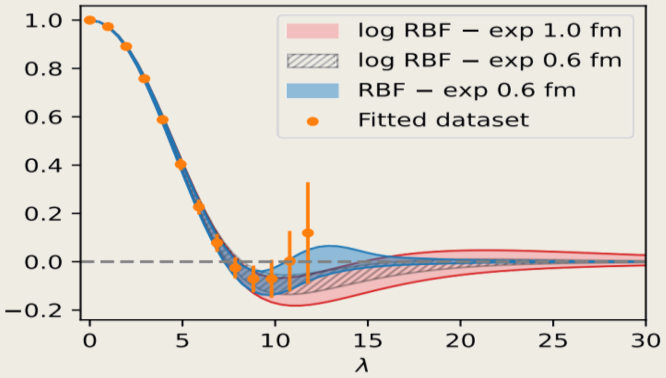
Improper renormalization method in 2504.17706

- The ratio scheme at large distance introduces uncontrolled non-perturbative effects that cannot be taken into account by modifying the perturbative matching kernel.

- As shown in the theoretical derivation, the leading exponential decay factor is irrelevant to the hadron momentum. Therefore, the ratio scheme cancels this exponential decay, which enlarges the error of asymptotic analysis.

Responses to 2504.17706

- Major argument in 2504.17706: the current lattice data are not precise enough for asymptotic analysis and one has to study an inverse problem.



- Our responses:
  - Poor data quality and improper analysis methods (including ratio-scheme renormalization and data-driven-only IP methods) leading to the conclusions in 2504.17706.
  - The existence of good-quality data for asymptotic analysis.
  - Potential to improve the data quality with the novel techniques, such as the kinematically-enhanced interpolation operator.
  - **Even if the data quality is not ideal, the asymptotic analysis is a better option to provide reliable error estimate rather than the Data-driven-only methods methods with little physical constraints used in 2504.17706.**

Data-driven-only methods could violate physical constraints

- Two fundamental issues:
  - Multi-solutions, hard to tell which one is physical
  - Potential violation of physical constraints in the asymptotic region

- e.g. the issues in 2504.17706
  - The Gaussian decay model violates the laws of physics, as the nucleon correlation functions decay exponentially at long range;
  - The GPR exp methods infer posterior distributions based on all data from $z = 0$ to 1.13~fm, despite the fact that exponential decay does not apply at small or moderate z;
  - The choice of RBF kernels and hyperparameters remains *ad hoc* and introduces biases in the data analysis procedure.

Asymptotic analysis provides reasonable error estimate regardless of data quality

- Recall the error bound derived in ANL/BNL, PRL (2022)

$$\delta\tilde{f}(x, P^z) < \frac{4N_x|h(z,P;\lambda_L)|_{\max}}{\pi x}$$

  where $|h(z,P;\lambda_L)|_{\max}$ is the maximum value of $\tilde{h}^{H\prime}(z,P_z)$ for $\lambda_L < \lambda$

  N_x is an integer at which the contribution from $\lambda > \lambda_L + N_x 2\pi/x$ is negligible

- Estimate the FT error with poor-quality data in 2504.17706

  set $|h(z,P;\lambda_L)|_{\max} = 0.1$, reflecting the amplitude in the range $0.75\text{fm} < z < 1.03\text{fm}$;

  At $x = 0.5$, taking $N_x = 1$ is sufficient to saturate the Fourier integral;

  Therefore, $\delta\tilde{f}(x = 0.5, P^z) < 0.25$ is a reasonable upper bound.

- Fig.~3 in 2504.17706 is consistent with the error bound

  Smaller x contains larger error

  At $x = 0.5$, the spread among central values is around 0.2, implying an uncertainty of about 0.1, which is still bounded by the estimate