Zero mode issue in the minus-minus component calculation of the transition form factors in the light-front dynamics

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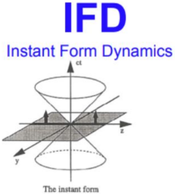
The instant form and light front form: Dirac's proposition

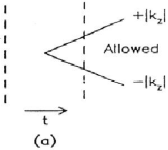
— P.A.M. Dirac, Rev. Mod. Phys. 21, 392 (1949)

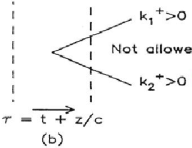


Energy-Momentum Dispersion Relations

$$p^0 = \sqrt{\vec{p}^{\,2} + m^2} \qquad\qquad p^- = \frac{p_\perp^{\,2} + m^2}{p^+}$$

Time-ordered Diagrams: Vacuum fluctuations





Except zero-modes

$$k_1^+ = k_2^+ = 0$$

The interpolation method between IFD and LFD

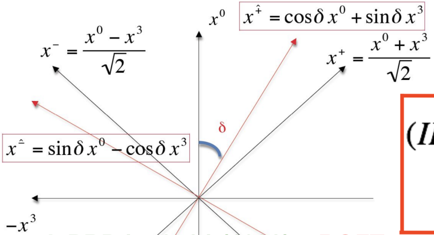
Define interpolating space-time coordinates

$$\begin{pmatrix} x^{\hat{+}} \\ x^{\hat{1}} \\ x^{\hat{2}} \\ x^{\hat{-}} \end{pmatrix} = \begin{pmatrix} \cos\delta & 0 & 0 & \sin\delta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sin\delta & 0 & 0 & -\cos\delta \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

with the short-hand notation $\mathbb{S} = \sin 2\delta$ and $\mathbb{C} = \cos 2\delta$, we have the metric written as

$$g^{\hat{\mu}\hat{\nu}} = g_{\hat{\mu}\hat{\nu}} = \begin{pmatrix} \mathbb{C} & 0 & 0 & \mathbb{S} \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ \mathbb{S} & 0 & 0 & -\mathbb{C} \end{pmatrix}$$

Works on the interpolation method between IFD and LFD



$$\boxed{\begin{array}{c} (IFD)\quad 0 \le \delta \le \dfrac{\pi}{4} \quad (LFD) \\ 1 \ge \quad C \equiv \cos(2\delta) \quad \ge 0 \end{array}}$$

K. Hornbostel, PRD45, 3781 (1992) – RQFT
C.Ji and S.Rey, PRD53,5815(1996) – Chiral Anomaly
C.Ji and C. Mitchell, PRD64,085013 (2001) – Poincare Algebra
C.Ji and A. Suzuki, PRD87,065015 (2013) – Scattering Amps
C.Ji, Z. Li and A. Suzuki, PRD91, 065020 (2015) – EM Gauges
Z.Li, M. An and C.Ji, PRD92, 105014 (2015) – Spinors
C.Ji, Z.Li, B.Ma and A.Suzuki, PRD98, 036017(2018) – QED
B.Ma and C.Ji, PRD104, 036004(2021) – QCD_{1+1}

Meson-photon transition form factors

Simple scalar model in 1+1 space-time dimension

Feynman rules for the scalar (covariant Bethe-Salpeter) model

- quark (scalar) propagator: $\frac{1}{k^2-m^2+i\epsilon}$
- photon $(p-k)$ - quark (p) - antiquark (k) vertex: derivative coupling $(p+k)^\mu$
- seagull term: $\sim\ g^{\mu\nu}$
- meson - quark - antiquark vertex: for now taken as 1, but can be upgraded

Meson $\to \gamma^*\gamma^*$ transition form factor in 1+1-d scalar model

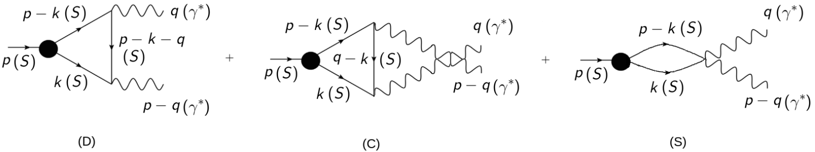


Figure: One-loop covariant Feynman diagrams that contribute to the $S \to \gamma^*\gamma^*$ transition form factor

The total amplitude consists of these three Feynman diagrams, i.e., the direct (D), crossed (C), and the seagull (S) diagrams, where p is the momentum of the incident meson, while q is the momentum of the emitted photon. As a result of momentum conservation, $q' = p - q$ is the momentum of the final photon.

Gauge invariant form factor

From gauge invariance, we can know that the total amplitude $\Gamma^{\mu\nu}$ is of the form

$$\Gamma^{\mu\nu} = F(q^2, q'^2) \left(g^{\mu\nu} q \cdot q' - q'^\mu q^\nu \right) , \tag{1}$$

which satisfies both

$$q_\mu \left(g^{\mu\nu} q \cdot q' - q'^\mu q^\nu \right) = 0 \tag{2}$$

and

$$q'_\nu \left(g^{\mu\nu} q \cdot q' - q'^\mu q^\nu \right) = 0, \tag{3}$$

so that the form factor can be obtained by

$$F(q^2, q'^2) = \frac{\Gamma^{\mu\nu}}{g^{\mu\nu} q \cdot q' - q'^\mu q^\nu} . \tag{4}$$

$q^+ \neq 0$ frame

- In this 1+1 dimensional case, the Drell-Yan-West frame ($q^+ = 0$) can not be taken since it results in $q^2 = 2q^+q^- = 0$, so q^2 could not be varied.
- As one must use a $q^+ \neq 0$ frame, one must include both the valence and non-valence light-front time-ordered diagrams for the calculation of the form factors.
- Choosing a $q^+ \neq 0$ frame, one can also directly access not only the space-like ($q^2 < 0$), but also the time-like ($q^2 > 0$) momentum regions, without resorting to analytic continuation. In contrast, in the 3+1 dimensional, Drell-Yan-West frame case, $q^2 = 2q^+q^- - \vec{q}_\perp^2 < 0$ is always in the space-like region.

Light-front time-ordered diagrams

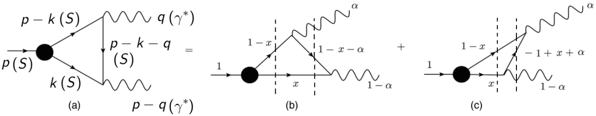


Figure: (Take the direct diagram as an example). The covariant diagram (a) is sum of the two LF x^+-ordered diagrams (b): Valence (V), $0 < x < 1 - \alpha < 1$ and (c): Non-valence (NV), $0 < 1 - \alpha < x < 1$.

- α is the momentum fraction q^+/p^+, and x is the momentum fraction k^+/p^+. Because of 1+1 dimension, α and q^2 has one-to-one correspondence.
- Usually people assume each individual LFTO diagram is only of the gauge invariant form, i.e. $\Gamma_i^{\mu\nu} = f_i(q^2, q'^2)\,(g^{\mu\nu} q \cdot q' - q'^\mu q^\nu)$, and obtain the LFTO contributions to the form factor from the ++ current only: $f_{(b)} = \frac{\Gamma^{++}_{(b)}}{g^{++} q\cdot q' - q'^+ q^+}$, $f_{(c)} = \frac{\Gamma^{++}_{(c)}}{g^{++} q\cdot q' - q'^+ q^+}$.

Democracy of the light-front components

However, we can define the contributions using other components

$$f_i^{(++)} = \frac{\Gamma_i^{++}}{g^{++} q \cdot q' - q'^{+} q^{+}}, \tag{5}$$

$$f_i^{(+-)} = \frac{\Gamma_i^{+-}}{g^{+-} q \cdot q' - q'^{+} q^{-}}, \tag{6}$$

$$f_i^{(-+)} = \frac{\Gamma_i^{-+}}{g^{-+} q \cdot q' - q'^{-} q^{+}}, \tag{7}$$

$$f_i^{(--)} = \frac{\Gamma_i^{--}}{g^{--} q \cdot q' - q'^{-} q^{-}}. \tag{8}$$

where i represents $D(b)$, $D(c)$, $C(b)$, $C(c)$, or S.

Individual LFTO form factors depend on the component

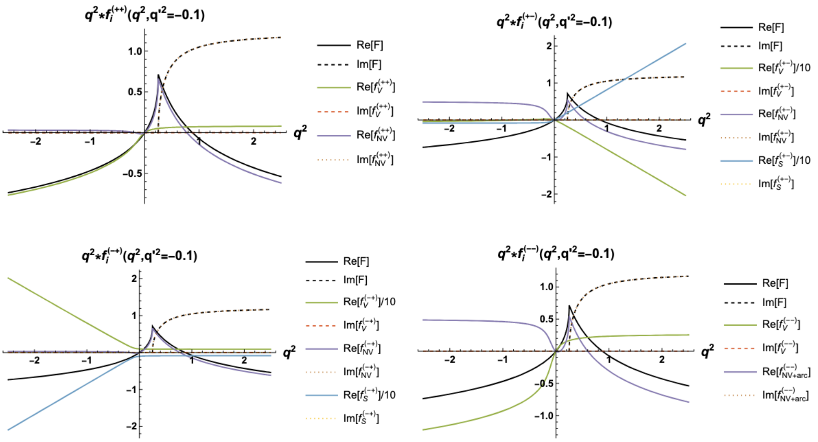


FIG. 14: LFTO diagram contributions to the transition form factor for all 4 components, for the case of $m = 0.25\ GeV$, $M = 0.14\ GeV$, and $q'^2 = -0.1\ GeV^2$, normalized to $F(q^2 = 0, q'^2 = -0.1) = 1$.

Each individual LFTO diagram is not gauge invariant

The most general way, is to write the LFTO amplitudes as 4 form factors

$$\Gamma_i^{\mu\nu} = f_i^A(q^2, q'^2)A^{\mu\nu} + f_i^B(q^2, q'^2)B^{\mu\nu} + f_i^C(q^2, q'^2)C^{\mu\nu} + f_i^D(q^2, q'^2)D^{\mu\nu}.$$

The four forms are found to be

$$\begin{aligned} A^{\mu\nu} &= g^{\mu\nu} q \cdot q' - q'^{\mu} q^{\nu}, \\ B^{\mu\nu} &= q^{\mu} q'^{\nu}, \\ C^{\mu\nu} &= q^{\mu}\left(q^{\nu} - \frac{q \cdot q'}{q'^2} q'^{\nu}\right), \\ D^{\mu\nu} &= \left(q'^{\mu} - \frac{q \cdot q'}{q^2} q^{\mu}\right) q'^{\nu}. \end{aligned}$$

where we select them so that each two out of the four are orthogonal.
Only $A^{\mu\nu}$ is gauge invariant, while $B^{\mu\nu}$, $C^{\mu\nu}$, and $D^{\mu\nu}$ are not.
Thus, they must satisfy

$$\sum_i f_i^B(q^2, q'^2) = \sum_i f_i^C(q^2, q'^2) = \sum_i f_i^D(q^2, q'^2) = 0. \tag{9}$$

Component-independent form factors

We can obtain the individual form factors by

$$\begin{pmatrix} f_i^A \\ f_i^B \\ f_i^C \\ f_i^D \end{pmatrix} = \begin{pmatrix} \frac{A_{++}}{q^2 q'^2} & \frac{A_{+-}}{q^2 q'^2} & \frac{A_{-+}}{q^2 q'^2} & \frac{A_{--}}{q^2 q'^2} \\ \frac{B_{++}}{q^2 q'^2} & \frac{B_{+-}}{q^2 q'^2} & \frac{B_{-+}}{q^2 q'^2} & \frac{B_{--}}{q^2 q'^2} \\ \frac{C_{++}}{q^2\left(q^2 - \frac{(q\cdot q')^2}{q'^2}\right)} & \frac{C_{+-}}{q^2\left(q^2 - \frac{(q\cdot q')^2}{q'^2}\right)} & \frac{C_{-+}}{q^2\left(q^2 - \frac{(q\cdot q')^2}{q'^2}\right)} & \frac{C_{--}}{q^2\left(q^2 - \frac{(q\cdot q')^2}{q'^2}\right)} \\ \frac{D_{++}}{q'^2\left(q'^2 - \frac{(q\cdot q')^2}{q^2}\right)} & \frac{D_{+-}}{q'^2\left(q'^2 - \frac{(q\cdot q')^2}{q^2}\right)} & \frac{D_{-+}}{q'^2\left(q'^2 - \frac{(q\cdot q')^2}{q^2}\right)} & \frac{D_{--}}{q'^2\left(q'^2 - \frac{(q\cdot q')^2}{q^2}\right)} \end{pmatrix} \times \begin{pmatrix} \Gamma_i^{++} \\ \Gamma_i^{+-} \\ \Gamma_i^{-+} \\ \Gamma_i^{--} \end{pmatrix}.$$

Spurious form factors

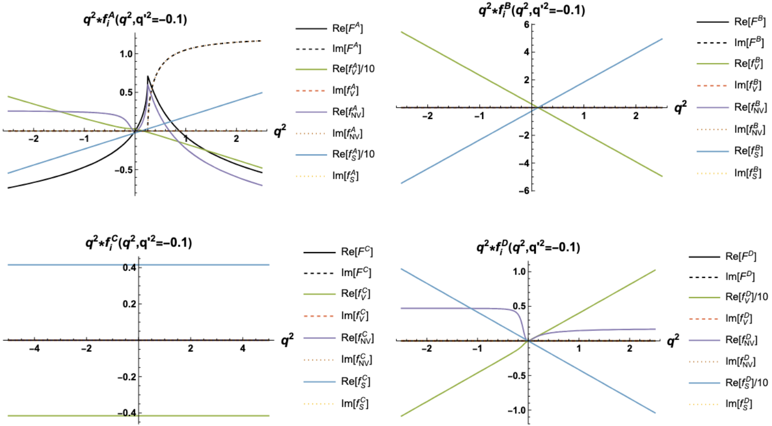


FIG. 15: LFTO diagram contributions to the gauge-invariant and spurious form factors. Note that the real parts of some individual contributions are divided by 10 to fit in the scale.

Minus-minus component of the meson-photon transition amplitude

Meson $\to \gamma^*\gamma^*$ TFF in 1+1-d scalar model: "$--$" component of LFTO calculation

The minus minus component of the transition amplitude is

$$\begin{aligned}\Gamma^{--} &= \Gamma_D^{--} + \Gamma_C^{--} \\ &= ie^2 g_s \int \frac{d^2k}{(2\pi)^2} \frac{(2p-2k-q)^-\,(p-2k-q)^-}{((p-k-q)^2-m^2)\,((p-k)^2-m^2)\,(k^2-m^2)} \\ &+ ie^2 g_s \int \frac{d^2k}{(2\pi)^2} \frac{(q-2k)^-\,(p-2k+q)^-}{((p-k)^2-m^2)\,(k^2-m^2)\,((q-k)^2-m^2)}.\end{aligned} \tag{10}$$

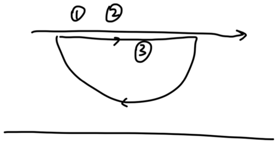
There is no seagull term contribution for the $--$ current. Now plugging in the kinematics, we get (I'm only showing the direct diagram calculation and omitting the similar crossed diagram calculation)

$$\begin{aligned}\Gamma_D^{--} = \frac{ie^2 g_s}{4\pi^2} \int dk^+ \int dk^- &\left(\frac{M^2}{2p^+} - 2k^- + \frac{q'^2}{2(1-\alpha)p^+}\right)\left(-2k^- + \frac{q'^2}{2(1-\alpha)p^+}\right) \\ &\cdot \left(2(p-k-q)^+(p-k-q)^- - m^2\right)^{-1} \left(2(p-k)^+(p-k)^- - m^2\right)^{-1} \\ &\cdot \left(2k^+k^- - m^2\right)^{-1}.\end{aligned} \tag{11}$$

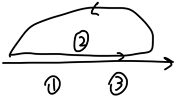
Now let us do the integration $\int dk^-$.
There are 3 poles

$$
\begin{aligned}
k_1^- &= p^- - q^- - \frac{m^2 - i\varepsilon}{2(p-k-q)^+} \\
k_2^- &= p^- - \frac{m^2 - i\varepsilon}{2(p-k)^+} \\
k_3^- &= \frac{m^2 - i\varepsilon}{2k^+}
\end{aligned}
\qquad (12)
$$

Region (b): $0 < x < 1-\alpha < 1$



Region (c): $0 < 1-\alpha < x < 1$



where the k_2^- pole is located at the upper half plane, k_3^- pole is at lower half plane, while k_1^- pole depending on the sign of $1 - x - \alpha$, when $1 - x - \alpha > 0$, it is at upper plane, and we call this region (b). For region (b), we enclose the contour for lower half plane and catch pole 3. When $1 - x - \alpha < 0$, it is at lower plane, and we call this region (c). For region (c), we enclose the contour for upper half plane and catch pole 2.

- However, this results in

$$f_V^{(--)} + f_{NV}^{(--)} \neq F_{cov}$$

- Upon inspection, we realize that for this "minus-minus" component case, there is enough power of k^- on the numerator for the contour integration to have contribution from the arc at infinity
- When $k^- = Re^{i\theta} \to \infty$, the contribution is

$$\Gamma_D^{--} = \frac{ie^2 g_s}{4\pi^2} \lim_{R\to\infty} \int dk^+ \int_0^{\pm\pi} iRe^{i\theta} d\theta \frac{4(Re^{i\theta})^2}{2(p-k-q)^+ 2(p-k)^+ 2k^+ (Re^{i\theta})^3}.$$

- We have to subtract this arc contribution from the residue
- But, this still results in disagreement between the "minus-minus" component calculation and the covariant one.

The interpolation view of the problem

In the interpolation form, the 3 denominators of Eq. (11) can be rewritten as

$$\begin{aligned} D_1 = \mathbb{C}\left(p_{\hat{+}} - k_{\hat{+}} - q_{\hat{+}}\right)^2 + 2\mathbb{S}\left(p_{\hat{+}} - k_{\hat{+}} - q_{\hat{+}}\right)\left(p_{\hat{-}} - k_{\hat{-}} - q_{\hat{-}}\right) \\ - \mathbb{C}\left(p_{\hat{-}} - k_{\hat{-}} - q_{\hat{-}}\right)^2 - m^2 + i\varepsilon, \end{aligned} \tag{13}$$

$$D_2 = \mathbb{C}\left(p_{\hat{+}} - k_{\hat{+}}\right)^2 + 2\mathbb{S}\left(p_{\hat{+}} - k_{\hat{+}}\right)\left(p_{\hat{-}} - k_{\hat{-}}\right) - \mathbb{C}\left(p_{\hat{-}} - k_{\hat{-}}\right)^2 - m^2 + i\varepsilon, \tag{14}$$

and

$$D_3 = \mathbb{C}k_{\hat{+}}^2 + 2\mathbb{S}k_{\hat{+}}k_{\hat{-}} - \mathbb{C}k_{\hat{-}}^2 - m^2 + i\varepsilon. \tag{15}$$

Now there is no arc contribution because there is now 6 powers of the integration variable on the denominator, instead of 3, while still 3 on the numerator.

There are 6 poles in total

$$k_{\hat{+}1u,1d} = p_{\hat{+}} - q_{\hat{+}} + \frac{\mathbb{S}}{\mathbb{C}}(p_{\hat{-}} - k_{\hat{-}} - q_{\hat{-}}) \mp \frac{\omega_1}{\mathbb{C}} \pm i\varepsilon \tag{16}$$

$$k_{\hat{+}2u,2d} = p_{\hat{+}} + \frac{\mathbb{S}}{\mathbb{C}}(p_{\hat{-}} - k_{\hat{-}}) \mp \frac{\omega_2}{\mathbb{C}} \pm i\varepsilon \tag{17}$$

$$k_{\hat{+}3d,3u} = -\frac{\mathbb{S}}{\mathbb{C}}k_{\hat{-}} \pm \frac{\omega_3}{\mathbb{C}} \mp i\varepsilon \tag{18}$$

where

$$\omega_1 = \sqrt{(p_{\hat{-}} - k_{\hat{-}} - q_{\hat{-}})^2 + \mathbb{C}m^2} \tag{19}$$

$$\omega_2 = \sqrt{(p_{\hat{-}} - k_{\hat{-}})^2 + \mathbb{C}m^2} \tag{20}$$

$$\omega_3 = \sqrt{k_{\hat{-}}^2 + \mathbb{C}m^2} \tag{21}$$

In the $\mathbb{C} \to 0$ limit, in each pair of the poles, one of them goes to infinity, the other goes to the light-front poles, Eq. (12). Keeping only the leading order in $\mathbb{C}$, the 6 poles become

$$k^-_{1reg} = p^- - q^- - \frac{m^2 - i\varepsilon}{2(p-k-q)^+} \tag{22}$$

$$k^-_{1inf} = \frac{p^+ - k^+ - q^+}{\mathbb{C}} \mp \frac{|p^+ - k^+ - q^+|}{\mathbb{C}} \pm i\varepsilon \tag{23}$$

$$k^-_{2reg} = p^- - \frac{m^2 - i\varepsilon}{2(p-k)^+} \tag{24}$$

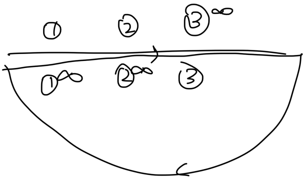
$$k^-_{2inf} = \frac{p^+ - k^+}{\mathbb{C}} \mp \frac{|p^+ - k^+|}{\mathbb{C}} \pm i\varepsilon \tag{25}$$

$$k^-_{3reg} = \frac{m^2 - i\varepsilon}{2k^+} \tag{26}$$

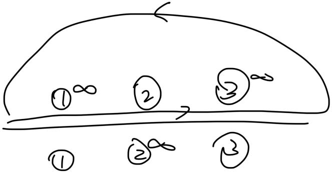
$$k^-_{3inf} = -\frac{k^+}{\mathbb{C}} \pm \frac{|k^+|}{\mathbb{C}} \mp i\varepsilon. \tag{27}$$

Now the pole structure becomes

Region (b) : $0 < x < 1-\alpha < 1$

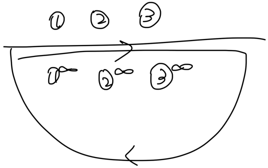


Region (c) : $0 < 1-\alpha < x < 1$

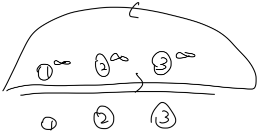


- Now, adding these two time-ordered contributions still does not give agreement to the covariant result.
- The reason is, regions of x outside of $[0, 1]$ give contributions.
- The pole structures in those regions are

region of $x \in (-\infty, 0)$



region of $x \in (1, +\infty)$



- These "poles at infinity" contributions must be included.

Calculating their residues, we obtain for the region of $x < 0$,

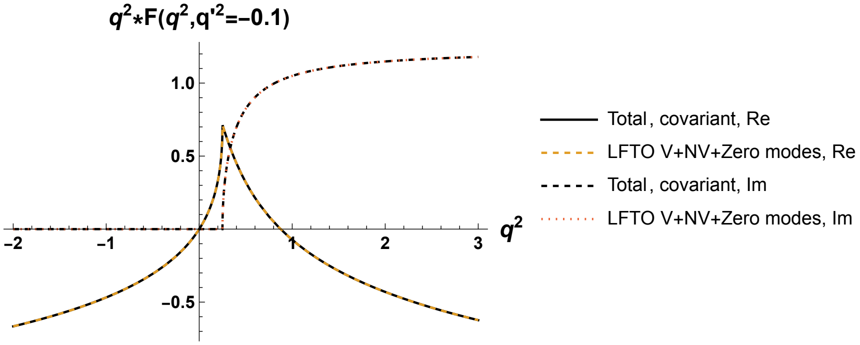
$$\Gamma^{--}_{D(x\in(-\infty,0))} = \frac{e^2 g_s}{4\pi p^+ p^+}\int_{-\infty}^{0} dx \left\{ \frac{1}{(1-x-\alpha)(-\alpha)(1-\alpha)} + \frac{1}{(1-x)\alpha} + \frac{1}{(-x)(1-\alpha)} \right\}. \tag{28}$$

Similarly for the region of $x > 1$, we obtain

$$\Gamma^{--}_{D(x\in(1,+\infty))} = \frac{e^2 g_s}{4\pi p^+ p^+}\int_{1}^{+\infty} dx \left\{ -\frac{1}{(1-x-\alpha)(-\alpha)(1-\alpha)} - \frac{1}{(1-x)\alpha} - \frac{1}{(-x)(1-\alpha)} \right\}. \tag{29}$$

Now, adding all 4 regions, we finally reached agreement between the "minus-minus" component form factor calculation and the covariant one.

Agreement of the LF "$--$" component and the covariant calculations obtained from the interpolation method



Types of zero modes from the view of interpolation method of the light-front dynamics

- Take into account the Non-Valence diagram due to $q^+ \neq 0$ frame.
- arc/"pole at infinity" contributions in the Valence $x \in (0, 1-\alpha)$ and Non-Valence $x \in (1-\alpha, 1)$ regions.
- the unusual momentum fraction regions $x \in (-\infty, 0)$ and $x \in (1, +\infty)$ give contribution because of arc/"pole at infinity".

Summary

- One can compute not only the good current $(++)$, but also the bad currents $(+-, -+, --)$ in the LFD, in order to define the individual LF time-ordered contributions to the form factor.
- In the computation of the "$--$" component of the current, special care is needed to take into account the contributions from regions of x outside of the usual $[0, 1]$.
- The individual LF time-ordered amplitudes depend on the component, thus are not gauge invariant, and 3 spurious form factors arise in this $1 + 1$-d scalar model case.

Questions

- Could spurious form factors arise in the GPD formulation?
- Are the gravitational form factors really component-independent?

Thank you for your attention!