Learning Parton Structure from Lattice QCD

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Parton Structure

For various flavors and spin combinations

Wigner Distribution/
Generalized Transverse Momentum
Distribution (GTMD)

$F(x, b_T, k_T)$

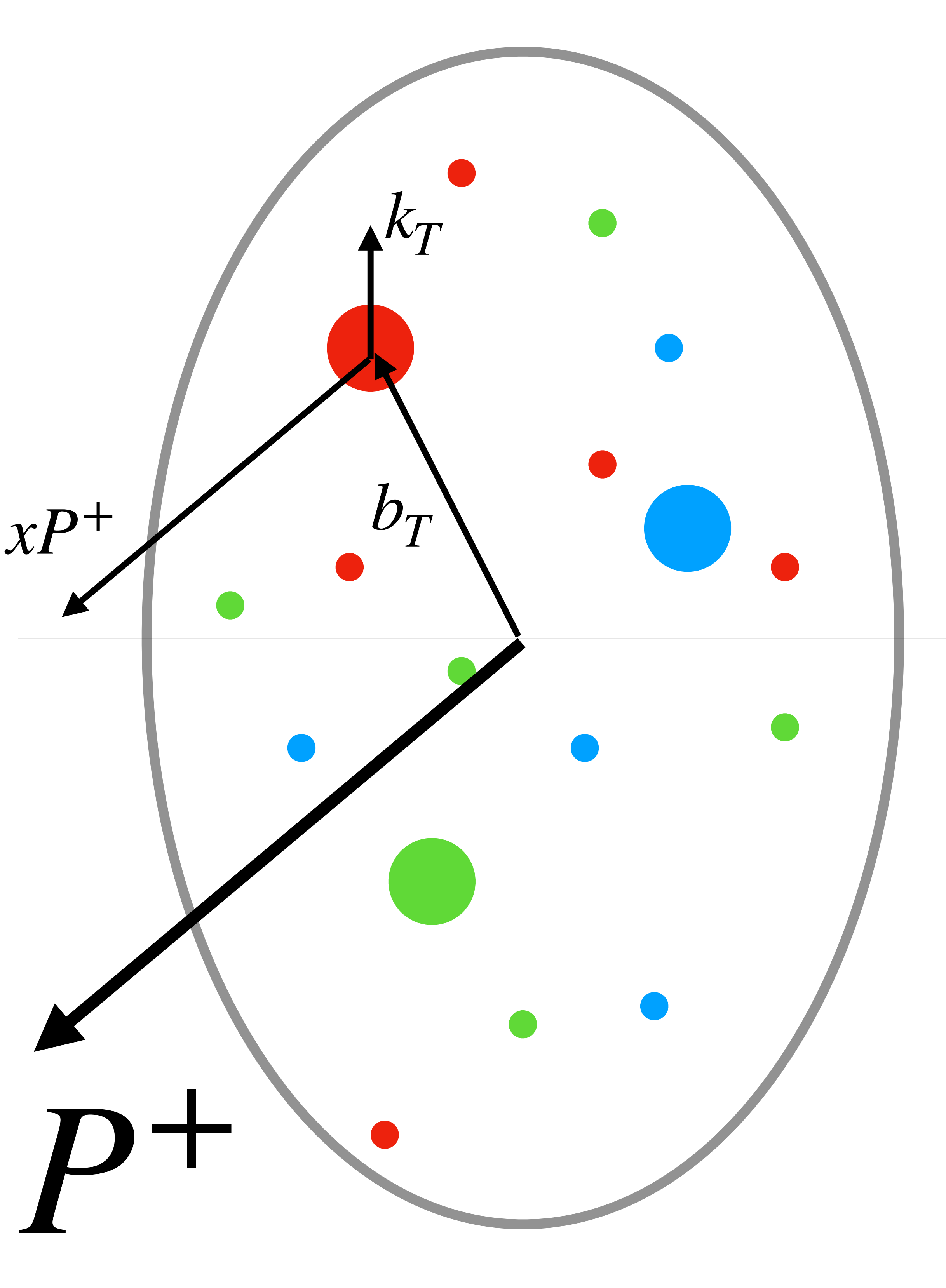
$\int d^2 b_t$ $\int d^2 k_t$

Transverse Momentum
Distribution (TMD)
$f(x, k_T)$

Generalized Parton
Distribution (GPD)
$f(x, b_T)$

$\int d^2 k_t$ $\int d^2 b_t$

Parton Distribution Function (PDF)
$f(x)$



Parton and Ioffe Time distributions

- Unpolarized Ioffe time distributions Ioffe time: $\nu = p \cdot z$

"Ioffe time distributions instead of parton momentum distributions in description of DIS"
V. Braun, P. Gornicki, L. Mankiewicz
Phys Rev D 51 (1995) 6036-6051

- $$I_q(\nu, \mu^2) = \frac{1}{2p^+} \langle p \,|\, \bar{\psi}_q(z^-) \gamma^+ W(z^-; 0) \psi_q(0) \,|\, p \rangle_{\mu^2}$$

$$z^2 = 0$$

$$I_g(\nu, \mu^2) = \frac{1}{(2p^+)^2} \langle p \,|\, F_{+i}(z^-) W(z^-; 0) F^i_+(0) \,|\, p \rangle_{\mu^2}$$

$$i = x, y$$

- Parton Distribution Functions

$$I_q(\nu, \mu^2) = \int_{-1}^{1} dx\, e^{ix\nu} f_q(x, \mu^2) \to f_q(x, \mu^2) = \int \frac{d\nu}{2\pi} e^{ix\nu} I_q(\nu, \mu^2)$$

$$I_g(\nu, \mu^2) = \int_{0}^{1} dx\, \cos(x\nu)\, x f_g(x, \mu^2) \to x f_g(x, \mu^2) = \int \frac{d\nu}{2\pi} \cos(x\nu) I_g(\nu, \mu^2)$$

Parton Distributions and the Lattice

- Parton Distributions are defined by operators with light-like separations

- Use space-like separations

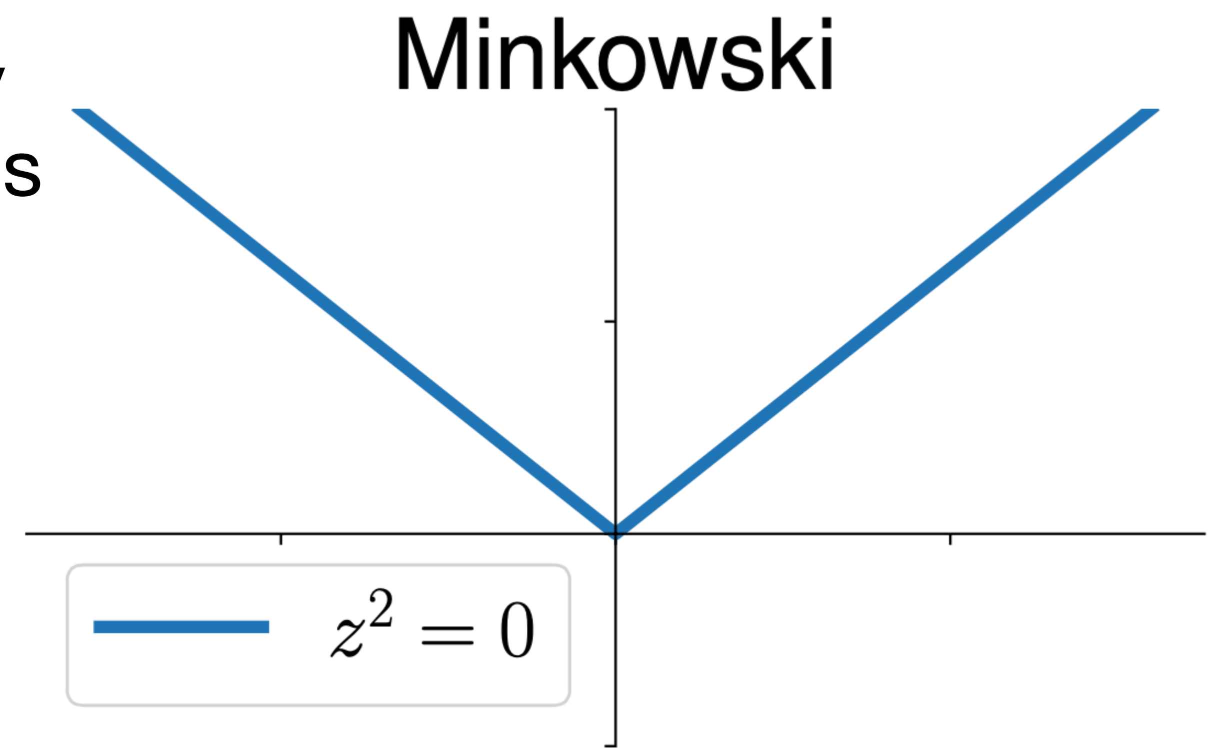
  X. Ji *Phys Rev Lett* 110 (2013) 262002

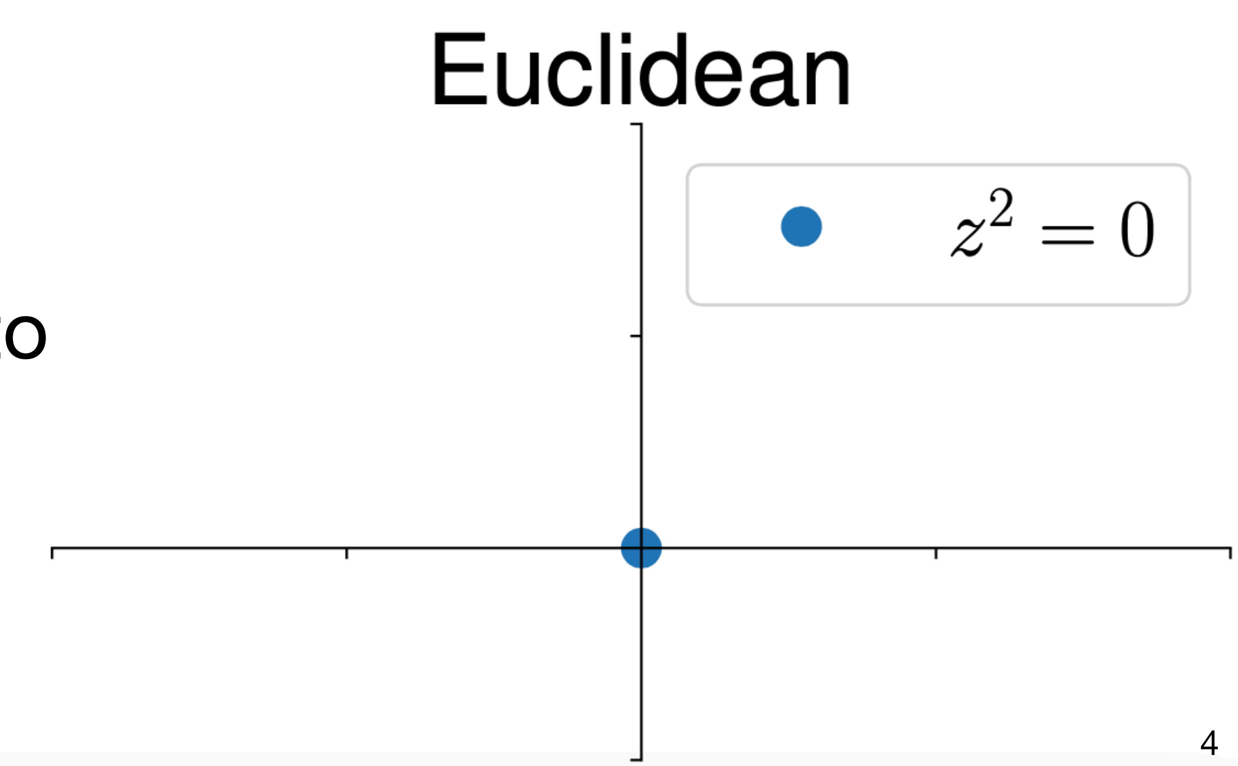
  - Wilson line operators

$$O_\Gamma^{\mathrm{WL}}(z) = \bar{\psi}(z)\Gamma W(z;0)\psi(0)$$

$$z^2 \neq 0$$

  - Factorizations exist analogous to cross sections





Wilson Line Matrix Elements

- More generic element $$M^\alpha(p,z) = \langle p\,|\,\bar{\psi}(z)\gamma^\alpha W(z;0)\psi(0)\,|\,p\rangle$$ $$= 2p^\alpha \mathcal{M}(\nu, z^2) + 2z^\alpha \mathcal{N}(\nu, z^2)$$
- PDF (given collinear divergence handled): $$f_q(x,\mu^2) = \int d\nu e^{ix\nu}\mathcal{M}(\nu,0)$$
- Quasi-PDF: $\tilde{q}(y,p_z^2) = \int d\nu e^{i\nu y}\mathcal{M}(\nu, \frac{\nu^2}{p_z^2})$ $\quad z^2 < 0$
- Large Momentum Effective Theory: X. Ji *Phys. Rev. Lett.* 110 (2013) 262002
- $$\tilde{q}(y,p_z^2) = \int \frac{dx}{|x|} K\left(\frac{y}{x}, \frac{\mu^2}{(xp_z)^2}\right) q(x,\mu^2) + O\left(\frac{\Lambda^2_{\rm QCD}}{(yp_z)^2}, \frac{\Lambda^2_{\rm QCD}}{((1-y)p_z)^2}\right)$$
- Pseudo-PDF: A. Radyushkin *Phys. Rev. D* 96 (2017) 3, 034025

$$\mathcal{M}(\nu, z^2) = \int_{-1}^{1} dx e^{i\nu x} P(x, z^2) = \int_{-1}^{1} dx\, C(x\nu, \mu^2 z^2) q(x,\mu^2) + O(\Lambda^2_{\rm QCD} z^2)$$

$$= \int_{-1}^{1} du C'(u, \mu^2 z^2) I_q(u\nu, \mu^2) + O(\Lambda^2_{\rm QCD} z^2)$$

The Role of Separation and Momentum

- In Structure Functions, quasi-PDF, and pseudo-PDF, variables have common roles

Scale:

$$Q^2 \,/\, p_z^2 \,/\, z^2$$

- Scale for factorization to PDF
- Scale in power expansion
- Keep away from Λ^2_{QCD}
- Technically only requires single value, use many to study systematics

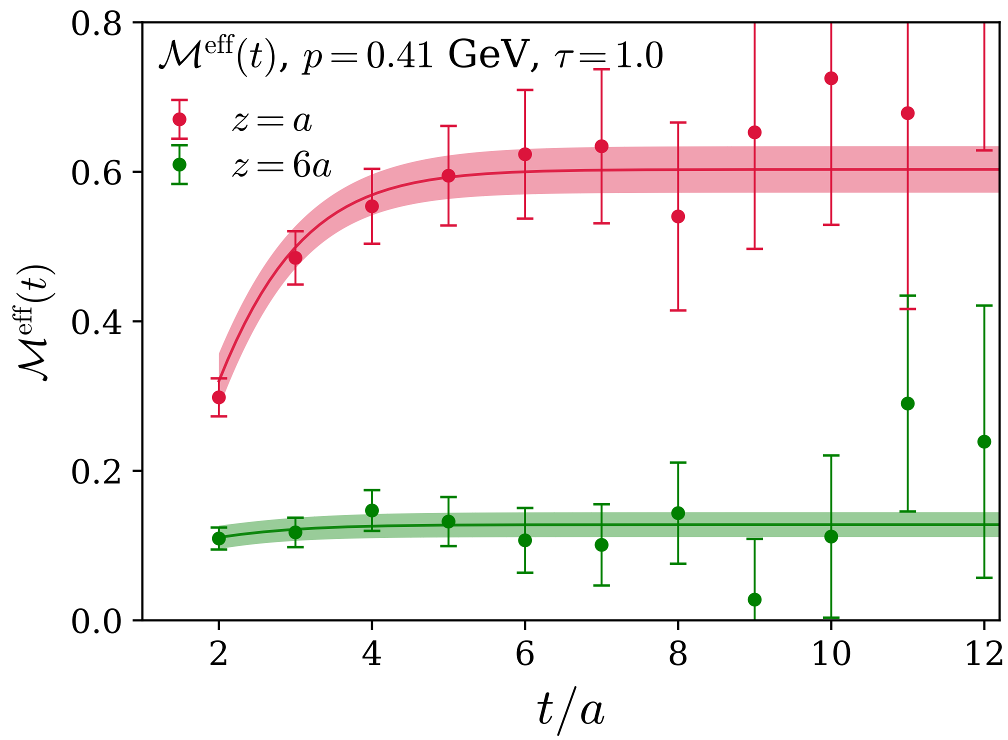
Dynamical variable:

$$x_B \,/\, z \,/\, p_z \text{ , or } \nu = p \cdot z$$

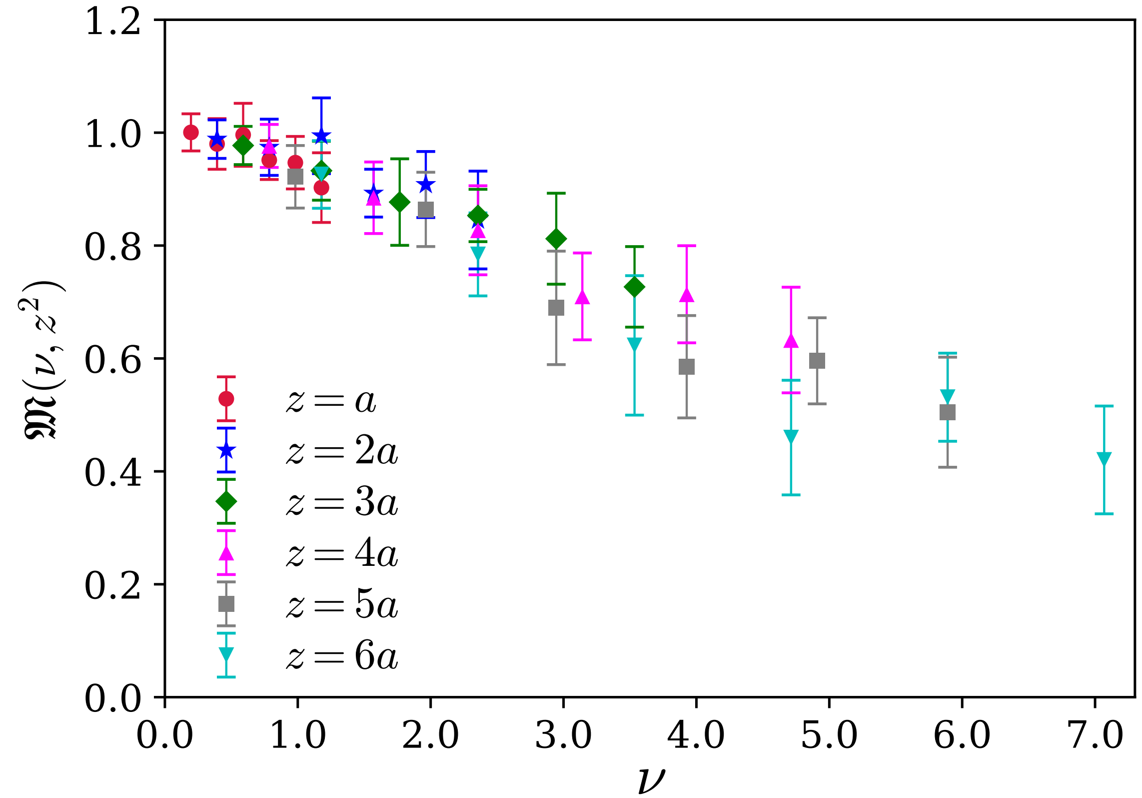
- Variable describes non-perturbative dynamics
- Can take large or small value
- Want as many as are available
- Wider range improves the inverse problem

From Lattice QCD to PDFs

Lattice Correlation Functions



Hadron Matrix Elements



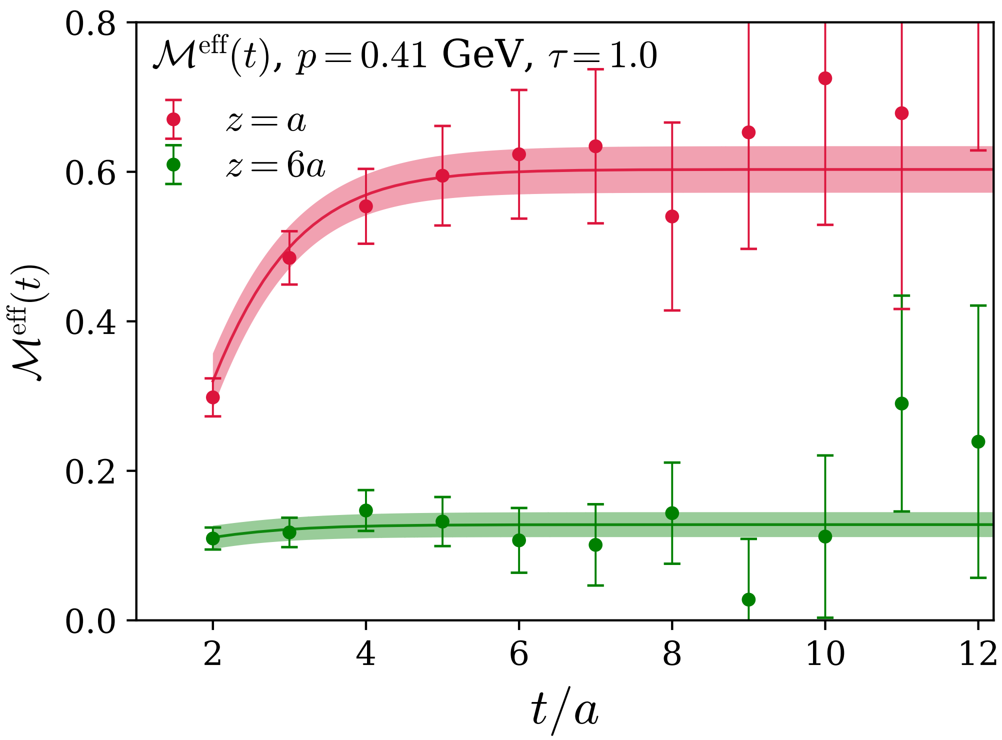
- Correlators (vacuum expectation values of time separated operators) are described as sums over an exponential for each energy eigenstate.
- Coefficients are matrix elements and exponential rates are energy levels
- Model and/or remove subdominant states by using large time but noise grows exponentially

Unpolarized Gluon PDF

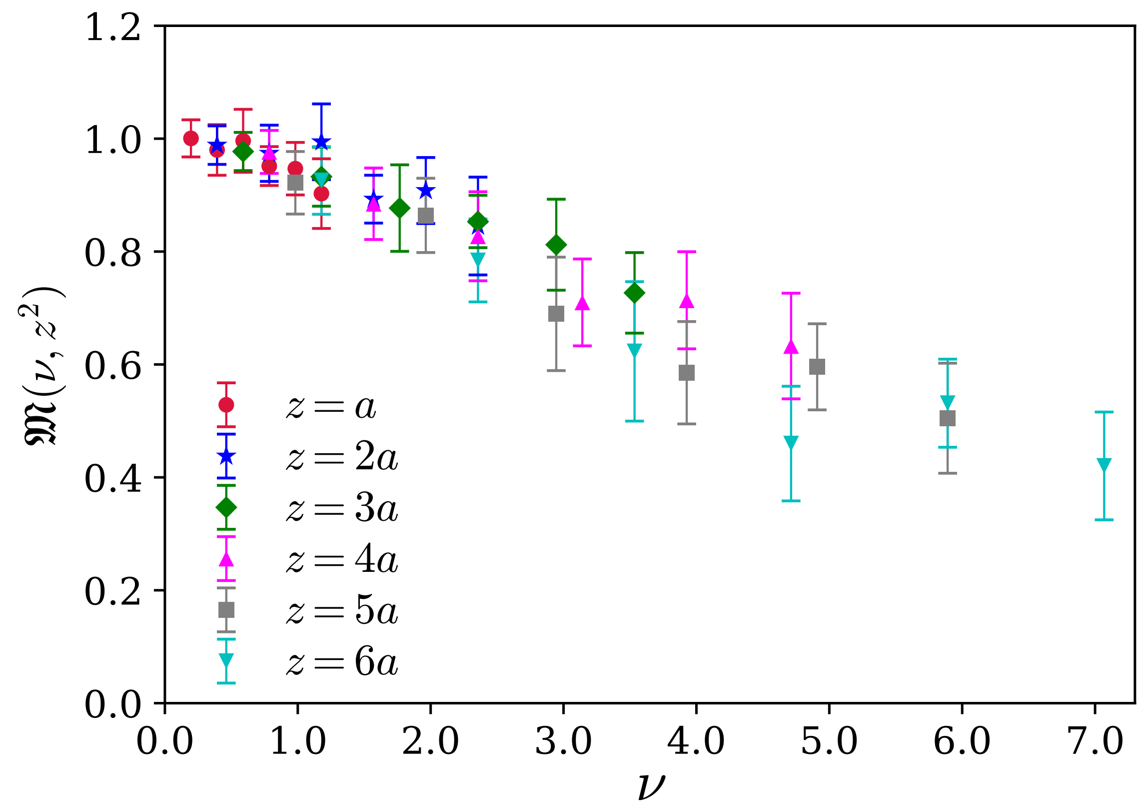
T. Khan, R. Sufian, JK, C. Monahan, C. Egerer, B. Joo, W. Morris, K. Orginos, A. Radyushkin, D. Richards, E. Romero, S. Zafeiropoulos
PRD 104 (2021) 9, 094516

From Lattice QCD to PDFs

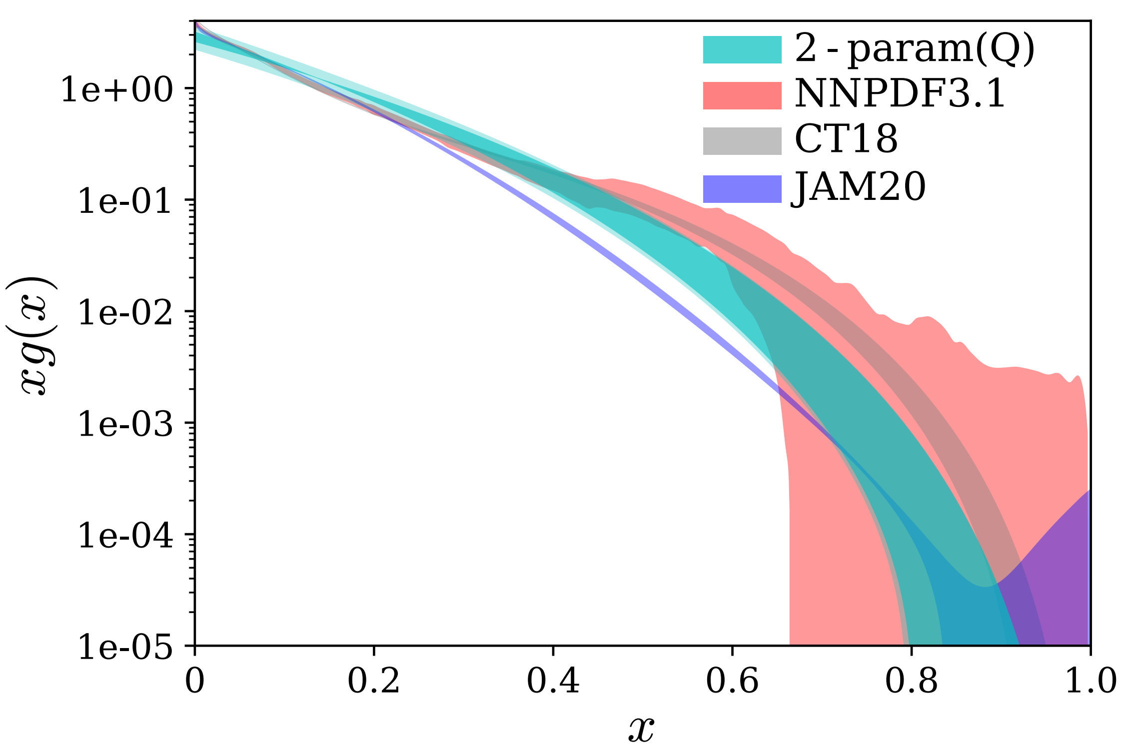
Lattice Correlation Functions



Hadron Matrix Elements



Parton Distributions



- Incomplete information gives integral inverse problem

$$M(\nu) = \int dx C(x\nu) xg(x)$$

$$xg(x) = x^a(1-x)^b / B(a+1, b+1)$$

- To more accurately infer PDF, we need larger ν

Unpolarized Gluon PDF

T. Khan, R. Sufian, JK, C. Monahan, C. Egerer, B. Joo, W. Morris, K. Orginos, A. Radyushkin, D. Richards, E. Romero, S. Zafeiropoulos
PRD 104 (2021) 9, 094516

Lattice QCD: Fake it till you make it

All systematics are improvable, but at what cost?

- Finite lattice spacing $a \sim 0.045 - 0.1$ fm

  Polynomial of a to model

- Finite volumes $L \sim 3 - 5$ fm and $m_\pi L \sim 4 - 6$

  Single hadron: Exponential decay in $m_\pi L$ to model
  Multi-hadron: Polynomial in L^{-1} which Luscher method removes

- Heavy quarks / pions $m_\pi \sim 140 - 600$ MeV

  Chiral PT gives polynomials and logs of m_π to model

- Excited state control $\Delta \sim 140 - 500$ MeV

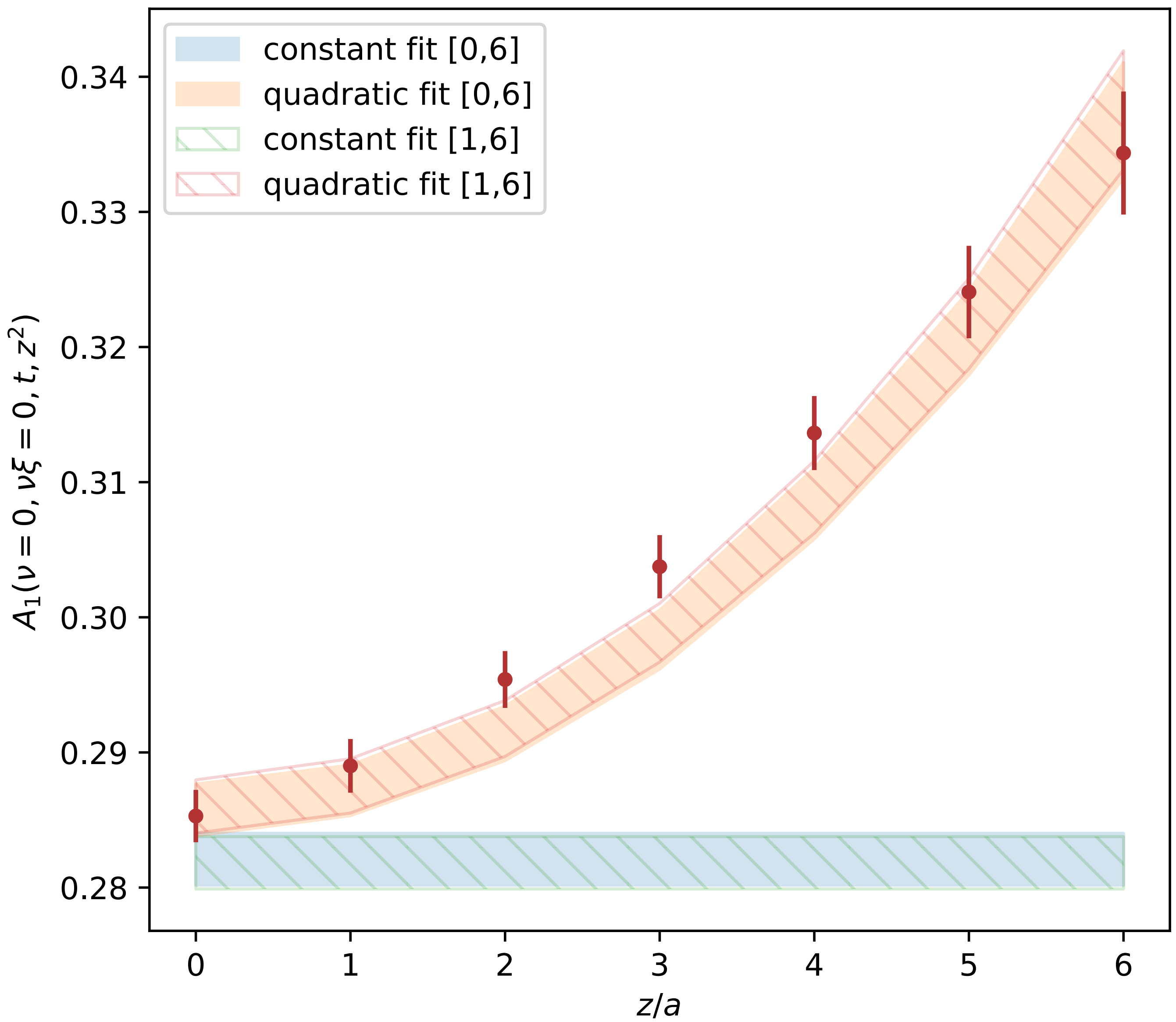
  Use larger T and do better fits
  Variational can separate lowest states

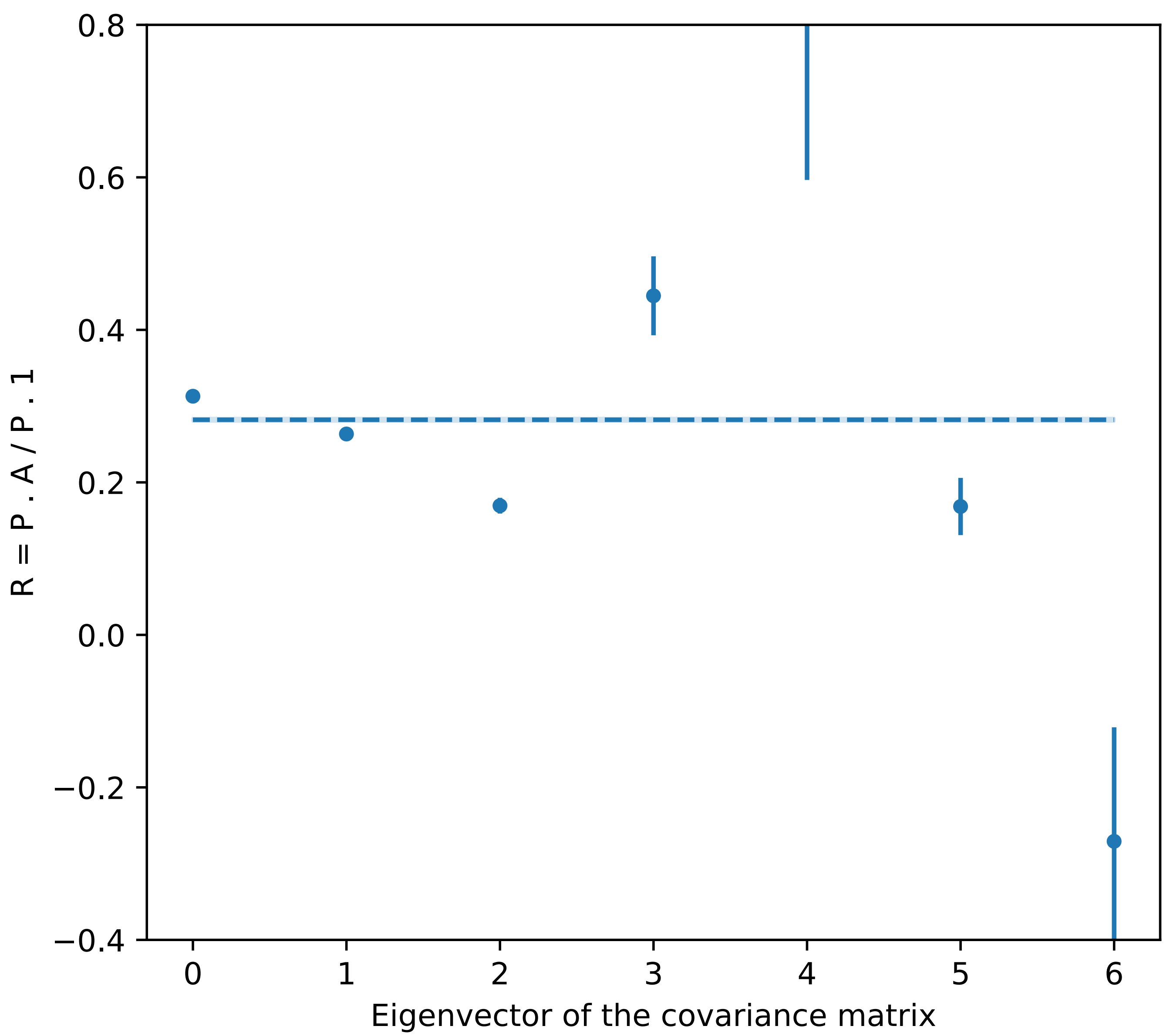
- Statistics **Always there**

  Beg the DOE for bigger computer

There is less than you think

- Lattice data are highly correlated and you have less information than you think
- What is the average of these 7 data points?





H. Dutrieux et al (HadStruc) JHEP 08 (2024) 162

Gaussian Process Priors

ETMC PRD 102 (2020) 9, 094508
A. Candido et al Eur. Phys. J. C 84 (2024) 7, 716
H. Dutrieux et al PRD 111 (2025) 3, 034515

$$P[q\,|\,M,I] = \frac{P[M\,|\,q]\,P[q\,|\,I]}{P[M\,|\,I]}$$

- Given a theory: $M(\nu) = \int dxB(\nu,x)q(x)$
- The data-likelihood is Gaussian $P[M\,|\,q] \propto \exp[-\frac{1}{2}\chi^2]$
- Assume the PDF has a Gaussian prior with given mean $g(x)$ and covariance kernel $K(x,x')$
- $$P[q\,|\,I] \propto \exp\left[-\frac{1}{2}\int dxdx'\big(q(x)-g(x)\big)K^{-1}(x,x')\big(q(x')-g(x')\big)\right]$$
- The posterior is Gaussian

$$P[q\,|\,M,I] \propto \exp\left[-\frac{1}{2}\int dxdx'\big(q(x)-\bar{q}(x)\big)H^{-1}(x,x')\big(q(x')-\bar{q}(x')\big)+\ldots\right]$$

Mean and Covariance

- Posterior is Gaussian

$$P[q \,|\, M, I] \propto \exp\left[-\frac{1}{2} \int dx dx' \big(q(x) - \bar{q}(x)\big) H^{-1}(x, x') \big(q(x') - \bar{q}(x')\big) + \ldots \right]$$

- In practice, integrals are broken into finite elements (pixels) and q and K are discretized to vector and matrix

- Given a sufficient grid, arbitrarily complicated function q can be learned from simple matrix operations

- Observables $\langle O \rangle = \int Dq\, O\big[q(x)\big] P[q \,|\, M, I]$

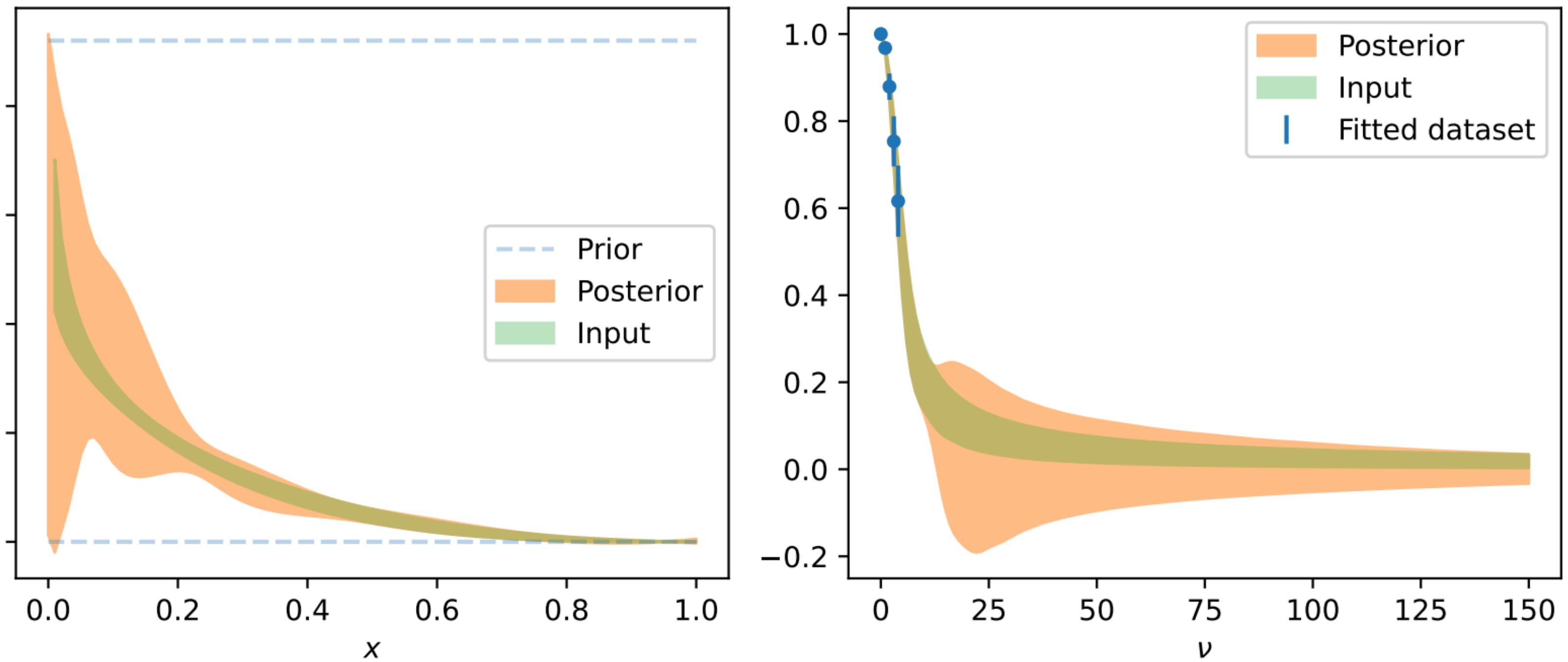
- Mean $\langle q \rangle = \bar{q} = g + KB^T \left[C + BKB^T\right]^{-1} \left[M - Bg\right]$

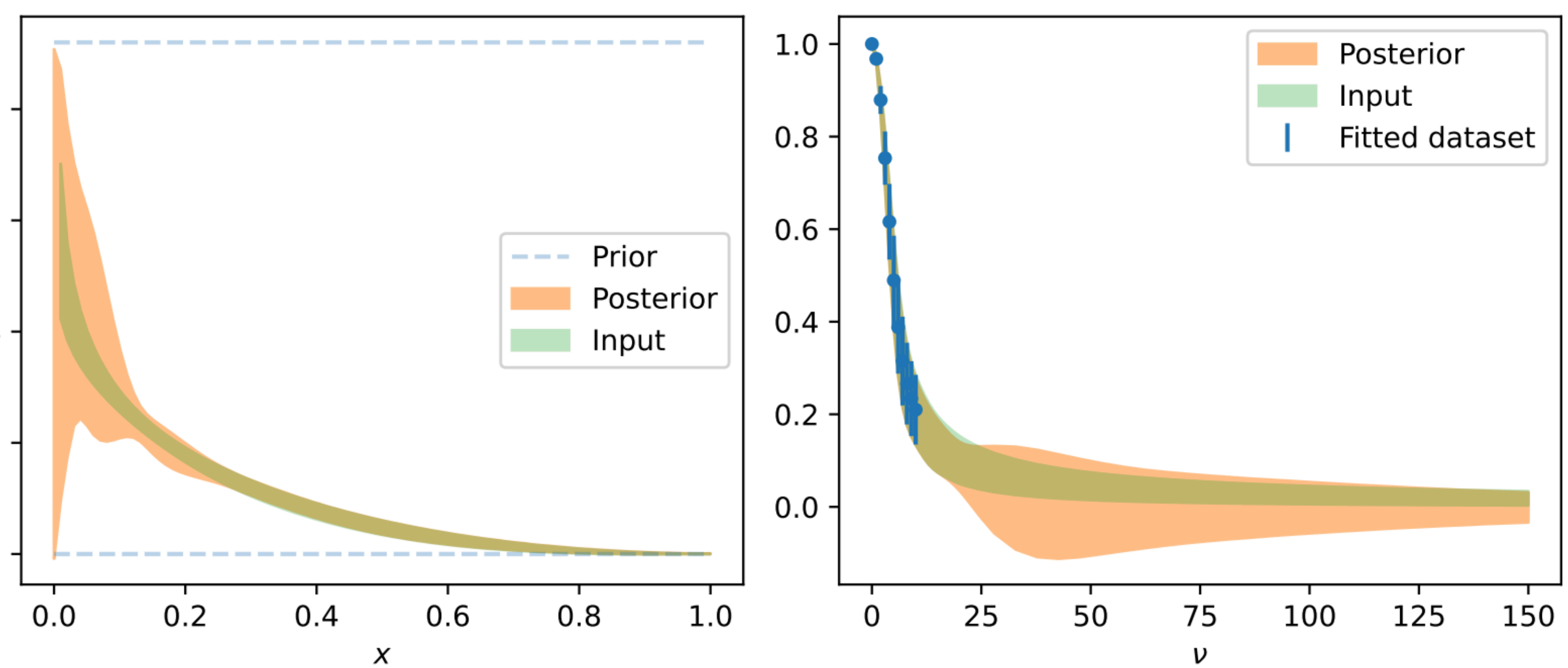
- Covariance

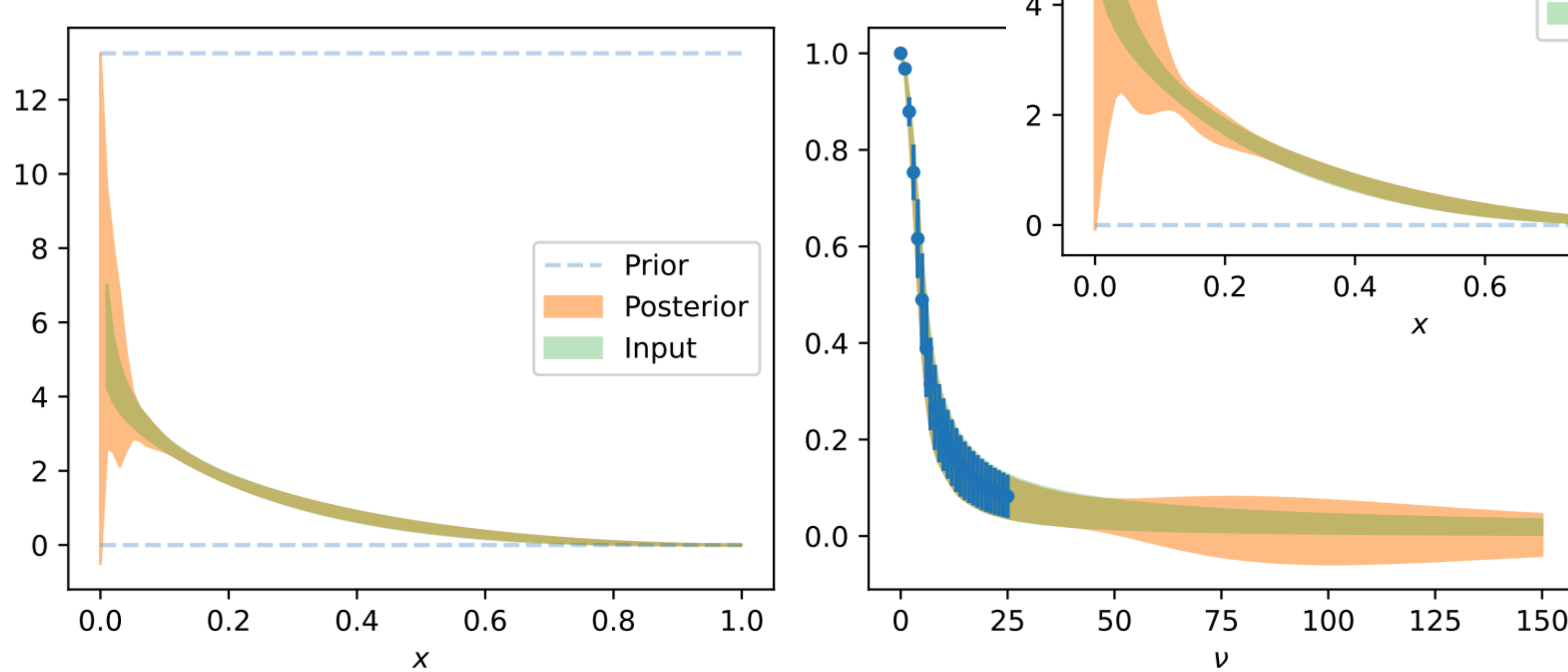
$$\langle Cov[q(x), q(x')] \rangle = H(x, x') = K - KB^T \left[C + BKB^T\right]^{-1} BK$$

Mock Data Closure Study

- The PDF is entirely set by prior at $x = 0$
- Reproduction at low x is improved as $\nu_{\max}$ increases
- Reconstruction error is 50% larger than true when $x \approx 2/\nu_{\max}$







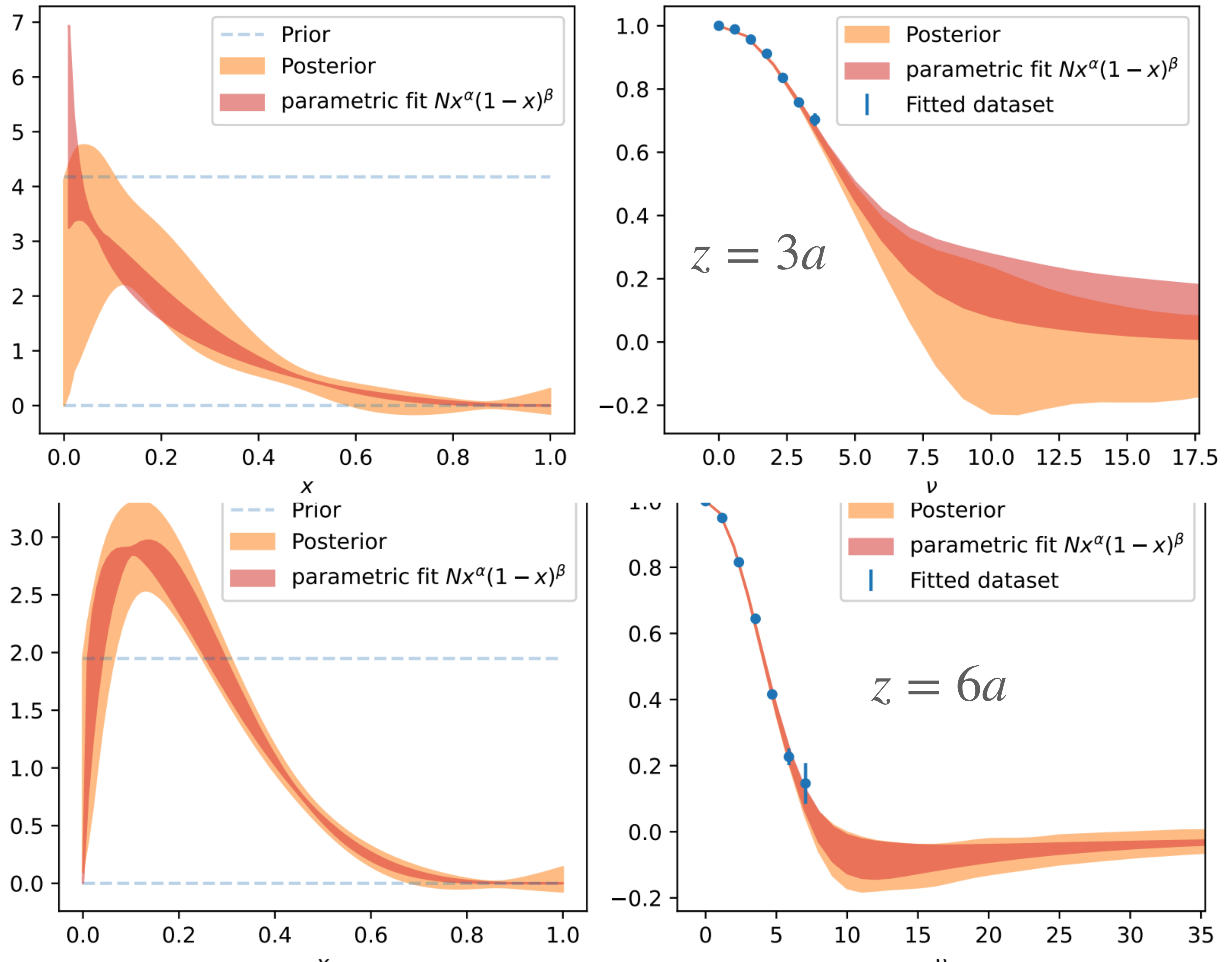
H. Dutrieux et al PRD 111 (2025) 3, 034515

Pseudo-PDF Study

C. Egerer et al (HadStruc) 2107.05199
H. Dutrieux et al PRD 111 (2025) 3, 034515

- Compare to fits to standard function, limit of $\sigma \to 0$ and optimizing the hyperparameters of mean g

- GPR extends the ability of parametric fits to allow fluctuations of result q from model g



H. Dutrieux et al PRD 111 (2025) 3, 034515

Quasi-PDF Study

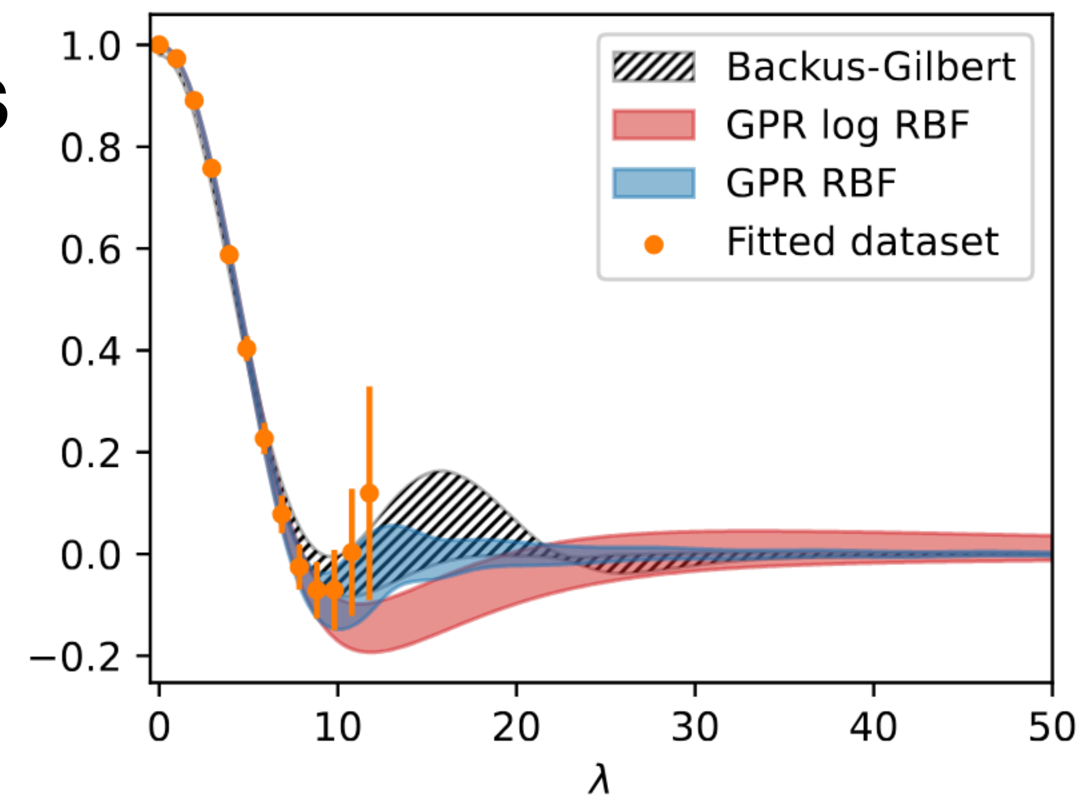
Impact of model assumptions

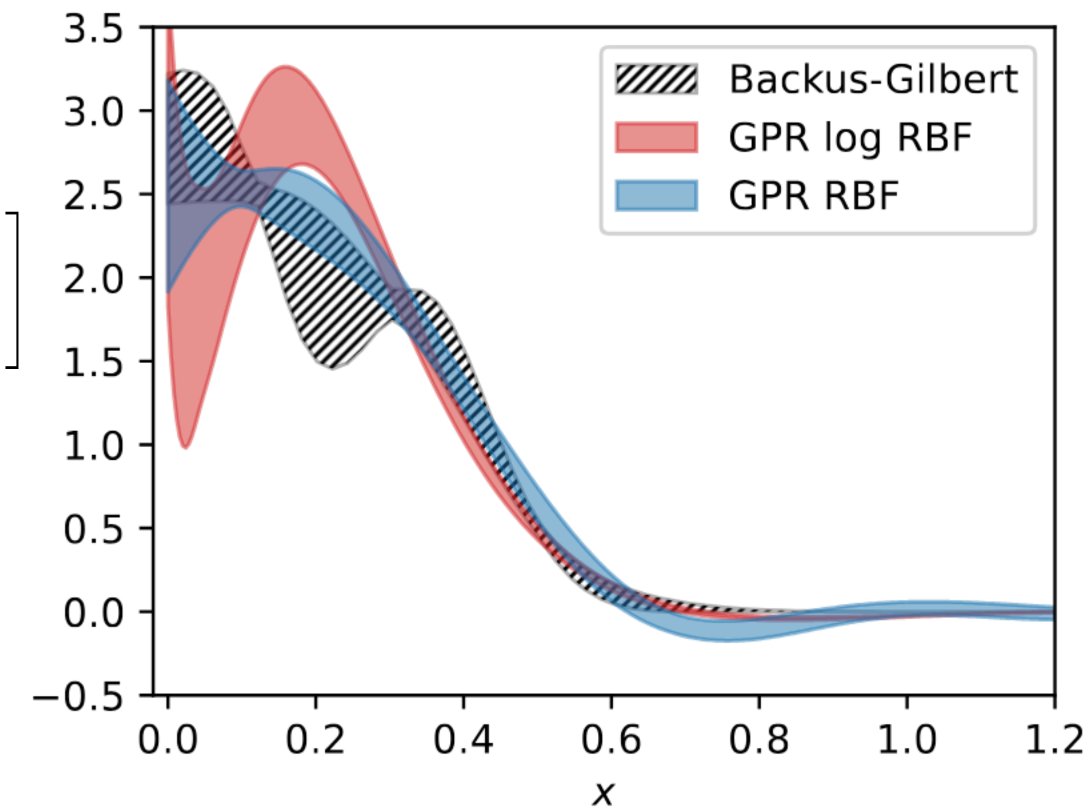
- Some quasi-PDF studies ***assume*** the data is in the asymptotic regime with decay like a single linear
- Add additional prior with mean 0 and exponentially shrinking covariance
- Intermediate z could be Gaussians from TMD

$$k_{\text{gauss}}((\lambda, z), (\lambda', z'); \Sigma^2, L^2, r) = \Sigma^2 \exp\left[-\frac{(\lambda - \lambda')^2}{2L^2} + \frac{z^2 + z'^2}{2r^2} \right]$$

- Asymptotic z is linear exponential

$$k_{\text{exp}}((\lambda, z), (\lambda', z'); \Sigma^2, L^2, r) = \Sigma^2 \exp\left[-\frac{(\lambda - \lambda')^2}{2L^2} - \frac{|z| + |z'|}{r} \right]$$





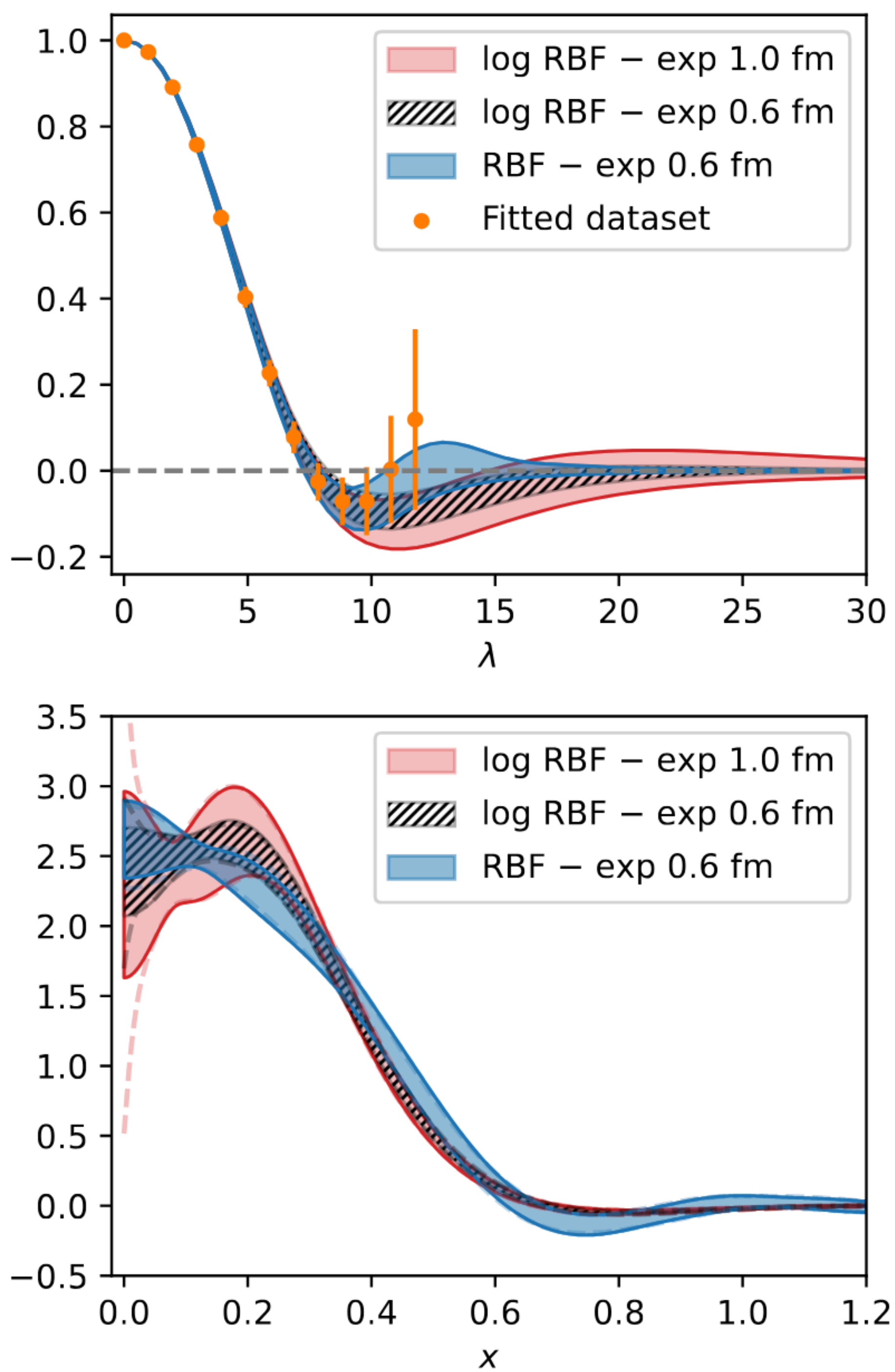
No exponential assumptions

H. Dutrieux et al PRD 111 (2025) 3, 034515

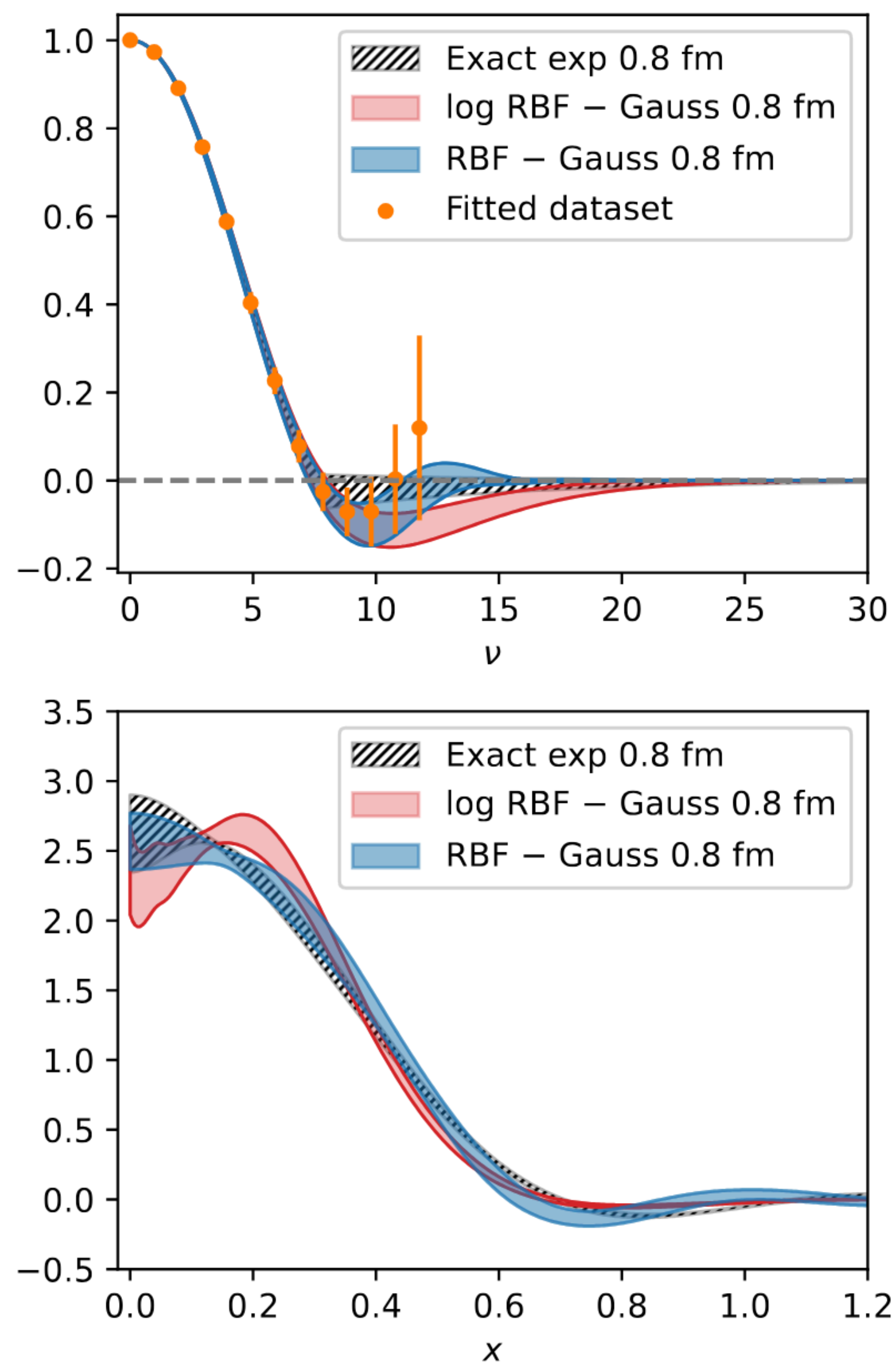
Quasi-PDF Study

Impact of model assumptions

Linear exponential



Gaussian assumption



Historical study of rest frame

“Exploring quark transverse momentum distributions with lattice QCD”

B. Musch et al PRD 83 (2011) 094507 arXiv:1011.1213

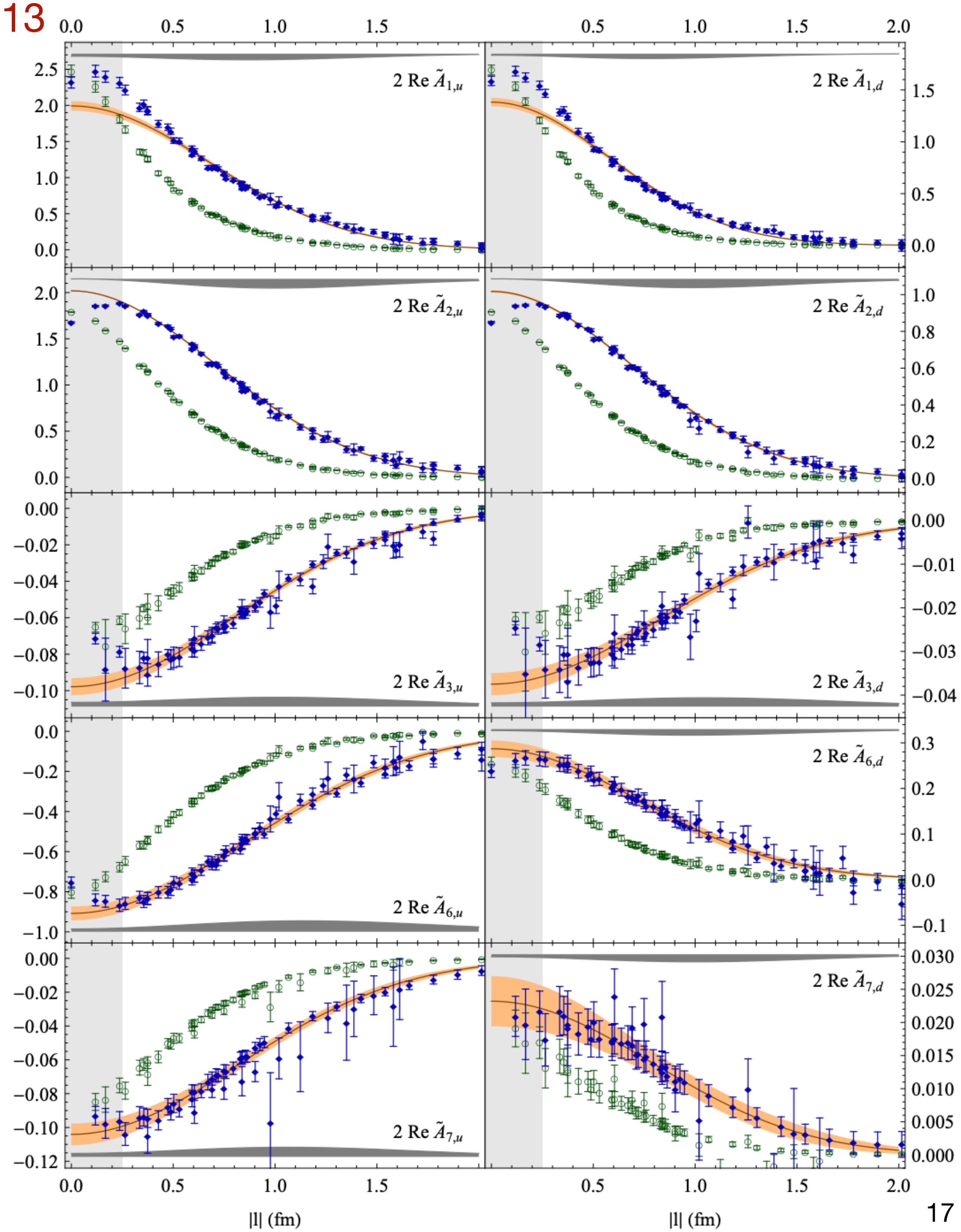
- pseudo-ITD and primordial TMD relation

$$\mathscr{M}(\nu, z_T^2) = \int dx e^{i\nu x} \int d^2 k_T e^{i k_T \cdot z_T} F(x, k_T^2)$$

- Given a Gaussian ansatz for $\mathscr{M}(0, z^2)$, the width is related to k_T^2 moments

$$\langle k_T^2 \rangle = \int dx \int d^2 k_T k_T^2 F(x, k_T^2)$$

- Analyzed for all gamma structures



Rest Frame in Cont limit

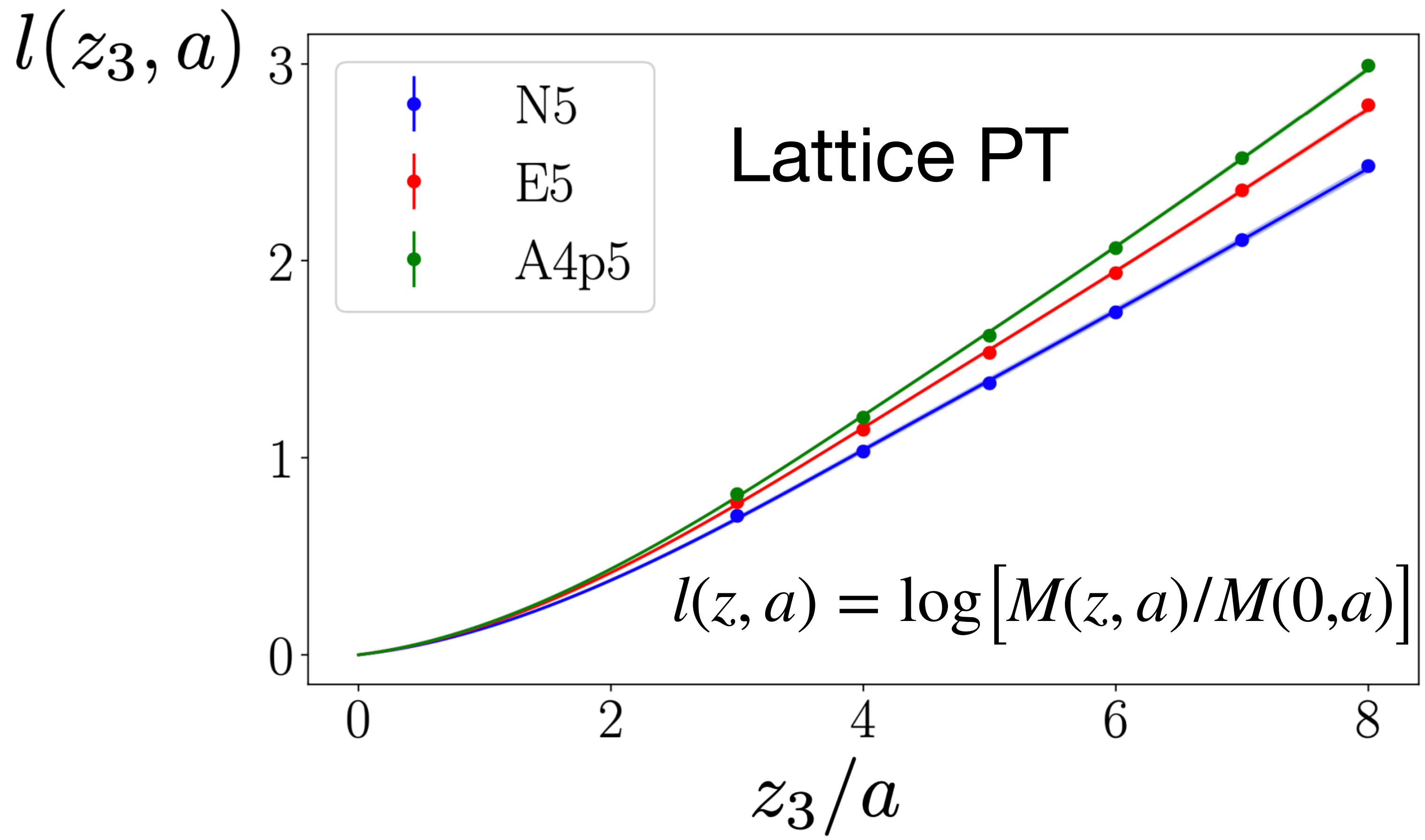
JK, C. Monahan, A. Radyushkin PRD 111 (2025) 5, 054503 arXiv:2407.16577

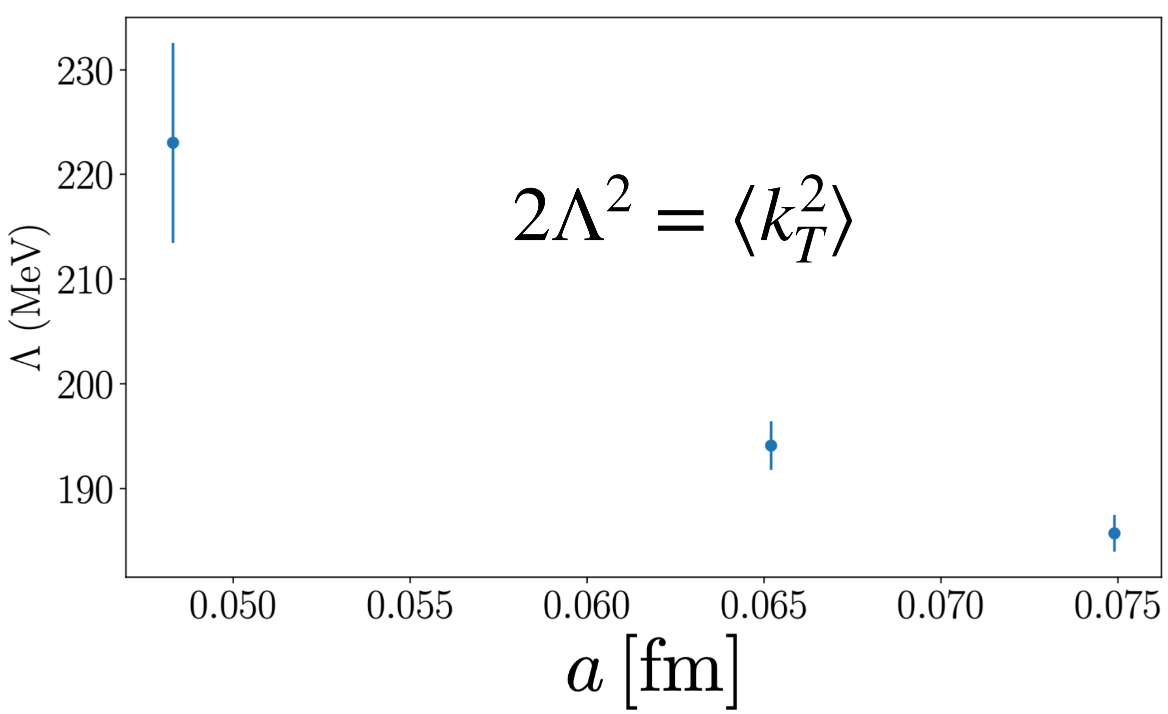
- Studied 3 lattice spacings at sub-percent stat precision

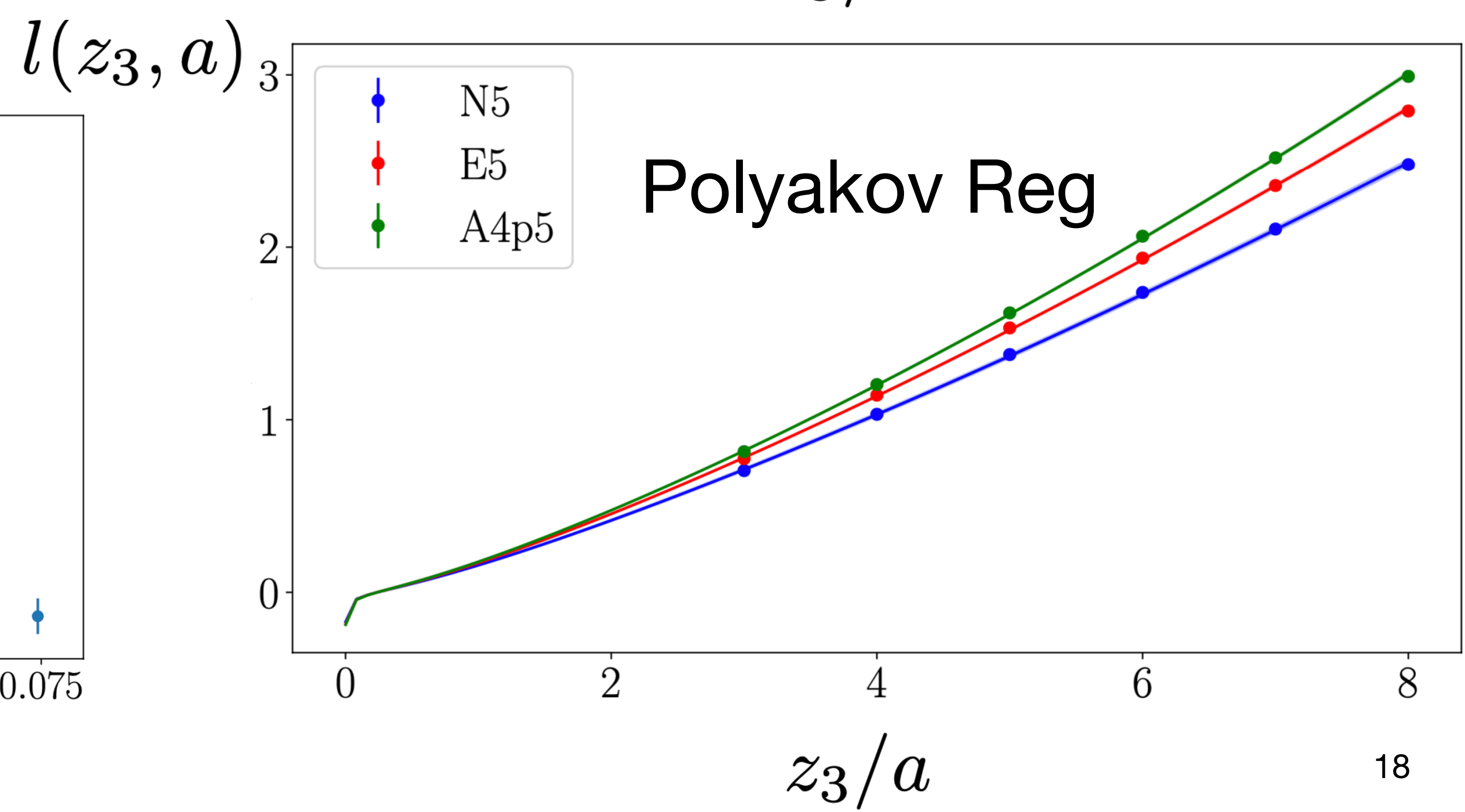
  $a \sim 0.045, 0.065, 0.075\ \mathrm{fm}$

- Agrees to 1-loop perturbation theory alone up to few percent

- Leading term in Taylor expansion of TMD (Gaussian model) gives major improvement





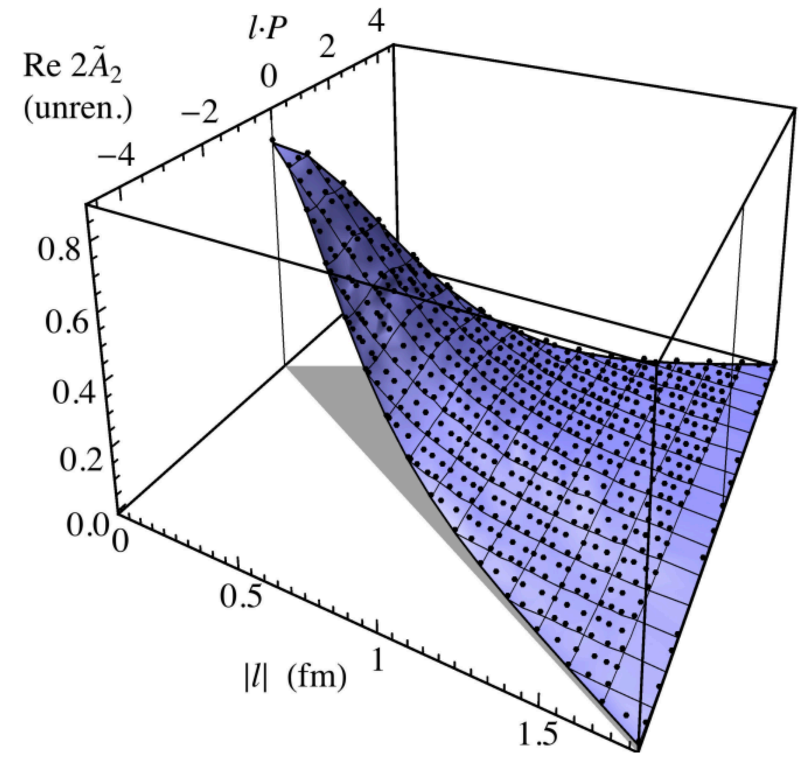


Really though read Musch et al

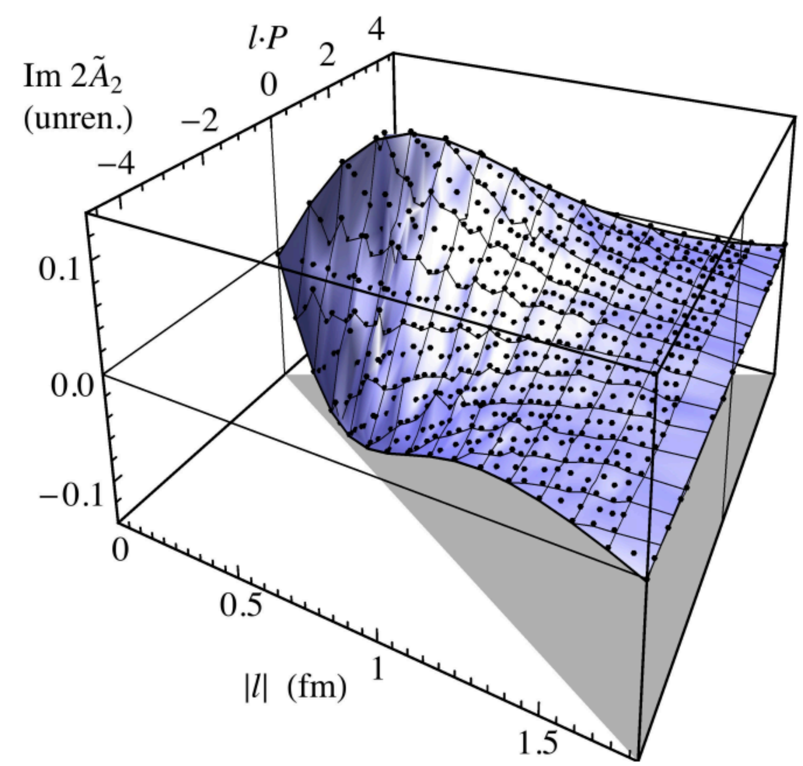
“Exploring quark transverse momentum distributions with lattice QCD”

B. Musch et al PRD 83 (2011) 094507 arXiv:1011.1213

- Discusses Lorentz invariance for identifying the right amplitudes
- Uses Wilson loops to learn linear divergence
- Proposed rest frame ratio to cancel renormalization constants
- Demonstrate uncanceled TMD effects in ratio are small up to 2 fm
- Studies all gammas and off axis separations for wide coverage and study of rotational symmetry breaking



(b)



What can we do beyond looking at nice quasi/pseudo-PDF fits?

What can lattice data do for you?

If PDFs are universal….

*If the **same** PDFs are factorizable from lattice and experiment, and if power corrections can be controlled for both*

Why not analyze both simultaneously?

- Factorization of hadronic cross sections

$$d\sigma_h = d\sigma_q \otimes f_{h/q} + P.C.$$

- Factorization of Lattice observables

$$M_h = M_q \otimes f_{h/q} + P.C.$$

Consider Lattice data as a theoretical prior to the experimental Global Fit

Complementarity in Lattice and Experiment

EXPERIMENT

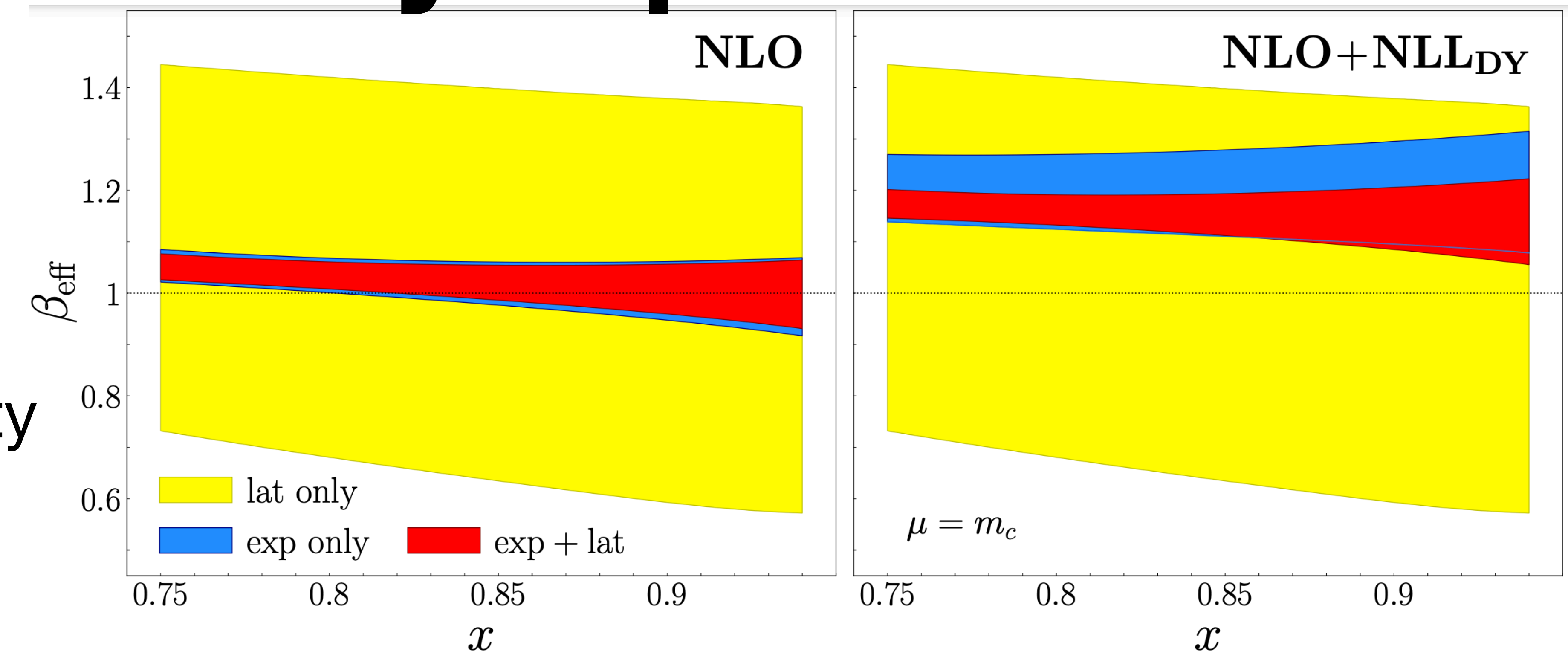
- Cross Sections limited to specific max but can reach low x_B
- Cross Section matching is integral from x_B to 1
  - Creates sensitivity of large x_B data to hard kernel in large x region
- Wealth of decades of experimental data outweigh modern lattice in both number and systematic error control

LATTICE

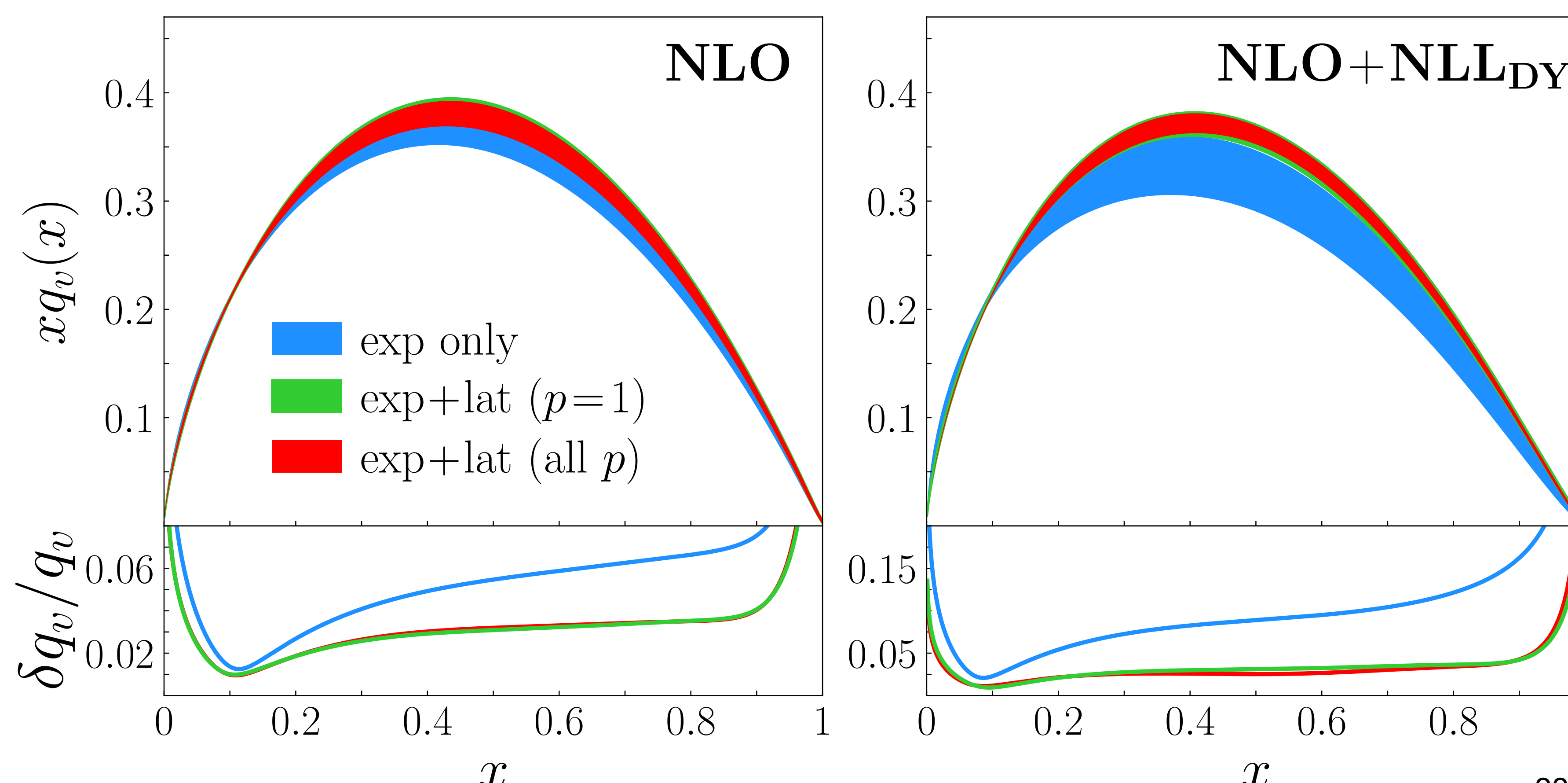
- Lattice limited to low ν, inverse Fourier gives to $x \gtrsim 0.2$, but higher sensitivity to larger x
- Lattice matching relation is integral over all x
- Low p_z data can reach high signal-to-noise compared to experiment
- Lattice can evaluate independently each spin, flavor, and hadron

Complementarity in pion PDF

- Lattice can readily access different hadrons
- Lattice lacks sensitivity to threshold logs and can be used to study theoretical kernels
- Improves precision in large x where experimental data does not exist
- Low momentum pion data are extremely precise

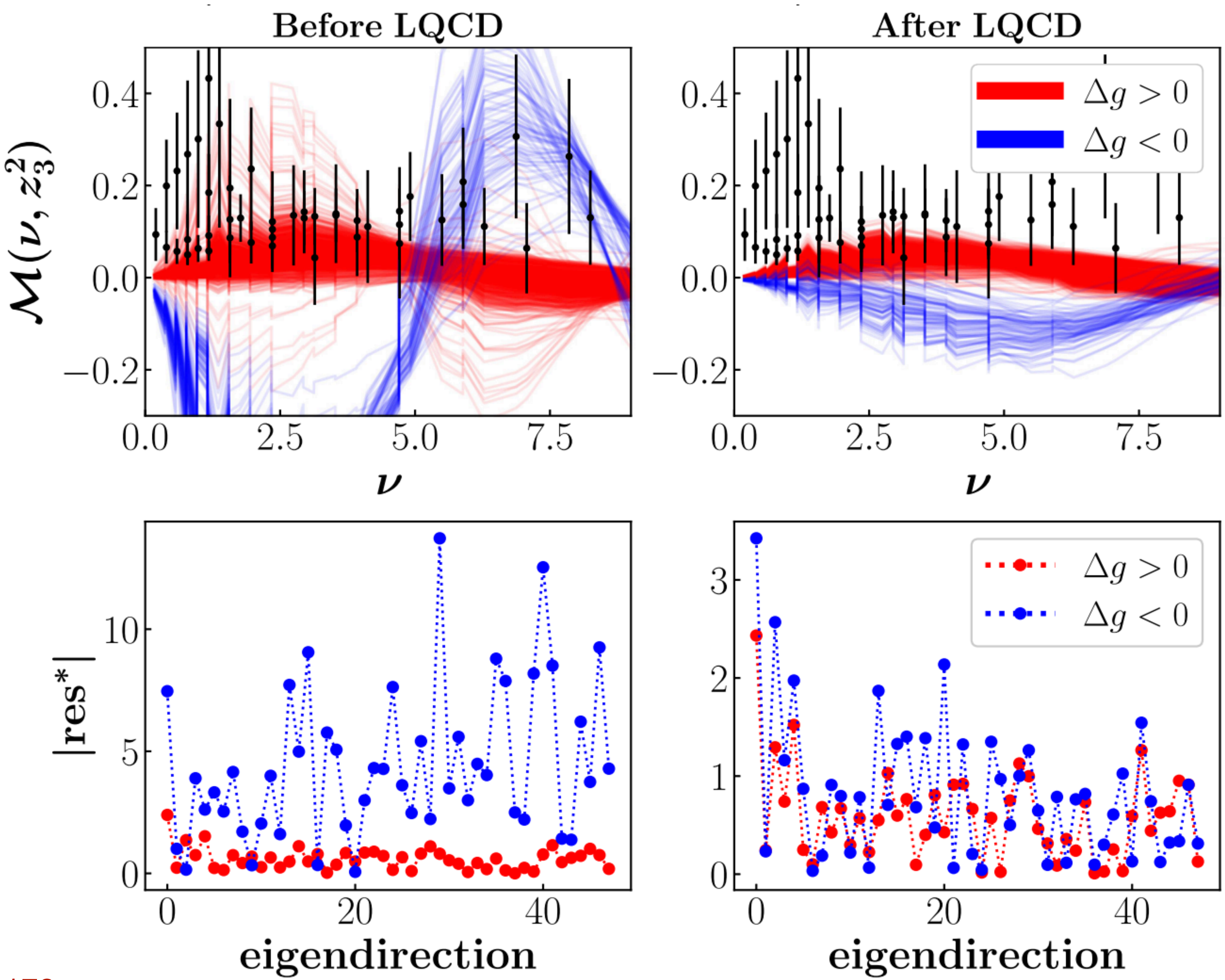


P. Barry et al, *Phys. Rev. D* 105 (2022) 11, 114051



Helicity Gluon PDF with LQCD

- Can this Lattice QCD data discriminate the red and blue solutions?



- Before LQCD:

$\chi^2/n \sim$ 0.65 30

- After LQCD:

$\chi^2/n \sim$ 0.65 1.5

JK et al arXiv:2310.18179

Wait, didn't you say Lattice people are faking it till they make it?

How do we know when lattice data are ready?

PRELIMINARY

Window Observables instead of x space comparison

- Lattice QCD data is more precise in the low ν (wide, slow Fourier modes) and lack any information at high ν (narrow, fast Fourier modes)
- The lattice results have this precision hiding in the covariance of the PDF

PRELIMINARY

Window Moments

$$a_n(x_-, x_+) = \int_{x_-}^{x_+} dx\, x^n f(x)$$

``Truncated moments" ($x_+ = 1$) have known evolution

Gaussian windows

$$g_n(x_-, x_+) = \int_{-1}^{1} dx \sqrt{\frac{n^2}{2\pi x_d^2}} \exp\left[-n^2 \frac{(x - x_0)^2}{2x_d^2}\right] f(x)$$

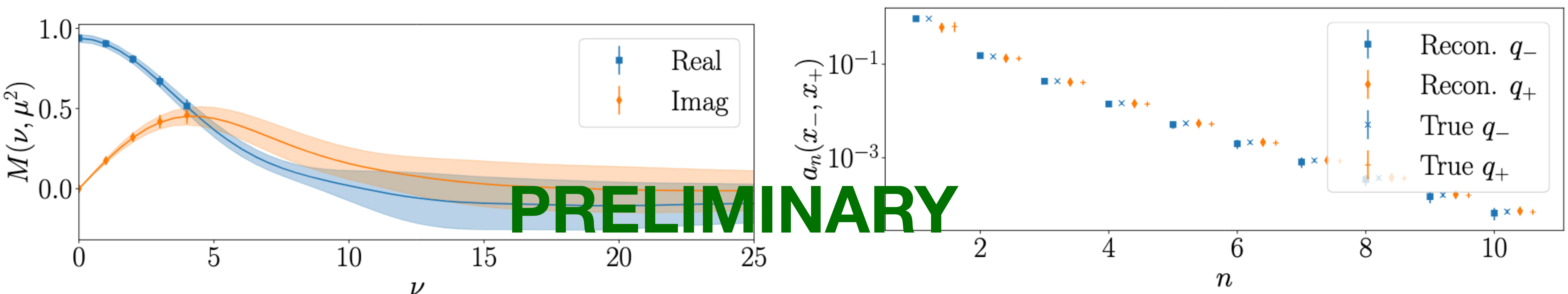
$$x_d = x_+ - x_- \qquad x_0 = \frac{x_+ + x_-}{2}$$

Window Observables instead of x space comparison

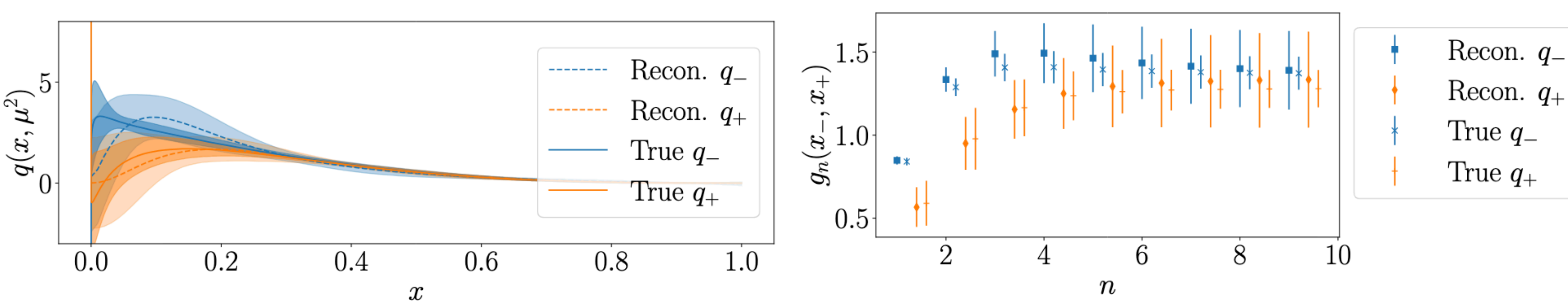
- GP reconstruction of mock data from JAM3D* transversity u-d PDFs

L. Gamberg et al arXiv:2205.00999
C. Cocuzza et al arXiv:2306.12998

- Precision of the low n (widest windows) is hiding in the covariance of the data



$x_- = 0.1 \quad x_+ = 0.5$



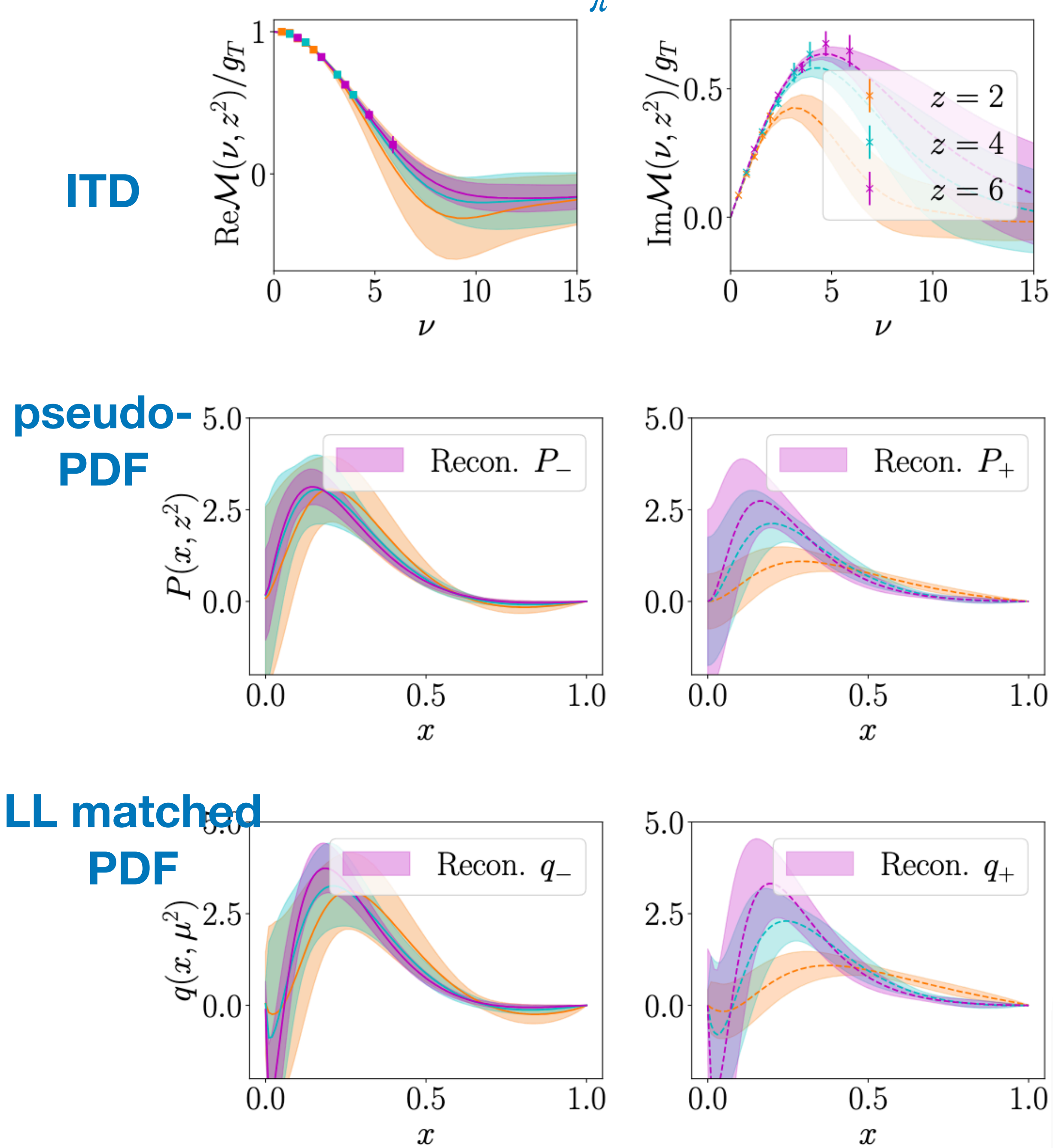
Window Observables instead of x space comparison

- GP reconstruction of HadStruc transversity u-d PDFs

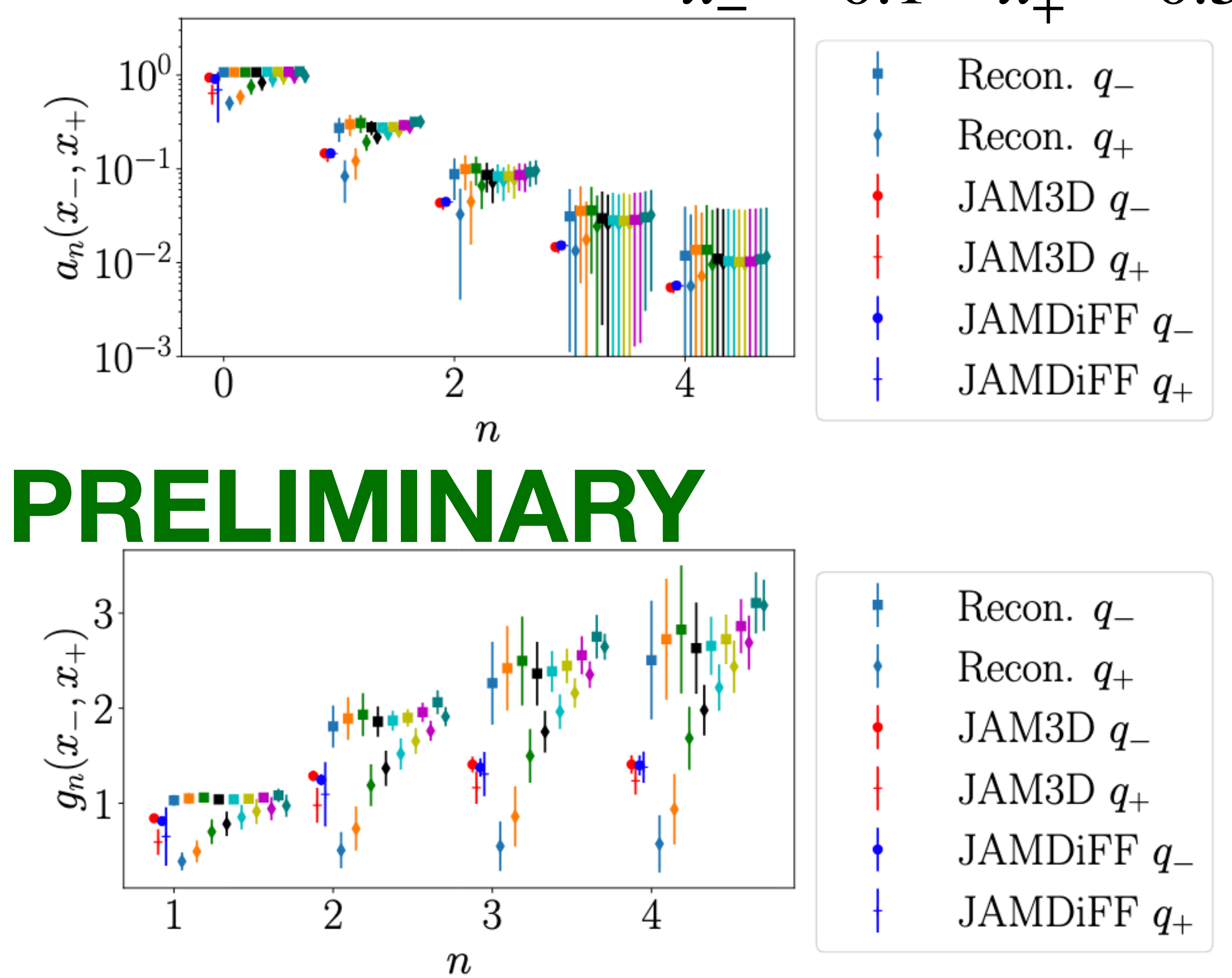
C. Egerer et al arXiv:2111.01808

- Precision of the low n (widest windows) is hiding in the covariance of the data

$a = 0.094\text{fm}\quad m_\pi = 358\text{MeV}$



$x_- = 0.1 \quad x_+ = 0.5$



- Lowest zs give bad reconstruction, but errors are promising

Conclusions

- Lattice matrix elements can be related to PDFs and their calculation have matured over the decade with new techniques for handling data analysis

- Including Lattice correlations is fundamental to correct error analysis and hypothesis testing and there is less precision information than you think

- Gaussian Processes provide a useful method for extracting pseudo-PDFs and quasi-PDFs with controllable assumptions

- Window observables can provide robust benchmarking between lattice and experiment to justify combined analyses

Back up slides

Helicity Gluon matrix element

I. Balitsky, W. Morris, A. Radyushkin JHEP 02 (2022) 193
C. Egerer et al (HadStruc) arXiv:2207.08733

- Helicity Gluon Matrix Element:

$$\widetilde{M}_{\mu\alpha;\nu\beta}(z,p,s) = \frac{1}{2}\epsilon_{\nu\beta\rho\sigma}M_{\mu\alpha;\rho\sigma} = \langle p,s\,|\,\mathrm{Tr}\big[F^{\mu\alpha}(z)W(z;0)\widetilde{F}^{\nu\beta}(0)\big]\,|p,s\rangle$$

- Useful Combination $\widetilde{\mathscr{M}}(z,p) = \big[\widetilde{M}_{ti;it} + \widetilde{M}_{ij;ij}\big]$
    - Gives **two** amplitudes, one has no leading twist contribution

Helicity Gluon matrix element

I. Balitsky, W. Morris, A. Radyushkin JHEP 02 (2022) 193
C. Egerer et al (HadStruc) arXiv:2207.08733

- Helicity Gluon Matrix Element:

$$\widetilde{M}_{\mu\alpha;\nu\beta}(z,p,s) = \frac{1}{2}\epsilon_{\nu\beta\rho\sigma}M_{\mu\alpha;\rho\sigma} = \langle p,s\,|\,\mathrm{Tr}\Big[F^{\mu\alpha}(z)W(z;0)\,\widetilde{F}^{\nu\beta}(0)\Big]\,|p,s\rangle$$

- Useful Combination $\widetilde{\mathscr{M}}(z,p) = \Big[\widetilde{M}_{ti;it} + \widetilde{M}_{ij;ij}\Big]$
  - Gives **two** amplitudes, one has no leading twist contribution
- Use ratio with finite continuum limit

$$\widetilde{\mathfrak{M}}(\nu, z^2) = i\frac{\Big[\widetilde{\mathscr{M}}(z,p)/p_z p_0\Big]/Z_L(z/a)}{\mathscr{M}(0,z^2)/m^2}$$

Helicity Gluon matrix element

I. Balitsky, W. Morris, A. Radyushkin JHEP 02 (2022) 193
C. Egerer et al (HadStruc) arXiv:2207.08733

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- Relation to gluon and quark singlet ITD

$$\langle x\rangle_g\,\widetilde{\mathfrak{M}}(\nu,z^2) = \int_0^1 \widetilde{C}^{gg}(u,\mu^2z^2)\widetilde{I}_g(u\nu,\mu^2) + \widetilde{C}^{qg}(u,\mu^2z^2)\widetilde{I}_s(u\nu,\mu^2)$$

Pol Gluon Lorentz decomposition

I. Balitsky, W. Morris, A. Radyushkin
JHEP 02 (2022) 193

$$\widetilde{M}^{(2)}_{\mu\alpha;\lambda\beta}(z,p) = (sz)\left(g_{\mu\lambda}p_\alpha p_\beta - g_{\mu\beta}p_\alpha p_\lambda - g_{\alpha\lambda}p_\mu p_\beta + g_{\alpha\beta}p_\mu p_\lambda\right)\widetilde{\mathcal{M}}_{pp}$$
$$+(sz)\left(g_{\mu\lambda}z_\alpha z_\beta - g_{\mu\beta}z_\alpha z_\lambda - g_{\alpha\lambda}z_\mu z_\beta + g_{\alpha\beta}z_\mu z_\lambda\right)\widetilde{\mathcal{M}}_{zz}$$
$$+(sz)\left(g_{\mu\lambda}z_\alpha p_\beta - g_{\mu\beta}z_\alpha p_\lambda - g_{\alpha\lambda}z_\mu p_\beta + g_{\alpha\beta}z_\mu p_\lambda\right)\widetilde{\mathcal{M}}_{zp}$$
$$+(sz)\left(g_{\mu\lambda}p_\alpha z_\beta - g_{\mu\beta}p_\alpha z_\lambda - g_{\alpha\lambda}p_\mu z_\beta + g_{\alpha\beta}p_\mu z_\lambda\right)\widetilde{\mathcal{M}}_{pz}$$
$$+(sz)\left(p_\mu z_\alpha - p_\alpha z_\mu\right)\left(p_\lambda z_\beta - p_\beta z_\lambda\right)\widetilde{\mathcal{M}}_{ppzz}$$
$$+(sz)\left(g_{\mu\lambda}g_{\alpha\beta} - g_{\mu\beta}g_{\alpha\lambda}\right)\widetilde{\mathcal{M}}_{gg}$$

$$\widetilde{M}^{(1)}_{\mu\alpha;\lambda\beta}(z,p) = \left(g_{\mu\lambda}s_\alpha p_\beta - g_{\mu\beta}s_\alpha p_\lambda - g_{\alpha\lambda}s_\mu p_\beta + g_{\alpha\beta}s_\mu p_\lambda\right)\widetilde{\mathcal{M}}_{sp}$$
$$+\left(g_{\mu\lambda}p_\alpha s_\beta - g_{\mu\beta}p_\alpha s_\lambda - g_{\alpha\lambda}p_\mu s_\beta + g_{\alpha\beta}p_\mu s_\lambda\right)\widetilde{\mathcal{M}}_{ps}$$
$$+\left(g_{\mu\lambda}s_\alpha z_\beta - g_{\mu\beta}s_\alpha z_\lambda - g_{\alpha\lambda}s_\mu z_\beta + g_{\alpha\beta}s_\mu z_\lambda\right)\widetilde{\mathcal{M}}_{sz}$$
$$+\left(g_{\mu\lambda}z_\alpha s_\beta - g_{\mu\beta}z_\alpha s_\lambda - g_{\alpha\lambda}z_\mu s_\beta + g_{\alpha\beta}z_\mu s_\lambda\right)\widetilde{\mathcal{M}}_{zs}$$
$$+(p_\mu s_\alpha - p_\alpha s_\mu)(p_\lambda z_\beta - p_\beta z_\lambda)\widetilde{\mathcal{M}}_{pspz}$$
$$+(p_\mu z_\alpha - p_\alpha z_\mu)(p_\lambda s_\beta - p_\beta s_\lambda)\widetilde{\mathcal{M}}_{pzps}$$
$$+(s_\mu z_\alpha - s_\alpha z_\mu)(p_\lambda z_\beta - p_\beta z_\lambda)\widetilde{\mathcal{M}}_{szpz}$$
$$+(p_\mu z_\alpha - p_\alpha z_\mu)(s_\lambda z_\beta - s_\beta z_\lambda)\widetilde{\mathcal{M}}_{pzsz}$$

Want: $M_{\Delta g}(\nu, z^2) = \left[\widetilde{\mathcal{M}}^{(+)}_{sp} - \nu\,\widetilde{\mathcal{M}}_{pp}\right]$

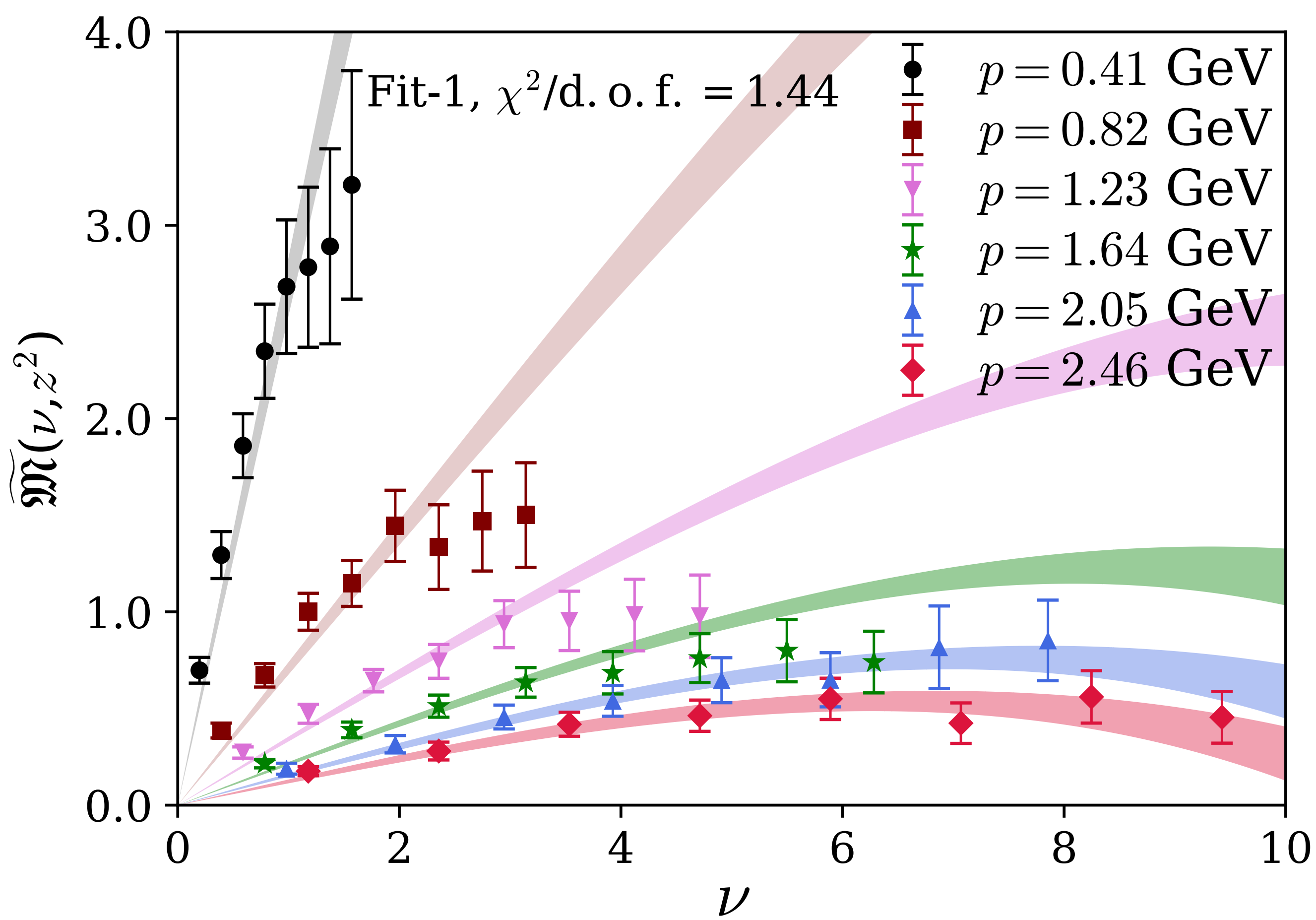
Can get: $\widetilde{\mathcal{M}}(z,p) = \left[\widetilde{M}_{ti;it} + \widetilde{M}_{ij;ij}\right]$

$$= M_{\Delta g} - \frac{m^2 z^2}{\nu}\widetilde{\mathcal{M}}_{pp}$$

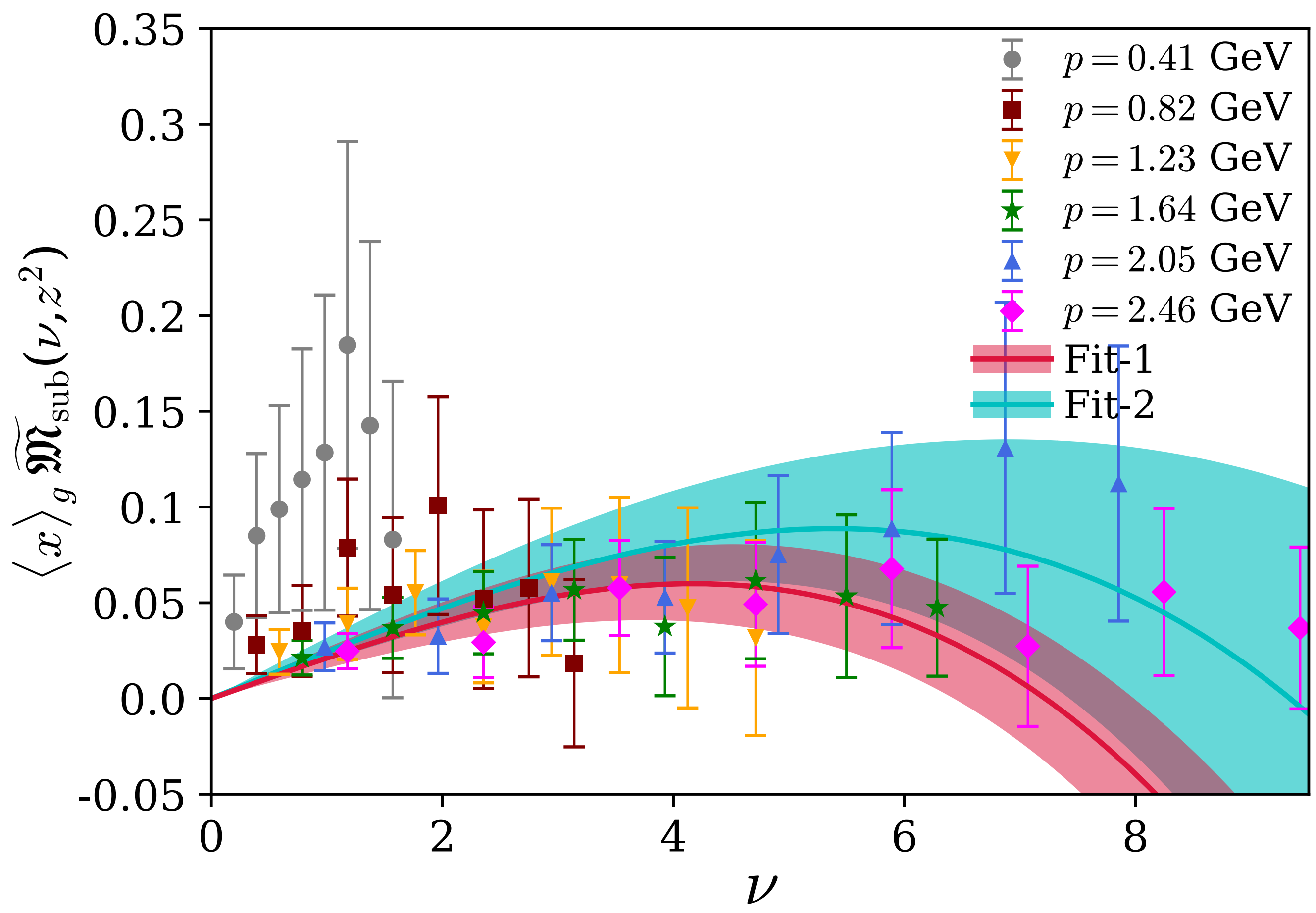
$$= M_{\Delta g} - \frac{m^2}{p_z^2}\nu\,\widetilde{\mathcal{M}}_{pp}$$

Helicity Gluon PDF

- Model both terms



- Subtract rest frame



$a = 0.094$ fm

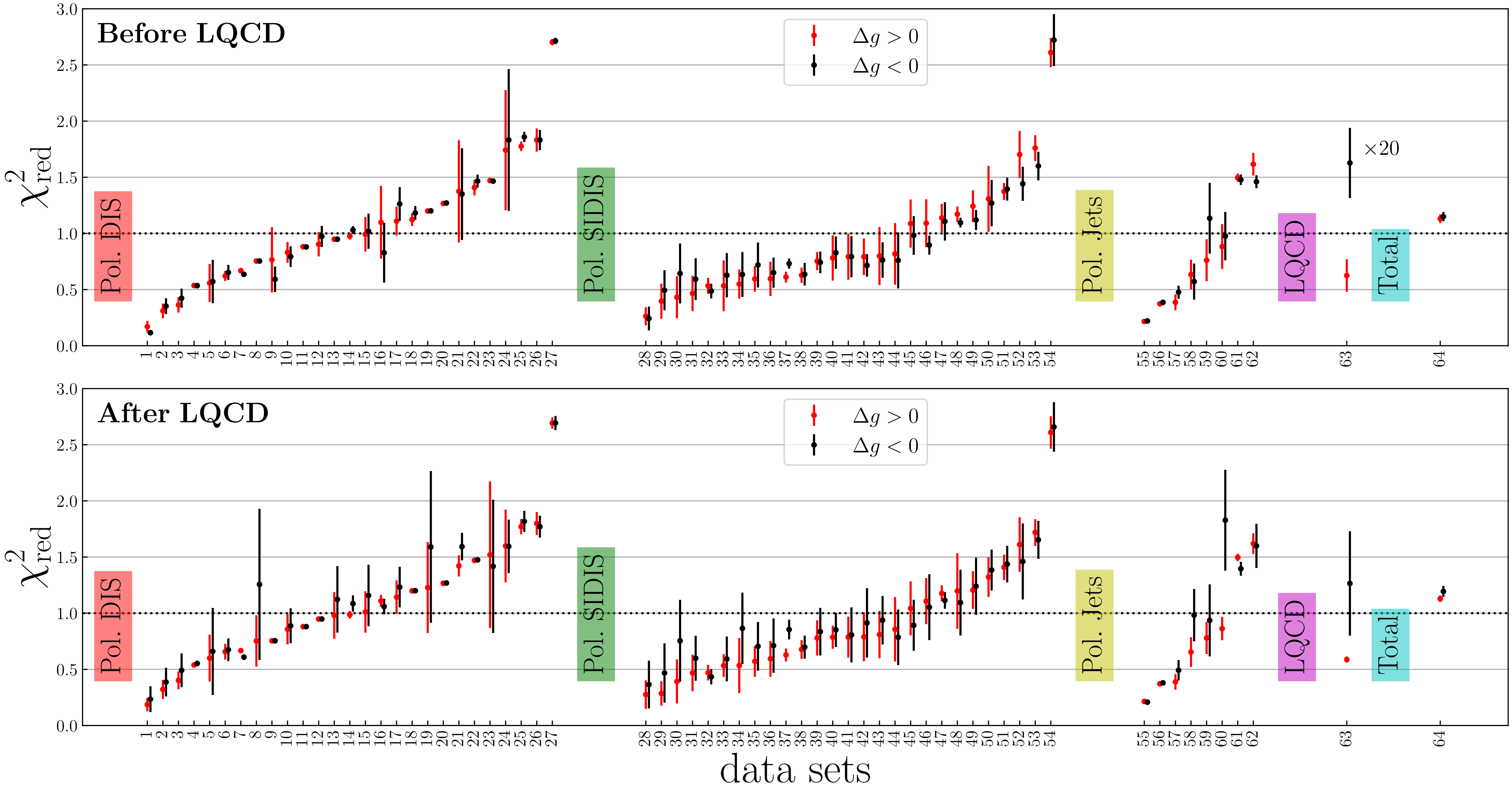
$m_\pi = 358$ MeV

C. Egerer et al (HadStruc) arXiv: 2207.08733

Helicity Gluon PDF with LQCD

JK et al arXiv:2310.18179

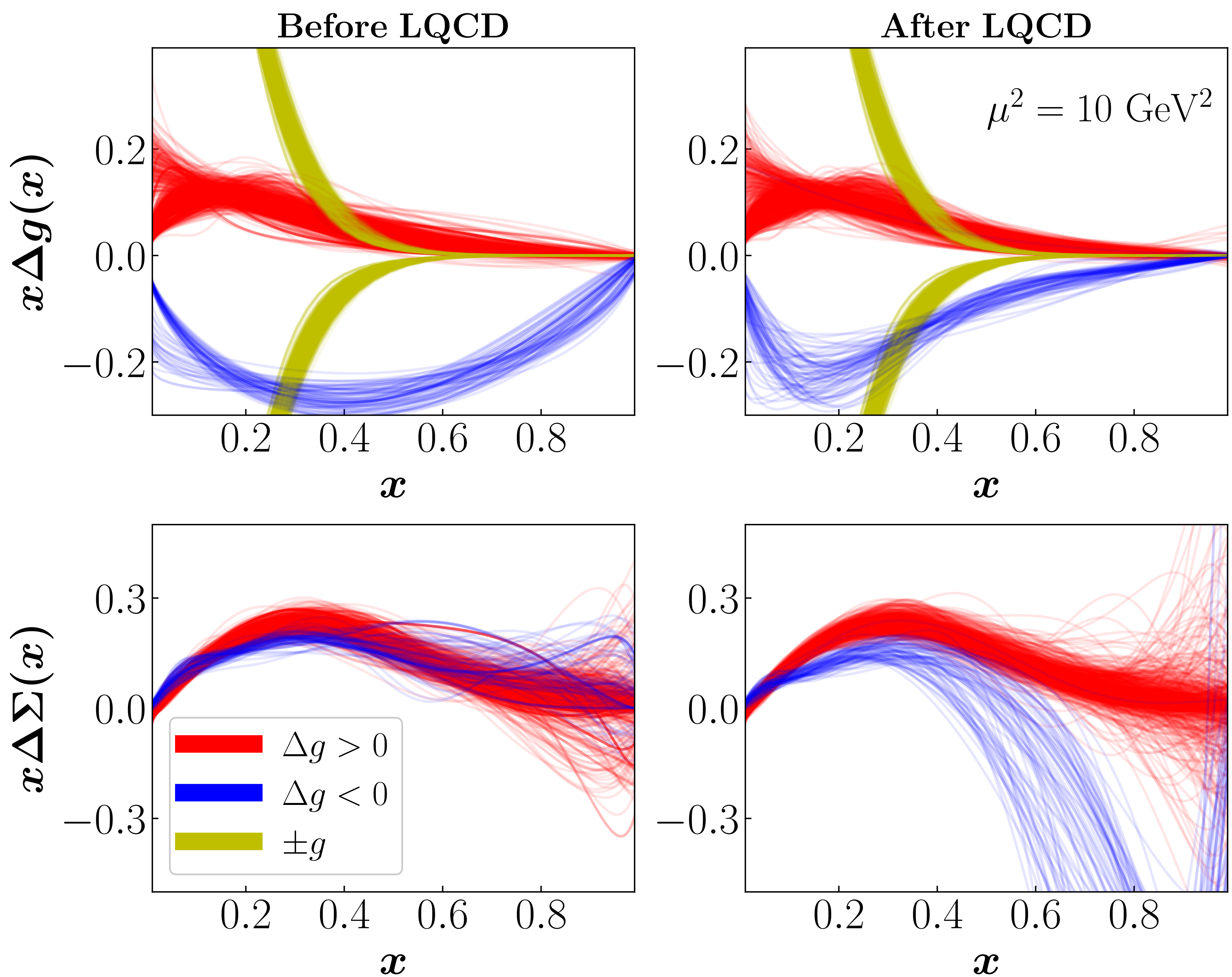
- Negative and positive Δg appear consistent with experiment and lattice



Lattice gluon data impacts quarks

C. Egerer et al (HadStruc) arXiv:2207.08733
JK et al arXiv:2310.18179

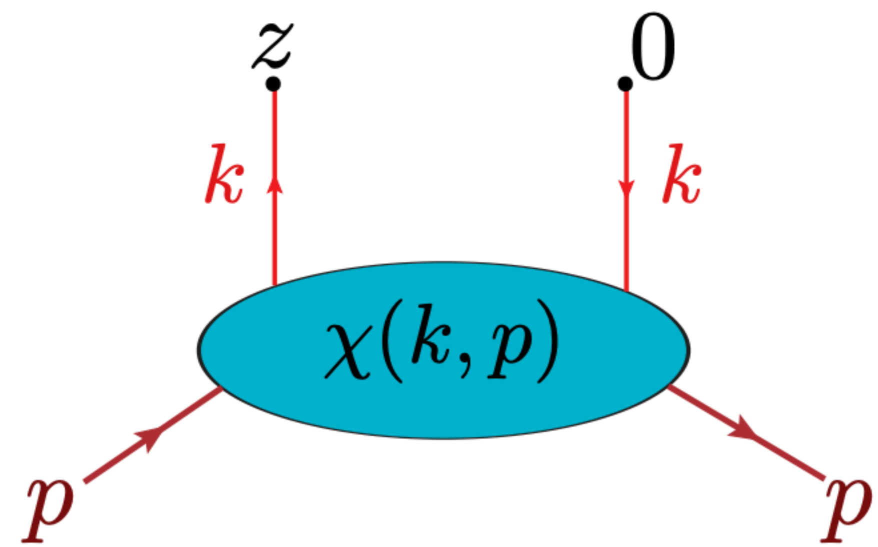
- Quark gluon mixing leads to impact on singlet
- Unexpected change in extrapolation region
- Compensates reduced magnitude of Δg in relation to cross sections



Other Faces of WL Matrix Element

- Some are Lorentz invariant interpretations
- These interpretations nor the functions' bounds require small z^2 , only relation to light cone PDF with $z^2 = 0$ and some other regulation

Review: A. Radyushkin (2019) 1912.04244



$$i\chi_{d_i}(k,p) = i^l \frac{P(\mathrm{c.c.})}{(4\pi i)^{2L}} \int_0^\infty \prod_{j=1}^{l} d\alpha_j [D(\alpha)]^{-2}$$
$$\times \exp\left\{ ik^2 \frac{A(\alpha)}{D(\alpha)} + i\frac{(p-k)^2 B_s(\alpha) + (p+k)^2 B_u(\alpha)}{D(\alpha)} \right\}$$
$$\times \exp\left\{ ip^2 \frac{C(\alpha)}{D(\alpha)} - i\sum_j \alpha_j (m_j^2 - i\epsilon) \right\},$$

$$i\chi(k,p) = \int_0^\infty d\sigma \int_{-1}^{1} dx e^{i\sigma\left[k^2 - 2x(k\cdot p) + i\epsilon\right]} V(x,\sigma)$$

$$\sigma_{d_i} = \frac{A_{d_i}(\alpha) + B_{s_{d_i}}(\alpha) + B_{u_{d_i}}(\alpha)}{D_{d_i}(\alpha)}$$

$$x_{d_i} = \frac{B_{s_{d_i}}(\alpha) - B_{u_{d_i}}(\alpha)}{A_{d_i}(\alpha) + B_{s_{d_i}}(\alpha) + B_{u_{d_i}}(\alpha)}$$

α_j are positive numbers and A, B_u, B_s, C, D are sums of products of α_j

Fourier transform to position space

$$\mathcal{M}(\nu, z^2) = \int \frac{d^4k}{(2\pi)^4} e^{ik\cdot z} \chi(k,p) = \int_{-1}^{1} dx e^{i\nu x} \int_0^\infty e^{-i\sigma(z^2-\epsilon)} V(x,\sigma)$$

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Review: A. Radyushkin (2019) 1912.04244

Virtuality Distribution Function
Lorentz invariant picture
σ^{-1} pole gives log z^2
Limits from nature of Feynman diagrams

$$\mathscr{M}(\nu, z^2) = \int_{-1}^{1} dx e^{i\nu x} \int_0^\infty d\sigma e^{-i\sigma(z^2-\epsilon)} V(x,\sigma)$$

pseudo-PDF
Lorentz invariant picture
log z^2 divergence from poles of TMD/VDF

$$\mathscr{M}(\nu, z^2) = \int_{-1}^{1} dx e^{i\nu x} P(x, z^2)$$

$$\tilde{q}(y, p_z^2) = \int dz \int_{-1}^{1} dx e^{ip_z z(x-y)} P(x, z^2)$$

Musch, Hagler, Negele, Schafer PRD 83 (2011) 094507

Straight Link / Primordial TMD
Frame dependent picture with nice interpretation
$1/k_T^2$ pole gives log z^2

$$z = (0, z^-, z_T) \qquad p = (p^+, \frac{m^2}{p^+}, 0_T)$$

$$\mathscr{M}(\nu, z^2) = \int_{-1}^{1} dx e^{i\nu x} \int d^2k_T e^{ik_T\cdot z_T} F(x, k_T^2)$$

Light cone PDF from regulated integral of TMD
Relate to the Lorentz invariant VDF
$1/k_T^2$ or σ^{-1} poles generate $\log \mu^2$ divergence

$$f(x, \mu^2) = \int^{\mu^2} d^2k_T F(x, k_T^2) = \int_0^\infty d\sigma \left[1 - e^{-\frac{i}{\sigma}(\mu^2 - i\epsilon)}\right] V(x,\sigma)$$

Inverse Problems in all Parton Physics

Wikipedia: An inverse problem in science is the process of calculating from a set of observations the causal factors that produce them

- **Structure Functions**

$$F_2(x, Q^2) = \sum_f \int_x^1 d\xi \, C_f(\xi, \frac{\mu^2}{Q^2}) \, q_f(\frac{x}{\xi}, \mu^2)$$

- **LaMET (on the lattice)**

$$M(p_z, z) = \int_{-\infty}^{\infty} dy \, e^{iyp_z z} \, \tilde{q}(y, p_z^2)$$

- **pseudo-Distributions / Good Lattice Cross Sections**

$$\mathscr{M}(\nu, z^2) = \int_{-1}^{1} dx \, C(x\nu, \mu^2 z^2) \, q(x, \mu^2)$$

- **Local Matrix elements / HOPE / OPE-without-OPE**

$$a_n(\mu^2) = \int_{-1}^{1} dx \, x^{n-1} \, q(x, \mu^2)$$

- **Hadronic Tensor**

$$\tilde{W}_{\mu\nu}(\tau) = \int d\nu \, e^{-\nu\tau} \, W_{\mu\nu}(\nu)$$

Integral Inverse Problems

General statement of the inverse problem [edit]

The inverse problem is the "inverse" of the forward problem: instead of determining the data produced by particular model parameters, we want to determine the model parameters that produce the data d_{obs} that is the observation we have recorded (the subscript **obs** stands for observed). Our goal, in other words, is to determine the model parameters p such that (at least approximately)

$$d_{\text{obs}} = F(p)$$

- If amount of unknown information (possibly infinite) is more than the known data then it will be ill-posed by lack of uniqueness
- Ill-posed inverse problems require addition information to add isolate solutions
- Integral Inverse problems are interpolations and/or extrapolations of a completely unknown function

$$M(\nu) = \int dx B(\nu, x) q(x)$$

Well-posed problem

Article Talk

From Wikipedia, the free encyclopedia

In mathematics, a **well-posed problem** is one for which the following properties hold:[a]

J. Hadamard (1902)

1. The problem has a solution
2. The solution is unique
3. The solution's behavior changes continuously with the initial conditions

    - Solving to linear systems or fits to exactly known models

- All other problems are ill-posed typically due to uniqueness
    - Lattice correlators where infinite series of energies must be truncated (ill-posedness gives different results as truncation changes)
    - Inverse integrals where extrapolation is required (extrapolation choices changes the answers)

Bayesian Solutions

- Integral Inverse problems require interpolations and/or extrapolations

$$M(\nu) = \int dx B(\nu, x) q(x)$$

- Regulate problem by having some prior information about unknown and some data sensitive to it
  - For Regression we want $\langle q(x) \rangle = \int Dq\, q(x)\, P[q\,|\,M, I]$

$$\langle Cov[q(x), q(x')] \rangle = \int Dq\, Cov[q(x), q(x')]\, P[q\,|\,M, I]$$

  - Bayes's Theorem

$$P[A\,|\,B, C] = \frac{P[B\,|\,A]\, P[A\,|\,C]}{P[B\,|\,C]}$$

- A is the function q we want to infer
- B is the data M we want to inform our inference
- C is the prior information I we wish to use to constrain the result

Components of the Posterior

- The inverse we wish to understand $M(\nu) = \int dx B(\nu, x) q(x)$

Data Likelihood: assumed by Central Limit Theorem

$$P[M\,|\,q] \propto \exp[-\frac{1}{2}\sum_{\nu\nu'}(M_\nu - M(\nu))C^{-1}_{\nu\nu'}(M_{\nu'} - M(\nu'))] = \exp[-\frac{1}{2}\chi^2]$$

Posterior

$$P[q\,|\,M, I] = \frac{P[M\,|\,q]\,P[q\,|\,I]}{P[M\,|\,I]}$$

Prior Distribution:
Human choice here decides how problem is regulated and a unique solution emerges

Evidence: Normalizing factor independent of q

$$P[M\,|\,I] = \int Dq\,P[M\,|\,q]P[q\,|\,I]$$

CHOICE OF PRIOR SETS TRADEOFF BETWEEN BIAS AND VARIANCE

Choosing Hyperparameters

Choice	Motivation
$K(x, x')$ eq. (11)	Enforce smoothness in x while decorrelating small x from large x
$l = \ln(2) \approx 0.693$	Set the flexibility of the x-dependence
$g(x) = \sigma$	Set the uncertainty to 100% at $x = 0$ and loosely enforce positivity
σ^2 such that the worst uncertainty in Fourier space is $\max(1.3\Delta M(\nu_{max}), 0.3\lvert M(\nu_{max})\rvert)$	Control the size of uncertainty in the extrapolation region in Fourier space