Quasifragmentation functions and quasiparton distributions in the massive Schwinger model

QCD Evolution 2025

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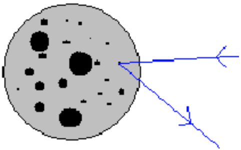
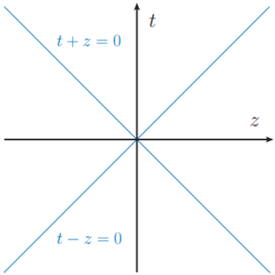
Why study PDFs and FFs?

- Parton distribution functions (PDFs): probability density to find partons in hadron as function of fraction x of the hadron's momentum (carried by parton).
- Fragmentation functions (FFs) describe how high-energy parton transforms into a jet of hadrons; counterpart of PDFs but describe "reverse" process: parton hadronizes
- PDFs and FFs crucial for understanding internal structure of hadrons and dynamics of partonic interactions
- PDFs and FFs central for analyses of most high energy processes in QCD (i.e. data from LHC, RHIC, EIC).

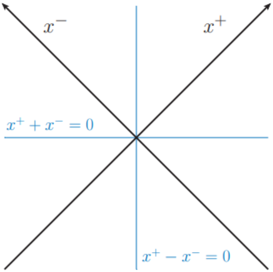


The Light Front

Light-front time: $x^+ = t + z$, Light-front-space: $x^- = t - z$



Minkowski coordinates



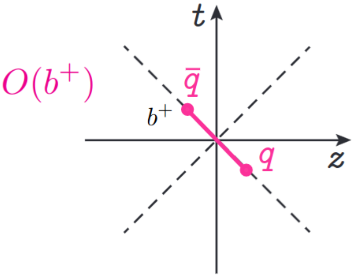
Light front coordinates

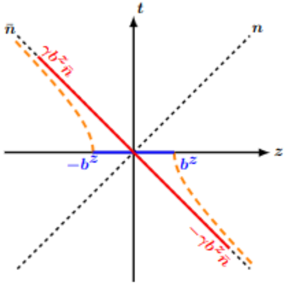
On light front:

- hadrons composed of frozen partons due to time dilation and asymptotic freedom.
- hard processes can be split into perturbatively calculable hard block times non-perturbative matrix elements like PDFs and FFs.

PDFs are real time quantities

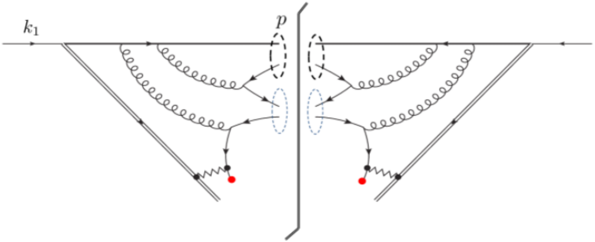
- PDFs inherently non-perturbative and valued on light front; hard to access in standard Euclidean lattice formulations → quasi-distributions [Ji; '13]: light-cone correlations of quarks and gluons calculated by boosting matrix elements of spatial correlations to large momentum
- In Hamiltonian time evolution can compute both. Goal: Benchmark qPDF vs PDF (in 1+1d)





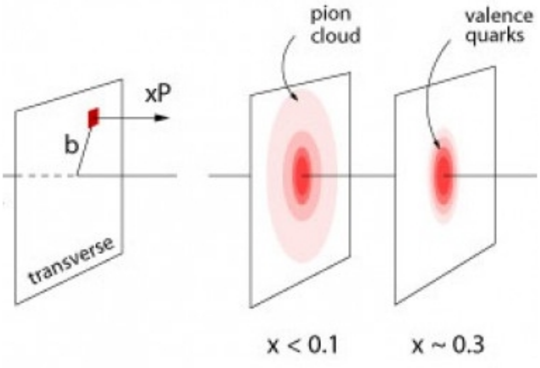
Quark fragmentation

- Light front formulation of fragmentation functions (FFs) was suggested by Collins and Soper.
- Formulation is fully gauge invariant but inherently non-perturbative.
- Collins and Soper FFs are still not accessible to first principle QCD lattice simulations, due to their inherent light front structure
- Introduce concept of quasi-FF
- Drell-Levy-Yan: FFs may be approximated from PDFs using crossing and analyticity symmetries (assuming factorization etc)
- Goal: Crosscheck DLY FF with qFF



Generalized parton distributions (GPDs)

- GPDs: more detailed info on partonic structure of hadrons: correlations between longitudinal parton momentum and transverse spatial position $\rightarrow$ 3d picture of partonic content of hadrons
- Here: Establish first non-perturbative analysis of the qGPDs in massive QED2



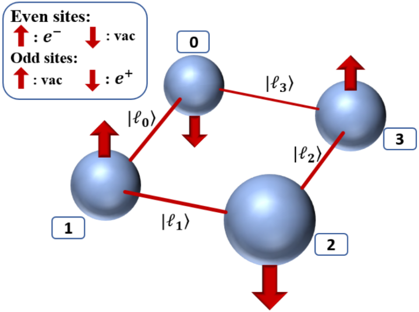
Building a computational framework

Idea:

Create controlled theoretical framework to benchmark performance and accuracy of quantum simulations in nuclear physics

0. Problem where 1+1d toy model can be generalized to QCD_4.
1. 1+1 d system that can be solved in the continuum limit
2. Solve corresponding discretized version using exact diagonalization and tensor networks
3. Design quantum circuit
4. Quantum simulation in $d = 1 + 1$
5. ... $d = 3 + 1$

Lattice Schwinger model in 1+1d



The massive Schwinger model: QED_2

Massive Schwinger model: [Schwinger; '62], [Coleman; '76]

$$S = \int d^2x \left(\frac{1}{4} F_{\mu\nu}^2 + \overline{\psi}(i\not{D} - m)\psi \right) \text{ with } \not{D} = \not{\partial} - ig\not{A}.$$

- ψ fermion field (Dirac spinor), $\bar{\psi} = \psi^\dagger \gamma^0$
- $D_\mu = \partial_\mu - igA$ covariant derivative: coupling of gauge field A_μ to fermion field. Charge of fermion: g.
- m mass of fermion (electron). Mass term breaks chiral symmetry explicitly ($m = 0 \rightarrow$ exactly solvable)
- Fermions interact with gauge field (E-field between charged fermions), leading to confinement (confines fermions into bound states, like mesons in QCD).
- Interaction between fermions and gauge field $\rightarrow$ charge screening (vacuum polarizes around charges); modifies vacuum significantly.

The massive Schwinger model: QED_2

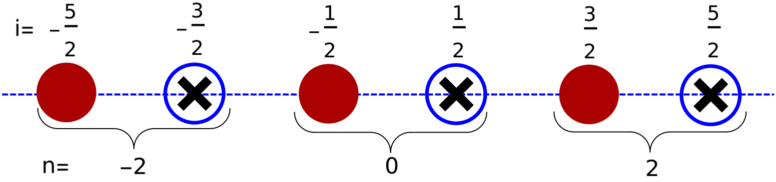
Staggered fermions: $\psi(0, z = na) = \frac{1}{\sqrt{a}} \begin{pmatrix} \psi_e(n) \\ \psi_o(n) \end{pmatrix} = \frac{1}{\sqrt{a}} \begin{pmatrix} \varphi_{n:\text{even}} \\ \varphi_{n+1:\text{odd}} \end{pmatrix}$

Optional: Jordan-Wigner map to spins $\varphi_n = \prod_{m<n}[+iZ_m]\frac{1}{2}(X_n - iY_n)$.

Spin-Hamiltonian:

$$H = \frac{1}{4a}\sum_{n=1}^{N-2}(X_n X_{n+1} + Y_n Y_{n+1}) + \frac{m}{2}\sum_{n=0}^{N-1}(-1)^n Z_n + \frac{ag^2}{2}\sum_{n=0}^{N-2} L_n^2$$

$L_n = L_0 + \sum_{m=0}^{n} \frac{Z_m + (-1)^m}{2}$.



First excited state

Use **open boundary conditions** + eliminate gauge field using Gauss's law; solve system with **exact diagonalization** and **tensor networks**

Consider mass gap m_η of first excited state $|\eta(0)\rangle$ (meson-like state).

Strong coupling $m/g \ll 1/\pi$**:**
(split in pseudo-scalar mass due to $U(1)$ anomaly + chiral condensate)

$$m_\eta^2 = m_S^2 + m_\pi^2 = \frac{g^2}{\pi} - 4\pi\, m\langle\overline{\psi}\psi\rangle_0,$$

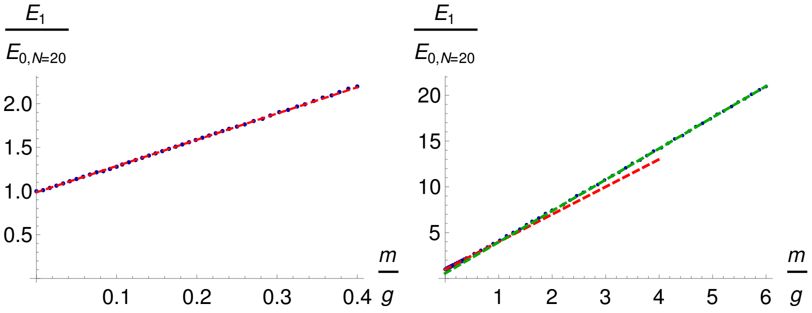
with chiral condensate $\langle\overline{\psi}\psi\rangle_0 = -\frac{e^{\gamma_E}}{2\pi} m_S$, where $\gamma_E = 0.577$.

$$\boldsymbol{\frac{m_\eta}{m_S}} = \left(1 + 2e^{\gamma_E}\frac{m}{m_S}\right)^{\frac{1}{2}} \approx 1 + e^{\gamma_E}\frac{m}{m_S} \approx \boldsymbol{1 + 1.78\frac{m}{m_S}}$$

Weak coupling $\frac{m}{g} \gg \frac{1}{\pi}$: $\boldsymbol{m_\eta \to 2m}$.

Mass gap of first excited state

Mass gap in finite spatial box receives finite size corrections
$E_0 = \sqrt{m_s^2 + \pi^2/L^2}$ with $L = N \cdot a$ and $m_s^2 = g^2/\pi$.



Red-dashed line fit to $\frac{E}{E_0} = 0.99 + 1.76 \frac{m}{E_0}$

green-dashed line $\frac{E}{E_0} = \frac{0.33+1.99\,m}{E_0}$. Crossing from strong to weak coupling at about $m/g \sim 1/3$.

Works well numerically (even for a small number of gridpoints).

Boost operator in QED2

Boost excited state at equal time toward light cone $\mathbb{K} = \int dx\, x\mathcal{H}$.

η' is the lowest massive meson in the spectrum at strong coupling

$$|\eta(\chi)\rangle = e^{i\chi\mathbb{K}}|\eta(0)\rangle,\ \ \chi \equiv \frac{1}{2}\ln\left(\frac{1+v}{1-v}\right),$$
$$:\mathbb{H}:|\eta(\chi)\rangle = m_\eta \cosh\chi|\eta(\chi)\rangle,$$
$$:\mathbb{P}:|\eta(\chi)\rangle = m_\eta \sinh\chi|\eta(\chi)\rangle.$$

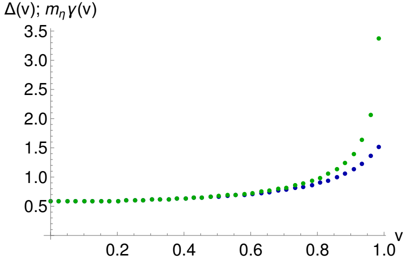
with $p^\mu = \gamma m_\eta(1, v)$, $\gamma = \cosh\chi = 1/\sqrt{1-v^2}$.

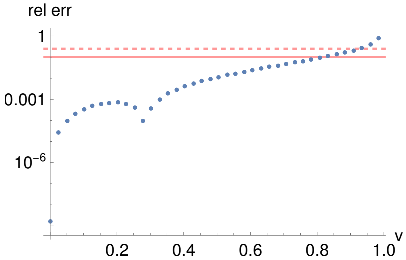
To benchmark the accuracy of the boost, consider

$$\Delta(v) \equiv \langle\eta(v)|:\mathbb{H}:|\eta(v)\rangle = \langle\eta(v)|\mathbb{H}|\eta(v)\rangle - E_0.$$

Boosted excited state: exact diagonalization

$\Delta(v) \equiv \langle \eta(v)| :\mathbb{H}: |\eta(v) \equiv m_\eta\, \gamma(\chi)$; fix $m_{\text{lat}}{=}0$, $N = 24, g = 1, a = 1$

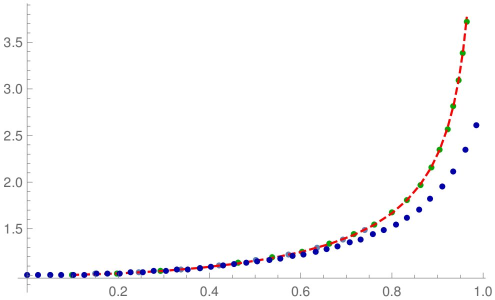




Error in excess of 10% (at around $v \gtrsim 0.83$), and in excess of 20% (at around $v \gtrsim 0.91$). Also, the overlap $\langle \eta(0)|0(v)\rangle$ is nonzero.

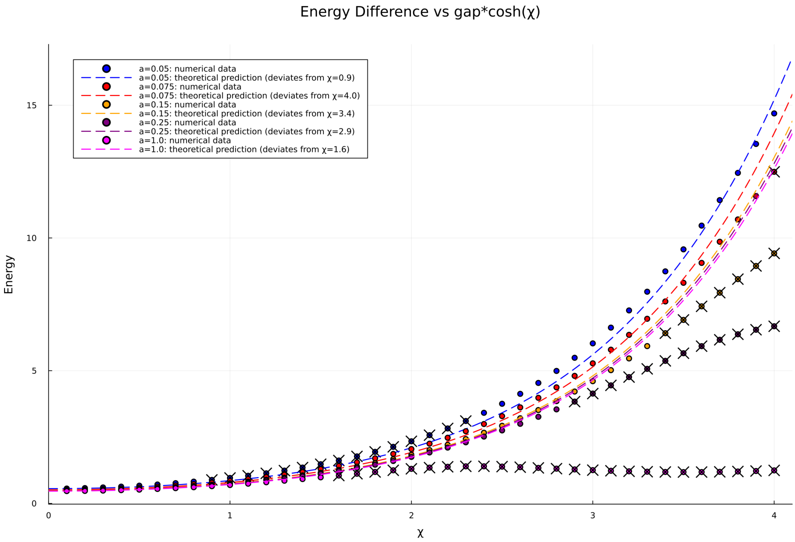
Large amount of resource needed (already in 1+1d). 24 gridpoints far too little ⇒ Quantum hardware needed eventually to study 3+1 d

Boosted excited state using matrix product states



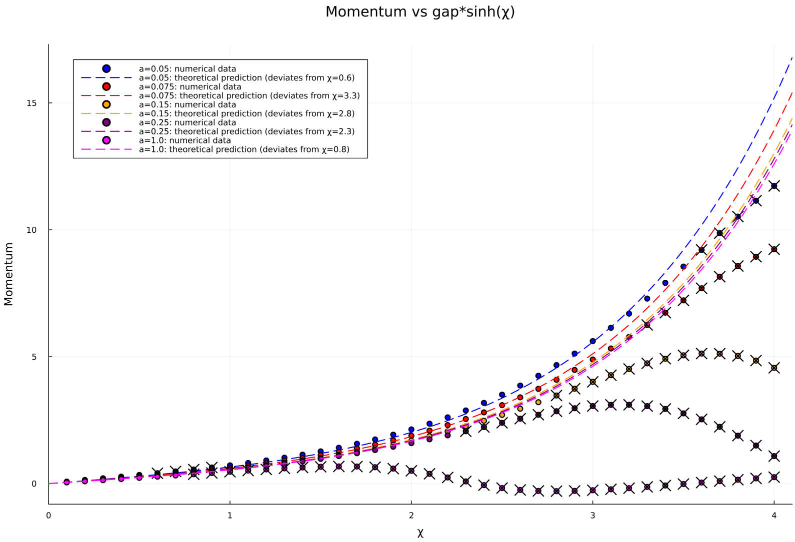
Tensor network calculation with $N = 180$ and lattice spacing $a = 0.33$. Largest symmetric error only 1.2%!

Where is the limit?



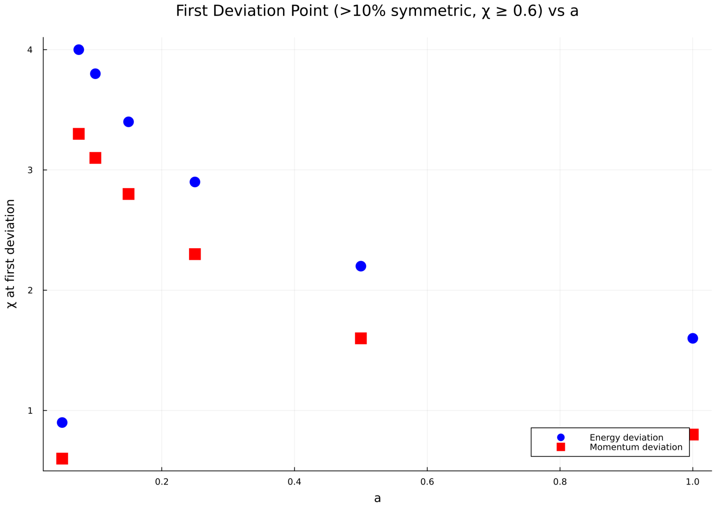
$$\chi = \frac{1}{2}\log\left(\frac{1+v}{1-v}\right); \chi = 2 \leftrightarrow v = 0.964; \chi = 3 \leftrightarrow v = 0.995; \chi = 4 \leftrightarrow 0.999$$

Where is the limit?



$$\chi = \frac{1}{2}\log\left(\frac{1+v}{1-v}\right); \chi = 2 \leftrightarrow v = 0.964; \chi = 3 \leftrightarrow v = 0.995; \chi = 4 \leftrightarrow 0.999$$

Summary for N=202



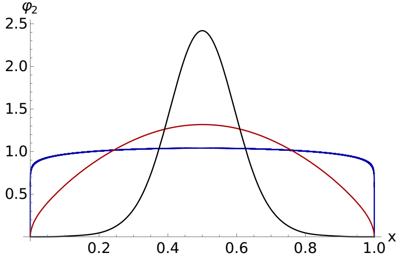
$$\chi = \frac{1}{2}\log\left(\frac{1+v}{1-v}\right); \chi = 2 \leftrightarrow v = 0.964; \chi = 3 \leftrightarrow v = 0.995; \chi = 4 \leftrightarrow 0.999$$

Light front wavefunctions

Light front wavefunctions $\varphi_n(\zeta)$ in 2-particle Fock-space approx solve:
(ζP symmetric momentum fraction of partons, $\zeta = 2x - 1$) [Bergknoff; '77]

$$M_n^2 \varphi_n(\zeta)$$
$$= \frac{1}{2} m_S^2 \int_{-1}^{1} d\zeta' \, \varphi_n(\zeta') + \frac{4m^2}{1-\zeta^2} \varphi_n(\zeta) - 2m_S^2 \, \mathrm{PP} \int_{-1}^{1} d\zeta' \, \frac{\varphi_n(\zeta') - \varphi_n(\zeta)}{(\zeta' - \zeta)^2}$$

't Hooft equation + U(1) anomaly; M_n is mass gap
Due to pole: $\varphi_n(\pm 1) \stackrel{!}{=} 0$, PDF: $q_\eta(x) = |\varphi(x)|^2$.
Expansion using orthonormal Jacobi polynomials $P_n^{2\beta,2\beta}$ [Mo, Perry; '93]



$\beta = 0.1\sqrt{3}/\pi$ (blue),
$\beta = \sqrt{3}/\pi$ (red),
$\beta = 10\sqrt{3}/\pi$ (black) using 13 Jacobi polynomials.

Boosted quasi-distributions

The partonic distribution function (PDF) for the boosted pseudo-scalar (in rest frame) is defined as

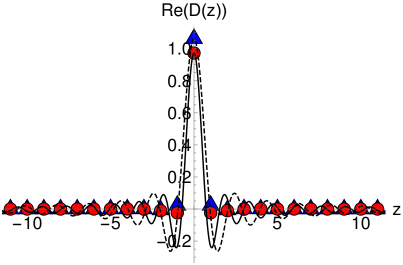
$$q_\eta(x, v) = \int_{-\infty}^{+\infty} \frac{dz}{4\pi} e^{-iz\zeta p^1} \langle \eta(0)| \, e^{-i\chi\mathbb{K}} \, \overline{\psi}(0, z)[z, -z]\gamma^+\gamma^5\psi(0, -z) \, e^{i\chi\mathbb{K}} \, |\eta(0)\rangle.$$

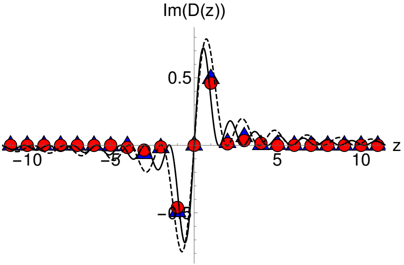
with $p^1 = \gamma m_\eta v$ and $\zeta = 2x - 1$ with x the parton fraction. Here $\gamma^+ = \gamma^0 + \gamma^1$, $[z, -z]$ is link along spatial direction. PDA similar.

Both defined at equal time for a fixed boost, reduce to Ji's light front partonic functions in the large rapidity limit $\chi \gg 1$. The PDF is

$$\begin{aligned} q_\eta(\zeta, v) =& \frac{1}{2\pi} \sum_n e^{-in\zeta aP(v)} \langle \eta(0)| e^{-i\chi(v)\mathbb{K}} (\varphi_n^\dagger + \varphi_{n+1}^\dagger)(\varphi_{-n} + \varphi_{-n+1}) e^{i\chi(v)\mathbb{K}} \, |\eta(0)\rangle \\ \equiv& \frac{1}{2\pi} \sum_n \, e^{-in\zeta aP(v)} \, D(na). \end{aligned}$$

Spatial quasi-distribution function using exact diagonalization





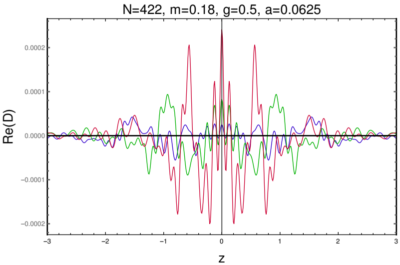
Parameters for strong coupling result: $v = 0.925$ and $m/m_s = 0.1$ (red disks) and improved mass m_{lat} (blue triangles), we fixed $N = 26$, with $a = g = 1$. Black lines are inverse Fourier transforms of light front wave function result (scaled to peak).

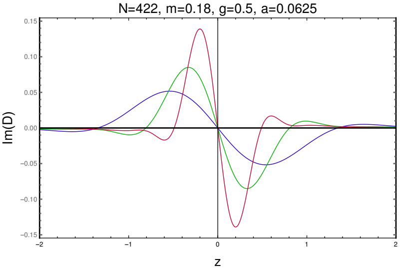
Large amount of resource needed (already in 1+1d). 26 gridpoints far too little $\Rightarrow$ Quantum hardware needed eventually to study 3+1 d

PDFs from matrix product states (tensor networks)

Now consider matrix element with vacuum expectation value subtracted, i.e.

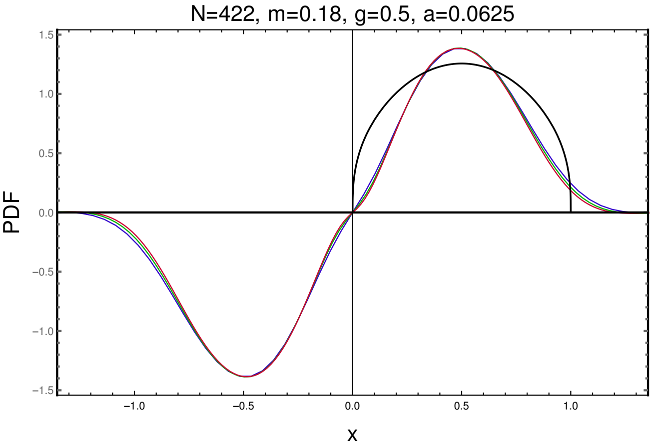
$$D(na) - \langle 0| e^{-i\chi(v)\mathbb{K}} (\varphi_n^\dagger + \varphi_{n+1}^\dagger)(\varphi_{-n} + \varphi_{-n+1}) e^{i\chi(v)\mathbb{K}} |0\rangle$$





Parameters: $m = 0.18$, $N = 422$, $a = 0.0625$, $g = 0.5$.

PDFs from tensor networks (preliminary)



(Pre-liminary) Tensor network results for v=0.995055 ($\chi = 3$) in red.
Black curve is two-particle Fock space solution.

Hamiltonian evolution to light front

Trade boost for Hamiltonian time evolution. Use the boost and "time" identities:

$$e^{-i\chi\mathbb{K}}\psi(0,-z)e^{i\chi\mathbb{K}} = e^{\chi\gamma^5/2}\psi(-\gamma vz,\gamma z)$$
$$\psi(-vz,z) = e^{-ivz\mathbb{H}}\psi(0,z)e^{ivz\mathbb{H}}$$

Resulting eventually in:

$$q_\eta = \frac{1}{2\pi}\sum_n e^{-in\zeta am_\eta v}\langle\eta(0)|\left(\varphi_n^\dagger + \varphi_{n+1}^\dagger\right)e^{-i2vn\mathbb{H}}\left(\varphi_{-n} + \varphi_{-n+1}\right)|\eta(0)\rangle$$

Problem so far: bond dimension in TN simulation is growing very fast during time evolution

Quasi-GPDs

Light front GPD for η' in QED2 is off-diagonal ME of analogue of leading twist-2 quark operator in QCD4

$$H(x,\xi,t) = \frac{1}{2}\int_{-\infty}^{+\infty} \frac{dz^-}{2\pi}\, e^{i\zeta P^+ z^-} \times$$
$$\Big\langle P+\frac{\Delta}{2}\Big|\overline{\psi}(-z^-)[-z^-,+z^-]_-\gamma^+\gamma^5\psi(+z^-)\Big|P-\frac{\Delta}{2}\Big\rangle.$$

where $[]_-$ is Wilson-line along light cone direction

2D: no transverse momentum; skewness is tied to momentum transfer through mass-shell condition $m_\eta^2 = \frac{t}{4}\left(1-\frac{1}{\xi^2}\right)$. In our language:

$$H(x,\xi,t,v) = \frac{1}{2}\int_{-\infty}^{+\infty} \frac{dz}{2\pi}\, e^{-iz\zeta P^1(v)} \times$$
$$\langle\eta(0)|\, e^{-i(\chi(v)+\xi_+)\mathbb{K}}\, \overline{\psi}(0,-z)[-z,+z]_S\gamma^+\gamma^5\psi(0,+z)\, e^{i(\chi(v)+\xi_-)\mathbb{K}}\, |\eta(0)\rangle.$$

Conclusions and Outlook

Conclusions:

- Introduced the concept of quasi-fragmentation functions
- Formulated quasi-distribution functions/amplitudes and quasi-fragmentation functions in language suitable for quantum computation

Outlook (in progress):

- qGPD (Generalized Parton Distribution) works analogous $\Rightarrow$ info about skewness
- Much finer lattices are needed for the comparison $\rightarrow$ tensor networks
- Check the proposal for the qFF versus the FF computed from DLY
- Multi-flavor case
- Set up the calculation on a quantum computer

Quark fragmentation

- Quark fragmentation (Field and Feynman): quark jet model to describe meson production in semi-inclusive processes
- Quark jet model independent parton cascade model: hard parton depletes its longitudinal momentum by emitting successive mesons through chain process (e.g. string breaking in Lund model)
- Jet fragmentation and hadronization important for collider experiments to extract partonic structure of matter, gluon helicity in nucleons and mechanism behind the production of diffractive dijets.
- FFs describe how a high-energy parton transforms into a jet of hadrons; counterpart of PDFs but describe "reverse" process: parton hadronizes

Collins-Soper fragmentation functions (I)

Measures the amount of meson outgoing from the quark.

On light front, gauge-invariant definition of the QCD quark fragmentation $Q \to Q + H$ was given by Collins and Soper. Introduce the **spatially symmetric qFF**

$$d_q^\eta(z,v) = \frac{1}{z}\int \frac{dZ}{4\pi} e^{-i(\frac{2}{z}-1)P(v)Z}$$
$$\mathrm{Tr}\left(\gamma^+\gamma^5\langle 0|\psi(-Z)[-Z,\infty]^\dagger a_{\mathrm{out}}^\dagger(P(v))a_{\mathrm{out}}(P(v))[\infty,Z]\overline{\psi}(Z)|0\rangle\right)$$

where $P(v) = \gamma(v)m_\eta v$ is momentum fraction carried by the emitted η from mother quark jet with momentum $P(v)/z$.

The asymptotic time limit implements the LSZ reduction on source field

$$a_{\mathrm{out}}^\dagger(P)a_{\mathrm{out}}(P) = \frac{2}{f^2}e^{i\mathbb{H}t}|\psi^\dagger\gamma_5\psi(0,P(v))|^2 e^{-i\mathbb{H}t}|_{t\to+\infty}.$$

Collins-Soper fragmentation function (II)

The **symmetric qFF** can be recast in terms of the spatial qFF correlator

$$d_q^{\eta}(z, v) = \frac{1}{z} \int \frac{dZ}{4\pi} e^{-i(\frac{2}{z}-1)P(v)Z}\, \mathbb{C}(Z, v, \infty),$$

$$\mathbb{C}(Z, v, t) = \frac{2}{f^2} \mathrm{Tr}\Big(\gamma^+\gamma^5 \langle 0|\psi(0, -Z)[-Z, \infty]^\dagger e^{i\mathbb{H}t} e^{i\chi(v)\mathbb{K}} |\psi^\dagger \gamma_5 \psi(0, m_\eta)|^2 \, e^{-i\chi(v)\mathbb{K}} e^{-i\mathbb{H}t} [\infty, Z] \overline{\psi}(0, Z)|0\rangle\Big).$$

Under combined boost and time evolution, the equal-time fermion field is now lying on the light cone.

Computed $\mathbb{C}(Z, v, \infty)$ in lattice model using exact diagonalization/tensor networks.

Discretized Lattice qFF

Recall:

$$\mathbb{C}(Z,v,t) = \frac{2}{f^2}\mathrm{Tr}\Big(\gamma^+\gamma^5\langle 0|\psi(0,-Z)[-Z,\infty]^\dagger e^{i\mathbb{H}t}e^{i\chi(v)\mathbb{K}}|\psi^\dagger\gamma_5\psi(0,0)|^2$$
$$e^{-i\chi(v)\mathbb{K}}e^{-i\mathbb{H}t}[\infty,Z]\overline{\psi}(0,Z)|0\rangle\Big).$$

Same discretization as for PDF. New element:

$$|\psi^\dagger\gamma_5\psi(0,0)|^2 = \frac{1}{a^2}\left|\sum_n(\sigma_n^+\sigma_{n+1}^- - \sigma_{n+1}^+\sigma_n^-)\right|^2,$$

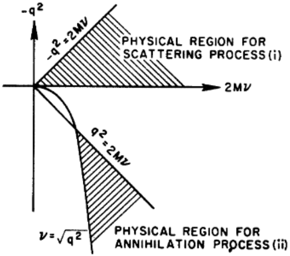
where $\sigma_n^\pm = \frac{1}{2}(X_n \pm iY_n)$. Discretized form of the symmetric spatial qFF:

$$\mathbb{C}(n,v,t) = \frac{4}{aF^2}\sum_{i,j=e,o} e^{in\gamma am_\eta}\langle 0|\psi_i(-n)e^{i\mathbb{H}t}|\psi^\dagger\gamma_5\psi(0,0)|^2 e^{-i\mathbb{H}t}\psi_j^\dagger(n)|0\rangle.$$

Drell-Levy-Yan relation

Crossing symmetry and charge conjugation: Estimate of the CS FF in terms of PDFs using the DLY

$$d_{DLY}(z,v) = z^{d-3}\, p_\eta\left(\frac{1}{z}, v\right)$$



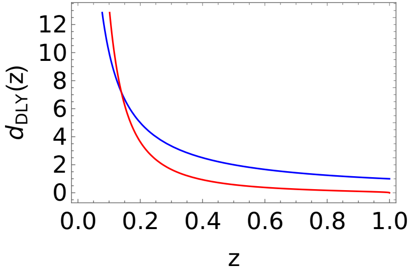
$p_\eta \equiv |\varphi_2|^2 \sim$ probability of finding parton of momentum fraction x in hadron $p(x)$. DLY: is related to the amount of meson spit out by parton with fraction of momentum z.

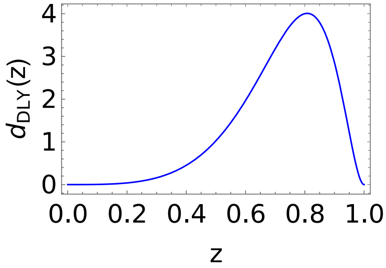
Using the EVP:

$$d_{DLY}(z,1) = \frac{\bar{z}^2}{z(\bar{z}\mu^2 + z^2\bar{\alpha})^2}\left(f - \int_0^1 dx \frac{\varphi(x)}{(x-1/z)^2}\right)^2$$

with $\mu^2 = M^2/m_S^2$ and $1+\bar{\alpha} = \alpha = m^2/m_S^2$.

DLY

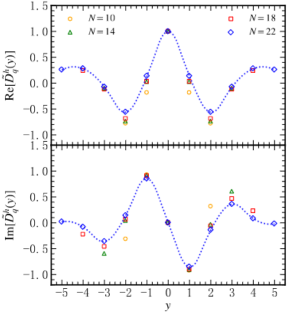


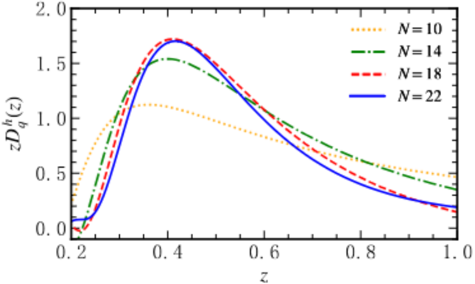


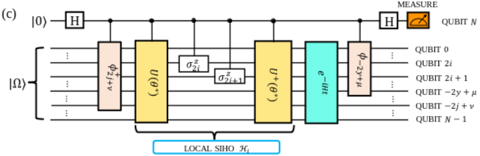
Strong coupling DLY fragmentation function (light quarks): $\beta = 0$ (blue) and $\beta = 0.2$ (red). The divergence for small masses (small β) is in agreement with the exact bosonization description of QED2

DLY fragmentation function for heavy quarks: FF is peaked in the forward (jet) direction, with a strong suppression as $z \to 0$ (vanishes for $1 = z = 0$).

Can it work on a QC? [Li, Xing, Zhang (QuNu Collaboration), arXiv:2406.05683]







In NJL model using QuSpin and projectQ