Latest Developments in the Theory of Multi-Hadron Fragmentation Functions

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Based on

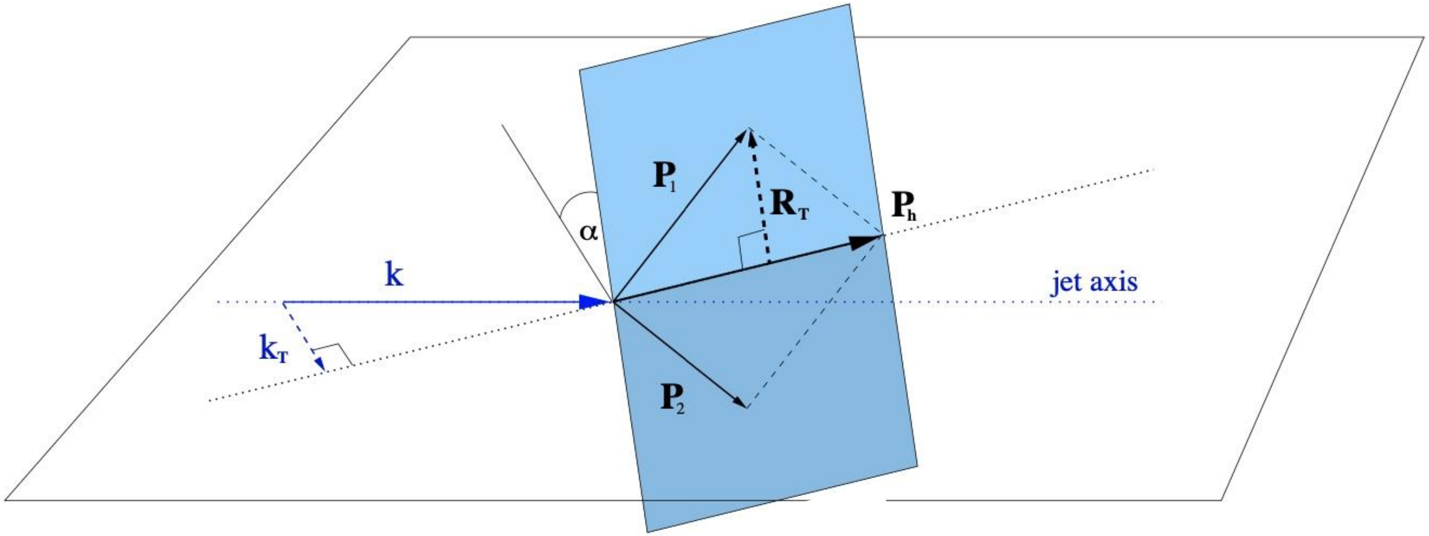
D. Pitonyak, C. Cocuzza, A. Metz, A. Prokudin and N. Sato,
“Number density interpretation of dihadron fragmentation functions,”
Phys. Rev. Lett. **132**, 011902 (2024) [arXiv:2305.11995 [hep-ph]].

D. Pitonyak, C. Cocuzza, A. Metz, A. Prokudin and N. Sato,
“Comment on “QCD factorization with multihadron fragmentation functions”,”
[arXiv:2502.15817 [hep-ph]], submitted to PRD.

Outline

- Background and motivation: why dihadron fragmentation?
- Recent DiFF (and multi-hadron FF) theory developments: new definition and its number density interpretation, sum rules, and evolution equations
- Comments on previous DiFF definitions, dihadron cross section results, and other claims in the literature (especially regarding the compatibility of our new definition with factorization)
- Summary

Background and Motivation



From Bianconi, et al. (2000)

Bianconi, et al. (2000); Bacchetta, Radici (2003, 2004), …

$$P_h = P_1 + P_2 \qquad R = (P_1 - P_2)/2 \qquad \xi_1 = P_1^-/k^- \quad \xi_2 = P_2^-/k^-$$

$$\xi = \xi_1 + \xi_2 \qquad \zeta = (P_1^- - P_2^-)/P_h^- = (\xi_1 - \xi_2)/\xi$$

$$P_1 = \left(\frac{M_1^2 + \vec{R}_T^2}{(1+\zeta)P_h^-}, \frac{1+\zeta}{2}P_h^-, \vec{R}_T\right) \qquad P_2 = \left(\frac{M_2^2 + \vec{R}_T^2}{(1-\zeta)P_h^-}, \frac{1-\zeta}{2}P_h^-, -\vec{R}_T\right)$$

$$\vec{R}_T^2 = \frac{1-\zeta^2}{4}M_h^2 - \frac{1-\zeta}{2}M_1^2 - \frac{1+\zeta}{2}M_2^2$$

Note: Sometimes the variable $\xi = (1+\zeta)/2$ has been used, which is different from the momentum fraction ξ above.

- Dihadron fragmentation involves more structures than single-hadron fragmentation (only unpolarized hadron FFs are shown below)

Single-hadron FFs

$$\Delta^{h/q}(\xi,\vec{k}_T) \longrightarrow D_1^{h/q}(\xi,\xi^2\vec{k}_T^2),\;\; -\frac{\epsilon_T^{ij}k_T^j}{M_h}H_1^{\perp h/q}(\xi,\xi^2\vec{k}_T^2)$$

Dihadron FFs

(Bianconi, et al. (2000);
Bacchetta, Radici (2003, 2004))

$$\Delta^{h_1h_2/q}(\xi,\zeta,\vec{k}_T,\vec{R}_T) \longrightarrow \left\{ \begin{array}{l} D_1^{h_1h_2/q}(\xi,\zeta,\vec{k}_T^2,\vec{R}_T^2,\vec{k}_T\cdot\vec{R}_T), \\ \dfrac{\epsilon_T^{ij}R_T^ik_T^j}{M_h^2}\,G_1^{\perp\, h_1h_2/q}(\xi,\zeta,\vec{k}_T^2,\vec{R}_T^2,\vec{k}_T\cdot\vec{R}_T), \\ -\dfrac{\epsilon_T^{ij}R_T^j}{M_h}H_1^{\sphericalangle\prime\, h_1h_2/q}(\xi,\zeta,\vec{k}_T^2,\vec{R}_T^2,\vec{k}_T\cdot\vec{R}_T), \\ -\dfrac{\epsilon_T^{ij}k_T^j}{M_h}H_1^{\perp\prime\, h_1h_2/q}(\xi,\zeta,\vec{k}_T^2,\vec{R}_T^2,\vec{k}_T\cdot\vec{R}_T) \end{array} \right.$$

- Dihadron fragmentation involves more structures than single-hadron fragmentation (only unpolarized hadron FFs are shown below)

Single-hadron FFs

$$\int d^2\vec{k}_T\,\Delta^{h/q}(\xi,\vec{k}_T)\longrightarrow D_1^{h/q}(\xi)$$

Dihadron FFs

$$\int d^2\vec{k}_T\,\Delta^{h_1h_2/q}(\xi,\zeta,\vec{k}_T,\vec{R}_T)\longrightarrow\begin{cases} D_1^{h_1h_2/q}(\xi,\zeta,\vec{R}_T^2), \\ -\dfrac{\epsilon_T^{ij}R_T^j}{M_h}H_1^{\sphericalangle h_1h_2/q}(\xi,\zeta,\vec{R}_T^2)\end{cases}$$

- Dihadron fragmentation involves more structures than single-hadron fragmentation (only unpolarized hadron FFs are shown below)

Single-hadron FFs

$$\int d^2\vec{k}_T\,\Delta^{h/q}(\xi,\vec{k}_T)\longrightarrow D_1^{h/q}(\xi)$$

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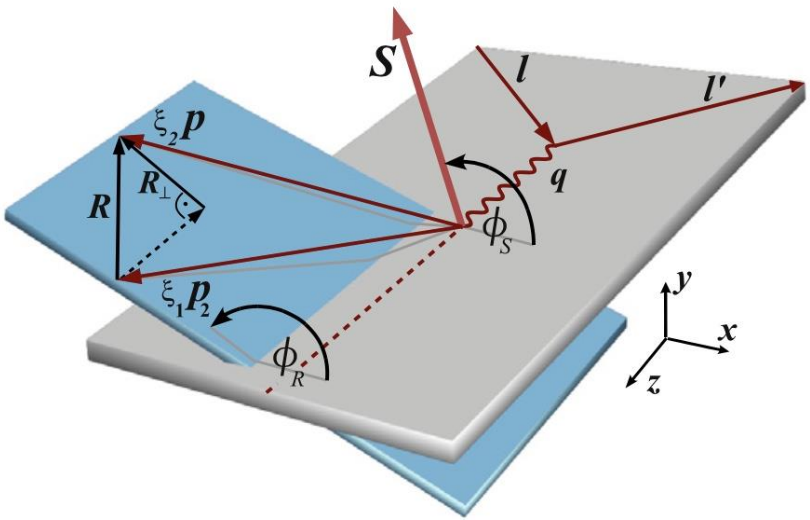
chiral-odd "interference" FF (IFF)

(Collins, et al. (1994); Bianconi, et al. (2000); Bacchetta, Radici (2003, 2004); Courtoy, et al. (2012); Matevosyan, et al. (2018); Radici, et al. (2013, 2015, 2018); Benel, et al. (2020), Courtoy, et al. (2014, 2022); Cocuzza, et al. (2024))

$$\ell\, N \to \ell\,(h_1 h_2)\, X$$



$$A_{UT}^{\sin(\phi_R+\phi_S)} = \frac{\sum_q e_q^2\, \boldsymbol{h_1^q(x)}\, \boldsymbol{H_1^{\sphericalangle,q}(z,M_h)}}{\sum_q e_q^2\, f_1^q(x)\, \boldsymbol{D_1^q(z,M_h)}}$$

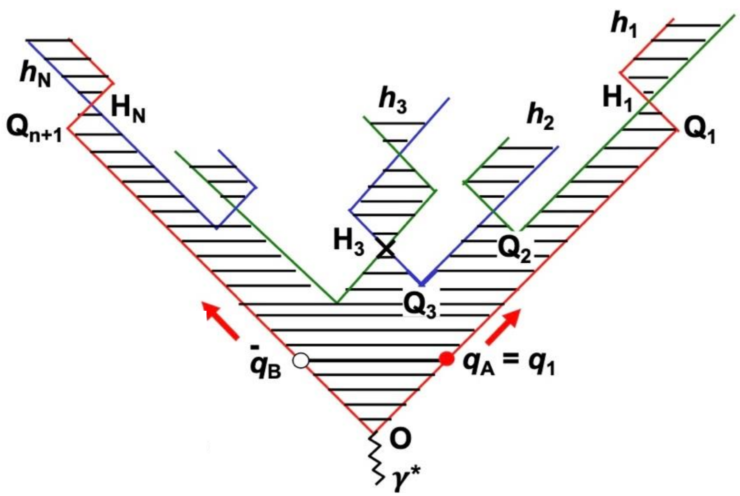
$h_1(x)$ = transversity PDF - connection to the tensor charges of the nucleon

$$A_{LU}^{\sin\phi_R} \sim \frac{\sum_q e_q^2 \left[x\,\boldsymbol{e^q(x)}\, \boldsymbol{H_1^{\sphericalangle,q}(z,M_h)} + \frac{M_h}{zM}\, f_1^q(x) \tilde{G}^{\sphericalangle,q}(z,M_h) \right]}{\sum_q e_q^2\, f_1^q(x)\, \boldsymbol{D_1^q(z,M_h)}}$$

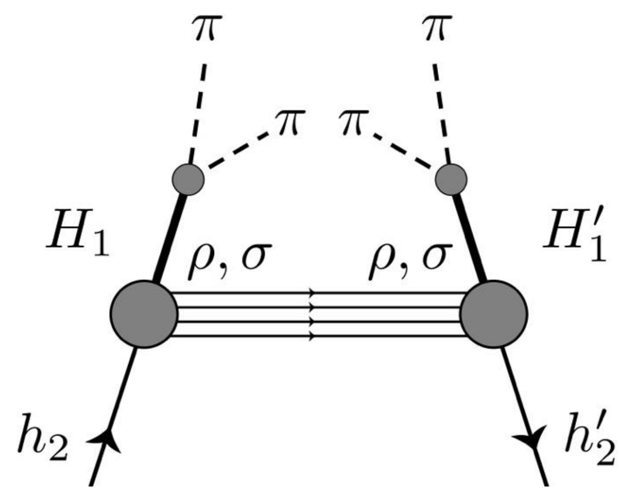
$e(x)$ = twist-3 PDF - connection to the decomposition of the nucleon mass

(Collins, et al. (1994); Bianconi, et al. (2000); Bacchetta, Radici (2003, 2004); Courtoy, et al. (2012); Matevosyan, et al. (2018); Radici, et al. (2013, 2015, 2018); Benel, et al. (2020), Courtoy, et al. (2014, 2022); Cocuzza, et al. (2024))

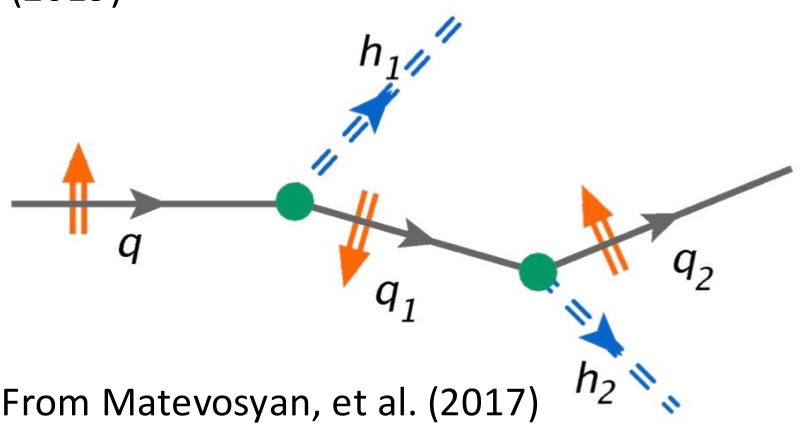
- DiFFs are interesting in their own right, e.g., one can test models for (un)polarized parton fragmentation/hadronization (Collins, Ladinsky (1994); Jaffe, et al. (1998); Bianconi, et al. (2000); Bacchetta, Radici (2006); Matevosyan, et al. (2017, 2018); Kerbizi, et al. (2019, 2023))



From Kerbizi, et al. (2019)



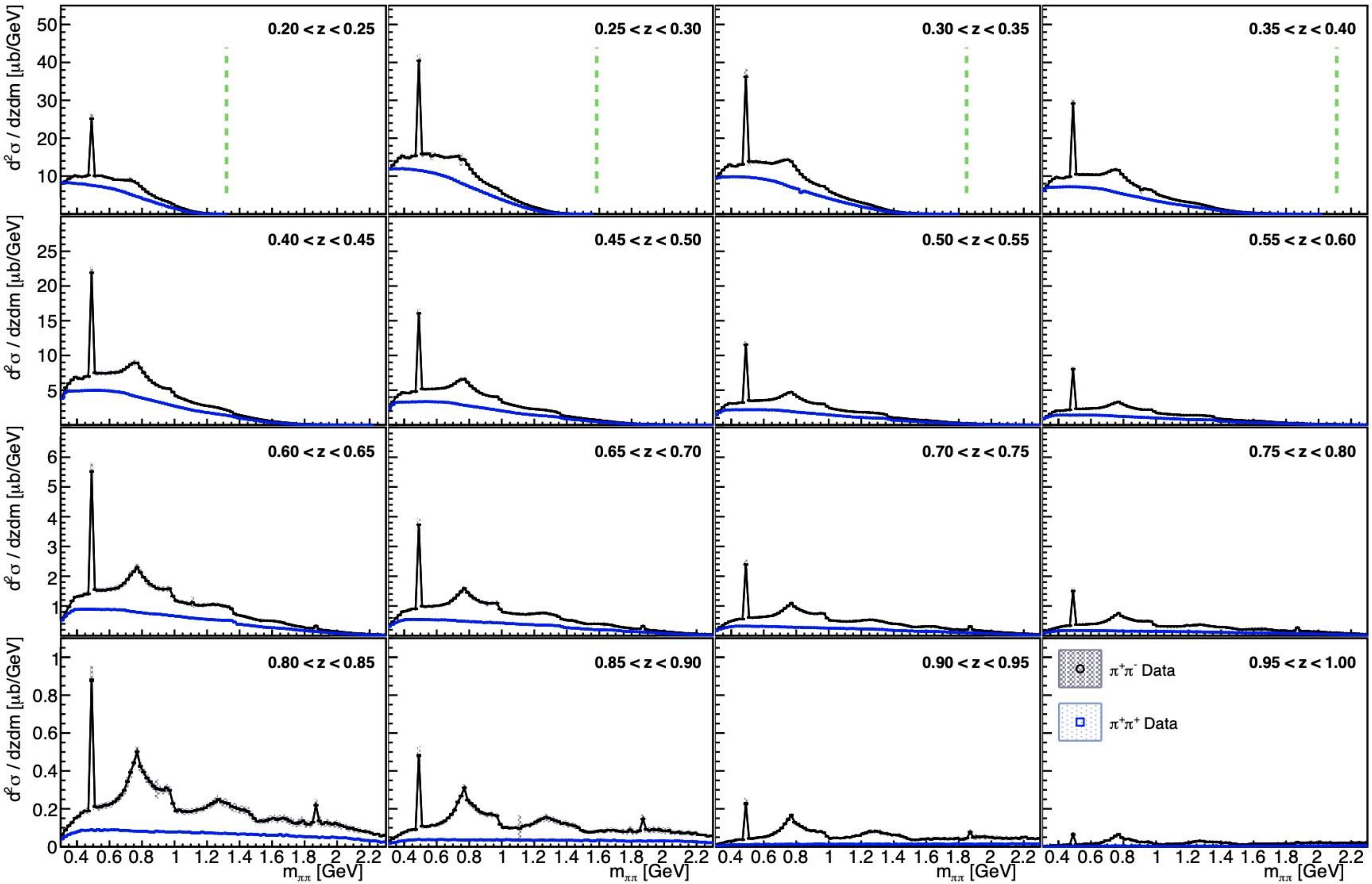
From Jaffe, et al. (1998)



From Matevosyan, et al. (2017)

- There is also a complicated/interesting resonance structure that can/must be analyzed

Belle (2017)



Recent DiFF Theory Developments

- An aside: notation and reference frames
  - For *n*-hadron FFs, $\xi_i = \dfrac{P_i^-}{k^-}$ and $\xi = \sum_{i=1}^{n} \xi_i$
  - The arguments of the FF will denote in which variables it is a number density, e.g., $D_1^{h_1 h_2/q}(\xi_1, \xi_2, \vec{P}_{1\perp}, \vec{P}_{2\perp})$ is a number density in $(\xi_1, \xi_2, \vec{P}_{1\perp}, \vec{P}_{2\perp})$
  - "Parton frame" (p): parton has no transverse momentum, hadron has transverse momentum $P_\perp$ - useful in the formulation of FFs as number densities and proofs of sum rules
  - "Hadron frame" (h): hadron has no transverse momentum, parton has transverse momentum k_T - more practical for phenomenology

$$V_{\rm p}^- = V_{\rm h}^- \equiv V^-$$
$$V_{\rm p}^+ = (\vec{k}_T/k^-)^2\, V^-/2 + V_{\rm h}^+ - \vec{k}_T \cdot \vec{V}_T/k^-$$
$$\vec{V}_\perp = -(\vec{k}_T/k^-)V^- + \vec{V}_T$$

- (TMD) PDFs and (single-hadron) FFs are defined in a way so that they are number densities in a parton model framework

Number sum rules

$$\sum_{i=u,d,s,..} \int_0^1 d\xi \, [f_1^{i/N}(\xi) - f_1^{\bar{i}/N}(\xi)] = \mathcal{B}$$

($\mathcal{B}$ is the baryon number, e.g.,= 3 for a proton)

$$\sum_h \int_0^1 d\xi \, D_1^{h/i}(\xi) = \langle \mathcal{N} \rangle$$

(<$\mathcal{N}$> is the expectation value for the total number of hadrons produced when the parton fragments)

Momentum sum rules

$$\sum_i \int_0^1 d\xi \, \xi \, f_1^{i/N}(\xi) = 1 \qquad \sum_h \int_0^1 d\xi \, \xi \, D_1^{h/i}(\xi) = 1$$

Note: Paper by Collins, Rogers (2024) has questioned sum rules for FFs, but their analysis does *not* affect the validity of the fundamental definition of single-hadron FFs, our DiFF (or *n*-hadron FF) definitions, nor their interpretations as a number densities.

$$D_1^{h/q}(\xi, \vec{P}_\perp) = \frac{1}{N_c}\,\frac{\mathbf{1}}{\mathbf{4\xi}} \sum_X \int \frac{dx^+ d^2\vec{x}_\perp}{(2\pi)^3}\, e^{ik^- x^+} \mathrm{Tr}\Big[\langle 0|\mathcal{W}(\infty, x)\psi_q(x^+, 0^-, \vec{x}_\perp)|P; X\rangle \times \langle P; X|\bar{\psi}_q(0^+, 0^-, \vec{0}_\perp)\mathcal{W}(0,\infty)|0\rangle\, \gamma^-\Big]$$

$$D_1^{h/q}(\xi,\vec{P}_\perp)=\frac{1}{N_c}\frac{\mathbf{1}}{\mathbf{4\xi}}\sum_X\int\frac{dx^+d^2\vec{x}_\perp}{(2\pi)^3}\,e^{ik^-x^+}\,\mathrm{Tr}\Big[\langle 0|\mathcal{W}(\infty,x)\psi_q(x^+,0^-,\vec{x}_\perp)|P;X\rangle \times\,\langle P;X|\bar{\psi}_q(0^+,0^-,\vec{0}_\perp)\mathcal{W}(0,\infty)|0\rangle\,\gamma^-\Big]$$

$$\sum_h\int_0^1 d\xi\int d^2\vec{P}_\perp D_1^{h/q}(\xi,\vec{P}_\perp)=\frac{1}{N_c}\frac{1}{2}\int dx^+d^2\vec{x}_\perp\,e^{ik^-x^+}\,\mathrm{Tr}\Big[\langle 0|\mathcal{W}(\infty,x)\psi_q(x^+,0^-,\vec{x}_\perp)\hat{N}\bar{\psi}_q(0^+,0^-,\vec{0}_\perp)\times\mathcal{W}(0,\infty)|0\rangle\,\gamma^-\Big]$$

where $$\hat{N}\equiv\sum_h\int\frac{dP^-d^2\vec{P}_\perp}{(2\pi)^3\,2P^-}\,\hat{a}_h^\dagger\hat{a}_h=\sum_h\int\frac{d\xi\,d^2\vec{P}_\perp}{(2\pi)^3\,2\xi}\,\hat{a}_h^\dagger\hat{a}_h$$

Introduce "good" quark fields $\psi_{-,q}\equiv\frac{1}{2}\gamma^+\gamma^-\psi_q$, insert their anticommutator, and use

$$\{\psi_{-,q}(x^+,0^-,\vec{x}_\perp),\psi^\dagger_{-,q}(0^+,0^-,\vec{0}_\perp)\}=\frac{1}{2\sqrt{2}}\gamma^+\gamma^-\delta(x^+)\delta^{(2)}(\vec{x}_\perp)$$

$$\sum_h\int_0^1 d\xi\int d^2\vec{P}_\perp\,D_1^{h/q}(\xi,\vec{P}_\perp^2)=\langle\mathcal{N}\rangle$$

$$\Delta_{\alpha\beta}^{h_1h_2/i}(\xi_1,\xi_2,\vec{P}_{1\perp},\vec{P}_{2\perp}) = \frac{1}{N_i}\sum_X\int\frac{dx^+d^2\vec{x}_\perp}{(2\pi)^3}\,e^{ik\cdot x}\,\mathcal{O}_{\alpha\beta}^{h_1h_2/i}(x)\Big|_{x^-=0}$$

quark fragmentation ($N_i = N_c$)

$$\mathcal{O}_{\alpha\beta}^{h_1h_2/q}(x) = \langle 0|\mathcal{W}(\infty,x)\psi_{q,\alpha}(x^+,0^-,\vec{x}_\perp)|P_1,P_2;X\rangle \times \langle P_1,P_2;X|\bar{\psi}_{q,\beta}(0^+,0^-,\vec{0}_\perp)\mathcal{W}(0,\infty)|0\rangle$$

gluon fragmentation ($N_i = N_c^2 - 1$)

$$\mathcal{O}_{\alpha\beta}^{h_1h_2/g}(x) = \langle 0|\mathcal{W}^{ba}(\infty,x)F^a_{+\alpha}(x^+,0^-,\vec{x}_\perp)|P_1,P_2;X\rangle \times \langle P_1,P_2;X|F^c_{+\beta}(0^+,0^-,\vec{0}_\perp)\mathcal{W}^{cb}(0,\infty)|0\rangle$$

NB: we will focus on quark fragmentation, but similar results hold for gluon fragmentation

$$\frac{1}{64\pi^3\xi_1\xi_2}\,\mathrm{Tr}\left[\Delta^{h_1h_2/q}(\xi_1,\xi_2,\vec{P}_{1\perp},\vec{P}_{2\perp})\gamma^-\right] = D_1^{h_1h_2/q}(\xi_1,\xi_2,\vec{P}_{1\perp},\vec{P}_{2\perp})$$

$$\frac{\mathbf{1}}{\mathbf{64\pi^3\xi_1\xi_2}}\mathrm{Tr}\Big[\Delta^{h_1h_2/q}(\xi_1,\xi_2,\vec{P}_{1\perp},\vec{P}_{2\perp})\gamma^-\Big] = D_1^{h_1h_2/q}(\xi_1,\xi_2,\vec{P}_{1\perp},\vec{P}_{2\perp})$$

$$\sum_{h_1}\sum_{h_2}\int d\xi_1\, d^2\vec{P}_{1\perp}\int d\xi_2\, d^2\vec{P}_{2\perp}\, D_1^{h_1h_2/q}(\xi_1,\xi_2,\vec{P}_{1\perp},\vec{P}_{2\perp})$$
$$= \frac{1}{N_c}\frac{1}{2}\int dx^+ d^2\vec{x}_\perp\, e^{ik^-x^+}\mathrm{Tr}\Big[\langle 0|\mathcal{W}(\infty,x)\psi_q(x^+,0^-,\vec{x}_\perp)\Big(\sum_{h_1}\sum_{h_2}\hat{N}_{h_1}\hat{N}_{h_2}-\sum_{h_1}\hat{N}_{h_1}\Big)\bar{\psi}_q(0^+,0^-,\vec{0}_\perp)$$
$$\times\, \mathcal{W}(0,\infty)|0\rangle\,\gamma^-\Big]$$

where
$$\hat{N}_{h_j}\equiv\int\frac{dP_j^-\,d^2\vec{P}_{j\perp}}{(2\pi)^3\,2P_j^-}\,\hat{a}_{h_j}^\dagger\hat{a}_{h_j} = \int\frac{d\xi_j d^2\vec{P}_{j\perp}}{(2\pi)^3\,2\xi_j}\,\hat{a}_{h_j}^\dagger\hat{a}_{h_j}$$

⋮

$$\sum_{h_1}\sum_{h_2}\int_0^1 d\xi_2\int_0^{1-\xi_2}d\xi_1\int d^2\vec{P}_{1\perp}\int d^2\vec{P}_{2\perp}\,D_1^{h_1h_2/q}(\xi_1,\xi_2,\vec{P}_{1\perp},\vec{P}_{2\perp}) = \langle\mathcal{N}(\mathcal{N}-1)\rangle$$

Expectation value for the total number of *hadron pairs* produced when the parton fragments

We can also show the number density interpretation of $D_1^{h_1 h_2/q}(\xi_1, \xi_2, \vec{P}_{1\perp}, \vec{P}_{2\perp})$ by starting from the operator definition and using expressions for the quark field operator $\psi(x)$ in terms of (quark) lightcone creation and annihilation operators to find

$$\sum_{h_1} \sum_{h_2} D_1^{h_1 h_2}(\xi_1, \xi_2, \vec{P}_{1\perp}, \vec{P}_{2\perp}) \langle \vec{k}_1 | \vec{k}_2 \rangle = \langle \vec{k}_1 | \frac{d\Big(\hat{N}(\hat{N}-1)\Big)}{d\xi_1 d\xi_2 d^2\vec{P}_{1\perp} d^2\vec{P}_{2\perp}} | \vec{k}_2 \rangle$$

For the single hadron case, the analogous result reads (Collins (2011))

$$\sum_h D_1^h(\xi, \vec{P}_\perp) \langle \vec{k}_1 | \vec{k}_2 \rangle \equiv \langle \vec{k}_1 | \frac{d\hat{N}}{d\xi d^2\vec{P}_\perp} | \vec{k}_2 \rangle$$

In both cases, the relevant number operator is differential in the momentum fractions and transverse momenta of the final-state hadrons.

$$\frac{1}{64\pi^3\xi_1\xi_2}\mathrm{Tr}\Big[\Delta^{h_1h_2/q}(\xi_1,\xi_2,\vec{P}_{1\perp},\vec{P}_{2\perp})\gamma^-\Big] = D_1^{h_1h_2/q}(\xi_1,\xi_2,\vec{P}_{1\perp},\vec{P}_{2\perp})$$

$$\frac{1}{64\pi^3\xi_1\xi_2}\mathrm{Tr}\Big[\Delta^{h_1h_2/q}(\xi_1,\xi_2,\vec{P}_{1\perp},\vec{P}_{2\perp})\gamma^-\gamma_5\Big] = -\frac{\epsilon_\perp^{ij}R_\perp^i P_{h\perp}^j}{zM_h^2}\,G_1^{\perp\,h_1h_2/q}(\xi_1,\xi_2,\vec{P}_{1\perp},\vec{P}_{2\perp})$$

$$\frac{1}{64\pi^3\xi_1\xi_2}\mathrm{Tr}\Big[\Delta^{h_1h_2/q}(\xi_1,\xi_2,\vec{P}_{1\perp},\vec{P}_{2\perp})i\sigma^{i-}\gamma_5\Big] = -\frac{\epsilon_\perp^{ij}R_\perp^j}{M_h}H_1^{\sphericalangle\prime\,h_1h_2/q}(\xi_1,\xi_2,\vec{P}_{1\perp},\vec{P}_{2\perp}) + \frac{\epsilon_\perp^{ij}P_{h\perp}^j}{zM_h}H_1^{\perp\prime\,h_1h_2/q}(\xi_1,\xi_2,\vec{P}_{1\perp},\vec{P}_{2\perp})$$

NB: number density interpretation holds not only for unpolarized quarks (γ^- projection) but also for longitudinally ($\gamma^-\gamma^5$ projection) and transversely ($i\sigma^{i-}\gamma^5$ projection) polarized quarks

Number sum rule

$$\sum_{h_1}\sum_{h_2}\int_0^1 d\xi_2 \int_0^{1-\xi_2} d\xi_1 \int d^2\vec{P}_{1\perp}\int d^2\vec{P}_{2\perp}\, D_1^{h_1 h_2/q}(\xi_1,\xi_2,\vec{P}_{1\perp},\vec{P}_{2\perp}) = \langle \mathcal{N}(\mathcal{N}-1)\rangle$$

$$\Longrightarrow \boldsymbol{D_1^{h_1 h_2/i}(w, x, \vec{Y}, \vec{Z})} \equiv \mathcal{J}\cdot D_1^{h_1 h_2/i}(\xi_1,\xi_2,\vec{P}_{1\perp},\vec{P}_{2\perp})$$

is a number density

Jacobian for the variable transformation from $(\xi_1,\xi_2,\vec{P}_{1\perp},\vec{P}_{2\perp})$ to $(w,x,\vec{Y},\vec{Z})$

Number sum rule

$$\sum_{h_1}\sum_{h_2}\int_0^1 d\xi_2\int_0^{1-\xi_2} d\xi_1\int d^2\vec{P}_{1\perp}\int d^2\vec{P}_{2\perp}\,D_1^{h_1h_2/q}(\xi_1,\xi_2,\vec{P}_{1\perp},\vec{P}_{2\perp}) = \langle\mathcal{N}(\mathcal{N}-1)\rangle$$

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Using this prescription, we can define a DiFF that is a density in <u>any</u> momentum variables of choice for the number of hadron pairs ($h_1 h_2$) fragmenting from the parton

Number sum rule

$$\sum_{h_1}\sum_{h_2}\int_0^1 d\xi_2\int_0^{1-\xi_2} d\xi_1\int d^2\vec{P}_{1\perp}\int d^2\vec{P}_{2\perp}\,D_1^{h_1h_2/q}(\xi_1,\xi_2,\vec{P}_{1\perp},\vec{P}_{2\perp}) = \langle\mathcal{N}(\mathcal{N}-1)\rangle$$

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is a number density

Jacobian for the variable transformation from $(\xi_1,\xi_2,\vec{P}_{1\perp},\vec{P}_{2\perp})$ to $(w,x,\vec{Y},\vec{Z})$

Momentum sum rule

$$\sum_{h_1}\int_0^{1-\xi_2} d\xi_1\int d^2\vec{P}_{1\perp}\,\xi_1\,D_1^{h_1h_2/i}(\xi_1,\xi_2,\vec{P}_{1\perp},\vec{P}_{2\perp}) = (1-\xi_2)\,D_1^{h_2/i}(\xi_2,\vec{P}_{2\perp})$$

Generalization to n-hadron fragmentation

$$\frac{1}{4(16\pi^3)^{n-1}\xi_1\cdots\xi_n}\mathrm{Tr}\left[\Delta^{\{h_i\}_n/q}(\{\xi_i\}_n,\{\vec{P}_{i\perp}\}_n)\gamma^-\right] = D_1^{\{h_i\}_n/q}(\{\xi_i\}_n,\{\vec{P}_{i\perp}\}_n)$$

$$\sum_{h_1}\cdots\sum_{h_n}\int d\xi_n\cdots d\xi_1\int d^2\vec{P}_{1\perp}\cdots d^2\vec{P}_{n\perp}\,D_1^{\{h_i\}_n/i}(\{\xi_i\}_n,\{\vec{P}_{i\perp}\}_n) = \left\langle\prod_{k=0}^{n-1}(\mathcal{N}-k)\right\rangle$$

- Connection to phenomenology/experiment - work in a frame where the dihadron has no transverse momentum and integrate over k_T (and perhaps $\boldsymbol{\zeta}$)

$$D_1^{h_1 h_2/i}(w, x, \vec{Y}, \vec{Z}) \equiv \mathcal{J} \cdot D_1^{h_1 h_2/i}(\xi_1, \xi_2, \vec{P}_{1\perp}, \vec{P}_{2\perp})$$

NB: The following are number densities in the respective function arguments

$$D_1^{h_1 h_2/q}(\xi, \zeta, \vec{R}_T) = \frac{\xi}{32\pi^3(1-\zeta^2)} \int d^2\vec{k}_T \, \Delta^{h_1 h_2/q}(\xi_1, \xi_2, \vec{P}_{1\perp}, \vec{P}_{2\perp})$$

$$D_1^{h_1 h_2/q}(\xi_1, \xi_2, \vec{R}_T) = \frac{\xi^2}{64\pi^3 \xi_1 \xi_2} \int d^2\vec{k}_T \, \Delta^{h_1 h_2/q}(\xi_1, \xi_2, \vec{P}_{1\perp}, \vec{P}_{2\perp})$$

$$D_1^{h_1 h_2/q}(\xi, M_h) = \frac{\xi M_h}{64\pi^2} \int d\zeta \int d^2\vec{k}_T \, \Delta^{h_1 h_2/q}(\xi_1, \xi_2, \vec{P}_{1\perp}, \vec{P}_{2\perp})$$

- Connection to phenomenology/experiment - work in a frame where the dihadron has no transverse momentum and integrate over k_T (and perhaps $\boldsymbol{\zeta}$)

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$$D_1^{h_1 h_2/q}(\xi_1, \xi_2, \vec{R}_T) = \frac{\xi^2}{64\pi^3 \xi_1 \xi_2} \int d^2\vec{k}_T \, \Delta^{h_1 h_2/q}(\xi_1, \xi_2, \vec{P}_{1\perp}, \vec{P}_{2\perp})$$

$$D_1^{h_1 h_2/q}(\xi, M_h) = \frac{\xi M_h}{64\pi^2} \int d\zeta \int d^2\vec{k}_T \, \Delta^{h_1 h_2/q}(\xi_1, \xi_2, \vec{P}_{1\perp}, \vec{P}_{2\perp})$$

This DiFF is especially relevant for phenomenology and analyzing experimental data. This operator definition appeared for the first time in our work in Phys. Rev. Lett. **132**, 011902 (2024).

- Another check of the correct definition as a number density is to perform parton model calculations of cross sections, e.g., in $e^+e^- \to (h_1 h_2)\,X$

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$$\frac{d\sigma}{dz\,dM_h} = \sum_q \frac{4\pi N_c \alpha_{\rm em}^2}{3Q^2}\, e_q^2\, D_1^{h_1 h_2/q}(z, M_h)$$

$e^+e^- \to h\,X$

$$\frac{d\sigma}{dz} = \sum_q \hat{\sigma}_0^q\, D_1^{h/q}(z)$$

total partonic cross section for $e^+e^- \to \gamma \to q\bar{q} \equiv \hat{\sigma}_0^q$

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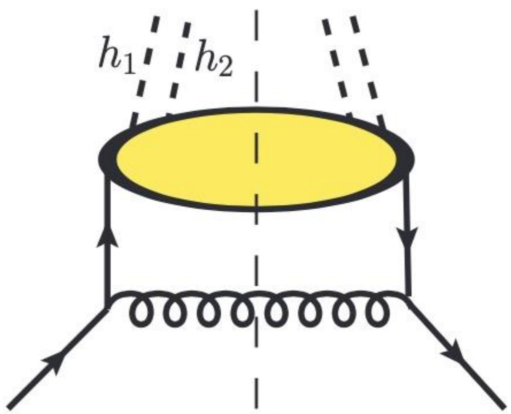
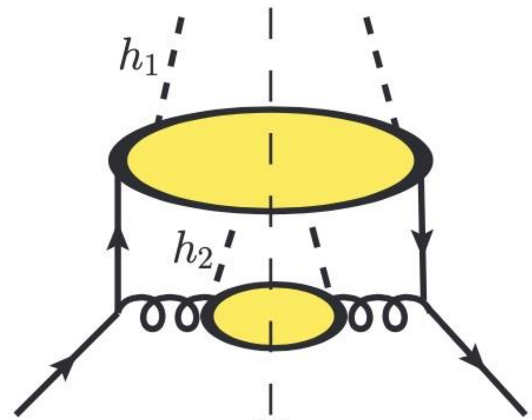
- Evolution equations for DiFFs



(a)

$$D_1^{h_1 h_2/i} \to D_1^{h_1 h_2/j}$$

"Homogeneous term"



(b)

$$D_1^{h_1 h_2/i} \to D_1^{h_1/j} \otimes D_1^{h_2/k}$$

"Inhomogeneous term"

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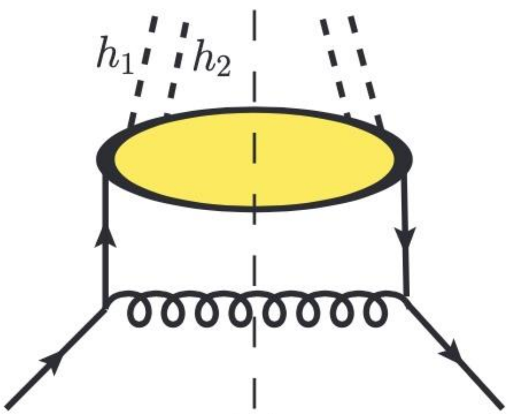

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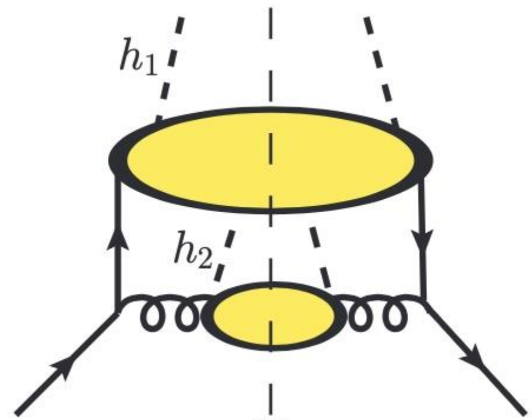

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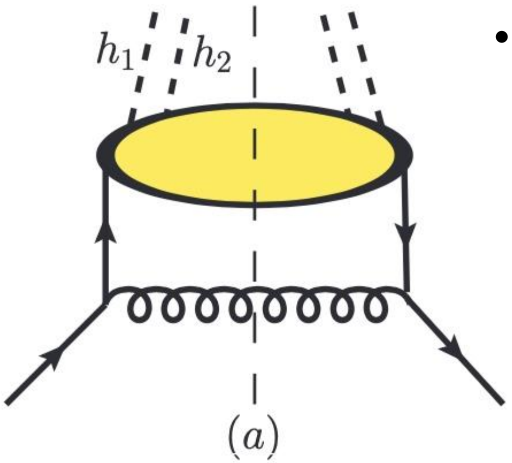
"Inhomogeneous term"

$$D_1^{h_1 h_2/q,(\mathrm{b})}(\xi_1,\xi_2,\vec{R}_T;\mu) = \frac{1}{\vec{R}_T^2}\frac{C_F\alpha_s}{2\pi^2}\int_{\xi_1}^{1-\xi_2}\frac{d\hat{z}}{\hat{z}(1-\hat{z})}D_1^{h_1/q}(\xi_1/\hat{z})D_1^{h_2/g}(\xi_2/(1-\hat{z}))\frac{1+\hat{z}^2}{1-\hat{z}}$$

The inhomogeneous terms are *not* UV divergent at $\mathcal{O}(\alpha_s)$ when one keeps the dependence on R_T (see also Ceccopieri, et al. (2007))

➢ Evolution equations for DiFFs



- Evolution is independent of the target (in the case of PDFs) or final state (in the case of FFs) (Collins (2011))

➡ The evolution equations for the DiFFs have the same splitting functions as single-hadron collinear FFs. The only potential change is in the integration measure of the convolution integral depending on which DiFF is under consideration.

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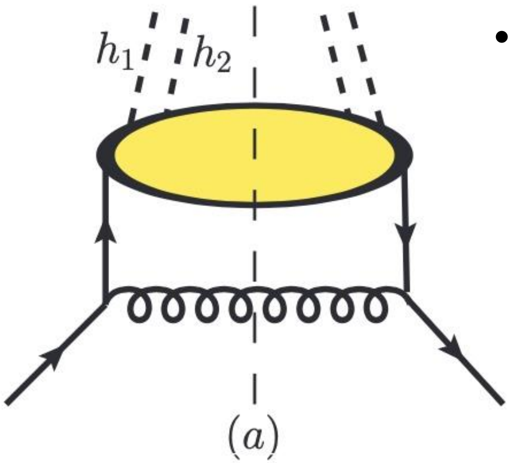

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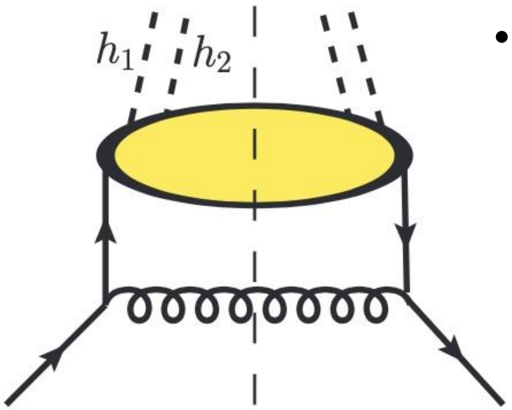
ζ dependence is not altered by evolution

$$D_1^{h/q}(\xi) = \frac{\xi}{4}\int d^2\vec{k}_T \,\mathrm{Tr}[\Delta^{h/q}(\xi,\vec{k}_T)\gamma^-]$$

- Evolution equations for DiFFs



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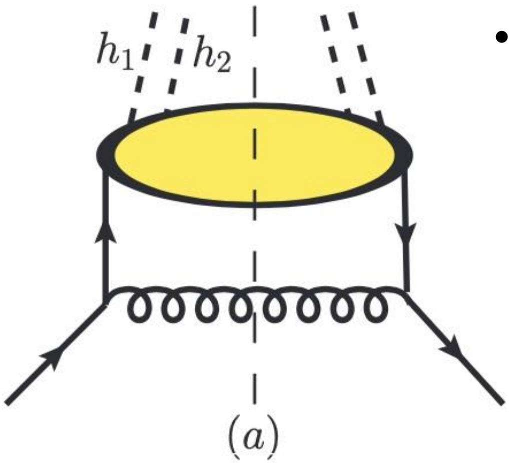
where $\mathcal{D} = D_1 \text{ or } H_1^{\sphericalangle}$

- D_1 → use unpolarized time-like splitting kernels
- $H_1^{\sphericalangle}$ → use transversely polarized splitting kernels

➢ Evolution equations for DiFFs



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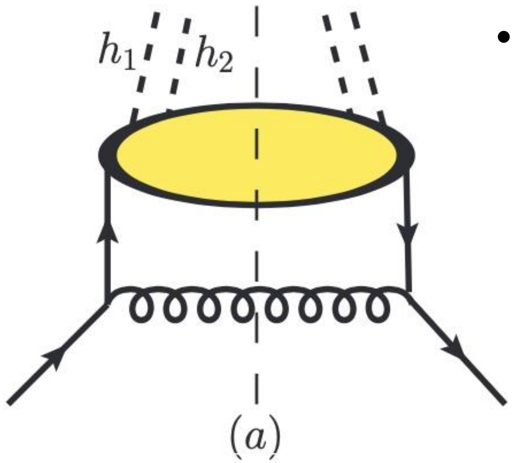

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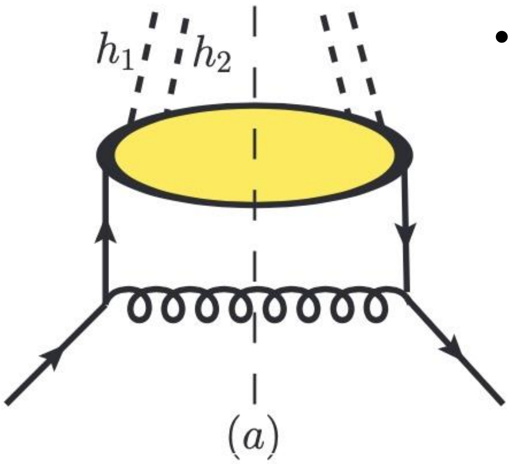
Using $D_1^{h_1 h_2/q}(\xi_1, \xi_2, \vec{R}_T) = \dfrac{2}{\xi} D_1^{h_1 h_2/q}(\xi, \zeta, \vec{R}_T)$

$$\frac{\partial D_1^{h_1 h_2/i}(\xi_1, \xi_2, \vec{R}_T; \mu)}{\partial \ln \mu^2} = \sum_{i'} \int_\xi^1 \frac{d\hat{z}}{\hat{z}^2} D_1^{h_1 h_2/i'}\Big(\frac{\xi_1}{\hat{z}}, \frac{\xi_2}{\hat{z}}, \vec{R}_T; \mu\Big) P_{i\to i'}(\hat{z})$$

Evolution equations for DiFFs



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➡ Agrees with Majumder, Wang (2004), Ceccopieri, et al. (2007), and de Florian, Vanni (2004)

Comments on Other Results and Claims in the Literature

- The original DiFF definition written down in Bianconi, et al. (2000) has the same prefactor as the single-hadron fragmentation case (see also Rogers, et al. (2025))

$$D_1^{h_1 h_2/q,\mathrm{BBJR}}(\xi, \zeta, \vec{k}_T^2, \vec{R}_T^2, \vec{k}_T \cdot \vec{R}_T) = \frac{\mathbf{1}}{\mathbf{4}\boldsymbol{\xi}} \mathrm{Tr}\Big[\Delta^{h_1 h_2/q}(\xi_1, \xi_2, \vec{P}_{1\perp}, \vec{P}_{2\perp})\gamma^-\Big]$$

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$$= 16\pi^3\,\frac{\xi_1\xi_2}{\xi}\,D_1^{h_1 h_2/q}(\xi_1,\xi_2,\vec{P}_{1\perp},\vec{P}_{2\perp})$$

$$= \left|\frac{\partial(\xi_1,\xi_2,\vec{P}_{1\perp},\vec{P}_{2\perp})}{\partial(\xi,\zeta,\xi\vec{k}_T,\vec{\widetilde{M}}_h)}\right|\,D_1^{h_1 h_2/q}(\xi_1,\xi_2,\vec{P}_{1\perp},\vec{P}_{2\perp})$$

$|\vec{\widetilde{M}}_h| = M_h/\sqrt{32\pi^3}$, $\phi_{M_h} = \phi_{R_T}$

➡ BBJR definition is a number density in $(\xi,\zeta,\xi\vec{k}_T,\vec{\widetilde{M}}_h)$ or any set of variables with unit Jacobian relative to those

- Rogers, et al. (2025) define a *n*-hadron FF with the same $\frac{1}{4\xi}$ prefactor and motivate its use as "fundamental" due to the fact that is arises in a (parton model) derivation of factorization for a small mass cluster of *n* hadrons, $e^+e^- \to (h_1 \cdots h_n)\,X$

$$d(\xi, -\xi\vec{k}_T, \{P_h\}) = \frac{1}{4\xi}\,\Delta^{\{h_i\}_n}$$

$$\Delta^{\{h_i\}_n} \equiv \mathrm{Tr}\sum_X \int \frac{dx^+ d^2\vec{x}_\perp}{(2\pi)^3}\, e^{ik\cdot x}\langle 0|\gamma^-\psi(x)|P_1,\ldots,P_n;X\rangle \times\langle P_1,\ldots,P_n;X|\bar{\psi}(0)|0\rangle\Big|_{x^-=0}$$

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No clear statement in which 3*n* variables this function is a number density

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$$\left(\prod_{i=1}^{n} \frac{2E_i(2\pi)^3}{d^3\vec{P}_i}\right) d\sigma = \frac{1}{z}\int_z^1 d\hat{z}\left(\frac{2E_{\hat{k}}(2\pi)^3 d\hat{\sigma}}{d^3\vec{\hat{k}}}\right)\left(\xi^2 \int d^2\vec{k}_T\, d(\xi, -\xi\vec{k}_T, \{P_h\})\right) + \text{p.s.}$$

Usual hard factor for the production of an on-shell massless parton

NB: $\hat{z} = \frac{z}{\xi} + \text{p.s.}\,, \;\; \frac{z_i}{\hat{z}} = \xi_i + \text{p.s.}$

- The claim in Rogers, et al. (2025) is that our *n*-hadron FF definition will not arise in a factorization formula with the same hard factors and splitting functions as single-hadron fragmentation

$$D_1^{\{h_i\}_n}(\{\xi_i\}_n, \{\vec{P}_{i\perp}\}_n) = \frac{1}{4(16\pi^3)^{n-1}\xi_1\cdots\xi_n}\,\Delta^{\{h_i\}_n}$$

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Note that $$1 = \frac{(16\pi^3)^{n-1}\,z_1\cdots z_n}{z} \times \frac{1}{\hat{z}^{n-1}} \times \frac{\xi}{(16\pi^3)^{n-1}\xi_1\cdots\xi_n}$$

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$$= \xi^2\int d^2\vec{k}_T\, D_1^{\{h_i\}_n}(\{\xi_i\}_n, \{\vec{P}_{i\perp}\}_n)$$

We have a factorization formula with our new definition that has the usual hard factor

NB: for *n* = 2, this agrees with the structure of the NLO calculation of de Florian, Vanni (2004) for $d\sigma/dz_1dz_2$

- We also mention an inconsistency in the literature between unpolarized cross section formulas for dihadron production in e^+e^- and SIDIS

  - Eq. (9) of Courtoy, et al. (2012) $e^+e^- \to h_1 h_2\, X$

$$\frac{d\sigma}{dz\, dM_h} = \sum_q \frac{4\pi N_c \alpha_{\rm em}^2}{3Q^2}\, e_q^2\, D_1^{h_1 h_2/q}(z, M_h)$$

This is the expected result if $D_1(z,M_h)$ is defined as a number density in (z,M_h), and also what we obtained

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- Eq. (2.5) of Radici, et al. (2015) $e\,N \to e'(h_1 h_2)\,X$

$$\frac{d\sigma}{dx\, dy\, dz\, dM_h} = \frac{4\pi\alpha_{em}^2}{yQ^2}(1 - y + y^2/2) \sum_q e_q^2\, f_1^{q/N}(x) \left[4\pi M_h\, D_1^{h_1 h_2/q}(z, M_h)|_{\rm RCBG15} \right]$$

This is NOT the expected result if $D_1(z,M_h)$ is defined as a number density in (z,M_h)

Therefore, Courtoy, et al. (2012) and Radici, et al. (2015) seem to be inconsistent in terms of the DiFF that is used

- Our results for the unpolarized cross section for dihadron production in e^+e^- and SIDIS give exactly the formulas one expects if $D_1(z,M_h)$ is defined correctly as a number density in (z,M_h)

$$e^+e^- \to h_1 h_2\, X$$

$$\frac{d\sigma}{dz\, dM_h} = \sum_q \frac{4\pi N_c \alpha_{\rm em}^2}{3Q^2}\, e_q^2\, D_1^{h_1 h_2/q}(z, M_h)$$

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Summary

- We have introduced a new definition of dihadron fragmentation functions, as well as a generalization to *n*-hadron fragmentation, that has a clear number density interpretation.
- This was justified by proving within a parton model framework certain number and momentum sum rules as well calculating cross sections in $e^+e^- \to h_1 h_2\, X$.
- We developed a simple prescription for how to define DiFF (and *n*-hadron FF) operators that are number densities in *any* variables of interest.
- We derived the $\mathcal{O}(\alpha_s)$ evolution of the DiFFs, which have the same splitting functions as for single-hadron FFs.
- We showed that our new definition arises in a factorization formula with the usual hard factors from single-hadron fragmentation.
- We addressed erroneous recent claims about our work.