GPDs through
Universal
Moment
Parametrization

A theoretical overview of the GUMP project

The GUMP Collaboration

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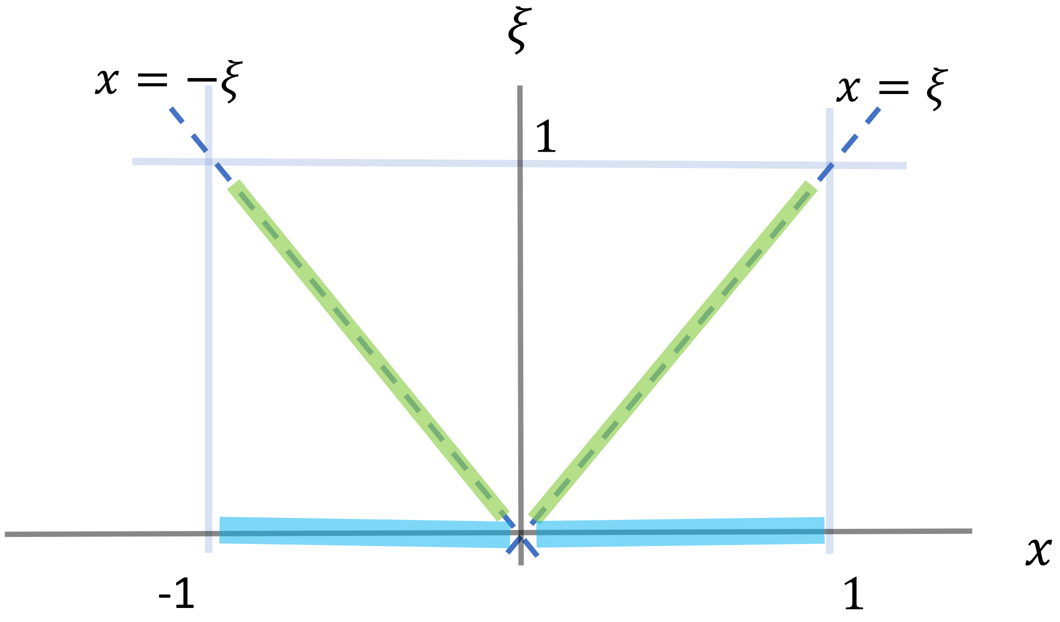
Experimental data is not sufficient to fully reconstruct the GPDs on the entire $x\,\xi$ −plane

Experimental data

$DVCS$
$DVMP$
TCS → $x = \pm\xi$ *

$PDFs$ → $\xi = 0$

$F\,(x, \xi, t)$



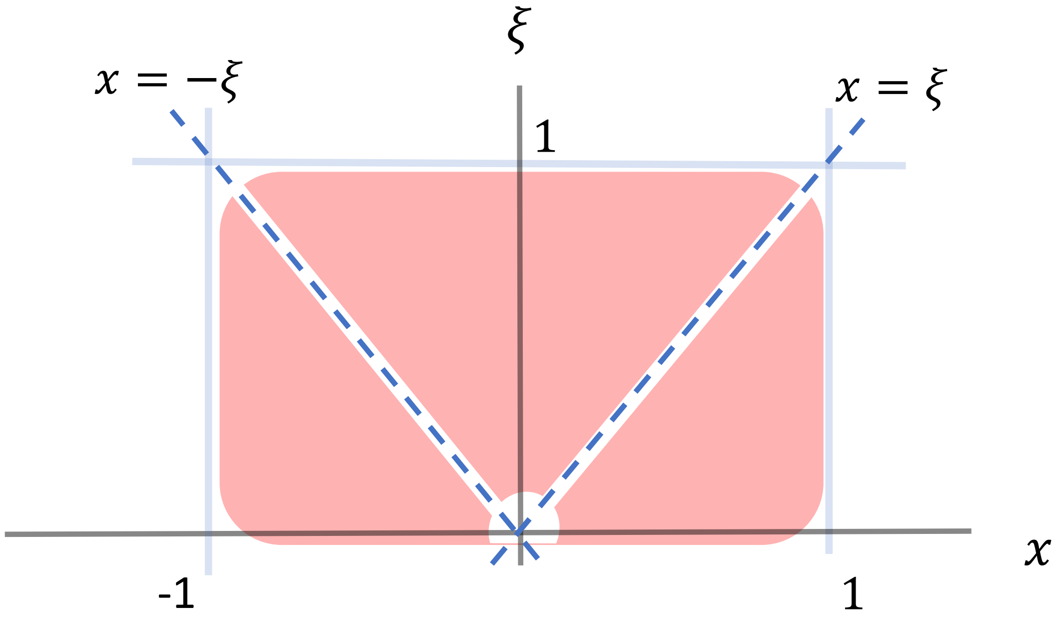
* Neglecting the real parts of the CFFs and TFFs

Lattice data has limitations on the $x = \pm\xi$ lines

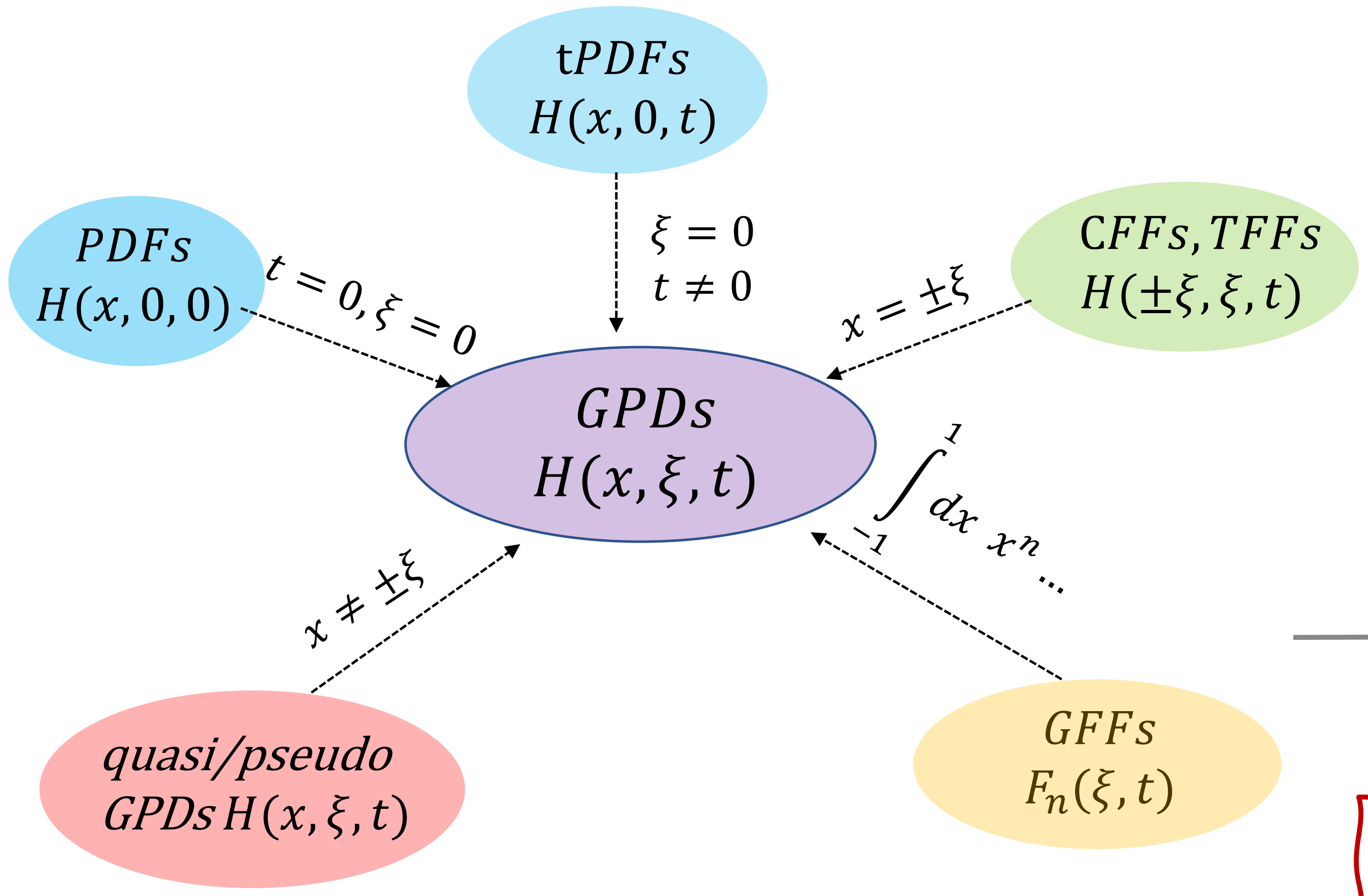
Lattice data

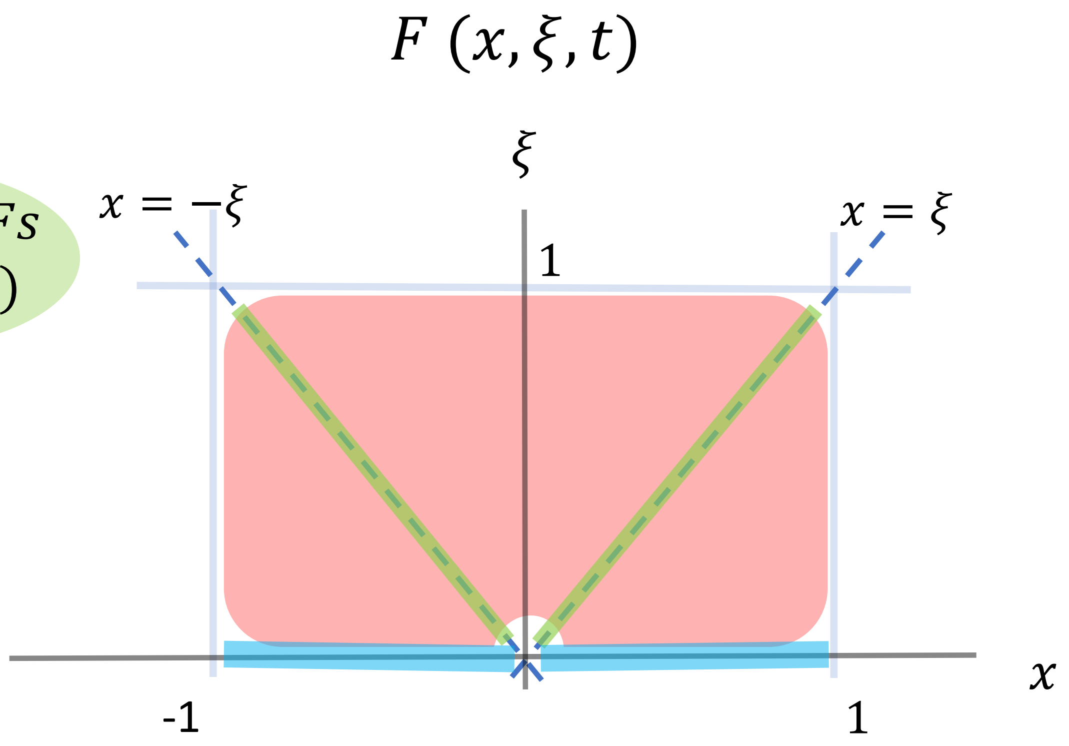
quasi/pseudo GPDs → $x \neq \pm\xi$

$F(x,\xi,t)$



GLOBAL ANALYSIS





Experimental data and Lattice data are complementary

PDF evolution in Mellin space

Evolution in x-space	**Evolution in Mellin space**
Integro-differential (difficult) $\frac{d\, f(x,Q^2)}{d\ln Q^2} = \frac{\alpha_s(Q^2)}{2\pi}\int_x^1 \frac{dy}{y} f(y,Q^2)P(x/y) + \mathcal{O}(\alpha_s(Q^2)^2) + \cdots$	Multiplicative (easy) $\frac{d\, f_n(Q^2)}{d\ln(Q^2)} = \frac{\alpha_s(Q^2)}{2\pi}\gamma_n f_n(Q^2) + \mathcal{O}(\alpha_s(Q^2)^2) + \cdots$

LO Evolution:
The evolution is **diagonal** in Mellin space, and there is **no mixing** of Mellin moments.

NLO and Beyond:
The evolution is **diagonal** in Mellin space, and there is **no mixing** of Mellin moments

$$\frac{d}{d\ln Q^2}\begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix} = \frac{\alpha_s(Q^2)}{2\pi}\begin{pmatrix} \gamma_1^{(0)} & 0 & 0 & \cdots & 0 \\ 0 & \gamma_2^{(0)} & 0 & \cdots & 0 \\ 0 & 0 & \gamma_3^{(0)} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \gamma_n^{(0)} \end{pmatrix}\begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix} + \left(\frac{\alpha_s(Q^2)}{2\pi}\right)^2\begin{pmatrix} \gamma_1^{(1)} & 0 & 0 & \cdots & 0 \\ 0 & \gamma_2^{(1)} & 0 & \cdots & 0 \\ 0 & 0 & \gamma_3^{(1)} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \gamma_n^{(1)} \end{pmatrix}\begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix} + \cdots$$

Multiplicative Renormalization, Diagonal Evolution

$$f_R = Z \circledast f$$

Distribution	Multiplicative renormalizability	Diagonal Evolution in
PDF	✓	Mellin space

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Distribution	Multiplicative renormalizability	Diagonal Evolution in
PDF	✓	Mellin space
GPD	?	?

Multiplicative Renormalization, Diagonal Evolution

$$f_R = Z \circledast f$$

Distribution	**Multiplicative renormalizability**	**Diagonal Evolution in**
PDF	✓	Mellin space
GPD	✗	✗

GPD evolution in conformal space

Evolution in x-space	Evolution in Conformal space
Integro-differential (difficult)	Multiplicative at LO (easy), Matrix multiplicative at NLO (easier)
$\frac{d\ F(x,\xi,t,Q^2)}{d\ln Q^2} = \frac{\alpha_s(Q^2)}{2\pi}\int_{-1}^{1}\frac{dx'}{\lvert\xi\rvert}\ V^{(0)}(\frac{x}{\xi},\frac{x'}{\xi})\ F(x',\xi,t,Q^2) + \mathcal{O}(\alpha(Q^2)^2) + \cdots$	$\frac{d\ F_n(\xi,t,Q^2)}{d\ln(Q^2)} = \frac{\alpha_s(Q^2)}{2\pi}\gamma_n^{(0)} F_n(\xi,t,Q^2) + \left(\frac{\alpha_s(Q^2)}{2\pi}\right)^2 \sum_m \gamma_{nm}^{(1)}\ F_m(\xi,t,Q^2) + \cdots$

LO Evolution:
The evolution is **diagonal** in conformal space, and there is **no mixing** of conformal moments.

NLO and Beyond:
The evolution is **non-diagonal** in conformal space, and there is **mixing** of conformal moments

$$\frac{d}{d\ln Q^2}\begin{pmatrix}\mathcal{F}_1\\ \mathcal{F}_2\\ \vdots\\ \mathcal{F}_n\end{pmatrix} = \frac{\alpha_s(Q^2)}{2\pi}\begin{pmatrix}\gamma_1^{(0)} & 0 & 0 & \dots & 0\\ 0 & \gamma_2^{(0)} & 0 & \dots & 0\\ 0 & 0 & \gamma_3^{(0)} & \dots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \dots & \gamma_n^{(0)}\end{pmatrix}\begin{pmatrix}\mathcal{F}_1\\ \mathcal{F}_2\\ \vdots\\ \mathcal{F}_n\end{pmatrix} + \left(\frac{\alpha_s(Q^2)}{2\pi}\right)^2\begin{pmatrix}\gamma_{11}^{(1)} & \gamma_{12}^{(1)} & \dots & \gamma_{1n}^{(1)}\\ 0 & \gamma_{22}^{(1)} & \dots & \gamma_{2n}^{(1)}\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & \gamma_{nn}^{(1)}\end{pmatrix}\begin{pmatrix}\mathcal{F}_1\\ \mathcal{F}_2\\ \vdots\\ \mathcal{F}_n\end{pmatrix} + \cdots$$

Conformal moments of the GPDs

$$\mathcal{F}_n(\xi,t) = \int_{-1}^{1} dx\, c_n(x,\xi) F(x,\xi,t)$$

$$c_n(x,\xi) = \xi^n \frac{\Gamma\left(\frac{3}{2}\right)\Gamma(1+n)}{2^n\Gamma\left(\frac{3}{2}+n\right)} C_n^{\frac{3}{2}}\left(\frac{x}{\xi}\right)$$

Gegenbauer polynomials

$$\int_{-1}^{1} \frac{dx'}{|\xi|} V^{(0)}\left(\frac{x}{\xi},\frac{x'}{\xi}\right) C_j^{3/2}\left(\frac{x'}{\xi}\right) = \gamma_j\, C_j\left(\frac{x}{\xi}\right)$$

In the forward limit ($\xi \to 0$) the conformal moments reduce to Mellin moments. $\lim_{\xi\to 0} c_n(x,\xi) = x^n.$

$$\mathcal{F}_n(\xi,t) = \int_{-1}^{1} dx\, c_n(x,\xi) F(x,\xi,t) \xrightarrow{\xi \to 0} \mathcal{F}_n = \int_{-1}^{1} dx\, x^n\, F(x)$$

Polynomiality condition for conformal moments

$$C_j^{(\lambda)}(x) = \sum_{k=0}^{j} c_{j,k}^{(\lambda)} x^k \, , \qquad x^j = \sum_{k=0}^{j} c_{j,k}^{-1,(\lambda)} C_k^{(\lambda)}(x) \, .$$

- *Polynomiality condition of Mellin moments leads to the polynomiality condition of conformal moments*

ξ dependence of Mellin moments goes like ξ^k

ξ dependence of conformal moments moments goes like ξ^k too!

$$F_n(\xi, t) = \sum_{k=0,\text{even}}^{n} \xi^k F_{n,k}(t) \, . \Longrightarrow \mathcal{F}_j(\xi, t) = \sum_{k=0,\text{even}}^{j+1} \xi^k \mathcal{F}_{j,k}(t)$$

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We parametrize the t-dependence!

Conformal moments and the classical moment problem

$$\mathcal{F}_n(\xi,t) = \int_{-1}^{1} dx\, c_n(x,\xi) F(x,\xi,t)$$

- *The goal is to invert this relationship and recover the GPD,* $F(x,\xi,t)$, *from its conformal moments,* $\mathcal{F}_n(\xi,t)$.

Conformal moments and the classical moment problem

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- *The goal is to invert this relationship and recover the GPD,* $F(x,\xi,t)$, *from its conformal moments,* $F_n(\xi,t)$.
- *The classical moment problem: Given the moments of a function, can we reconstruct the function itself?*

Conformal moments and the classical moment problem

$$\mathcal{F}_n(\xi,t) = \int_{-1}^{1} dx\, c_n(x,\xi) F(x,\xi,t)$$

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- *The classical moment problem: Given the moments of a function, can we reconstruct the function itself?*

  Yes, if you know the asymptotic behaviour!

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- The classical moment problem: Given the moments of a function, can we reconstruct the function itself?

  Yes, if you know the asymptotic behaviour!

- This is done by expressing the function as a series expansion using a set of orthogonal polynomials

$$F(x,\xi,t) = \sum_{n=0}^{\infty} (-1)^n \; p_n(x,\xi) \; \mathcal{F}_n(\xi,t)$$

Gegenbauer polynomials are orthogonal only in the ERBL region!

Conformal wave function: $p_n(x,\xi) = (-1)^n \xi^{-n-1} \frac{2^n \Gamma\left(\frac{5}{2}+n\right)}{\Gamma\left(\frac{3}{2}\right)\Gamma(3+n)} \left[1 - \left(\frac{x}{\xi}\right)^2\right] C_n^{\frac{3}{2}}\left(\frac{x}{\xi}\right) \theta(\xi - |x|) \quad \text{for } |x| < \xi$

Conformal moments and the classical moment problem

Compare it to

Mellin moments → *Inverse Mellin transform*

$$f_n := \int_0^1 \mathrm{d}x x^{n-1} f(x) \;\Rightarrow\; f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} f_{n+1} \delta^{(n)}(x) \;\Rightarrow\; f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} x^{-j} f_j \, \mathrm{d}j$$

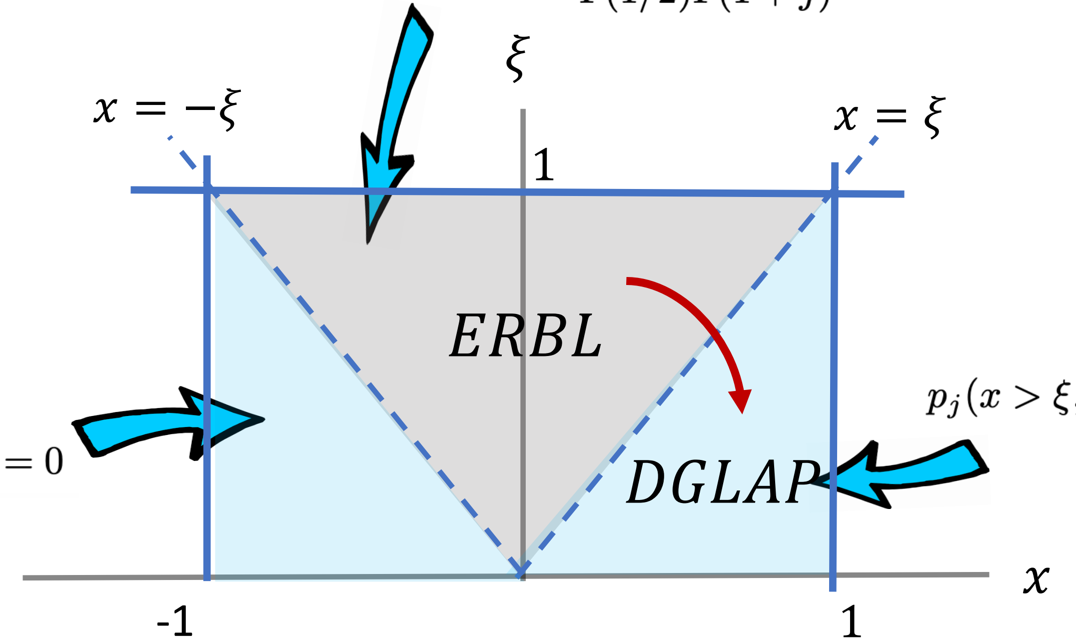
Conformal moments → *Inverse conformal transform*

$$\mathcal{F}_n(\xi,t) = \int_{-1}^{1} dx\, c_n(x,\xi) F(x,\xi,t) \;\Rightarrow\; F(x,\xi,t) = \sum_{n=0}^{\infty} (-1)^n \, p_n(x,\xi) \, \mathcal{F}_n(\xi,t) \;\Rightarrow\; ?$$

Inverse conformal transform

$$F_q(x,\xi,t) = \frac{1}{2i}\int_{c-i\infty}^{c+i\infty} dj\, \frac{p_j(x,\xi)}{\sin(\pi[j+1])}\mathcal{F}_j(\xi,t)$$

$$p_j(|x|\leq\xi,\xi) = \frac{2^{j+1}\Gamma(5/2+j)\xi^{-j-1}}{\Gamma(1/2)\Gamma(1+j)}(1+x/\xi)\,{}_2F_1\left(-1-j,j+2,2\,\middle|\,\frac{\xi+x}{2\xi}\right)$$



$$p_j(x\leq-\xi,\xi)=0$$

$$p_j(x>\xi,\xi) = \frac{\sin(\pi[j+1])}{\pi}x^{-j-1}\,{}_2F_1\left(\begin{matrix}(j+1)/2,(j+2)/2\\ 5/2+j\end{matrix}\,\middle|\,\frac{\xi^2}{x^2}\right)$$

Parametrization of the t dependence

Inverse conformal transform

$$F_q(x,\xi,t) = \frac{1}{2i}\int_{c-i\infty}^{c+i\infty} dj\, \frac{p_j(x,\xi)}{\sin(\pi[j+1])}\mathcal{F}_j(\xi,t)$$

polynomiality

$$\mathcal{F}_j(\xi,t) = \sum_{k=0,\text{even}}^{j+1} \xi^k \mathcal{F}_{j,k}(t)$$

parametrization

Parametrization of the t dependence

Scattering amplitudes	**GPDs**
Partial wave expansion $T(s,t,u) = 16\pi \sum_{J=0}^{\infty} (2J+1) P_j(\cos\theta_t) \mathcal{T}_j(t,u)$	Conformal wave expansion $F(x,\xi,t) = \sum_{j=0}^{\infty} (-1)^j \, p_j(x,\xi) \, \mathcal{F}_j(\xi,t)$
s channel and t channel	DGLAP region and ERBL region
Analytic continuation between the channels	Analytic continuation between the regions

Parametrization of the t dependence

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s channel and t channel	DGLAP region and ERBL region
Analytic continuation between the channels	Analytic continuation between the regions
$\mathcal{T}_j(t,u) = \sum \frac{r(t,u)}{J-\alpha(t)}$	$\mathcal{F}_{j,k}(t) = ?$

$$\mathcal{F}_{j,k}(t) \propto \sum \frac{r_{j,k}(t)}{J-\alpha(t)} \to \mathcal{F}_{j,k}(t) \propto \sum \frac{r_{j,k}(t)}{j+1-k-\alpha(t)}$$

$$J \leftrightarrow \; j+1-k$$

Regge inspired parametrization of the t dependence

$$\mathcal{F}_{j,k}(t) \propto \sum \frac{r_{j,k}(t)}{J-\alpha(t)} \to \mathcal{F}_{j,k}(t) \propto \sum \frac{r_{j,k}(t)}{j+1-k-\alpha(t)}$$

In the forward limit the conformal moment should reduce the Mellin moment

$$\mathcal{F}_{j,0}(t=0) = \int \mathrm{d}x\ x^j f(x)$$

The common PDF ansatz

$$f(x) = \sum_{i=1}^{i_{\max}} N_i x^{-\alpha_i}(1-x)^{\beta_i}$$

$$\mathcal{F}_{j,0}(t=0) = \int \mathrm{d}x\ x^j f(x) = \sum_{i=1}^{i_{\max}} N_i B(j+1-\alpha_i, 1+\beta_i)$$

$$\mathcal{F}_{j,k}(t) = \sum_{i=1}^{i_{\max}} N_{i,k} B(j+1-\alpha_{i,k}, 1+\beta_{i,k}) \frac{r'_{i,j,k}(t)}{j+1-k-\alpha_{i,k}(t)}$$

Summary

Inverse conformal transform

$$F_q(x,\xi,t) = \frac{1}{2i}\int_{c-i\infty}^{c+i\infty} dj\, \frac{p_j(x,\xi)}{\sin(\pi[j+1])}\mathcal{F}_j(\xi,t)$$

polynomiality

$$\mathcal{F}_j(\xi,t) = \sum_{k=0,\mathrm{even}}^{j+1} \xi^k \mathcal{F}_{j,k}(t)$$

parametrization

$$\mathcal{F}_{j,k}(t) = \sum_{i=1}^{i_{\max}} N_{i,k} B(j+1-\alpha_{i,k}, 1+\beta_{i,k}) \frac{r'_{i,j,k}(t)}{j+1-k-\alpha_{i,k}(t)}$$

Summary

- We are making GLOBAL ANALYSIS using both experimental and lattice data
- We are working with conformal moments of GPDs
- We parametrize the t-dependence of the conformal moments
- The parametrization is Regge theory inspired and flexible

References

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- On convergence properties of GPD expansion through Mellin/conformal moments and orthogonal polynomials, Hao-Cheng Zhang, Xiangdong Ji (2024)