Observables for scattering on targets with arbitrary spin

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Motivation

Matrix elements for Operators of composite particles with arbitrary spin

- Covariant decomposition of matrix element in independent non-perturbative objects

$$\langle p', s' | j^\mu | p, s \rangle = \bar{u}_{(p',s')} \Gamma^\mu_{(p',p)} u_{(p,s)} \underset{\text{spin}1/2}{\longrightarrow} \bar{u}_{(p',s')} \left[F_{1(t^2)} \gamma^\mu - F_{2(t^2)} \frac{i}{2m} \sigma^{\mu\nu} q_\nu \right] u_{(p,s)}$$

$$\langle d' | j^\mu | d \rangle = - \left(G_1 \left(Q^2\right) \left[\varepsilon'^* \cdot \varepsilon\right] - G_3 \left(Q^2\right) \frac{\left(q \cdot \varepsilon'^*\right) \left(q \cdot \varepsilon\right)}{2m_d^2} \right) 2P^\mu + G_M \left(Q^2\right) \left[\left(q \cdot \varepsilon'^*\right) \varepsilon^\mu - \left(q \cdot \varepsilon\right) \varepsilon'^{*\mu} \right]$$

Spin-j fields embedded in objects with $> 2j + 1$ components

- Polarization four-vector (spin 1), Rarita Schwinger (spin 3/2), Fierz-Pauli (spin 2)
- Need for constraints, Kinematical singularities

Use $(2j + 1)$-component (chiral) spinors: $(j, 0)$ & $(0, j)$

`[Joos; Barut-Muzinich-Williams 63; Weinberg's 64-65]`

Motivation

Advantages

- Same formalism for any spin j → systematic approach
- "Basic" algebraic construction → $su(2) \rightarrow su(2j+1) \rightarrow sl(2,C)$
- Covariant multipole basis emerges → physical interpretation
- Parity conserving interactions → generalized Dirac algebra
- Easy to implement different types of spin → (canonical, helicity, light front)
- Exact degrees of freedom → no need for constraints

Introduction: Lorentz Group Basics

- Algebra for Generators of the Lorentz group

$$[\mathbb{J}_l, \mathbb{J}_m] = i\epsilon_{lmn}\mathbb{J}_n \ , \quad [\mathbb{J}_l, \mathbb{K}_m] = i\epsilon_{lmn}\mathbb{K}_n \ , \quad [\mathbb{K}_l, \mathbb{K}_m] = -i\epsilon_{lmn}\mathbb{J}_n$$

- Two independent su(2) subalgebras → irreps (j_A, j_B)

$$\mathbb{A}_m = \frac{1}{2}(\mathbb{J}_m + i\mathbb{K}_m) \quad , \quad \mathbb{B}_m = \frac{1}{2}(\mathbb{J}_m - i\mathbb{K}_m)$$

$$[\mathbb{A}_l, \mathbb{A}_m] = i\epsilon_{lmn}\mathbb{A}_n \ , \quad [\mathbb{B}_l, \mathbb{B}_m] = i\epsilon_{lmn}\mathbb{B}_n \ , \quad [\mathbb{A}_l, \mathbb{B}_m] = 0$$

- Simplest irreps that contain spin-j → $(2j + 1$ components$)$
  - Right-handed $(j, 0)$: $\mathbb{K}_m \to -i\mathbb{J}_m$
  - Left-handed $(0, j)$: $\mathbb{K}_m \to +i\mathbb{J}_m$

Introduction: Chiral Reps

Some Representations constructed out of the Chiral ones

- $(0,0)$ $\rightarrow$ Scalar

- $(1/2,0)$ & $(0,1/2)$ $\rightarrow$ Right & Left Chiral spinors
- $(1/2,0)\bigoplus(0,1/2)$ $\rightarrow$ Dirac (spin 1/2) spinors (Extended by Parity / direct sum)
- $(1/2,1/2)$ $\rightarrow$ Vector

- $(1,0)$ & $(0,1)$ $\rightarrow$ Right & Left Chiral spinors
- $(1,0)\bigoplus(0,1)$ $\rightarrow$ Dirac (spin 1) spinors (Extended by Parity / direct sum)
- $(1,1)$ $\rightarrow$ Tensor

Causal chiral fields (massive, left- right-handed)

- Lorentz invariant S-matrix:

$$U_{[\Lambda,a]}\psi_{\sigma\,(x)}U^{-1}_{[\Lambda,a]} = \sum_{\sigma'}\left(D^{(j)}_{[\Lambda^{-1}]}\right)_{\sigma\sigma'}\psi_{\sigma'\,(\Lambda x+a)}$$

- No EoM for chiral fields (only obey KG eq.)

- Spinors appearing in the fields (not invariants, depend on boost choice)

Canonical $\rightarrow$

$$D^{(j)}_{[L(p)]} = e^{-\hat{p}\cdot\vec{J}^{(j)}\theta}$$

$$\bar{D}^{(j)}_{[L(p)]} = e^{+\hat{p}\cdot\vec{J}^{(j)}\theta}$$

Introduction: Propagators and Spinors: t-tensors

- Propagator numerator (invariant)

$$\Pi^{(j)}_{\sigma\sigma'}(\vec{p},\omega) = m^{2j} D^{(j)}_{\sigma\sigma'}[L(\vec{p})] \left(D^{(j)}_{\sigma'\sigma''}[L(\vec{p})]\right)^{\dagger} = m^{2j}\left(e^{-2\hat{p}\cdot\vec{J}^{(j)}\theta}\right)_{\sigma\sigma'}$$

$$\bar{\Pi}^{(j)}_{\sigma\sigma'}(\vec{p},\omega) = m^{2j} \bar{D}^{(j)}_{\sigma\sigma'}[L(\vec{p})] \left(\bar{D}^{(j)}_{\sigma'\sigma''}[L(\vec{p})]\right)^{\dagger} = m^{2j}\left(e^{2\hat{p}\cdot\vec{J}^{(j)}\theta}\right)_{\sigma\sigma'}$$

Introduction of 2j-rank t-tensors (symmetric & traceless)

- Central role of t-tensors

  boosts/spinors and more ...

$$\Pi^{(j)}_{\sigma\sigma'}(\vec{p},\omega) = t^{\mu_1\mu_2\ldots\mu_{2j}}_{\sigma\sigma'} p_{\mu_1} p_{\mu_2} \ldots p_{\mu_{2j}}$$

$$D^{(j)}_{[L(p)]} = t^{\mu_1\mu_2\ldots\mu_{2j}} \tilde{p}_{\mu_1} \tilde{p}_{\mu_2} \ldots \tilde{p}_{\mu_{2j}}$$

$\tilde{p}^{\mu}$ Parameters NOT 4-vector

Same for any Spin!

Canonical: $\tilde{p}^{\mu}_{C} = \sqrt{\frac{1}{2m(m+p^0)}}(p^0 + m, \vec{p}\,)$

Similar for Helicity and LF spinors (but $\mathbb{C}$-numbers)

Introduction: Bi-Spinors $(j,0)\bigoplus(0,j)$

- For Parity conserving interactions $\rightarrow$ Gamma matrices (Weyl rep.)
  (like the spin 1/2 case)

$$\gamma^{\mu_1\cdots\mu_{2j}} = \begin{pmatrix} 0 & t^{\mu_1\cdots\mu_{2j}} \\ \bar{t}^{\mu_1\cdots\mu_{2j}} & 0 \end{pmatrix}, \; \beta = \gamma^{0\cdots 0} = \begin{pmatrix} 0 & \mathbf{1}^{(j)} \\ \mathbf{1}^{(j)} & 0 \end{pmatrix}, \; \gamma_5 = \begin{pmatrix} -\mathbf{1}^{(j)} & 0 \\ 0 & \mathbf{1}^{(j)} \end{pmatrix}$$

- Bi-spinor satisfy the Dirac eq.
  (like the spin 1/2 case)
  (called Weinberg-Joos eq.)

$$\left(\gamma^{\mu_1\cdots\mu_{2j}} p_{\mu_1}\cdots p_{\mu_{2j}} - m^{2j}\right) u^{(j)}_{(p,s)} = 0$$

$$\bar{u}^{(j)}_{(p,s)} \left(\gamma^{\mu_1\cdots\mu_{2j}} p_{\mu_1}\cdots p_{\mu_{2j}} - m^{2j}\right) = 0$$

- Product of Gamma matrices involve alternating “barring” pattern: $t\bar{t}t\cdots$
  (like the spin 1/2 case)

$$\gamma^{\mu_1\cdots\mu_{2j}}\gamma^{\nu_1\cdots\nu_{2j}} = \begin{pmatrix} t^{\mu_1\cdots\mu_{2j}}\bar{t}^{\nu_1\cdots\nu_{2j}} & 0 \\ 0 & \bar{t}^{\mu_1\cdots\mu_{2j}} t^{\nu_1\cdots\nu_{2j}} \end{pmatrix}$$

Constructing the t-tensors

- Generalization of $\sigma^\mu = (1, \boldsymbol{\sigma})$ & $\bar{\sigma}^\mu = (1, -\boldsymbol{\sigma})$ to arbitrary spin

- Intertwining map:
$$\begin{array}{ll} (j,0)\otimes(0,j) & [\text{rank-2 in SL(2,}\mathbb{C})] \\ & \quad\quad\updownarrow \\ (j,j) & [\text{rank-}2j \text{ symm. traceless in SO(3,1)}] \end{array}$$

- Recursion relation for higher spins (Clebsch-Gordan)

$$t^{\mu_1\mu_2\cdots\mu_{2j}}_{\sigma\dot{\tau}} = \langle j\sigma | j-\tfrac{1}{2}\sigma_1 \tfrac{1}{2}\sigma_2\rangle \, \langle j\dot{\tau} | j-\tfrac{1}{2}\dot{\tau}_1 \tfrac{1}{2}\dot{\tau}_2\rangle \, t^{\mu_1\mu_2\cdots\mu_{2j-1}}_{\sigma_1\dot{\tau}_1} \, t^{\mu_{2j}}_{\sigma_2\dot{\tau}_2}$$

Efficient numerical implementation

$$\left(t^{+_1\cdots+_a -_1\cdots-_b R_1\cdots R_c L_1\cdots L_d}\right)_{\sigma\dot{\sigma}'} = 2^{2j}\,\frac{\sqrt{(a+c)!(a+d)!(b+c)!(b+d)!}}{(2j)!}\,\delta_{\sigma,\frac{a-b+c-d}{2}}\,\delta_{\dot{\sigma}',\frac{a-b-c+d}{2}}$$

$$v^\pm = v^0 \pm v^3\,,\quad v^{R/L} = v^1 \pm \mathrm{i}v^2\,,\qquad a,b,c,d\in\{0,\cdots,2j\}\,,\quad a+b+c+d=2j$$

- Contain a basis of su(N=$2j+1$) used to expand: $\langle\lambda'|\hat{O}|\lambda\rangle$.

t-tensor for Spin 1/2 & 1

- Spin 1/2 (Pauli matrices)

0-th powers of J_i: $t^0 = \mathbf{1}$

Linear in J_i: $t^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; t^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \; t^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

- Spin 1 **0-th powers in J_i:** $t^{00} = \mathbf{1}$

Linear in J_i: $t^{01} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \; t^{02} = \frac{i}{\sqrt{2}}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \; t^{03} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

Quadratic in J_i: $t^{11} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \; t^{22} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \; t^{33} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$t^{12} = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \; , t^{13} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}, \; t^{23} = \frac{i}{\sqrt{2}}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$

Algebra of t-tensors: Reduction of Quadratic Products

- Central role of the covariant t-tensors (spinors, boosts, propagators, gamma matrices)
- Bilinear calculus involve products with alternating "barring" pattern: $t\bar{t}t\cdots$
- Matrices in t-tensors form a basis of $\mathbf{su(2j+1)}$ → Products can be linearized
- $t^{\mu_1\cdots\mu_{2j}}\bar{t}^{\rho_1\cdots\rho_{2j}} = \frac{1}{(2j)!^2}\underset{\{(\rho)\}}{\mathcal{S}}\sum_{m=0}^{2j}\left[\binom{2j}{m}\underset{\{(\mu\rho)\}}{\mathcal{S}}\left(\prod_{l=1}^{m}\mathcal{Q}_{\text{red}}^{\mu_l\rho_l\alpha_l}\ \prod_{k=m+1}^{2j} g^{\mu_k\rho_k}\eta^{\alpha_k}\right)\right]t_{\alpha_1\cdots\alpha_{2j}}$

  $\mathcal{Q}_{\text{red}}^{\mu\rho\alpha} = -g^{\rho\alpha}\eta^{\mu} + g^{\mu\alpha}\eta^{\rho} + i\epsilon^{\mu\rho\sigma\alpha}\eta_{\sigma}$

  $\eta^{\mu} = (1,0,0,0)$

  each $0 \leq m \leq 2j$ corresponds to a Lorentz independent tensor

  $\mathrm{T}_m^{(\mu\rho)} \equiv \prod_{l=1}^{m}\mathcal{Q}_{\text{red}}^{\mu_l\rho_l\alpha_l}\prod_{r=m+1}^{2j}\eta^{\alpha_r}t_{\alpha_1\cdots\alpha_{2j}} = \prod_{l=1}^{m}\mathcal{Q}_{\text{red}}^{\mu_l\rho_l\alpha_l}t_{\alpha_1\cdots\alpha_m 0\cdots 0}$
- Trade matrix multiplication by number multiplication

Basis for Operators

- $\mathrm{T}_m^{(\mu\rho)} = \prod_{l=1}^{m} \mathcal{Q}_{\rm red}^{\mu_l\rho_l\alpha_l} t_{\alpha_1\cdots\alpha_m 0\cdots 0} \quad , \quad \mathcal{Q}_{\rm red}^{\mu\rho\alpha} = -g^{\rho\alpha}\eta^\mu + g^{\mu\alpha}\eta^\rho + i\epsilon^{\mu\rho\sigma\alpha}\eta_\sigma$

  $t_{\alpha_1\cdots\alpha_{2j}} \rightarrow$ Basis for Hermitian matrices

  $\mathrm{T}_m^{(\mu\rho)} \rightarrow$ Basis for General matrices

- Relation between Lorentz Gen & t-tensors & T-tensors:

$$\mathbb{M}^{\mu\rho} = \mathrm{i}\mathcal{Q}_{\rm red}^{\mu\rho\alpha}(j) t_{\alpha 0\cdots 0} = \mathrm{i}(j)\mathrm{T}_1^{(\mu\rho)}$$

- $\mathrm{T}_m^{(\mu\rho)}$ are covariantly independent

$$\left[\mathbb{M}^{\nu\sigma}, \mathrm{T}_m^{(\mu\rho)}\right] = \mathrm{i}\sum_{n=1}^{m}\Big(g^{\mu_n\sigma}\mathrm{T}_m^{\mu_1\rho_1,\ldots,\mu_{n-1}\rho_{n-1},\nu\rho_n,\mu_{n+1}\rho_{n+1},\ldots,\mu_m\rho_m} - g^{\mu_n\nu}\mathrm{T}_m^{\mu_1\rho_1,\ldots,\mu_{n-1}\rho_{n-1},\sigma\rho_n,\mu_{n+1}\rho_{n+1},\ldots,\mu_m\rho_m} + g^{\rho_n\sigma}\mathrm{T}_m^{\mu_1\rho_1,\ldots,\mu_{n-1}\rho_{n-1},\mu_n\nu,\mu_{n+1}\rho_{n+1},\ldots,\mu_m\rho_m} - g^{\rho_n\nu}\mathrm{T}_m^{\mu_1\rho_1,\ldots,\mu_{n-1}\rho_{n-1},\mu_n\sigma,\mu_{n+1}\rho_{n+1},\ldots,\mu_m\rho_m}\Big)$$

Orthogonal Basis for Operators

- $\mathrm{T}_m^{(\mu\rho)} = \prod_{l=1}^{m} \mathcal{Q}_{\text{red}}^{\mu_l \rho_l \alpha_l} t_{\alpha_1 \cdots \alpha_m 0 \cdots 0} \quad , \quad \mathcal{Q}_{\text{red}}^{\mu\rho\alpha} = -g^{\rho\alpha}\eta^{\mu} + g^{\mu\alpha}\eta^{\rho} + i\epsilon^{\mu\rho\sigma\alpha}\eta_{\sigma}$

  $t_{\alpha_1 \cdots \alpha_{2j}}$ $\rightarrow$ Basis for Hermitian matrices

  $\mathrm{T}_m^{(\mu\rho)}$ $\rightarrow$ Basis for General matrices

- Orthogonalization

  $$\mathcal{T}_m^{\mu_1\rho_1\cdots\mu_m\rho_m} = \mathrm{T}_m^{(\mu\rho)} - \sum_{n=(m \bmod 2)}^{m} N_{m,n} \underset{\{(\mu\rho)\}}{\mathcal{S}} \mathcal{T}_n^{\mu_1\rho_1\cdots\mu_n\rho_n} \prod_{a=n+1,n+3,\cdots}^{m-1} \mathcal{C}_{\text{red}}^{\mu_a\rho_a\mu_{a+1}\rho_{a+1}}$$

  Lowest rank Invariant: $\mathcal{C}_{\text{red}}^{\mu_1\rho_1\mu_2\rho_2} = -g^{\mu_1\mu_2}g^{\rho_1\rho_2} + g^{\mu_1\rho_2}g^{\rho_1\mu_2} + \mathrm{i}\epsilon^{\mu_1\rho_1\mu_2\rho_2}$
  (+ correct symmetry)

Meaning of this Orthogonal Basis for Operators: $sl(2,\mathbb{C})$ Multipoles

- $sl(2,\mathbb{C})$ Multipole of order m: $\quad \mathcal{M}_m^{\mu_1\rho_1,\cdots,\mu_m\rho_m} = \frac{1}{m!} \underset{\{(\mu\rho)\}}{\mathcal{S}} \prod_{r=1}^m \mathbb{M}^{\mu_r\rho_r} - (\text{Traces})$

- Multipoles *vs.* Orthogonal $\mathcal{T}$-tensors: $\quad \mathcal{M}_m^{\mu_1\rho_1,\cdots,\mu_m\rho_m} = \frac{\mathrm{i}^m}{2^m} m! \binom{2j}{m} \mathcal{T}_m^{\mu_1\rho_1,\cdots,\mu_m\rho_m}$

- Multipoles up to order 3: $\quad \mathcal{M}_0 = 1^{(j)} = \mathcal{T}_0$

$$\mathcal{M}_1^{\mu\rho} = \mathbb{M}^{\mu\rho} = i\mathcal{Q}_{\text{red}}^{\mu\rho\alpha_1} \left(\prod_{s=2}^{2j} \eta^{\alpha_s}\right)(j)t_{\alpha_1\cdots\alpha_{2j}}$$

$$\mathcal{M}_2^{\mu_1\rho_1,\mu_2\rho_2} = \frac{1}{2}j(2j-1)\left(-\mathcal{Q}_{\text{red}}^{\mu_1\rho_1\beta_1}\mathcal{Q}_{\text{red}}^{\mu_2\rho_2\beta_2}\, t_{\beta_1\beta_2 0\cdots 0} + \frac{1}{3}\mathcal{C}_{\text{red}}^{\mu_1\rho_1\mu_2\rho_2}\mathbf{1}^{(j)}\right)$$

- Decompose operators with physical interpretation for each term

  $\rightarrow$ monopole, dipole, quadrupole, ...

See also [Cotogno, Lorcé, Lowdon, Morales PRD 2020)]

Dirac Bilinear Calculus

- **Generalized Bilinears:**

$$\bar{u}^{(j)}_{(p_f,s_f)}\Gamma u^{(j)}_{(p_i,s_i)} = \mathring{u}^{(j)\dagger}_{s_f}\begin{pmatrix} 0 & t^{\beta_1\cdots}\tilde{p}^f_{\beta_1\cdots} \\ \bar{t}^{\beta_1\cdots}(\tilde{p}^f_{\beta_1\cdots})^* & 0 \end{pmatrix}\Gamma\begin{pmatrix} t^{\alpha_1\cdots}\tilde{p}^i_{\beta_1\cdots} & 0 \\ 0 & \bar{t}^{\bar{\alpha}_1\cdots}(\tilde{p}^i_{\alpha_1\cdots})^* \end{pmatrix}\mathring{u}^{(j)}_{s_i}$$

Generalized Dirac basis: $\Gamma \to \mathbf{1}\,,\,\gamma_5\,,\,\gamma^{\mu_1\cdots\mu_{2j}}\,,\,\gamma^{\mu_1\cdots\mu_{2j}}\gamma_5\,,\,\mathsf{G}_m{}^{\mu_1\rho_1\cdots\rho_m\mu_m}\,,\;1\leq m\leq 2j$

$$\gamma^{\mu_1\cdots\mu_{2j}}\gamma^{\rho_1\cdots\rho_{2j}} \to \mathsf{G}_m{}^{\mu_1\rho_1\cdots\rho_m\mu_m} = \begin{pmatrix} \prod_{l=1}^m \mathcal{Q}^{\mu_l\rho_l\alpha_l}_{\text{red}}\, t_{\alpha_1\cdots\alpha_m 0\cdots 0} & 0 \\ 0 & \prod_{l=1}^m \bar{\mathcal{Q}}^{\mu_l\rho_l\alpha_l}_{\text{red}}\,\bar{t}_{\alpha_1\cdots\alpha_m 0\cdots 0} \end{pmatrix}$$

- $2j$-rank Tensor bilinear $\widetilde{P} = \frac{1}{2}\left(\tilde{p}_f + \tilde{p}_i\right),\quad \widetilde{\Delta} = \tilde{p}_f - \tilde{p}_i$

$$\bar{u}_f\,\gamma^{\mu_1\cdots\mu_{2j}}\,u_f = m^{2j}\prod_{l=1}^{2j}\left[2\left(\widetilde{P}^{\mu_l}\widetilde{P}^{\tau_l} - \tfrac{1}{4}\widetilde{\Delta}^{\mu_l}\widetilde{\Delta}^{\tau_l}\right) - \left(\widetilde{P}^2 - \tfrac{1}{4}\widetilde{\Delta}^2\right)g^{\mu_l\tau_l} + i\varepsilon^{\mu_l\tau_l\widetilde{P}\widetilde{\Delta}}\right]\langle\lambda_f|t_{\tau_1\cdots\tau_{2j}}|\lambda_i\rangle$$

$$+m^{2j}\prod_{l=1}^{2j}\left[2\left(\widetilde{P}^{\mu_l}\widetilde{P}^{\tau_l} - \tfrac{1}{4}\widetilde{\Delta}^{\mu_l}\widetilde{\Delta}^{\tau_l}\right) - \left(\widetilde{P}^2 - \tfrac{1}{4}\widetilde{\Delta}^2\right)g^{\mu_l\tau_l} + i\varepsilon^{\mu_l\tau_l\widetilde{P}\widetilde{\Delta}}\right]^*\langle\lambda_f|\bar{t}_{\tau_1\cdots\tau_{2j}}|\lambda_i\rangle$$

- **Generalized Gordon Identities**: Reduces number of independent bilinears

Generalization On-Shell (Gordon) Identities

- Using Dirac equation $\left(\gamma^{\mu_1\cdots\mu_{2j}} p_{\mu_1}\ldots p_{\mu_{2j}} - m^{2j}\right) u_p^s = 0$

$$\bar{u}_{p'}^{s'}\left(\Gamma\right) u_p^s = u_{p'}^{s'}\left(\left\{\not{P}^{(j)},\Gamma\right\} + \frac{1}{2}\left[\not{\Delta}^{(j)},\Gamma\right]\right) u_p^s$$

$$0 = \bar{u}_{p'}^{s'}\left(\frac{1}{2}\left\{\not{\Delta}^{(j)},\Gamma\right\} + \left[\not{P}^{(j)},\Gamma\right]\right) u_p^s$$

$$P_{\mu_1\ldots\mu_{2j}} = \tfrac{1}{2}\left(p'_{\mu_1}\ldots p'_{\mu_{2j}} + p_{\mu_1}\ldots p_{\mu_{2j}}\right)$$

$$\Delta_{\mu_1\ldots\mu_{2j}} = p'_{\mu_1}\ldots p'_{\mu_{2j}} - p_{\mu_1}\ldots p_{\mu_{2j}}$$

$$P^{\mu_1\cdots\mu_{2j}}\Delta_{\mu_1\ldots\mu_{2j}} = 0$$

$$P_{(p',p)}^{\mu_1\cdots\mu_{2j}} = P_{(p,p')}^{\mu_1\cdots\mu_{2j}}$$

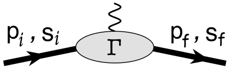
$$\Delta_{(p',p)}^{\mu_1\cdots\mu_{2j}} = -\Delta_{(p,p')}^{\mu_1\cdots\mu_{2j}}$$

- Useful to reduce independent Dirac structures

EM Current: Spin-1

- Local current: $\langle p_f, \lambda_f \,| j^\mu(0) |\, p_i, \lambda_i \rangle = \bar{u}(p_f, \lambda_f)\, \Gamma^\mu(P, \Delta)\, u(p_i, \lambda_i)$

(using all constraints)



$$\Gamma^\mu = P^\mu \left(F_C(\Delta^2)\mathcal{M}_0 + F_Q(\Delta^2)\mathcal{M}_2^{\nu\rho,\xi\sigma} g_{\rho\sigma} \frac{\Delta_\nu \Delta_\xi}{M^2} \right) + \frac{i}{2M} F_D(\Delta^2)\, \mathcal{M}_1^{\mu\rho} \Delta_\rho$$

- Monopole $\mathcal{M}_0 = \begin{pmatrix} \mathbf{1}^{(j)} & 0 \\ 0 & \mathbf{1}^{(j)} \end{pmatrix} = \begin{pmatrix} t_{00} & 0 \\ 0 & \bar{t}_{00} \end{pmatrix}$

- Dipole $\mathcal{M}_1^{\mu\rho} = \begin{pmatrix} \mathbb{M}^{\mu\rho} & 0 \\ 0 & \bar{\mathbb{M}}^{\mu\rho} \end{pmatrix} = \begin{pmatrix} \mathrm{i}(j)\mathcal{Q}_{\rm red}^{\mu\rho\alpha}\, t_{\alpha 0} & 0 \\ 0 & -\mathrm{i}(j)\bar{\mathcal{Q}}_{\rm red}^{\mu\rho\alpha}\, \bar{t}_{\alpha 0} \end{pmatrix}$

- Quadrupole

$$\mathcal{M}_2^{\mu_1\rho_1,\mu_2\rho_2} = -\frac{j(2j-1)}{2} \begin{pmatrix} \mathcal{Q}_{\rm red}^{\mu_1\rho_1\beta_1}\mathcal{Q}_{\rm red}^{\mu_2\rho_2\beta_2}\, t_{\beta_1\beta_2} + \frac{1}{3}\mathcal{C}_{\rm red}^{\mu_1\rho_1\mu_2\rho_2}\mathbf{1} & 0 \\ 0 & \bar{\mathcal{Q}}_{\rm red}^{\mu_1\rho_1\beta_1}\bar{\mathcal{Q}}_{\rm red}^{\mu_2\rho_2\beta_2}\, \bar{t}_{\beta_1\beta_2} + \frac{1}{3}\bar{\mathcal{C}}_{\rm red}^{\mu_1\rho_1\mu_2\rho_2}\mathbf{1} \end{pmatrix}$$

- Bilinear expressions are evaluated using t-algebra relations.

Summary

- Construction allows for efficient and manifestly covariant calculations
- Central role of covariant t-tensors $\rightarrow$ spinors, boosts, propagators, gamma matrices
- Simple/basic ingredients $\rightarrow$ reps. of generators of rotations
- Covariant $sl(2,\mathrm{C})$-multipole basis for operators $\rightarrow$ transparent interpretation
- Unique framework for any spin $\rightarrow$ intuition from spin-1/2 carries over
- Avoid calculations with (Dirac) matrices.
  Everything reduces to number multiplication $\rightarrow$ $\mathcal{C}^{\mu\rho\sigma\alpha}$, $\mathcal{Q}^{\mu\rho\alpha}$

Thank You For Your Time!

Questions?

Backup Slides

Algebra of t-tensors: Reduction for Cubic Monomials

- Central role of the covariant t-tensors (spinors, boosts, propagators, gamma matrices)
- Bilinear calculus involve products with alternating "barring" pattern: $t\bar{t}t\cdots$
- Matrices in t-tensors form a basis of $\mathbf{su(2j+1)}$ $\rightarrow$ Products can be linearized
- Cubic products are reduced with an Invariant Tensor

$$t^{\mu_1\cdots\mu_{2j}}\bar{t}^{\rho_1\cdots\rho_{2j}}t^{\sigma_1\cdots\sigma_{2j}} = \frac{1}{[(2j)!]^2}\underset{\{\rho_1\ldots\rho_{2j}\}}{\mathcal{S}}\underset{\{\sigma_1\ldots\sigma_{2j}\}}{\mathcal{S}}\left(\prod_{l=1}^{2j}\mathcal{C}^{\mu_l\rho_l\sigma_l\alpha_l}\right)t_{\alpha_1\cdots\alpha_{2j}}$$

$$\bar{t}^{\mu_1\cdots\mu_{2j}}t^{\rho_1\cdots\rho_{2j}}\bar{t}^{\sigma_1\cdots\sigma_{2j}} = \frac{1}{[(2j)!]^2}\underset{\{\rho_1\ldots\rho_{2j}\}}{\mathcal{S}}\underset{\{\sigma_1\ldots\sigma_{2j}\}}{\mathcal{S}}\left(\prod_{l=1}^{2j}\bar{\mathcal{C}}^{\mu_l\rho_l\sigma_l\alpha_l}\right)\bar{t}_{\alpha_1\cdots\alpha_{2j}}$$

$$\mathcal{C}^{\mu\rho\alpha\beta} = g^{\mu\rho}g^{\alpha\beta} - g^{\mu\alpha}g^{\rho\beta} + g^{\mu\beta}g^{\rho\alpha} + i\epsilon^{\mu\rho\alpha\beta} \qquad \text{(Lorentz Invariants)}$$

- Trade matrix multiplication by number multiplication

Algebra of t-tensors: Reduction for Quadratic Monomials

- Central role of the covariant t-tensors (spinors, boosts, propagators, gamma matrices)
- Since, $t^{0\cdots 0} = \bar{t}^{0\cdots 0} = 1 \quad \rightarrow \quad t^{\mu_1\cdots\mu_{2j}}\bar{t}^{\nu_1\cdots\nu_{2j}} = t^{\mu_1\cdots\mu_{2j}}\bar{t}^{\nu_1\cdots\nu_{2j}}\left(t^{\rho_1\cdots\rho_{2j}}\eta_{\rho_1}\cdots\eta_{\rho_{2j}}\right)$

$$t^{\mu_1\cdots\mu_{2j}}\bar{t}^{\rho_1\cdots\rho_{2j}} = \frac{1}{(2j)!}\underset{\{\rho_1\ldots\rho_{2j}\}}{\mathcal{S}}\left(\prod_{l=1}^{2j}\mathcal{C}^{\mu_l\rho_l\sigma_l\alpha_l}\eta_{\sigma_l}\right)t_{\alpha_1\cdots\alpha_{2j}}$$

$\eta^\mu = (1,0,0,0)$

$\mathcal{Q}^{\mu\rho\alpha} = \mathcal{C}^{\mu\rho\sigma\alpha}\eta_\sigma = g^{\mu\rho}\eta^\alpha - g^{\rho\alpha}\eta^\mu + g^{\mu\alpha}\eta^\rho + i\epsilon^{\mu\rho\sigma\alpha}\eta_\sigma$ (Rotational Invariant)

- General result $\left(\mathcal{Q}^{\mu\rho\alpha}_{\text{red}} \equiv \mathcal{C}^{\mu\rho\sigma\alpha}\eta_\sigma - g^{\mu\rho}\eta^\alpha\right)$

$$t^{\mu_1\cdots\mu_{2j}}\bar{t}^{\rho_1\cdots\rho_{2j}} = \sum_{m=0}^{2j}\frac{1}{(2j)!}\underset{\{\rho_1\ldots\rho_{2j}\}}{\mathcal{S}}\left[\sum_{n=1}^{B^{2j}_m}\left(\prod_{l\in\pi_{m,n}}\mathcal{Q}^{\mu_l\rho_l\alpha_l}_{\text{red}}\ \prod_{k\in\pi^{\complement}_{m,n}}g^{\mu_k\rho_k}\eta^{\alpha_k}\right)\right]t_{\alpha_1\cdots\alpha_{2j}}$$

each $0 \leq m \leq 2j$ corresponds to a Lorentz independent tensor