NC STATE UNIVERSITY

The chiral anomaly in polarized DIS: Wess-Zumino-Witten contributions and chiral Ward identities for finite quark mass

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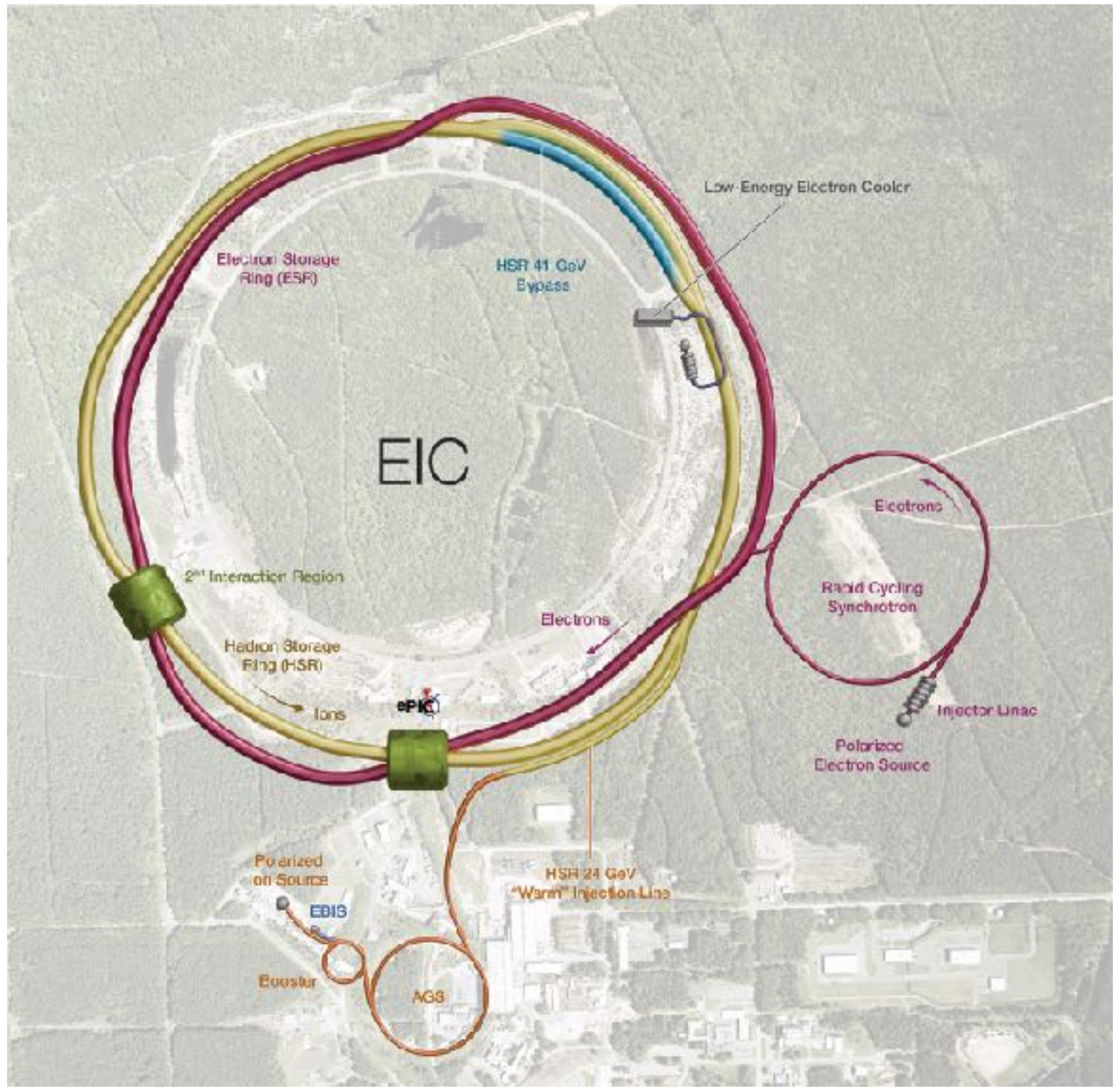
Based on Andrey Tarasov and Raju Venugopalan
arXiv:2501.10519,
Phys. Rev. D 102 (2020) 11, 114022 (arXiv:2008.08104),
Phys. Rev. D 105 (2022) 1, 014020 (arXiv:2109.10370)

POETIC 2025, February 26, 2025

How does the spin of the nucleon arise?

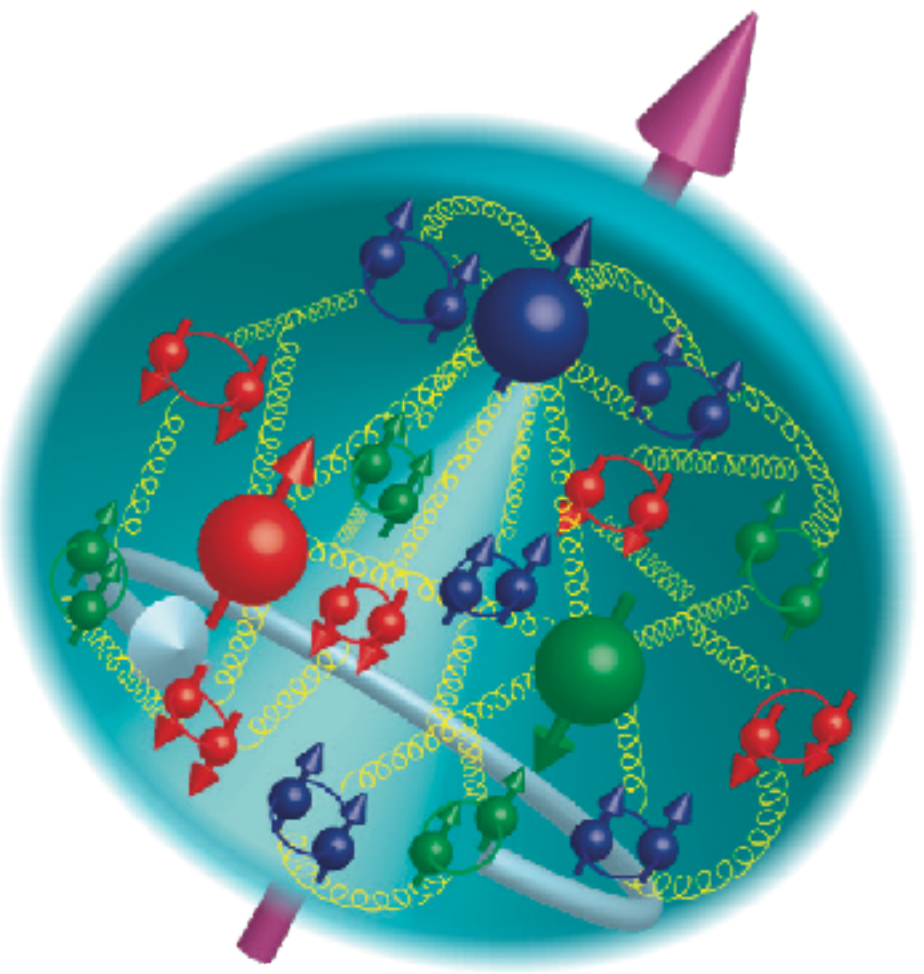
$$\frac{1}{2} = \frac{1}{2}\Delta\Sigma + \Delta G + L_q + L_g$$

Proton spin | Quark helicity | Gluon helicity | Orbital angular momentum



Spin puzzle: quarks carry only about 30% of the proton's spin: $\Delta\Sigma \approx 0.33$, which is much smaller than predicted by the quark model $\Delta\Sigma \approx 0.6$

> The small value of $\Delta\Sigma$ can be explained as due to the interplay between parton dynamics and the topology of the QCD vacuum in the helicity structure of the proton



→ Physics Opportunities at EIC: by measuring a certain class of spin dependent observables in the high-energy scattering we get access to fundamental topological properties of QCD

Deep inelastic scattering

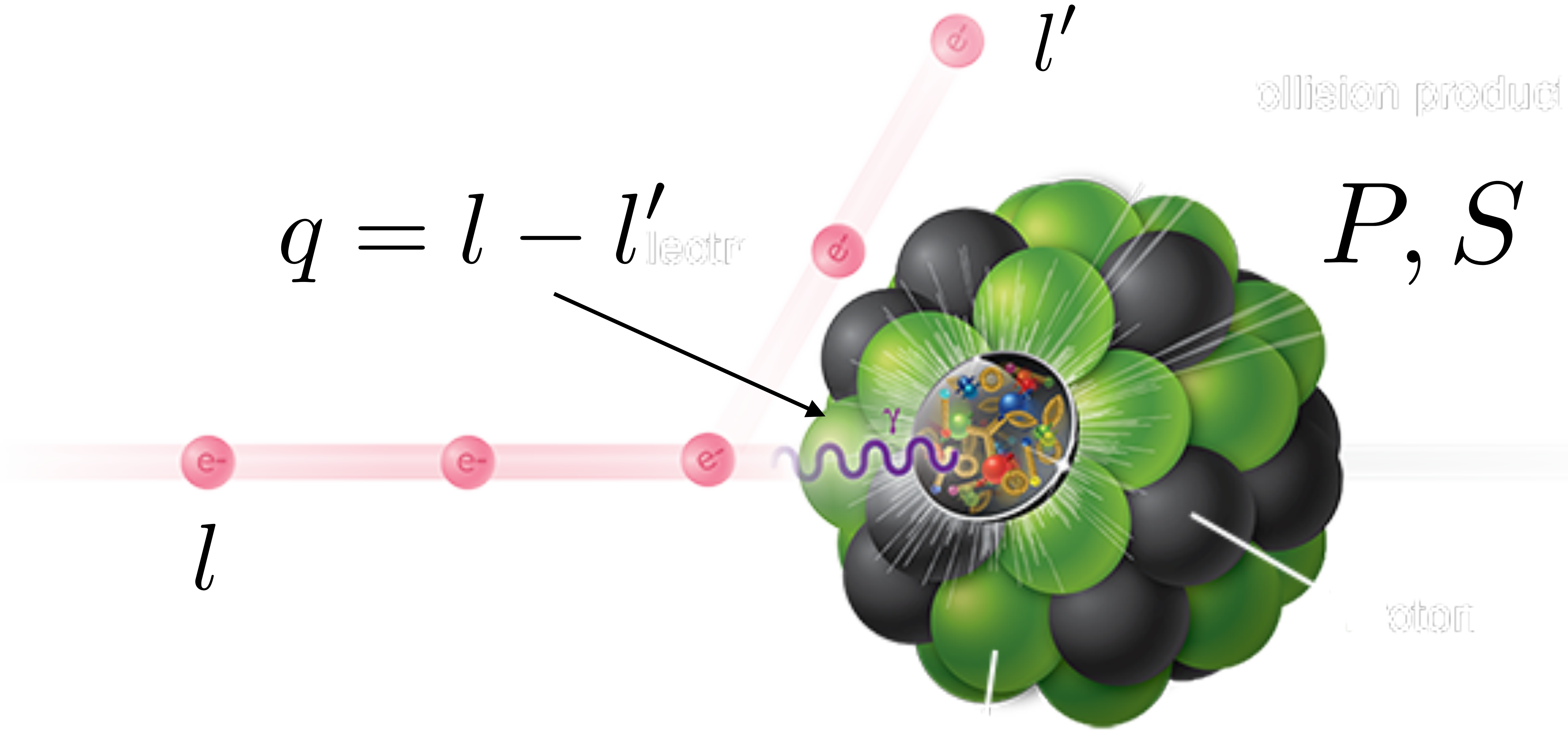
The helicity structure of the proton which can be measured in the polarized DIS

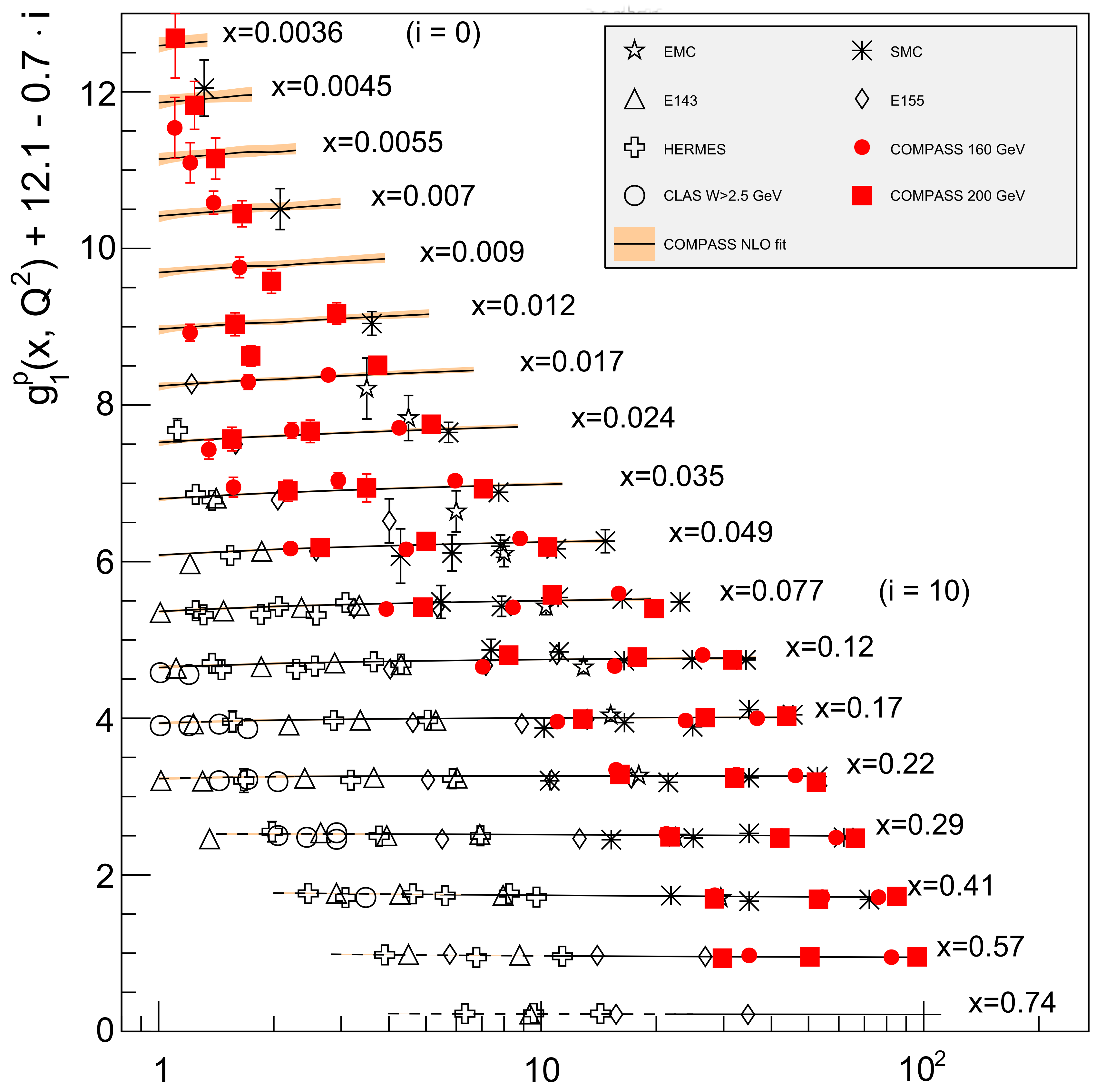
$$e(l) + N(P,S) \to e(l') + X$$

The process is characterized by its virtuality $Q^2 = -\,q^2$ and Bjorken variable $x_B = Q^2/(2P \cdot q)$.

A key observable to study the proton helicity structure is the polarized structure function $g_1(x_B, Q^2)$:

$$\frac{1}{2}\left[\frac{\mathrm{d}^2\sigma^{\leftrightarrows}}{\mathrm{d}x_B \mathrm{d}Q^2} - \frac{\mathrm{d}^2\sigma^{\rightrightarrows}}{\mathrm{d}x_B \mathrm{d}Q^2}\right] \simeq \frac{4\pi\alpha^2}{Q^4} y(2-y) g_1(x_B, Q^2)$$





C.Adolph et al. (COMPASS), PLB 753 (2016)

First moment of the g_1 structure function

The helicity can be extracted from the first moment of the g_1 structure function

$$\int_0^1 dx g_1^{p,n}(x,Q^2) = \left(1 - \frac{\alpha_s(Q^2)}{\pi}\right)\left[\pm \frac{1}{12}A_3 + \frac{1}{36}A_8 + \frac{1}{9}\Delta\Sigma(Q^2)\right] + \mathcal{O}(\alpha_s^2)$$

$$A_3 = F + D$$

$$A_8 = 3F - D$$

Formally $\Delta\Sigma$ can be define as a matrix element of the isosinglet axial vector current J_5^μ operator between proton states:

$$S^\mu \Delta\Sigma = \frac{1}{2M_N}\sum_f \langle P,S|\bar{\psi}_f \gamma^\mu \gamma_5 \psi_f|P,S\rangle$$

The anomaly equation

The fundamental property of the J_5^μ current is the anomaly equation.

Related to the first moment of g_1

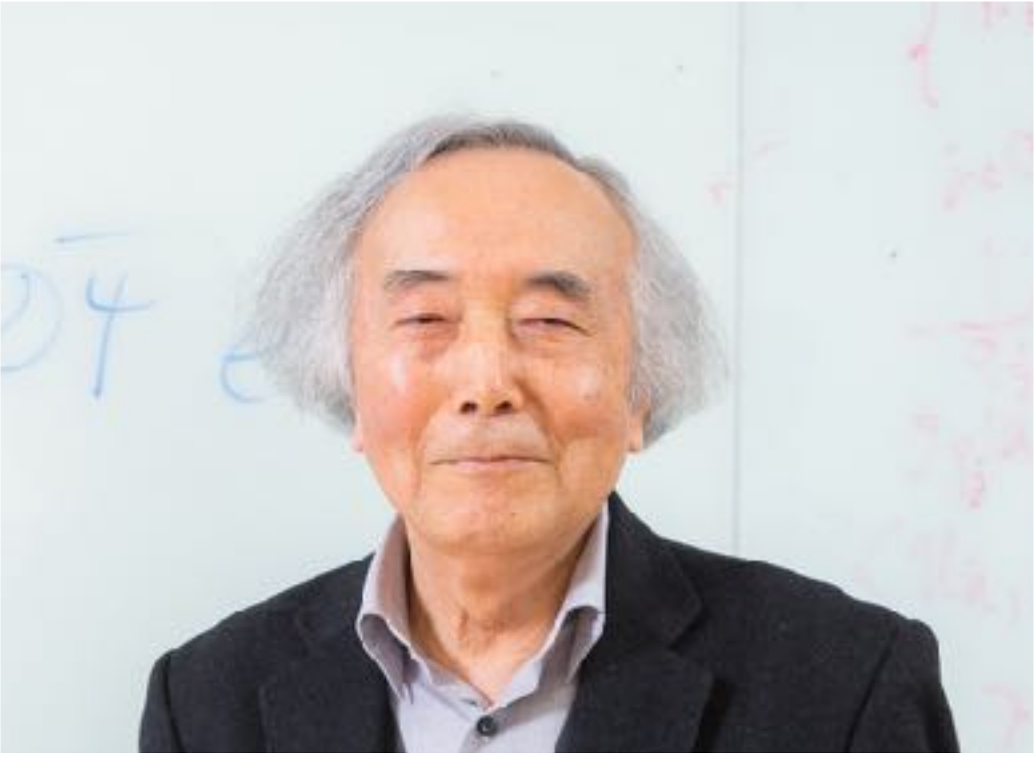
$$\partial^\mu J_\mu^5(x) = \frac{n_f \alpha_s}{2\pi}\ \mathrm{Tr}\left(F_{\mu\nu}(x)\tilde{F}^{\mu\nu}(x)\right) = 2\, n_f\, \partial_\mu K^\mu$$

The isosinglet current couples to the topological charge density in the polarized proton!

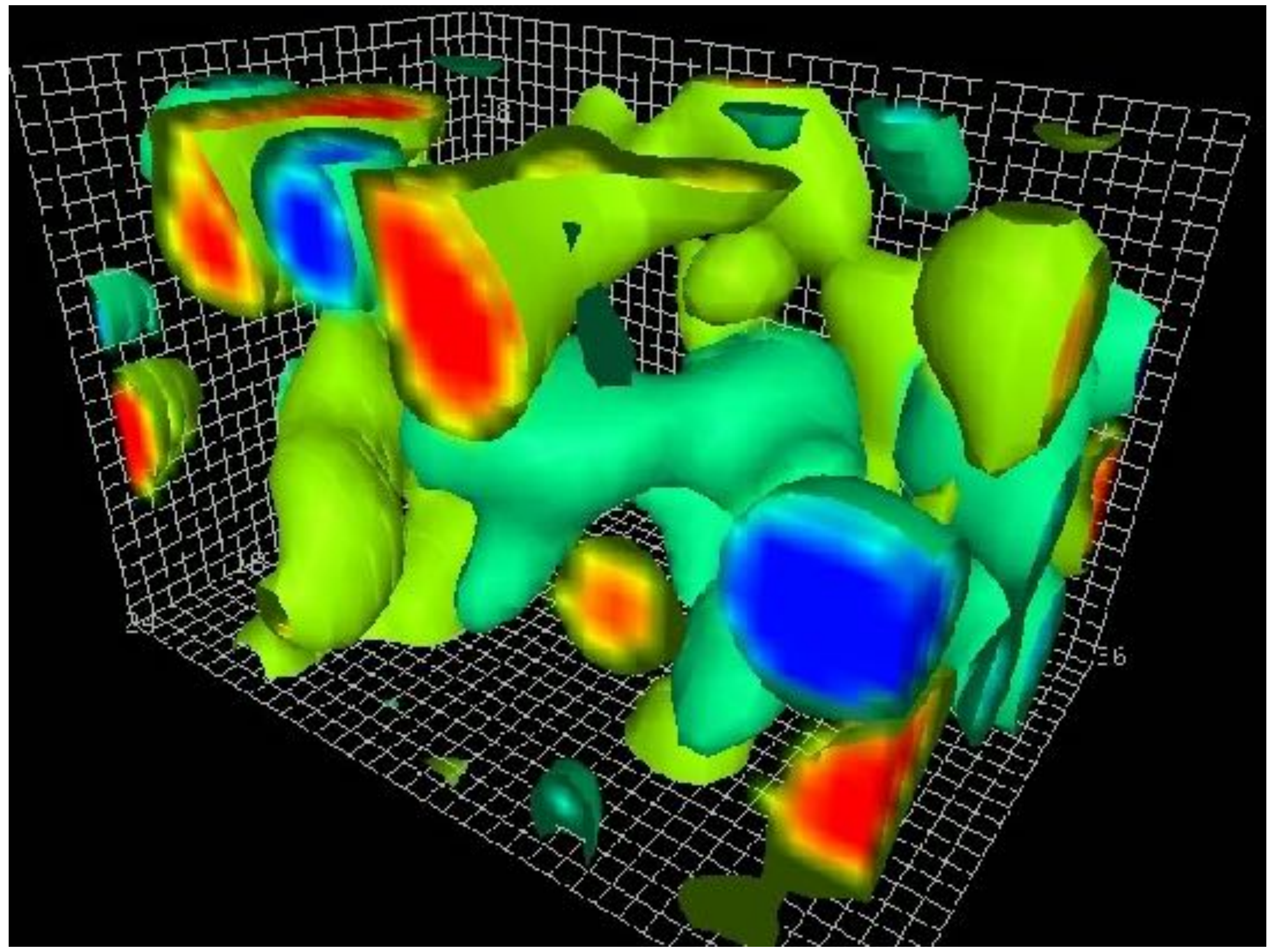
The anomaly arises from the non-invariance of the path integral measure under chiral (γ_5) rotations. Topological properties of the QCD vacuum! **K. Fujikawa, PRL. 42, 1195 (1979)**

Chern-Simons current:

$$K_\mu = \frac{\alpha_S}{4\pi}\epsilon_{\mu\nu\rho\sigma}\left[A_a^\nu\left(\partial^\rho A_a^\sigma - \frac{1}{3} g f_{abc} A_b^\rho A_c^\sigma\right)\right]$$



Kazuo Fujikawa

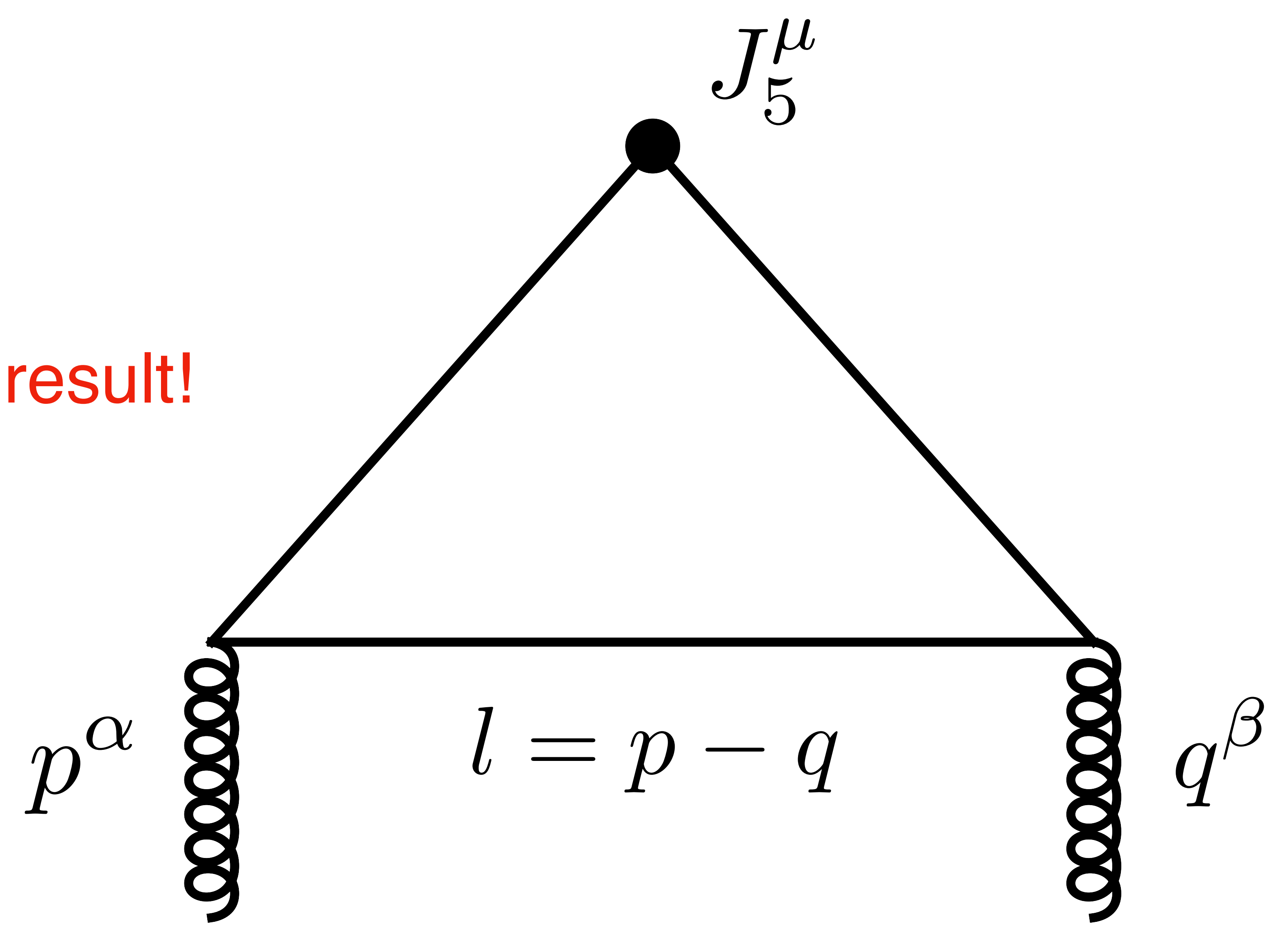


Topological charge density

http://www.physics.adelaide.edu.au/theory/staff/leinweber/VisualQCD/QCDvacuum/

The triangle diagram

In the leading order the anomaly is generated by the triangle diagram and can be seen from the structure of the diagram in the the off-forward limit:



Exact result!

$$\langle P', S | J_5^\mu | P, S \rangle = -i \boxed{\frac{l^\mu}{l^2}} \frac{\alpha_s n_f}{2\pi} \langle P', S | \boxed{\mathrm{Tr}(F\tilde{F})} | P, S \rangle$$

infrared
(anomaly) pole

topological charge
density

R. L. Jaffe, A. Manohar
Nucl. Phys., B337, 509 (1990)
Shore, Veneziano (1990)
Narison, Shore, Veneziano,
hep-ph/9812333
K.-F. Liu (1992)

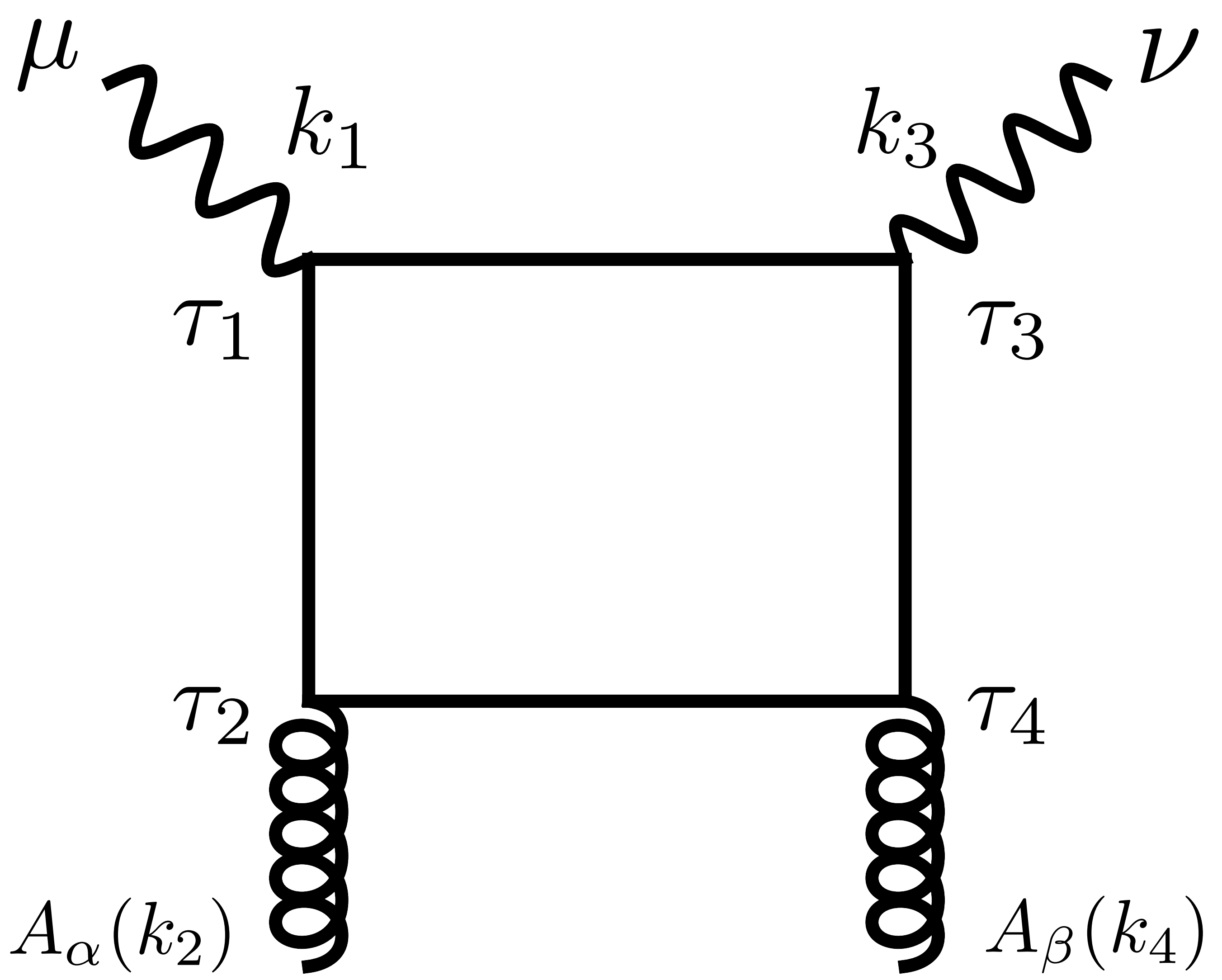
Adler-Bell-Jackiw anomaly

The triangle diagram is not local! The anomaly manifests itself as an infrared pole.
Taking a divergence we obtain the anomaly equation

$$\partial^\mu J_\mu^5 = i l^\mu J_\mu^5 = \frac{n_f \alpha_s}{2\pi} \mathrm{Tr}\left(F_{\mu\nu}\tilde{F}^{\mu\nu}\right)$$

Infrared pole in the box diagram

Similarly the infrared pole was observed in the off-forward box diagram
$\rightarrow g_1$ structure function, GPDs (talk by S. Bhattacharya)



$$= \sum_f e_f^2 \frac{\alpha_s}{i\pi M_N} \int_{x_B}^{1} \frac{dx}{x} \left(1 - \frac{x_B}{x}\right)$$

$$\times \int \frac{d\xi}{2\pi} e^{-i\xi x} \boxed{\frac{l^\mu}{l^2}} \langle P', S| \boxed{\mathrm{Tr_c} F_{\alpha\beta}(\xi n) \tilde{F}^{\alpha\beta}(0)} |P, S\rangle + \text{non-pole terms}$$

infrared pole

Tarasov, Venugopalan (2020-2022)
Bhattacharya, Hatta, Vogelsang (2023)

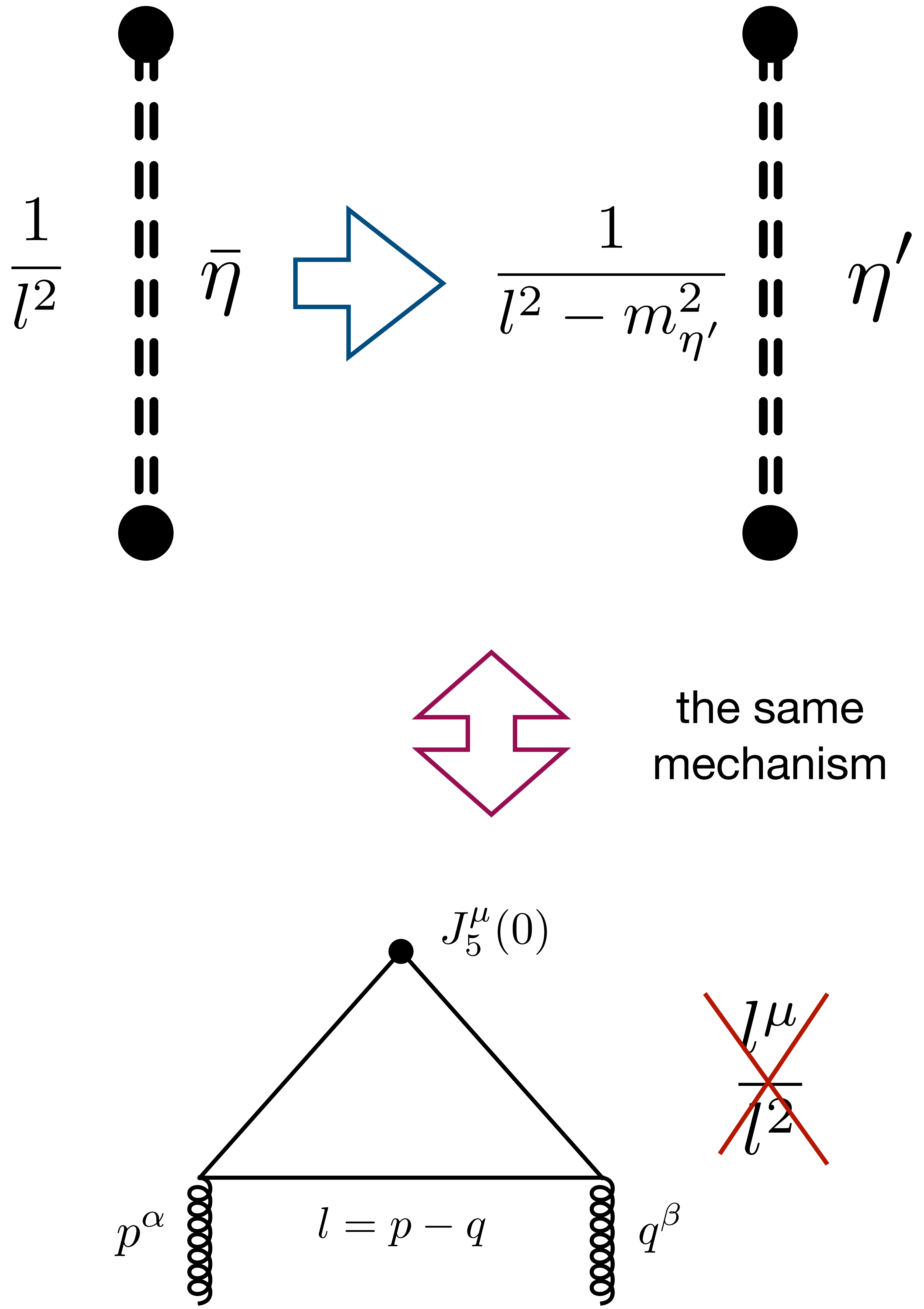
Anomaly pole and the $U_A(1)$ problem

The resolution of the anomaly pole is deeply related to the famous $U_A(1)$ problem in QCD: instead of an infrared pole $1/l^2$ of a "primordial" ninth Goldstone boson $\bar{\eta}$ there is a heavy η' ($m_{\eta'} \approx 957 MeV$)

There is no Goldstone pole just as there is no anomaly pole in the QCD spectrum

The dynamical interplay between the physics of the anomaly, and that of the isosinglet pseudoscalar $U_A(1)$ sector of QCD resolves both problems simultaneously: the lifting of the $\bar{\eta}$ pole by topological mass generation of the η' and the cancellation of the anomaly pole

Tarasov, Venugopalan (2022), see also Bhattacharya, Hatta, Schoenleber (2024) in the context of GPDs



This mechanism relates the helicity $\Delta\Sigma$ to the topology of the QCD vacuum

Topological screening

$$\boxed{\Delta\Sigma|_{m=0} = \frac{N_f}{M_N}\sqrt{\chi'_{\mathrm{QCD}}(0)}\, g_{\bar{\eta}NN}}$$

Shore, Veneziano (1992)
Tarasov, Venugopalan (2022)

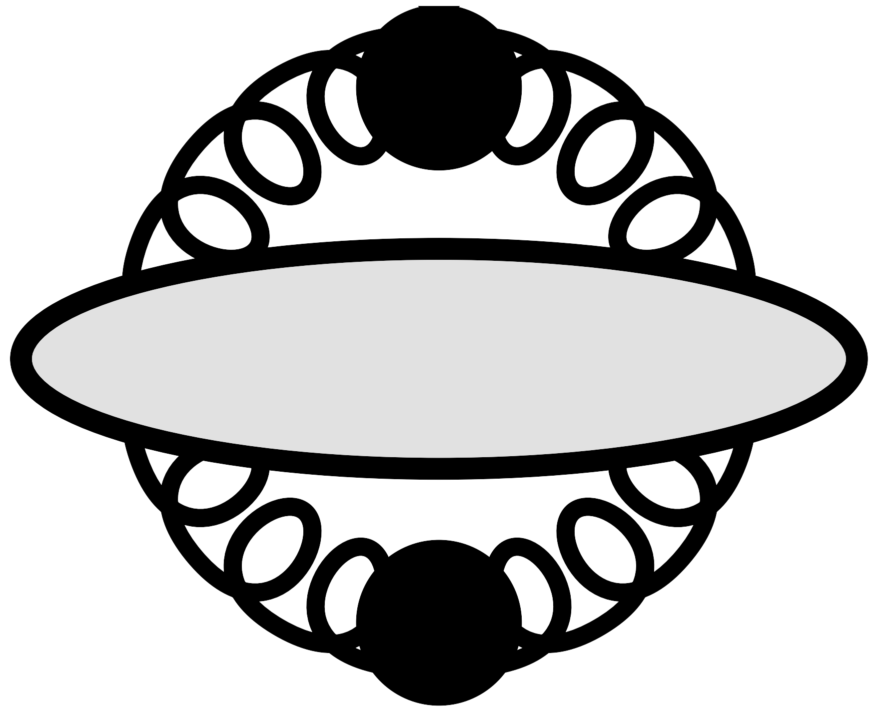
Shore

Veneziano

$$\Omega = \frac{\alpha_s}{4\pi}\mathrm{Tr}\left(F_{\mu\nu}\tilde{F}^{\mu\nu}\right)$$

Topological susceptibility:

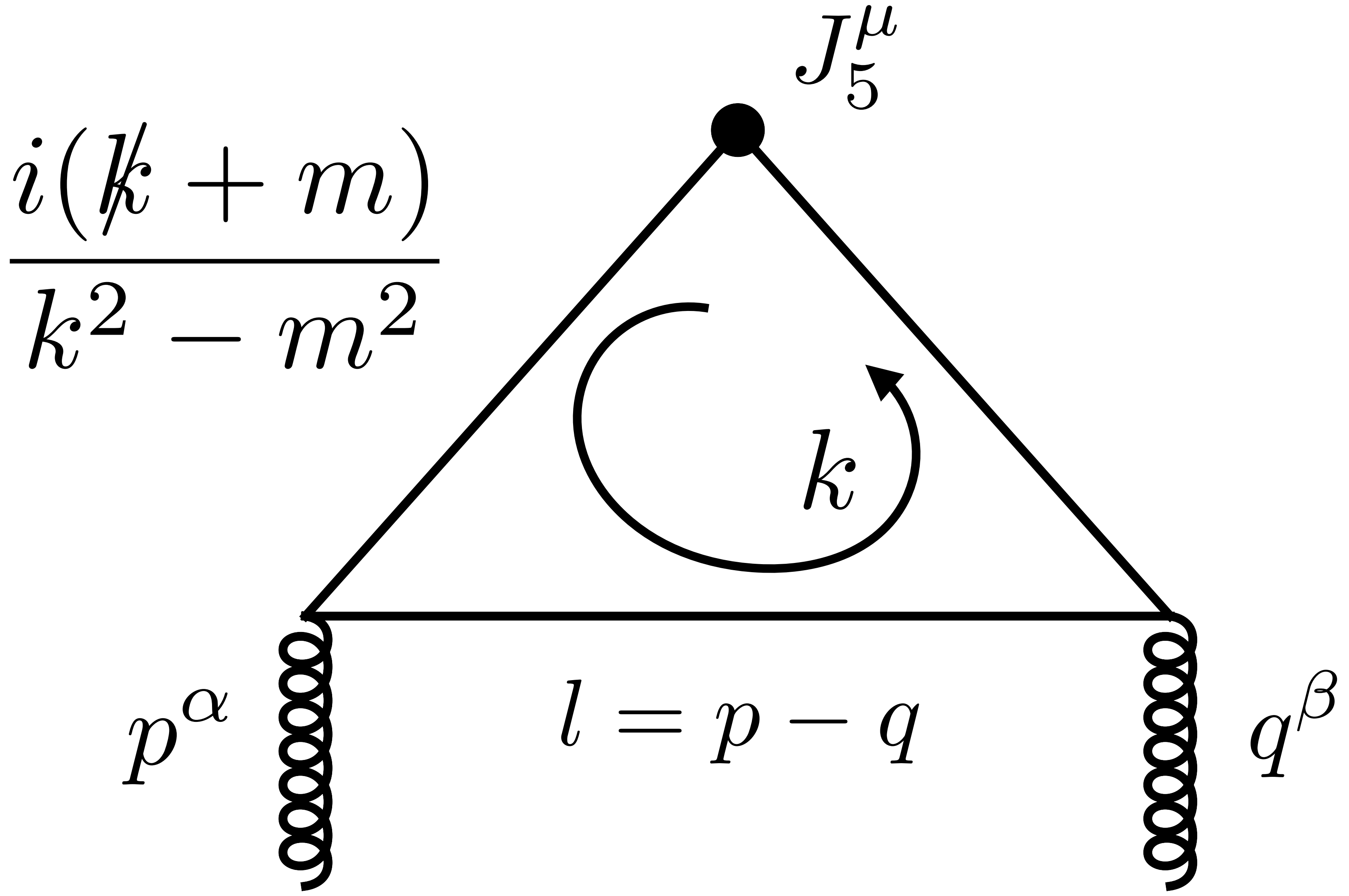
$$\chi_{\mathrm{QCD}}(l^2) = \int d^4x\, e^{ilx}\, \langle 0|T\Omega(x)\Omega(0)|0\rangle_{\mathrm{QCD}}$$



$$\Omega = \frac{\alpha_s}{4\pi}\mathrm{Tr}\left(F_{\mu\nu}\tilde{F}^{\mu\nu}\right)$$

What about finite quark mass?

The previous results where obtained in the chiral limit.
Can the finite quark mass resolve the infrared pole?



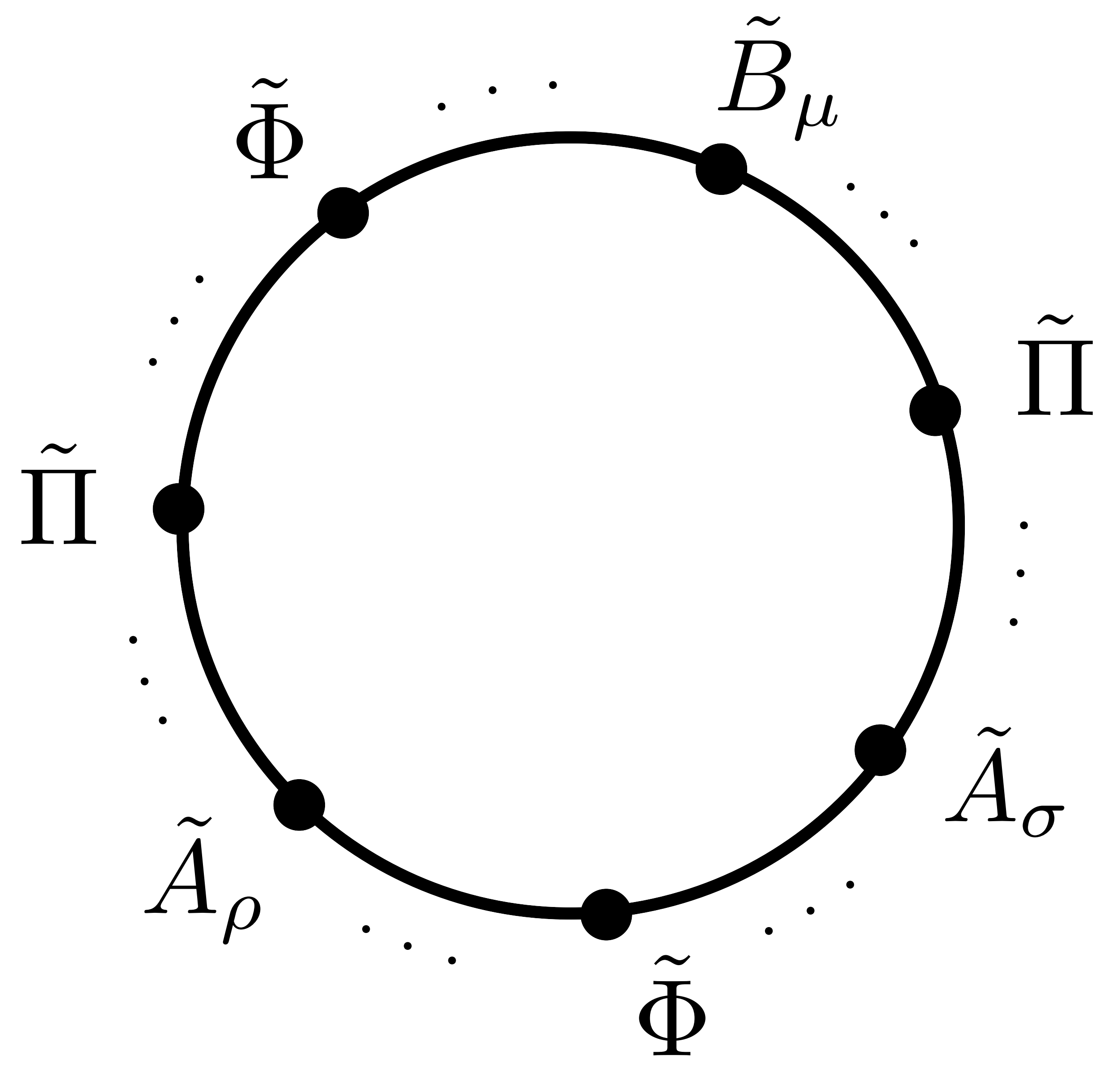
A similar effect occurs in Eq. (5.35) if one includes the quark mass. Namely,the anomalous gluon term $\sim \Delta g$ disappears and one formally recovers the naive Eq. (5.33). Thus the contribution to g_1 from the anomaly is hidden in the massive theory (as well as in the $\overline{MS}$ subtraction scheme). This result will be derived in detail in the next section.

Lampe, Reya "Spin Physics and Polarized Structure Functions" (1998)

Triangle diagram at finite quark mass

It is convenient to perform the calculation in the formalism of the effective action

$$e^{i\mathcal{W}[\Phi,\Pi,A,B]} = \int \mathcal{D}\bar{\Psi}\mathcal{D}\Psi e^{iS_{\text{fermion}}[\bar{\Psi},\Phi,\Pi,A,B,\Psi]}$$

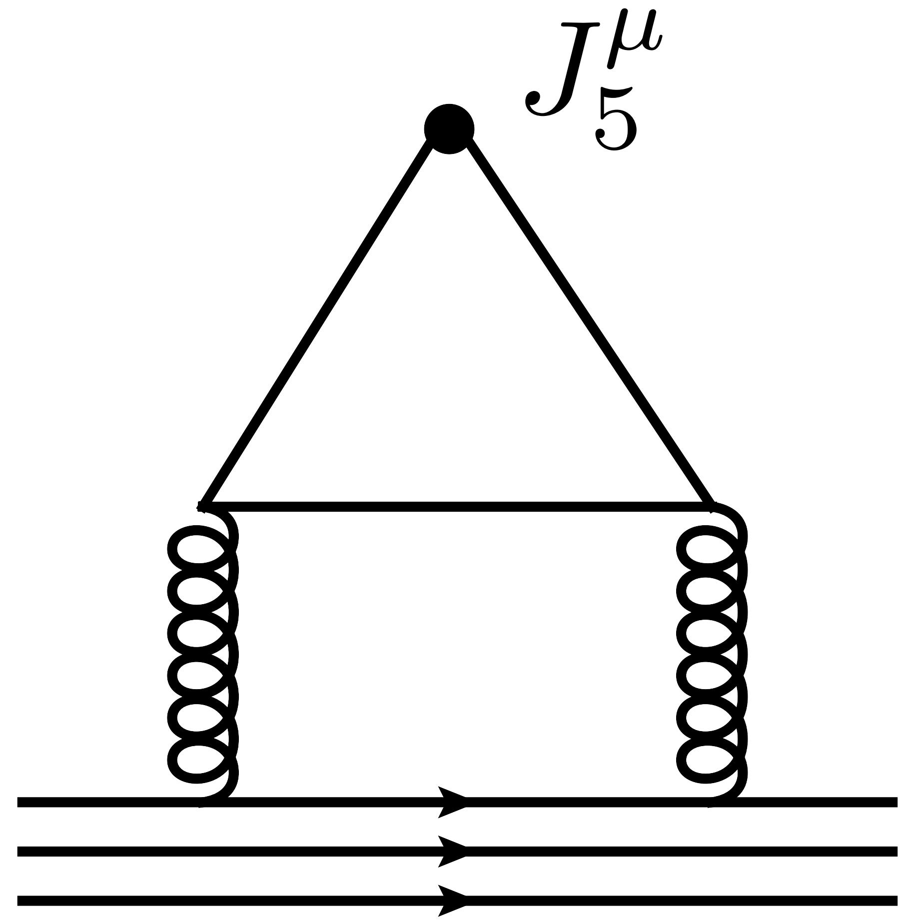


where the action contains coupling of the quark fields to the scalar (Φ), pseudoscalar (Π), vector (A_μ), and axial-vector (B_μ) fields:

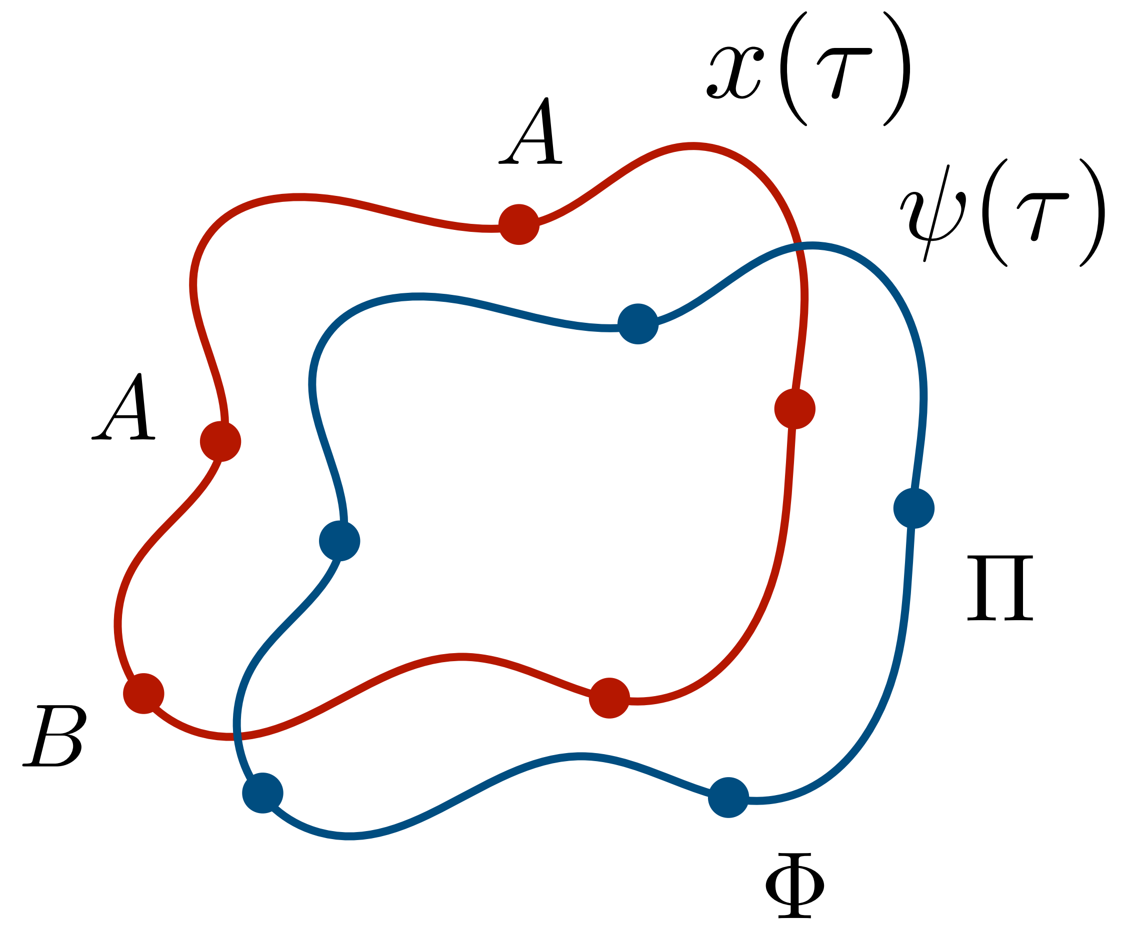
$$S_{\text{fermion}}[\bar{\Psi},\Phi,\Pi,A,B,\Psi] = \int d^4x\, \bar{\Psi}^I \left[i\not{\partial} - \Phi + i\gamma^5\Pi + \not{A} + \gamma^5\not{B}\right]^{IJ} \Psi^J$$

The expectation value of J_5^μ in the proton can be obtained by taking the functional derivative of W:

$$\langle P', S|J_5^\mu|P,S\rangle = \langle P', S| \int d^4y\, \frac{\partial \mathcal{W}_I[A,B]}{\partial B_\mu(y)}\Big|_{B_\mu=0} e^{ily}|P,S\rangle$$



Worldline representation of the effective action



In practice, for the calculation we use the worldline representation of the effective action: integration over trajectories $x(\tau)$ and $\psi(\tau)$ of the quark instead of the quantum fields

$$\mathcal{W}_{\mathcal{I}} = -\frac{i\mathcal{E}}{64}\int_{-1}^{1} d\alpha \int_0^{\infty} dT\mathcal{N}\int_P \mathcal{D}x\mathcal{D}\psi \ \mathrm{tr}\ \chi\bar{\omega}(0)\exp\left[-\int_0^T d\tau\mathcal{L}_{(\alpha)}(\tau)\right]$$

D'Hoker, Gagne (1996)

where the worldline Lagrangian:

$$\mathcal{L}_{(\alpha)}(\tau)\Big|^{\mathrm{singlet}} = \left(\frac{\dot{x}^2}{2\mathcal{E}} + \frac{1}{2}\psi\dot{\psi} - i\dot{x}^{\mu}A_{\mu} + \frac{i\mathcal{E}}{2}\psi^{\mu}\psi^{\nu}(\partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}) + \frac{\mathcal{E}}{2}\alpha^2\Phi^2 + \frac{\mathcal{E}}{2}\Pi^2\right)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$
$$+\left(-i\alpha\dot{x}^{\mu}B_{\mu} + \alpha\frac{i\mathcal{E}}{2}\psi^{\mu}\psi^{\nu}(\partial_{\mu}B_{\nu} - \partial_{\nu}B_{\mu})\right)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$
$$-\alpha\mathcal{E}\psi^{\mu}\psi_5\left(\partial_{\mu}\Phi - \{B_{\mu},\Pi\}\right)\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix} + i\mathcal{E}\psi^{\mu}\psi_5\left(\partial_{\mu}\Pi + \alpha^2\{B_{\mu},\Phi\}\right)\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}.$$

Defines interaction of the worldline (quark) with external fields

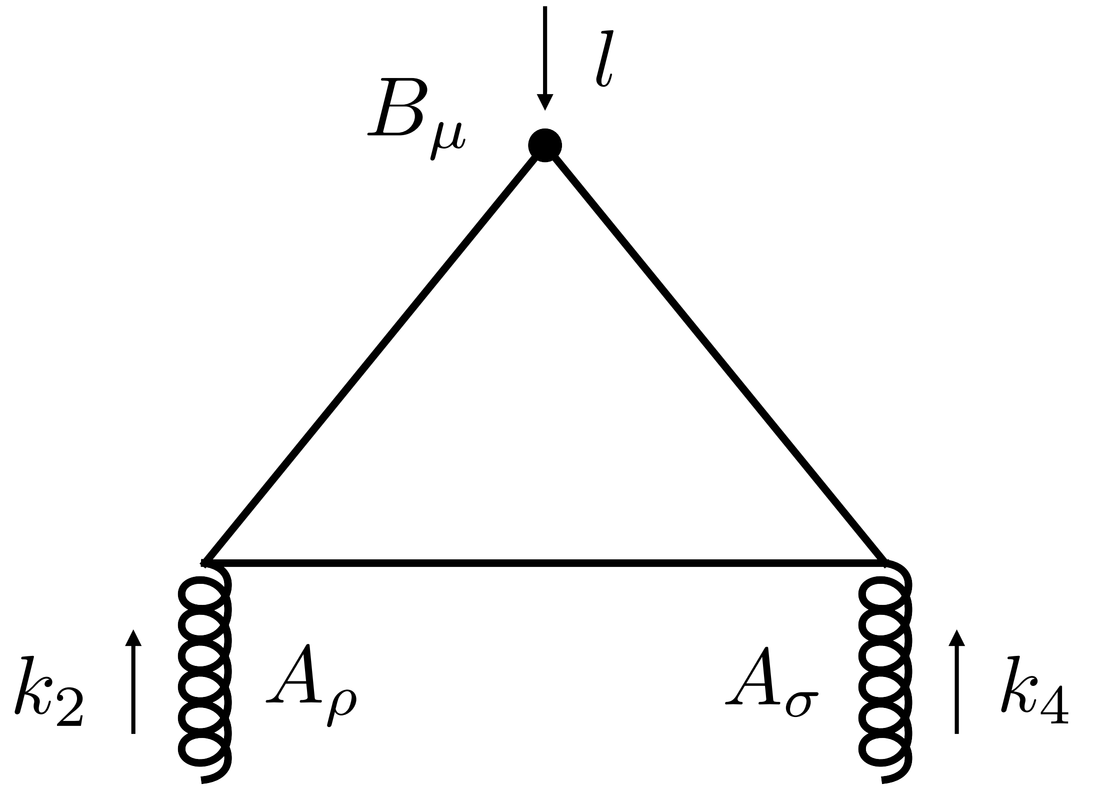
The scalar field can be rewritten as

$$\boxed{\Phi(x) = m - \mathcal{S}(x)}$$

here $S(x)$ is a source of the quark condensate, for the triangle we set it to zero (as in the case of QED)

Triangle diagram at finite mass (AVV)

the triangle in the chiral limit



$$\int d^4x\, e^{ilx}\partial_\mu \frac{\delta W_{\mathcal{I}}}{\delta B_\mu(x)}\Big|^{\text{singlet}}_{B_\mu=0;\Pi=0;\Phi=m} = -\frac{1}{4\pi^2}\int d^4x\, e^{ilx} F_{\alpha\rho}(x)\tilde{F}^{\alpha\rho}(x)$$

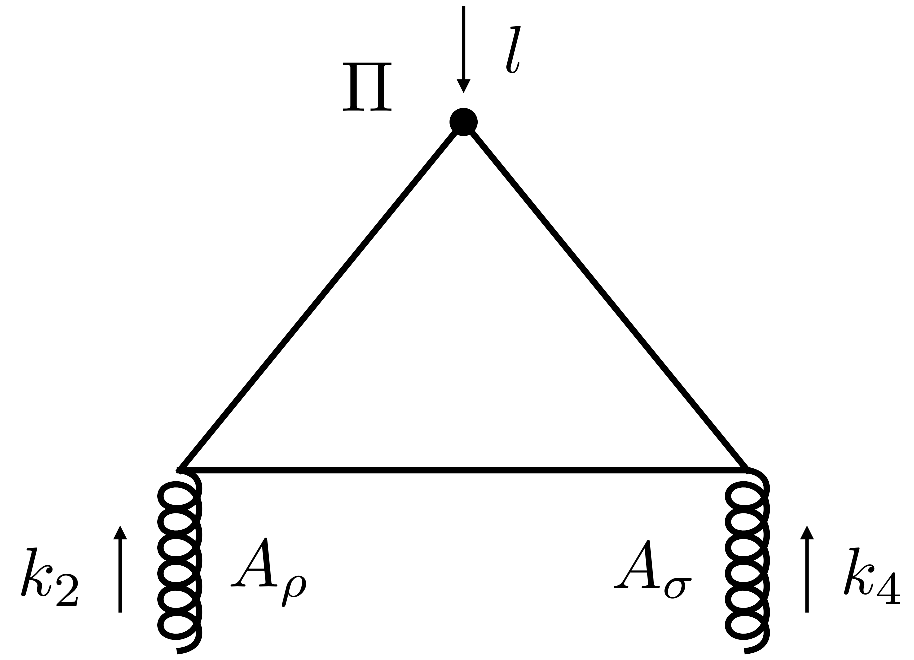
$$+m^2\Big(\int_0^1 du_2 \int_0^1 du_4 \frac{1}{\mathcal{G}(u_0,u_2,u_4)+m^2}\Big)\frac{1}{4\pi^2}\int d^4x e^{ilx} F_{\alpha\rho}(x)\tilde{F}^{\alpha\rho}(x)$$

finite mass correction to the chiral limit

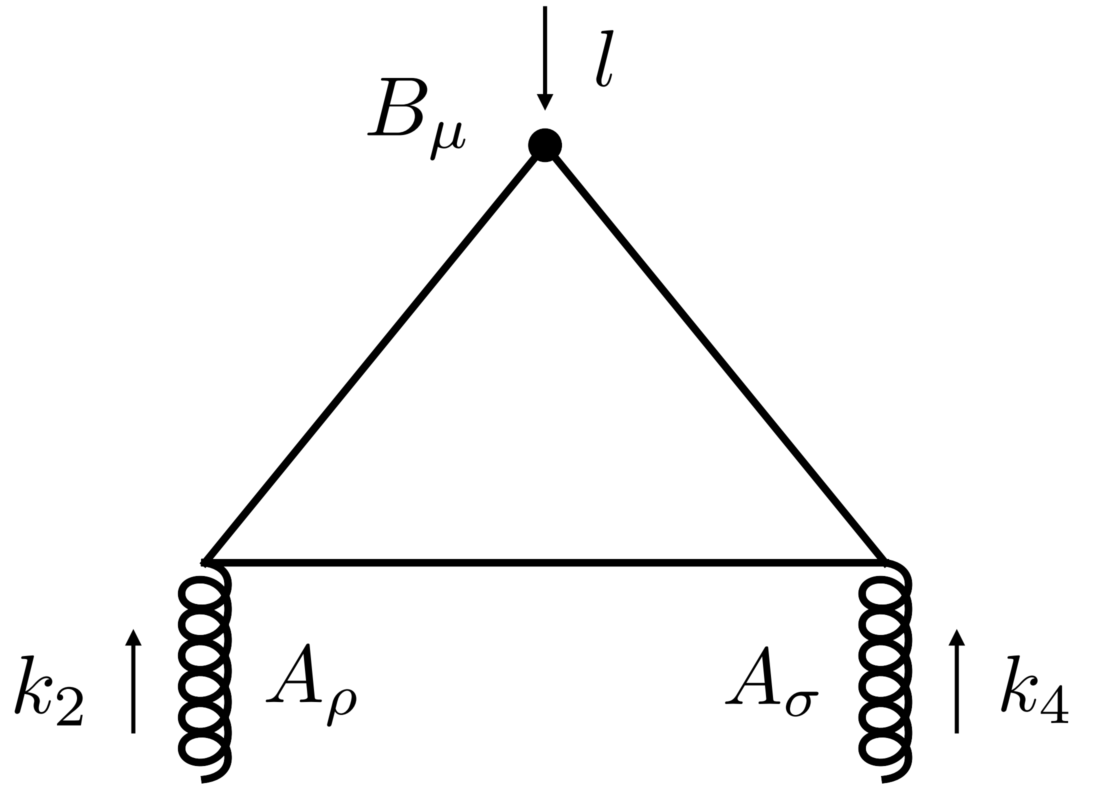
same expression

compare the finite mass term with the derivative of the effective action with Π (PVV):

$$\int d^4x\, e^{ilx} \frac{\delta W_{\mathcal{I}}}{\delta \Pi(x)}\Big|^{\text{singlet}}_{B_\mu=0;\Pi=0;\Phi=m} = -\frac{m}{2}\Big(\int_0^1 du_2 \int_0^1 du_4 \frac{1}{\mathcal{G}(u_0,u_2,u_4)+m^2}\Big)\frac{1}{4\pi^2}\int d^4x e^{ilx} F_{\alpha\rho}(x)\tilde{F}^{\alpha\rho}(x)$$



Triangle diagram at finite mass

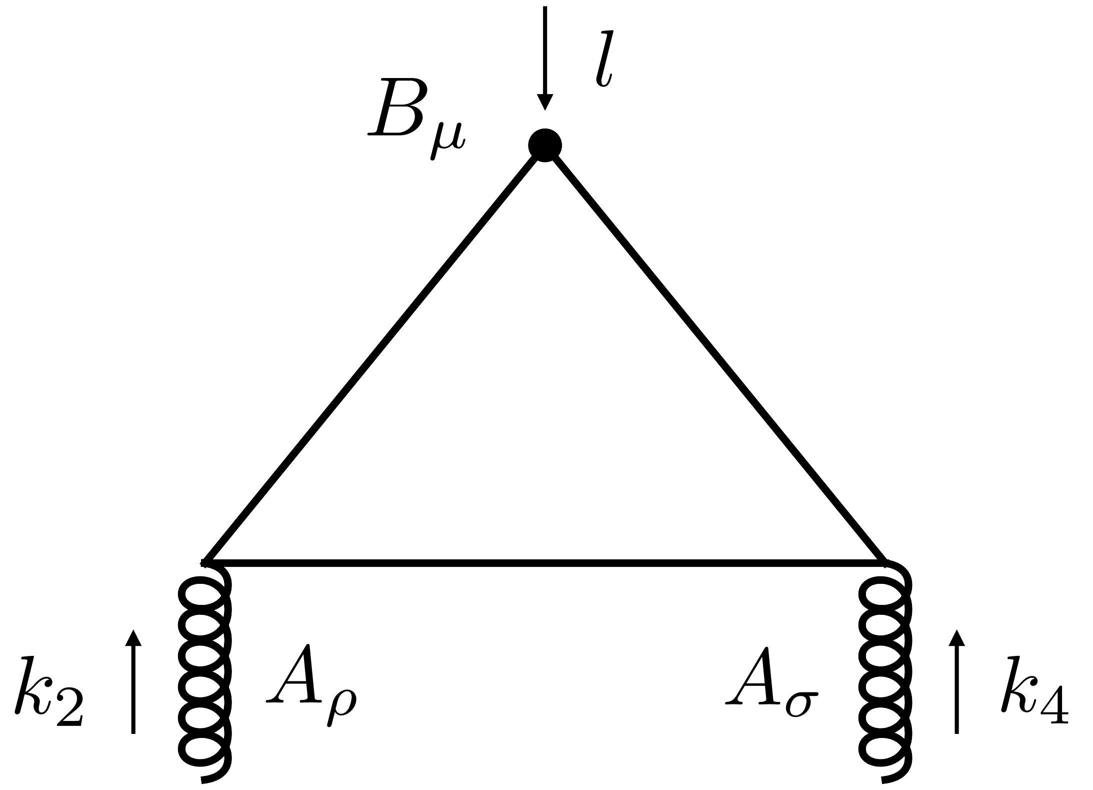


$$\int d^4x\, e^{ilx} \partial_\mu \frac{\delta W_{\mathcal{I}}}{\delta B_\mu(x)}\Big|^{\text{singlet}}_{B_\mu=0;\Pi=0;\Phi=m} = -\frac{1}{4\pi^2}\int d^4x\, e^{ilx} F_{\alpha\rho}(x)\tilde{F}^{\alpha\rho}(x) -2m\int d^4x\, e^{ilx}\frac{\delta W_{\mathcal{I}}}{\delta\Pi(x)}\Big|^{\text{singlet}}_{B_\mu=0;\Pi=0;\Phi=m}$$

As expected the result is consistent with the anomaly equation at finite mass:

$$\partial^\mu J^5_{\mu,f} = \frac{\alpha_S}{2\pi}\mathrm{Tr}\left(F_{\mu\nu}\tilde{F}^{\mu\nu}\right) + 2im_f\,\bar{q}_f\gamma_5 q_f$$

Triangle diagram at finite mass in the forward limit



$$\int d^4x\, e^{ilx} \partial_\mu \frac{\delta W_{\mathcal{I}}}{\delta B_\mu(x)}\Big|^{\text{singlet}}_{B_\mu=0;\Pi=0;\Phi=m} = -\frac{1}{4\pi^2}\int d^4x\, e^{ilx} F_{\alpha\rho}(x)\tilde{F}^{\alpha\rho}(x)$$

$$+m^2\left(\int_0^1 du_2 \int_0^1 du_4 \frac{1}{\mathcal{G}(u_0,u_2,u_4)+m^2}\right)\frac{1}{4\pi^2}\int d^4x e^{ilx} F_{\alpha\rho}(x)\tilde{F}^{\alpha\rho}(x)$$

In the forward limit $l^2 = 0$: $\mathcal{G}(u_0,u_2,u_4)\Big|_{l^2=0} = 0$

$$\int d^4x\, e^{ilx} \partial_\mu \frac{\delta W_{\mathcal{I}}}{\delta B_\mu(x)}\Big|^{\text{singlet}}_{B_\mu=0;\Pi=0;\Phi=m} \overset{l^2\to 0}{=} -\frac{1}{4\pi^2}\int d^4x\, e^{ilx} F_{\alpha\rho}(x)\tilde{F}^{\alpha\rho}(x)$$

$$+\frac{1}{4\pi^2}\int d^4x\, e^{ilx} F_{\alpha\rho}(x)\tilde{F}^{\alpha\rho}(x)$$

Triangle diagram at finite mass in the forward limit

$$\int d^4x\, e^{ilx} \partial_\mu \frac{\delta W_{\mathcal{I}}}{\delta B_\mu(x)}\Big|^{\text{singlet}}_{B_\mu=0;\Pi=0;\Phi=m}\Big|_{l^2=0} = 0$$

Lampe, Reya (1998)
Adler, Bardeen (1969)
G. ’t Hooft (1976)
see also Castelli, Freese, Lorce, Metz, Pasquini, Rodini (2024)

- The ``anomaly" pole from the AVV triangle is exactly canceled by the PVV pole in the forward limit. This is a well known result in QED.
- There is no anomaly pole in QED. The pole is regulated by the finite electron mass.
- The cancellation would not occur for massless electrons $m=0$. Massless QED is not a well-defined theory

However, the mechanism of resolution of the anomaly pole in QCD is dramatically different. In the chiral limit the QCD is a well-defined theory!

The QCD generating functional

Why the QED resolution of the anomaly pole doesn't work in the case of QCD? Confinement, formation of the quark condensate, coupling to the proton etc.

To take into account this effects we write the full QCD generating functional in the presence of the external sources

$$e^{iZ[B,\mathcal{S},\Pi,\Theta]} = \int \mathcal{D}A \int \mathcal{D}\bar{\Psi}\mathcal{D}\Psi e^{iS[A,\bar{\Psi},\Psi,B,\mathcal{S},\Pi,\Theta]}$$

$$S[A,\bar{\Psi},\Psi,B,\mathcal{S},\Pi,\Theta] = \int d^4x \left(\mathcal{L}_{QCD} + B_\mu \bar{\Psi}\gamma^\mu\gamma_5\Psi + \mathcal{S}\bar{\Psi}\Psi + \Pi\bar{\Psi}i\gamma_5\Psi + \Theta\Omega\right)$$

the quark condensate responsible for spontaneous chiral symmetry breaking in QCD

$$\langle\phi\rangle \equiv \frac{\delta Z}{\delta S}\Big|_{\mathcal{S}=0} = \langle\bar{\Psi}\Psi\rangle$$

$$\Omega = \frac{\alpha_s}{4\pi}\mathrm{Tr}\left(F_{\mu\nu}\tilde{F}^{\mu\nu}\right)$$

The fundamental chiral Ward identity for the QCD generating functional

$$\underbrace{\partial_\mu \frac{\delta Z}{\delta B_\mu} - 2N_f \frac{\delta Z}{\delta \Theta} - 2m \frac{\delta Z}{\delta \Pi}} + 2\mathcal{S} \frac{\delta Z}{\delta \Pi} - 2\Pi \frac{\delta Z}{\delta \mathcal{S}} = 0$$

Shore, Veneziano (1992)

quark condensate

$$\partial^\mu J^5_{\mu,f} = \frac{\alpha_S}{2\pi} \mathrm{Tr}\left(F_{\mu\nu}\tilde{F}^{\mu\nu}\right) + 2im_f\, \bar{q}_f \gamma_5 q_f$$

new terms in QCD

If we assume the QED mechanism in QCD, the $\delta Z/\delta\Theta$ and $2m\delta Z/\delta\Pi$ terms cancel each other, but it still doesn't address the last two terms in the equation

The mechanism of regularization of the anomaly pole in QCD is different!

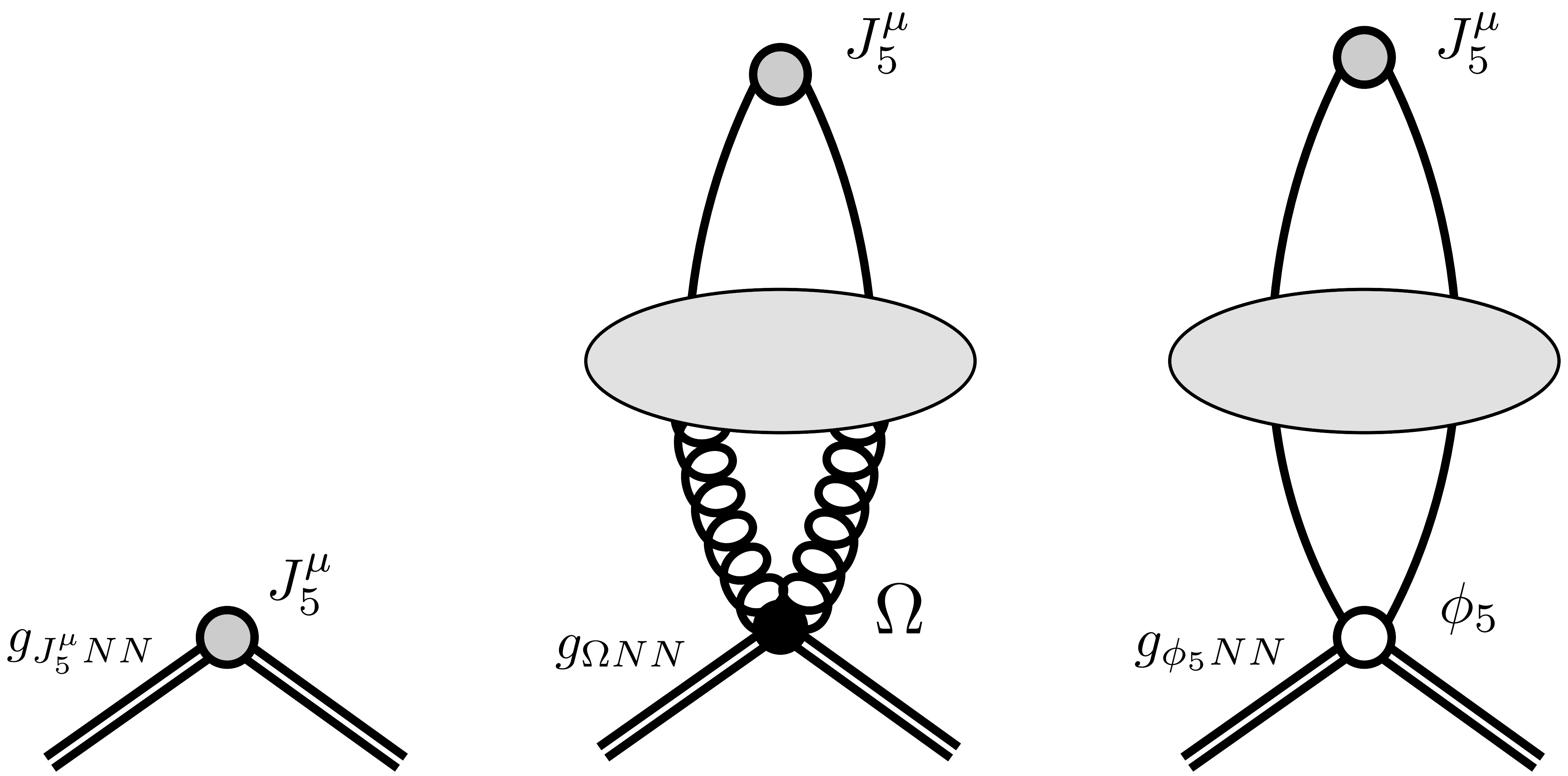
The QCD generating functional and the matrix element of the current

The anomaly pole in QCD is regulated by non-perturbative effects, the mechanism can be formalized in the effective action formalism. Using this formalism we can represent the matrix element of the current as

$$\langle P', S | J_5^\mu | P, S \rangle = -i g_{J_5^\mu NN}(l^2)\, \bar{u}(P', S)\gamma^\mu \gamma_5 u(P, S)$$
$$-i \frac{\delta^2 Z}{\delta B_\mu \delta \Theta}(l^2)\, g_{\Omega NN}(l^2)\, \bar{u}(P', S)\gamma_5 u(P, S) - i \frac{\delta^2 Z}{\delta B_\mu \delta \Pi}(l^2)\, g_{\phi_5 NN}(l^2)\, \bar{u}(P', S)\gamma_5 u(P, S)$$

The anomaly can be absorbed into the contribution of the two-point Green functions

The equation has a nice diagrammatic representation



The QCD generating functional and the matrix element of the current

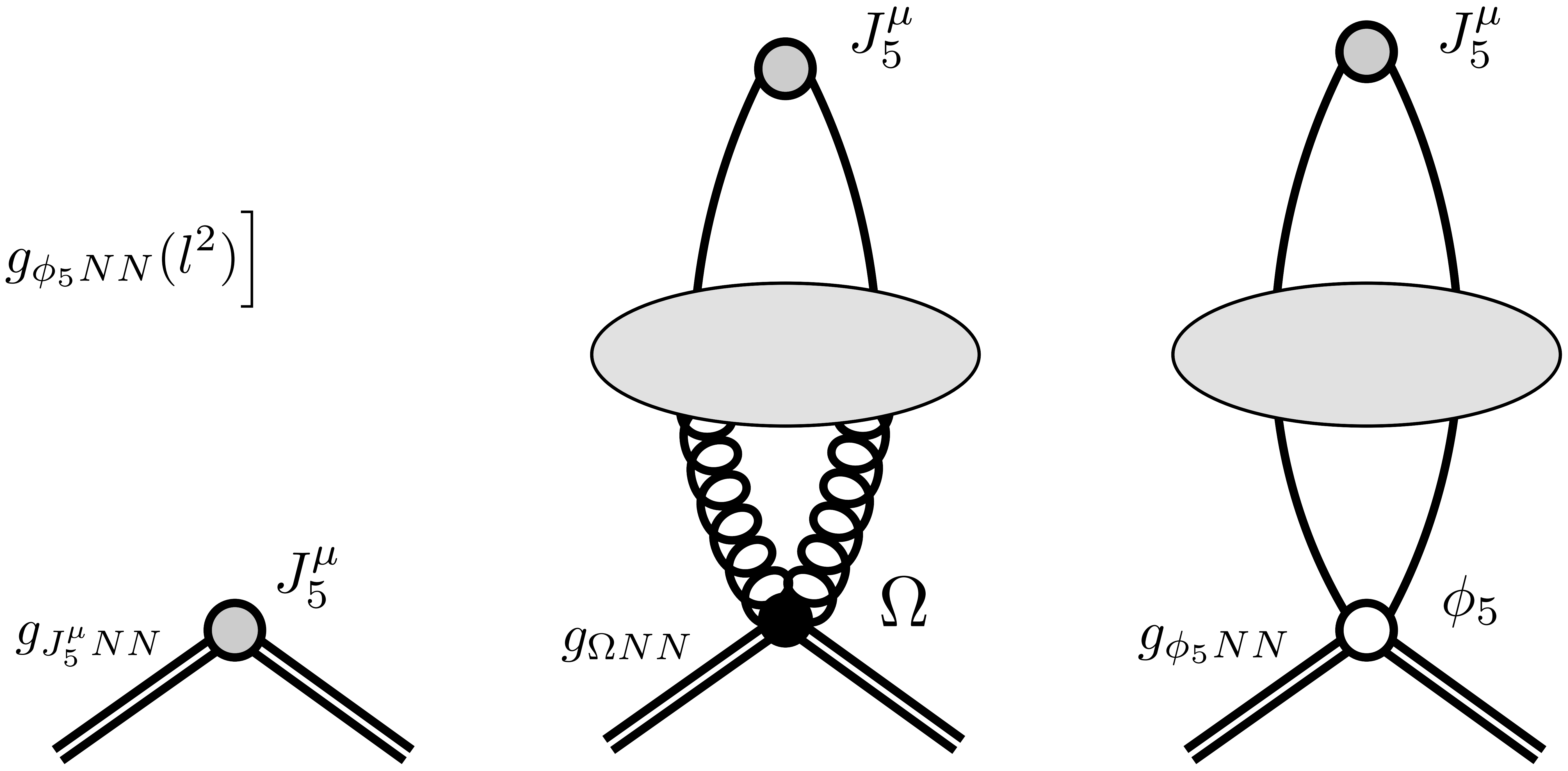
$$\langle P', S | J_5^\mu | P, S \rangle = \bar{u}(P', S) \Big[\gamma^\mu \gamma_5 G_A(l^2) + l^\mu \gamma_5 G_P(l^2) \Big] u(P, S)$$

the representation of the form factors

$$G_A(l^2) = -i g_{J_5^\mu NN}(l^2)$$

$$G_P(l^2) = -i \frac{l_\mu}{l^2} \Big[\frac{\delta^2 Z}{\delta B_\mu \delta \Theta}(l^2)\, g_{\Omega NN}(l^2) + \frac{\delta^2 Z}{\delta B_\mu \delta \Pi}(l^2)\, g_{\phi_5 NN}(l^2) \Big]$$

The anomaly in the two-point Green functions should be regulated by the non-perturbative effects, i.e. $\bar{\eta}$ exchanges. The functions should vanish in the forward limit. How can we realize that?



What do we know about the two-point Green functions?

$$\frac{\delta^2 Z}{\delta B_\mu \delta\Theta}(l^2) = i\int d^4x\, e^{ilx}\, \langle 0|T J_5^\mu(x)\Omega(0)|0\rangle \qquad \frac{\delta^2 Z}{\delta B_\mu \delta\Pi}(l^2) = i\int d^4x\, e^{ilx}\, \langle 0|T J_5^\mu(x)\phi_5(0)|0\rangle$$

The two-point functions are constraint by the chiral Ward identities. There are relations between different types of the two-point functions:

Any form of the two-point functions should satisfy the chiral Ward identities relations

$$il_\mu \frac{\delta^2 Z}{\delta B_\mu \delta\Theta}(l^2) - 2N_f \frac{\delta^2 Z}{\delta\Theta\delta\Theta}(l^2) - 2m\frac{\delta^2 Z}{\delta\Pi\delta\Theta}(l^2) = 0$$

$$il_\mu \frac{\delta^2 Z}{\delta B_\mu \delta\Pi}(l^2) - 2N_f \frac{\delta^2 Z}{\delta\Theta\delta\Pi}(l^2) - 2m\frac{\delta^2 Z}{\delta\Pi\delta\Pi}(l^2) - 2\langle\phi\rangle = 0$$

anomaly

mass term

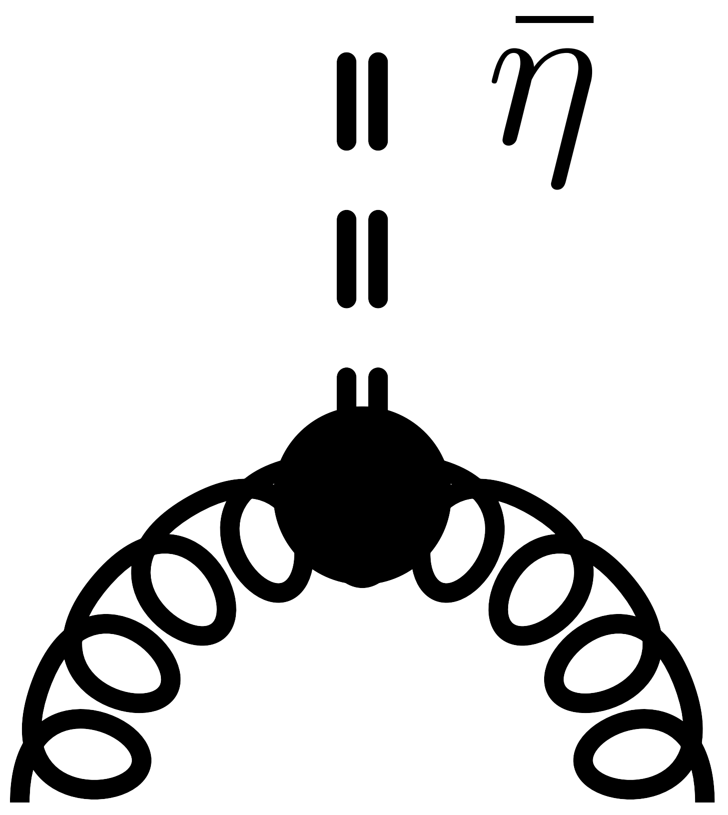
quark condensate

Can we adjust $\frac{\delta^2 Z}{\delta\Theta\delta\Theta}$, $\frac{\delta^2 Z}{\delta\Theta\delta\Pi}$, $\frac{\delta^2 Z}{\delta\Pi\delta\Pi}$ in such a way that the two-point Green functions $\frac{\delta^2 Z}{\delta B_\mu\delta\Theta}$, $\frac{\delta^2 Z}{\delta B_\mu\delta\Pi}$ vanish in the forward limit?

Wess-Zumino-Witten contributions

Can we say more about the form of the functions $\frac{\delta^2 Z}{\delta\Theta\delta\Theta}, \frac{\delta^2 Z}{\delta\Theta\delta\Pi}, \frac{\delta^2 Z}{\delta\Pi\delta\Pi}$? The form of two-point functions can be reconstructed taking into account the WZW coupling between $\bar{\eta}$ and topological charge density Ω:

$$S^{\bar{\eta}}_{\mathrm{WZW}} = -i\,\frac{\sqrt{2N_f}}{F_{\bar{\eta}}}\int d^4x\,\bar{\eta}\,\Omega$$



$$\Omega = \frac{\alpha_s}{4\pi}\mathrm{Tr}\left(F_{\mu\nu}\tilde{F}^{\mu\nu}\right)$$

Leutwyler (1996); Herrara-Sikody et al (1997);
Leutwyler-Kaiser (2000)

WZW and two-point Green functions

QCD topological susceptibility:

$$\frac{\delta^2 Z}{\delta\Theta\delta\Theta}(l^2)\Big|_{\theta=0} \equiv \chi_{\mathrm{QCD}}(l^2)$$

Using the WZW coupling we can construct the form of the two-point functions:

$$\chi_{\mathrm{QCD}}(l^2) = \chi_{\mathrm{YM}}(l^2)$$

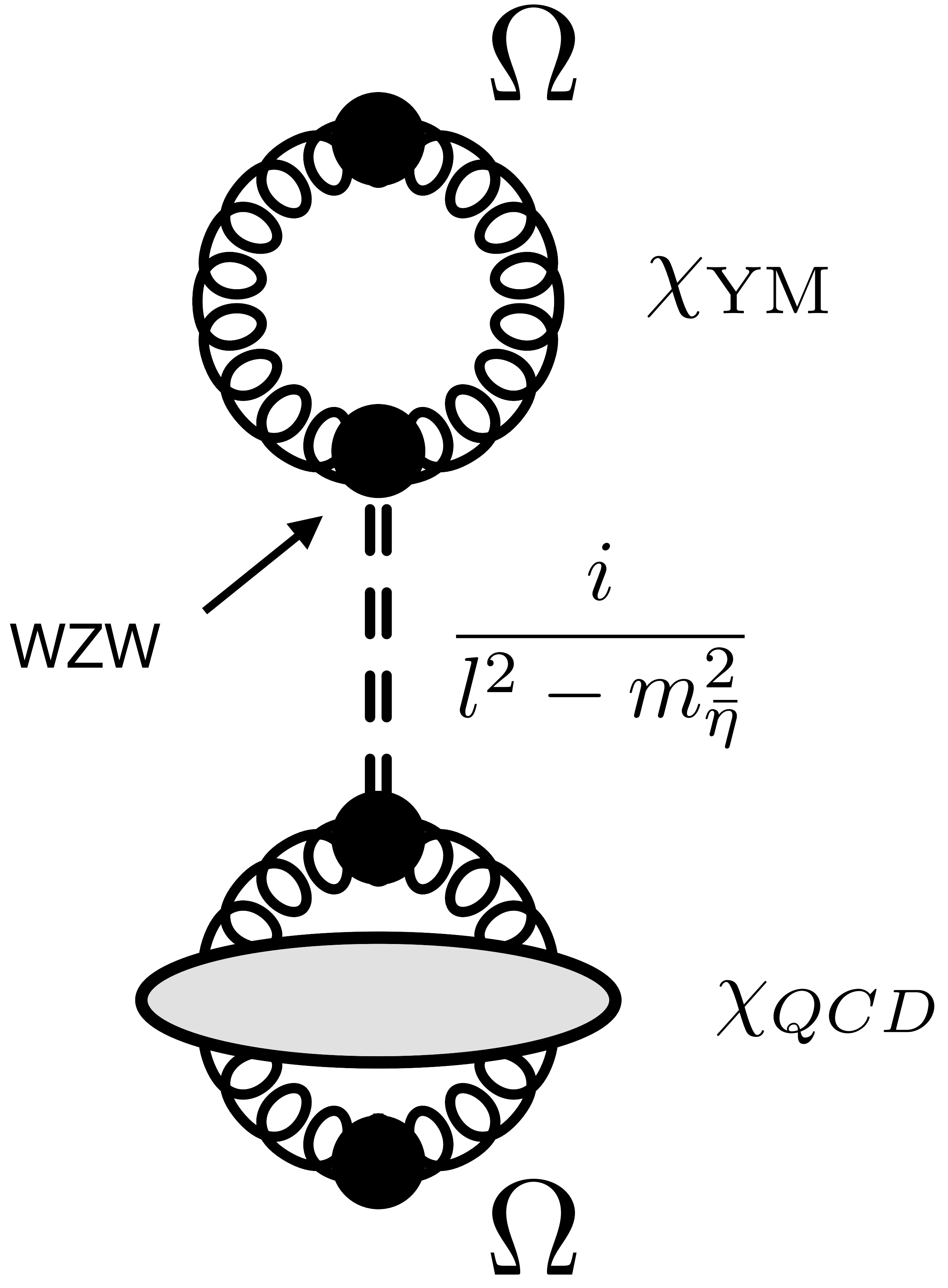
$$+\,i\Big[-i\chi_{\mathrm{YM}}(l^2)\Big]\,\Big[-i\frac{\sqrt{2N_f}}{F_{\bar\eta}}b(l^2)\Big]\,\frac{i}{l^2-m_{\bar\eta}^2}\,\Big[-i\frac{\sqrt{2N_f}}{F_{\bar\eta}}b(l^2)\Big]\,\Big[-i\chi_{\mathrm{QCD}}(l^2)\Big]$$

WZW

$\bar\eta$ propagation

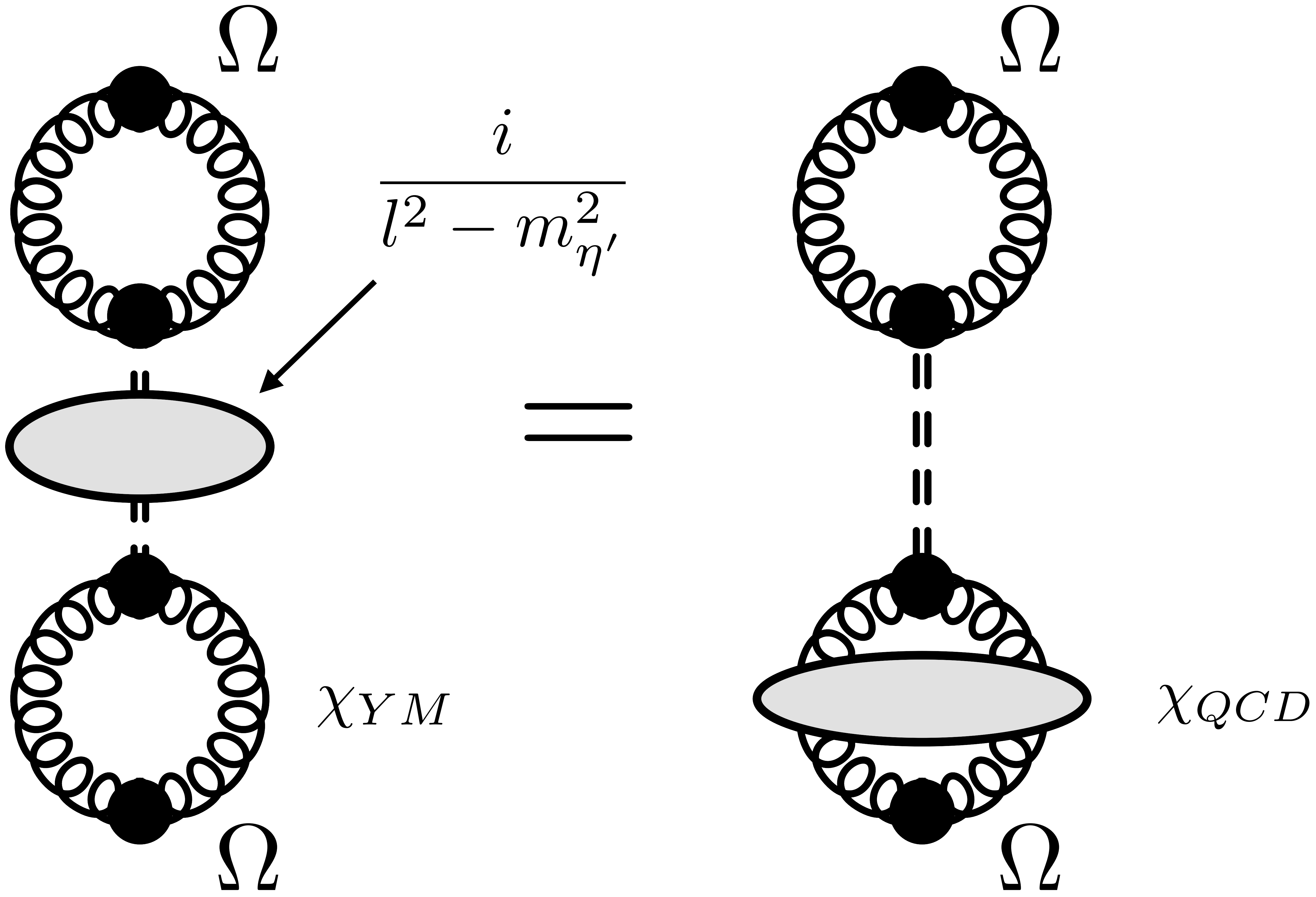
WZW

non-perturbative function to describe possible deviations from WZW



Similar expressions can be obtained for other two-point functions

Why is it useful?



In the chiral limit this relations yield:

$$\chi_{\mathrm{QCD}}|_{m=0}(0) = 0$$

$$\boxed{F_{\bar{\eta}}^2 = 2\,N_f\,\chi'_{\mathrm{QCD}}|_{m=0}(0)}$$

At finite quark mass:

$$m_{\eta'}^2 = -\frac{2N_f}{F_{\bar{\eta}}^2}\chi_{\mathrm{YM}}(0) + m_{\bar{\eta}}^2$$

Witten-Veneziano

finite mass correction

Regularization of the anomaly pole in QCD

By adjusting parameters of the two-point functions $\frac{\delta^2 Z}{\delta\Theta\delta\Theta}, \frac{\delta^2 Z}{\delta\Theta\delta\Pi}, \frac{\delta^2 Z}{\delta\Pi\delta\Pi}$, through the chiral Ward identities, we can get resolve the anomaly poles in $\frac{\delta^2 Z}{\delta B_\mu \delta\Theta}$ and $\frac{\delta^2 Z}{\delta B_\mu \delta\Pi}$:

$$\lim_{l^2\to 0} il_\mu \frac{\delta^2 Z}{\delta B_\mu \delta\Theta}(l^2) = 0 \qquad \lim_{l^2\to 0} il_\mu \frac{\delta^2 Z}{\delta B_\mu \delta\Pi}(l^2) = 0$$

This "adjustment" leads to a number of important relations between parameters:

$$m_{\bar\eta}^2 = -\frac{2\,m\langle\phi\rangle}{N_f F_{\bar\eta}^2}$$

Dashen-Gell-Mann-Oakes-Renner (DGMOR) relation

$$F_{\bar\eta}^2 = 2N_f\chi'_{\mathrm{QCD}}(0)\left(1 - m_{\bar\eta}^2\frac{2\chi'_{\mathrm{QCD}}(0) - \chi'_{\mathrm{YM}}(0)}{\chi_{\mathrm{YM}}(0)}\right) + O(m_{\bar\eta}^4)$$

chiral limit

finite mass correction

Goldberger-Treiman (GT) identity

To satisfy the anomaly equation the pseudovector and pseudoscalar sectors has to be related → GT identity:

$$2M_N \, G_A(0) = \sqrt{2N_f} \, F_{\bar{\eta}} \, g_{\bar{\eta}NN}$$

The GT identity leads to:

$$\Delta\Sigma = \frac{\sqrt{2N_f}}{2M_N} \, F_{\bar{\eta}} \, g_{\bar{\eta}NN}$$

or using relations from the analysis of the two-point Green functions:

$$\Delta\Sigma = \frac{2N_f}{2M_N} \sqrt{\chi'_{\mathrm{QCD}}(0)} \left(1 - m_{\bar{\eta}}^2 \frac{2\chi'_{\mathrm{QCD}}(0) - \chi'_{\mathrm{YM}}(0)}{\chi_{\mathrm{YM}}(0)}\right)^{1/2} g_{\bar{\eta}NN}$$

chiral limit

finite mass introduces $\sim 10\,\%$ correction

Phenomenology. Back of the envelope estimate

In the OZI (Okubo–Zweig–Iizuka) approximation: $\Delta\Sigma^{\mathrm{OZI}} \equiv G_A^{\mathrm{OZI}} = 0.579 \pm 0.021$

$$\sqrt{\chi'_{\mathrm{QCD,OZI}}(0)} = F_\pi/\sqrt{6} \approx 32\,\mathrm{MeV}$$

Looking at the ration:

$$\Delta\Sigma^{\mathrm{expt.}} = \Delta\Sigma^{\mathrm{OZI}} \frac{\sqrt{\chi'_{\mathrm{QCD}}(0)}}{\sqrt{\chi'_{\mathrm{QCD,OZI}}(0)}}$$

and substituting $\Delta\Sigma^{\mathrm{expt.}} \simeq 0.33$

we obtain

$$\sqrt{\chi'_{\mathrm{QCD}}(0)} \approx 18\,\mathrm{MeV} \qquad \sqrt{\chi'_{\mathrm{YM}}(0)} \approx 36\,\mathrm{MeV}$$

to be compared with the recent lattice result $\sqrt{\chi'_{\mathrm{YM}}(0)} \approx 17.1\,\mathrm{MeV}$ **Bonanno (2024)**

The fact that the $\sqrt{\chi'_{\mathrm{YM}}(0)}$ estimates are in the same range is very encouraging and calls for systematic implementation of chiral perturbation theory as well as further lattice computations of the slope of the topological susceptibility in full QCD

Thank you for your attention!