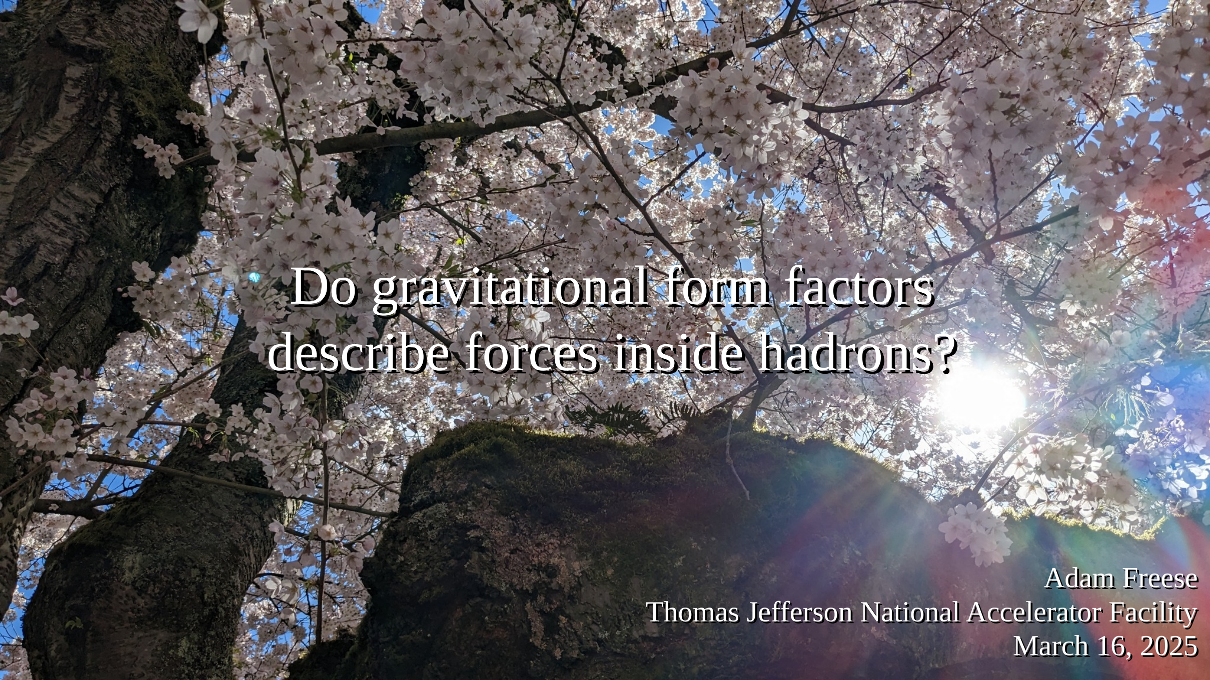

Do gravitational form factors describe forces inside hadrons?

Adam Freese
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March 16, 2025

Hadronic stress tensor

- Pressure in the proton has become a hot topic.
- Promises of much interesting physics:
  - Force distributions in the proton
  - Mechanical stability conditions
  - Understanding confinement?
- Empirical extractions happening at JLab!
  - Burkert, Elouadrhiri & Girod, Nature (2018)
  - Burkert, Elouadrhiri & Girod, 2104.02031
  - Duran &al., Nature (2023)

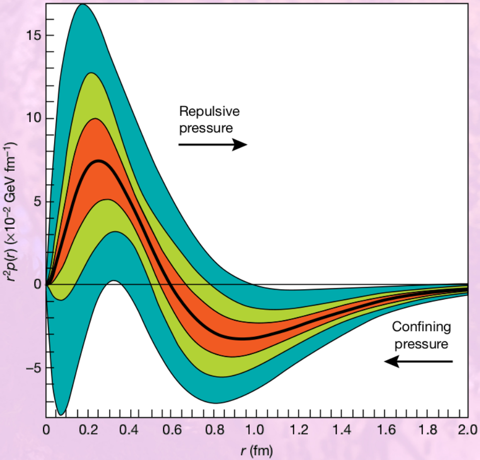
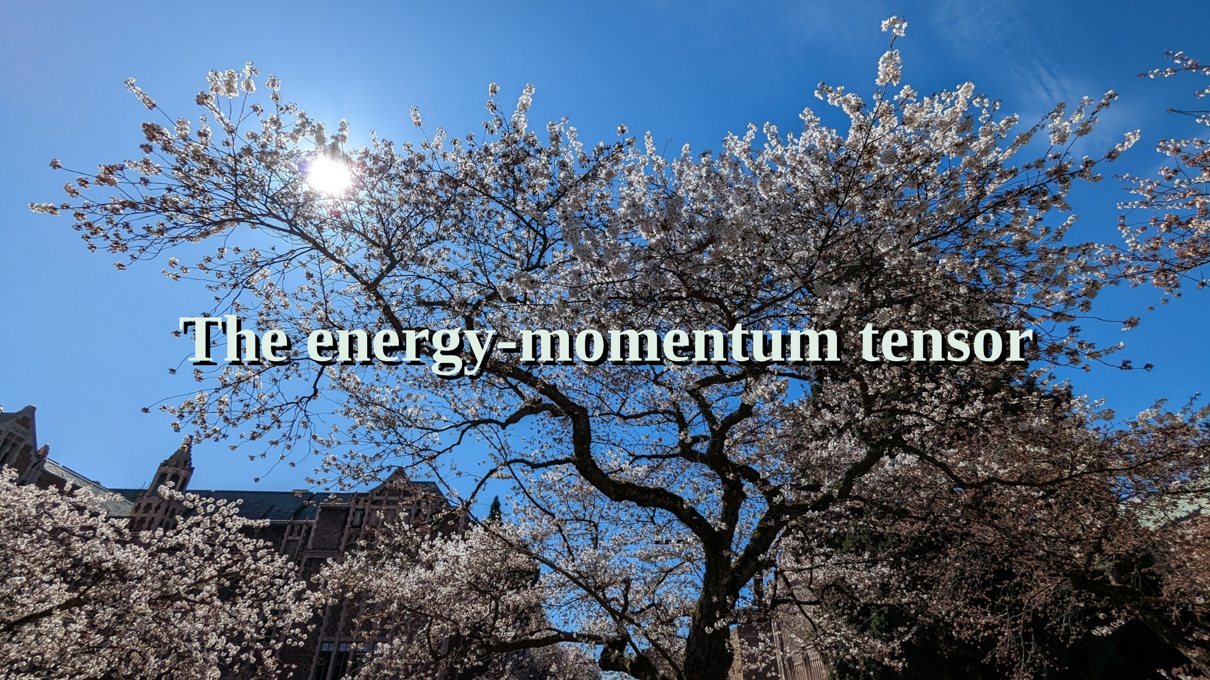


Figure: Burkert, Elouadrhiri & Girod, Nature (2018)

Outline

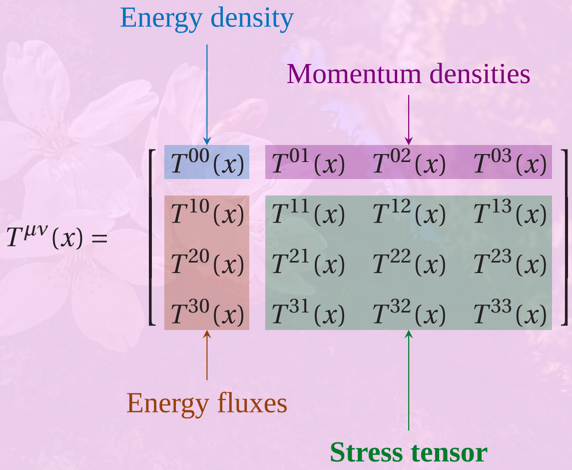
1. What is the hadronic stress tensor?
2. What are ~~gravitational~~ *mechanical* form factors?
3. How do they entail stress/“pressure” distributions?
4. What do these stress/“pressure” distributions mean?



The energy-momentum tensor

The energy-momentum tensor

- The energy-momentum tensor describes **density** and **flow** of energy & momentum.



Energy density

Momentum densities

$$T^{\mu\nu}(x) = \begin{bmatrix} T^{00}(x) & T^{01}(x) & T^{02}(x) & T^{03}(x) \\ T^{10}(x) & T^{11}(x) & T^{12}(x) & T^{13}(x) \\ T^{20}(x) & T^{21}(x) & T^{22}(x) & T^{23}(x) \\ T^{30}(x) & T^{31}(x) & T^{32}(x) & T^{33}(x) \end{bmatrix}$$

Energy fluxes

Stress tensor

Stress as momentum flux density

- Continuity equation for *closed system*:

$$\partial_\mu T^{\mu\nu}(x) = 0$$

  - Energy and momentum are transmitted locally.

- Integral form for spatial components:

$$\frac{\mathrm{d}P^j(V)}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left[\int_V \mathrm{d}^3x\, T^{0j}(\boldsymbol{x},t)\right] = -\int_V \mathrm{d}^3x\, \nabla_i T^{ij}(\boldsymbol{x},t) = -\oint_{\partial V} \mathrm{d}S\, \hat{n}_i T^{ij}(\boldsymbol{x},t)$$

  - Stress tensor tells us how momentum enters or leaves a region.
  - Can happen by particle flow or local force transmission.

Particle flux vs. forces

Particle flux

$$T^{ij}(\boldsymbol{x}) = v^i p^j \delta^{(3)}(\boldsymbol{x} - \boldsymbol{q})$$

Force

$$T^{ij}(\boldsymbol{x}) = \hat{n}^i \hat{n}^j \frac{F}{A}$$

- Momentum can enter/leave region because particles enter/leave.
- Momentum can enter/leave region because of forces.
  - Can produce positive or negative stress.
- Stress tensor includes both.
  - Is the hadronic stress tensor due to motion or forces? (Or both?)

Positive or negative pressure?

Positive pressure

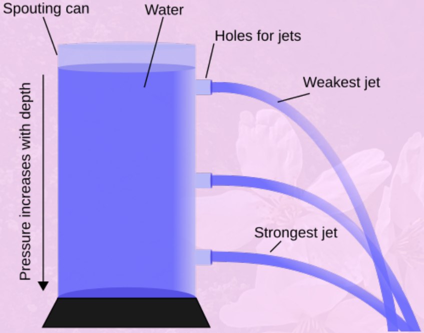
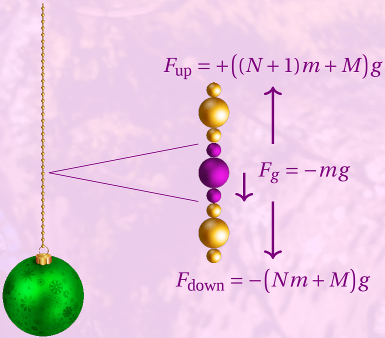


Image from Wikimedia

Negative pressure



- Same attractive force—gravity—in both cases.

$$\nabla_i T^{ij} = -\rho g \hat{z}^j = f^j_{\mathrm{grav}}$$

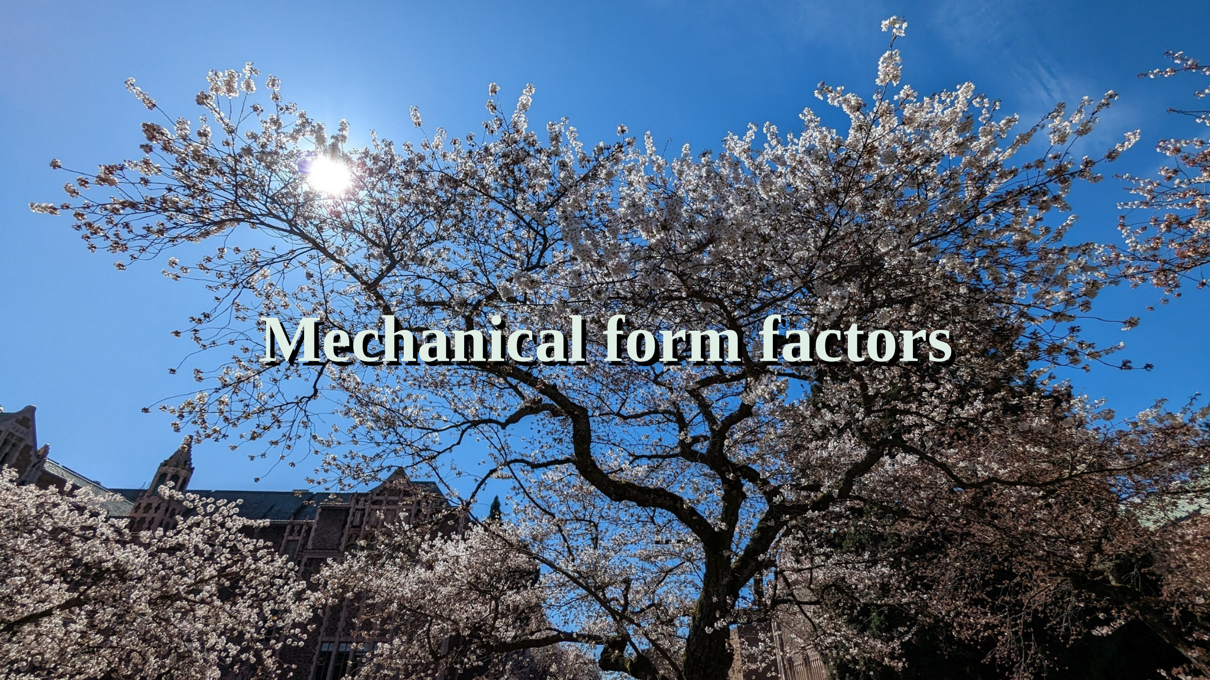
- Sign of pressure has nothing to do with attraction/repulsion!
  - Positive stress is compressive (pushing).
  - Negative stress is tensile (pulling).

Cauchy’s first law of motion

- **Cauchy’s first law of motion**: for a static, *open* system:

$$f^{j}_{\text{net}}(\boldsymbol{x}) = f^{j}_{\text{external}}(\boldsymbol{x}) - \nabla_i T^{ij}(\boldsymbol{x}) = 0$$

- Stress tensor can tell us how system responds to external forces.
  - **Important**: external is force not part of the stress tensor!
- Works for fluids, solids, or any other classical continuum system.
- Several authors have proposed using this for quantum systems!
  - Polyakov & Son, JHEP 09 (2018) 156
  - Won, Kim & Kim, JHEP 05 (2024) 173
  - AF, PRD 111 (2025) 034047



Mechanical form factors

Gravitational form factors

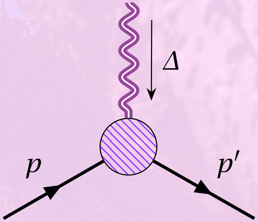
- Energy-momentum tensor parametrized using **gravitational form factors**
- Form factor breakdown (spin-zero example):

$$\langle \boldsymbol{p}' | \hat{T}_a^{\mu\nu}(0) | \boldsymbol{p} \rangle = 2P^\mu P^\nu A_a(\Delta^2) + \frac{\Delta^\mu \Delta^\nu - g^{\mu\nu}\Delta^2}{2} D_a(\Delta^2) + 2M^2 g^{\mu\nu} \bar{c}_a(\Delta^2)$$

  - $a = q, g$ labels constituent

$$P^\mu = \frac{1}{2}(p + p')^\mu$$

$$\Delta^\mu = (p' - p)^\mu$$



Gravitational form factors?

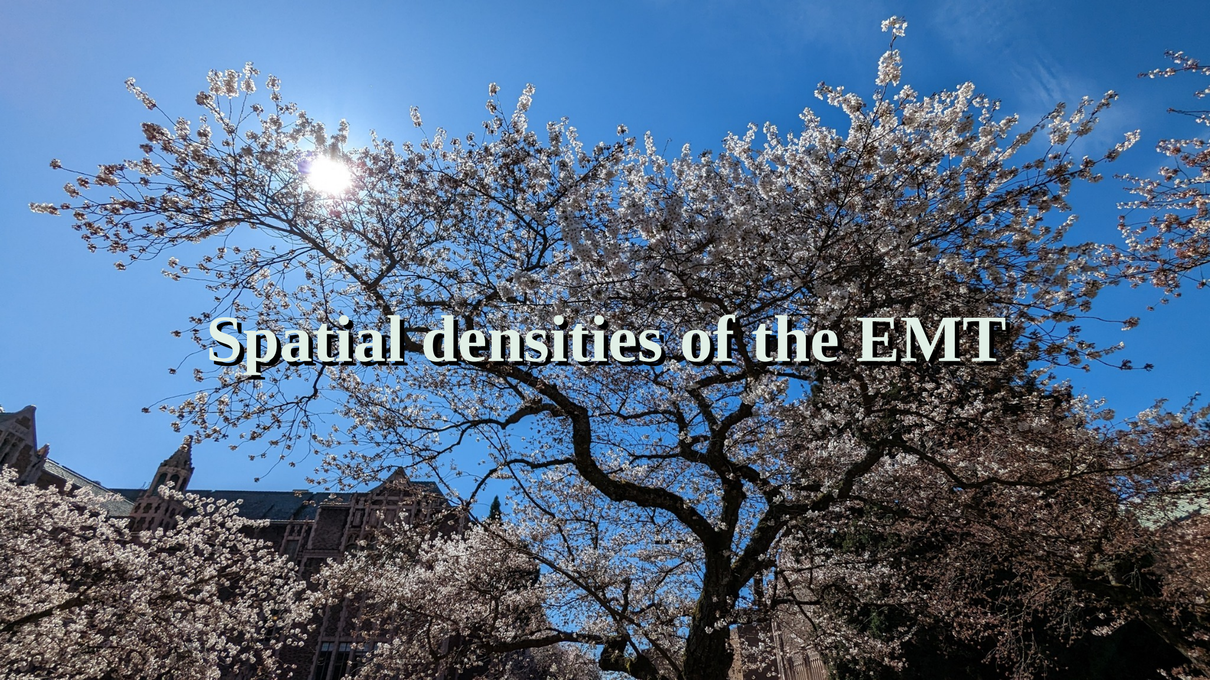
- Energy-momentum tensor the source of gravitation in general relativity:

$$R^{\mu\nu} - \frac{1}{2} R g^{\mu\nu} = 8\pi G T^{\mu\nu}$$

  - They describe distribution of stuff that gravitates.
  - It's technically accurate, but may be unintentionally misleading.

- But we're not really doing gravitational physics.
  - These form factors aren't measured using gravity.
  - We're not characterizing gravitational force.

- Maybe we should call them **mechanical form factors**.
  - They characterize *mechanical* properties.
  - We use them to study the *strong nuclear force*.



Spatial densities of the EMT

Spatial densities: non-relativistic

$$\langle\Psi|\hat{T}^{ij}_a(\boldsymbol{x},t)|\Psi\rangle = \int \mathrm{d}^3\boldsymbol{R}\,\Psi^*(\boldsymbol{R},t)\left\{-\underset{\substack{\uparrow\\ \textbf{Dynamic stress}}}{\frac{\overleftrightarrow{\nabla}^i_{(R)}\overleftrightarrow{\nabla}^j_{(R)}}{M}}\mathfrak{A}_a(\boldsymbol{x}-\boldsymbol{R}) + \underset{\substack{\uparrow\\ \textbf{Internal stress}}}{\mathfrak{t}^{ij}_a}(\boldsymbol{x}-\boldsymbol{R})\right\}\Psi(\boldsymbol{R},t)$$

- **Dynamic stress** due to barycentric motion and wave packet dispersion.
  - Smeared by the internal probability density.
- **Internal stress** due to internal motions or forces.
  - Smeared by barycenter probability density.
  - Conveniently related to the Breit frame Fourier transform:

$$\mathfrak{t}^{ij}_a(\boldsymbol{b}) = \int \frac{\mathrm{d}^3\boldsymbol{\Delta}}{(2\pi)^3}\frac{\left\langle\frac{\boldsymbol{\Delta}}{2}\middle|\hat{T}^{ij}_a(0)\middle|-\frac{\boldsymbol{\Delta}}{2}\right\rangle}{2M}\mathrm{e}^{-\mathrm{i}\boldsymbol{\Delta}\cdot\boldsymbol{b}}$$

- **Strictly non-relativistic** breakdown.
  - Afforded by Galilean symmetry (absoluteness of simultaneity).
  - Relativistic generalization a source of controversy.

Perspectives on relativistic EMT densities

- **Polyakov & Schweitzer:** *define* the static EMT densities in terms of Breit frame:

$$\mathrm{t}^{\mu\nu}_{\text{static}}(\boldsymbol{b}) \equiv \int \frac{\mathrm{d}^3\boldsymbol{\Delta}}{(2\pi)^3} \frac{\langle \frac{\boldsymbol{\Delta}}{2} \big| \hat{T}^{\mu\nu}(0) \big| -\frac{\boldsymbol{\Delta}}{2} \rangle}{2M\sqrt{1+\frac{\boldsymbol{\Delta}^2}{4M^2}}} \mathrm{e}^{-\mathrm{i}\boldsymbol{\Delta}\cdot\boldsymbol{b}}$$

  - Most widely-used approach.
  - hep-ph/0207153, PLB 555 (2003) 57, IJMPA (2018) 1830025

- **Lorcé *et al.*:** use Wigner phase-space formalism to set $\boldsymbol{R}=0$ and $\boldsymbol{P}=0$.
  - Gives Polyakov & Schweitzer's static EMT.
  - EPCJ 79 (2019) 89

- **Yang Li *et al.*:** expand expectation values as tower of multipole moment densities:

$$\langle \Psi | \hat{T}^{\mu\nu}(\boldsymbol{x},t) | \Psi \rangle \approx \int \mathrm{d}^3\boldsymbol{R} \left(\Psi^*(\boldsymbol{R},t)\, \mathrm{i}\overleftrightarrow{\partial_t}\, \Psi(\boldsymbol{R},t) \right) \mathrm{t}^{\mu\nu}_{\text{static}}(\boldsymbol{b}) + \text{corrections}$$

  - Use spatially diffuse wave packet, zero average momentum.
  - Gives Polyakov & Schweitzer's static EMT as leading contribution.
  - Corrections negligible if wave packet larger than Compton wavelength.
  - PLB 838 (2023) 137676, 2405.06892

More perspectives on relativistic EMT densities

- **Lorcé *et al.* / AF & Miller**: use light front densities.

$$\mathrm{t}^{\mu\nu}_{\mathrm{LF}}(\boldsymbol{b}_\perp) \equiv \int \frac{\mathrm{d}^2\boldsymbol{\Delta}_\perp}{(2\pi)^2} \frac{\left\langle P^+, \frac{\boldsymbol{\Delta}_\perp}{2} \middle| \hat{T}^{\mu\nu}(0) \middle| P^+, -\frac{\boldsymbol{\Delta}_\perp}{2} \right\rangle}{2P^+} \mathrm{e}^{-\mathrm{i}\boldsymbol{\Delta}\cdot\boldsymbol{b}}$$

  - Only $i, j \in \{1,2\}$; only get 2D densities.
  - Galilean symmetry allows barycenter/internal separation.
  - EPCJ 79 (2019) 89, PRD 103 (2021) 094023

- **Panteleeva *et al.***: use localized, zero average momentum wave packets.
  - Similar result to light front, but restore third dimension.
  - EPJC 83 (2023) 617, JHEP 07 (2023) 237

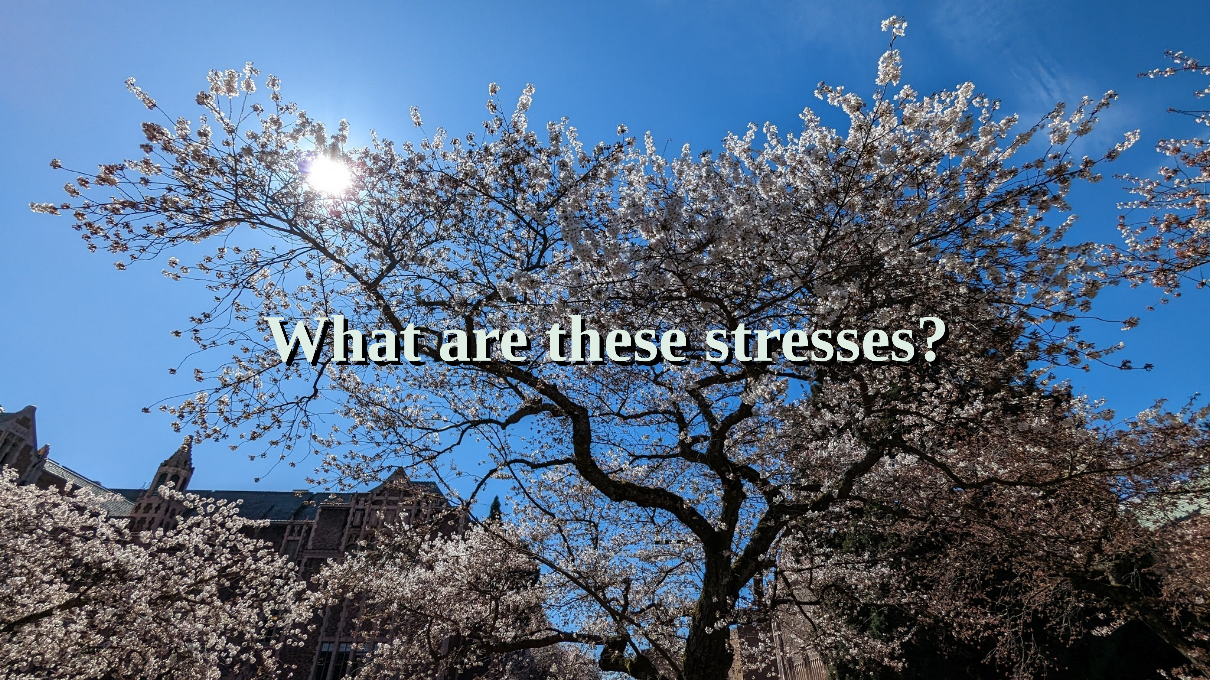
- **AF & Miller**: use light front time + Cartesian space.
  - Still only 2D densities, but get 3×3 stress tensor.
  - Separate wave packet & internal densities with factorization/smearing relations.
  - PRD 107 (2023) 074036, PRD 108 (2023) 094026

Status of relativistic EMT densities

- Controversy is unresolved, but static EMT is *good enough* for practical purposes.
  - Justified in wave packet approach of Li *et al.*
  - Corrections negligible for realistic wave packets (AF & Miller, PRD 108 (2023) 034008)

- Given the static EMT though, what does the **stress tensor** actually mean?
  - Formula for spin-half target:

$$\mathrm{t}_a^{ij}(\boldsymbol{b}) = \int \frac{\mathrm{d}^3\boldsymbol{\Delta}}{(2\pi)^3} \left(\frac{\Delta^i \Delta^j - \delta^{ij} \boldsymbol{\Delta}^2}{4M} D_a(\boldsymbol{\Delta}^2) - M\delta^{ij} \bar{c}_a(\boldsymbol{\Delta}^2) \right) \mathrm{e}^{-\mathrm{i}\boldsymbol{\Delta}\cdot\boldsymbol{b}}$$



What are these stresses?

Stress as momentum flux density

- Recall: stress tensor is **momentum flux density**

$$\frac{\mathrm{d}P^j(V)}{\mathrm{d}t} = -\oint_{\partial V} \mathrm{d}S\,\hat{n}_i\,T^{ij}(\boldsymbol{x},t)$$

  - Momentum can flow through particle movement *or* local force transmission

Particle flux

$$T^{ij}(\boldsymbol{x}) = v^i p^j \delta^{(3)}(\boldsymbol{x}-\boldsymbol{q})$$

Force

$$T^{ij}(\boldsymbol{x}) = \hat{n}^i\hat{n}^j\frac{F}{A}$$

Forces between subcomponents

- For an *open system*

$$f^j_{\mathrm{net}}(\boldsymbol{x},t) - \frac{\mathrm{d}p^j(\boldsymbol{x},t)}{\mathrm{d}t} = \nabla_i T^{ij}(\boldsymbol{x},t)$$

  - **Cauchy's first law of motion**

- Individual parton flavors are open systems!

$$\nabla_i \mathrm{t}^{ij}_a(\boldsymbol{b}) = -M\nabla^j \int \frac{\mathrm{d}^3\boldsymbol{\Delta}}{(2\pi)^3} \bar{c}_a(\boldsymbol{\Delta}^2)\,\mathrm{e}^{-\mathrm{i}\boldsymbol{\Delta}\cdot\boldsymbol{b}}$$

  - **We can map out the average force on quarks in a target.**
  - The $\bar{c}_q(\boldsymbol{\Delta}^2)$ form factor could probe the QCD force law!
    - Polyakov & Son, JHEP 09 (2018) 156
    - Won, Kim & Kim, JHEP 05 (2024) 173
    - AF, PRD 111 (2025) 034047
      shows Coulomb force can be recovered in hydrogen atom via $\bar{c}_e(\boldsymbol{\Delta}^2)$!
  - $D_a(\boldsymbol{\Delta}^2)$ has zero divergence; its contribution vanished

What about the D-term?

$D(\boldsymbol{\Delta}^2) \approx 0$

$D(\boldsymbol{\Delta}^2) \ll 0$

- Both donkeys have the same the same $\bar{c}(\boldsymbol{\Delta}^2)$.

$$\nabla_i T^{ij} = -\rho g \hat{z}^j$$

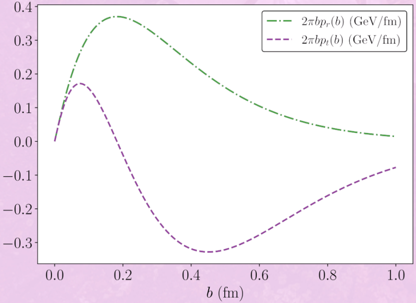
- Only their *D*-term differs.
- *D*-term encodes (potentially large) mutually-cancelling internal stresses.

Radial and tangential stresses

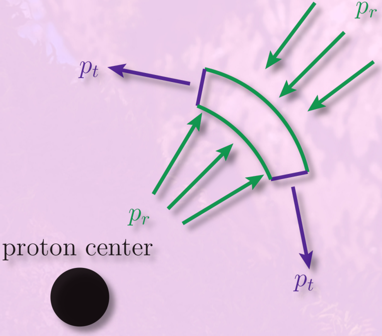
- Normal projections of D-term give *mutually-balancing* forces in any direction

$$p_n(\boldsymbol{b}) = \hat{n}_i \hat{n}_j \int \frac{\mathrm{d}^3\boldsymbol{\Delta}}{(2\pi)^3} \frac{\Delta^i \Delta^j - \delta^{ij}\boldsymbol{\Delta}^2}{4M} D(\boldsymbol{\Delta}^2)\, \mathrm{e}^{-\mathrm{i}\boldsymbol{\Delta}\cdot\boldsymbol{b}}$$

- Usually **radial** and **tangential** stresses are examined.
  - Positive means compressive (pushing)—not repulsion
  - Negative means tensile (pulling)—not attraction



(Light front calculation, AF & Miller 2021)

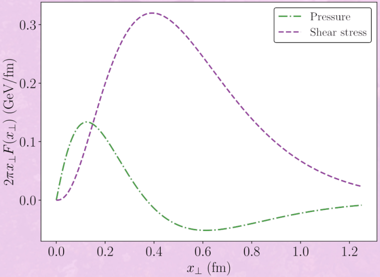


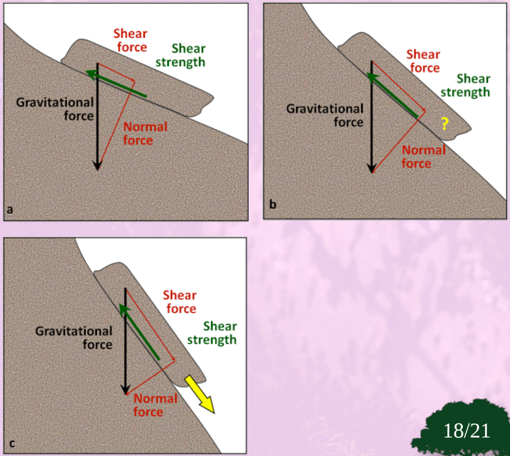
Pressure and shear?

- Isotropic **pressure** is just the average normal stress:

$$p_{\text{iso}} = \frac{p_r + p_\theta + p_\phi}{3} = \frac{p_x + p_y + p_z}{3}$$

- **Shear stress** is a pressure anisotropy.
  - Measures how momentum flows in a direction orthogonal to the momentum itself.

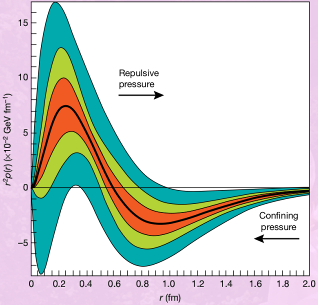




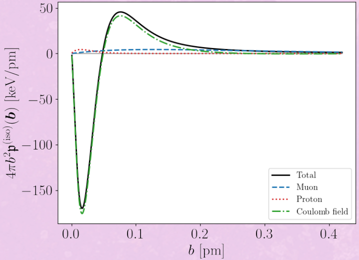
(Fig above: light front calculation, AF & Miller 2021)
(Fig on right: Steven Earle, *Physical Geology*)

Sign of the D-term

Pressure in proton: $D(0) < 0$ ... (BEG Nature)



Pressure in H atom: $D(0) > 0$ ... (AF PRD)



- **von Laue condition**: integral of isotropic pressure zero:

$$\int \mathrm{d}^3\boldsymbol{b}\, p_{\mathrm{iso}}(\boldsymbol{b}) = 0$$

  - Automatically satisfied for every hadron
- Sign of D-term related to balance of compression & tension:

$$D(0) = M \int \mathrm{d}^3\boldsymbol{b}\, \boldsymbol{b}^2\, p_{\mathrm{iso}}(\boldsymbol{b})$$

  - $D(0) < 0$ would mean the pulling pressure is further way.
- **Polyakov's conjecture**: $D(0) < 0$ for hadrons
  - Violated by electron
    Metz, Pasquini & Rodini, PLB 820 (2021) 136501
  - Violated by photon
    AF & Cosyn, PRD 106 (2022) 114014
  - Violated by hydrogen atom
    Ji, Yang & Liu, PRD 110 (2024) 114045
    AF, PRD 111 (2025) 034047

Wrapping up

Summary

- Stress tensor describes momentum flux densities.
  - Most authors (but not all!) think they're actually stresses too.
  - Positive & negative stresses have nothing to do with attractive vs. repulsive forces.
- Divergence of stress tensor gives forces between sub-components.
  - This makes $\bar{c}_q(\boldsymbol{\Delta}^2)$ the most exciting form factor.
  - We could potentially map out the QCD force law!
- Status of $D(0) < 0$ stability condition in question.
  - Counterexamples are electromagnetic, few-body systems.
  - Need understanding of *why* $D(0) < 0$ for hadrons.
- No consensus on proper relativistic densities to use.
  - Most authors seem to accept Polyakov's static EMT (Breit frame density) though.

Credits

QuantOm Collaboration

- SciDAC award: *Femtoscale Imaging of Nuclei using Exascale Platforms*

Jefferson Lab

- Jefferson Science Associates
- DOE contract No. DE-AC05-06OR23177

Thank you for your time!