The 11th workshop of the APS Topical Group on Hadronic Physics

Three-body amplitudes for the analysis of lattice data and experiment

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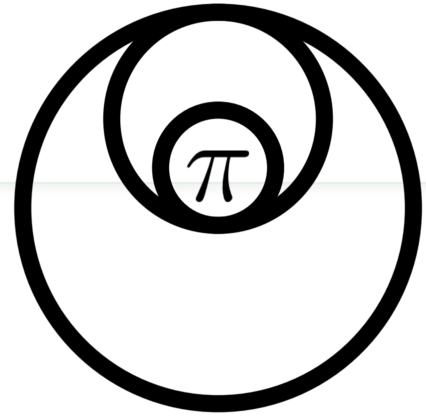
With slide material from Y. Feng and M. Mai

Overview



(Y. Feng)

Review 2B-lattice: [Briceno]
Reviews 3B-lattice: [Hansen] [Mai]
Review hadron resonances: [Mai]

Key publications Finite-Volume Unitary (FVU) approach:

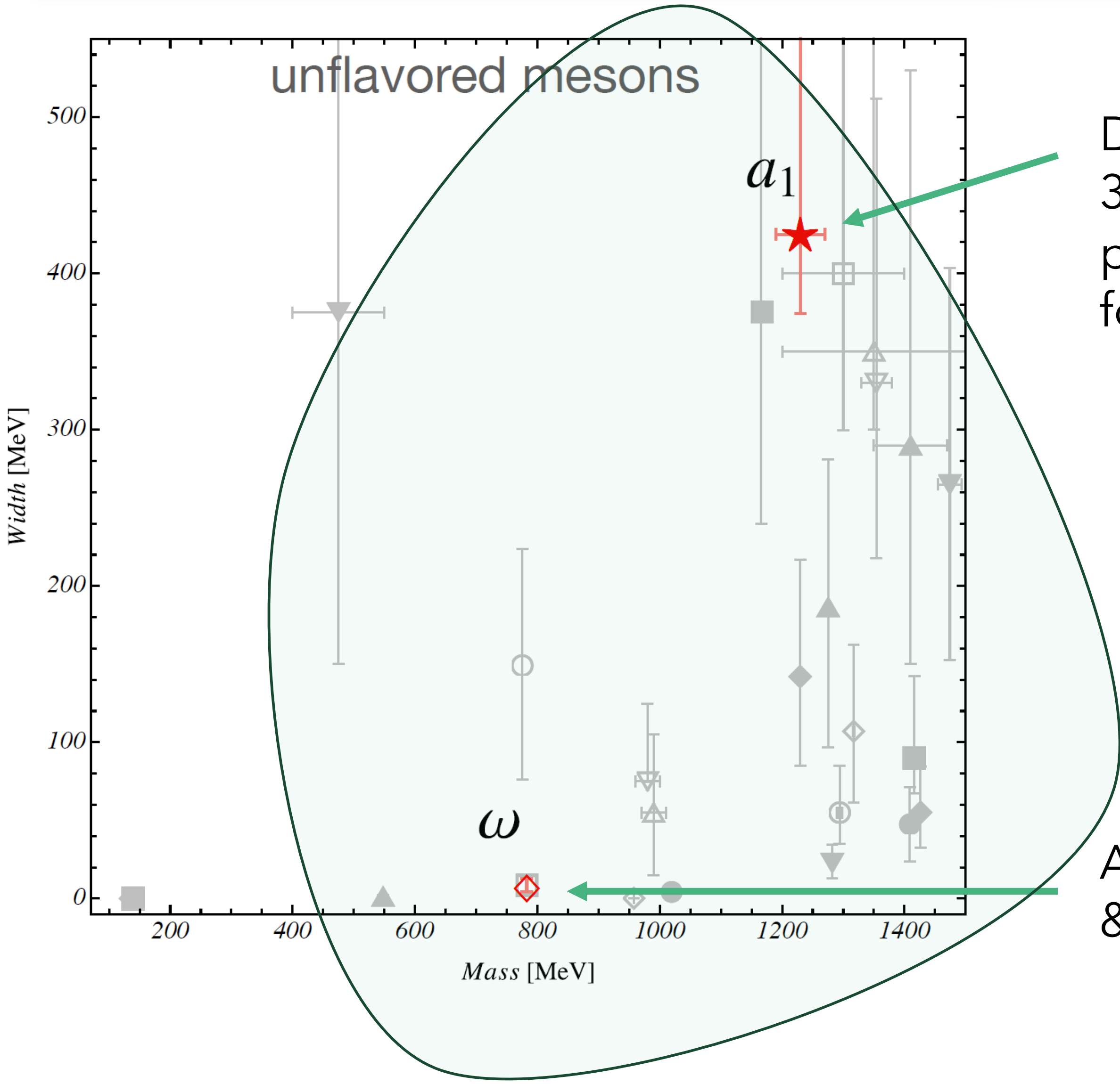
- Three-body unitarity [Mai/JPAC]
- Three-body unitarity finite volume [Mai]
- a_1 in finite volume & results from lQCD [Mai]

Talk outline:

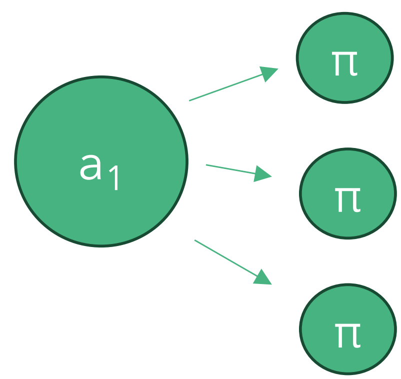
- 3-body unitarity
- a_1 in infinite volume
- a_1 in finite volume
- Recent extensions: channel space & applications

Work supported by: National Science Foundation Grant No. PHY 2012289
ExoHad Exotic Hadrons Topical Collaboration DE-SC0023598

>2-body meson decays



Decays almost 100% to 3 pions (decay to 2 pseudoscalar mesons forbidden)



Almost stable; lattice results & chiral extrapolation

[Yan, Mai et al., 2407.16659]

Three-body unitarity with isobars *

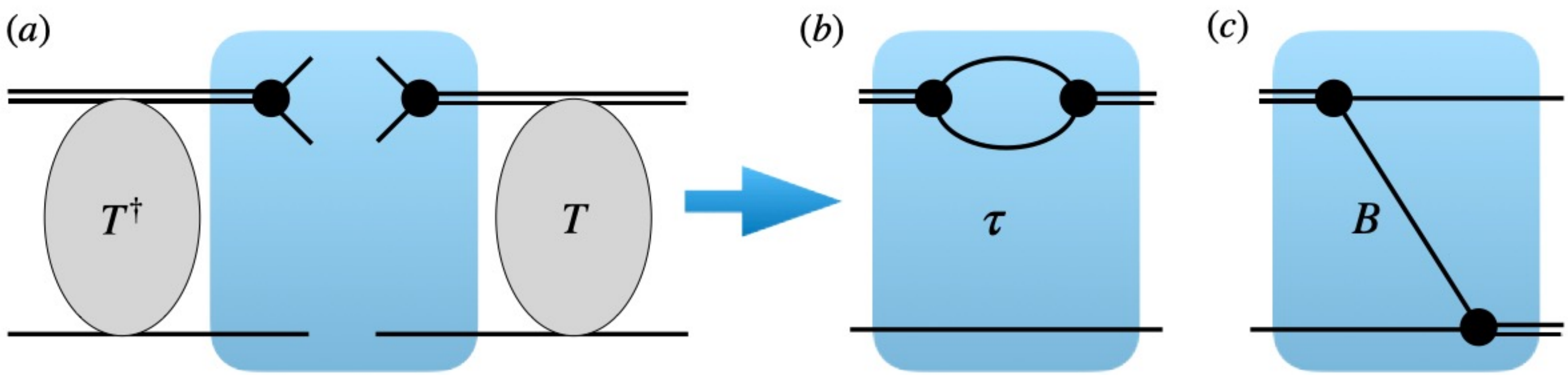
[Mai 2017]

$$\langle q_1, q_2, q_3|(\hat{T}-\hat{T}^\dagger)|p_1, p_2, p_3\rangle = i\int_P \langle q_1, q_2, q_3|\hat{T}^\dagger|k_1, k_2, k_3\rangle\langle k_1, k_2, k_3|\hat{T}|p_1, p_2, p_3\rangle$$

$$\times \prod_{\ell=1}^{3}\left[\frac{d^4 k_\ell}{(2\pi)^4}(2\pi)\delta^+(k_\ell^2 - m^2)\right](2\pi)^4\delta^4\left(P - \sum_{\ell=1}^{3} k_\ell\right)$$

Idea: To construct a 3B amplitude, start directly from unitarity (based on ideas of 60's); match a general amplitude to it

delta function sets all intermediate particles on-shell

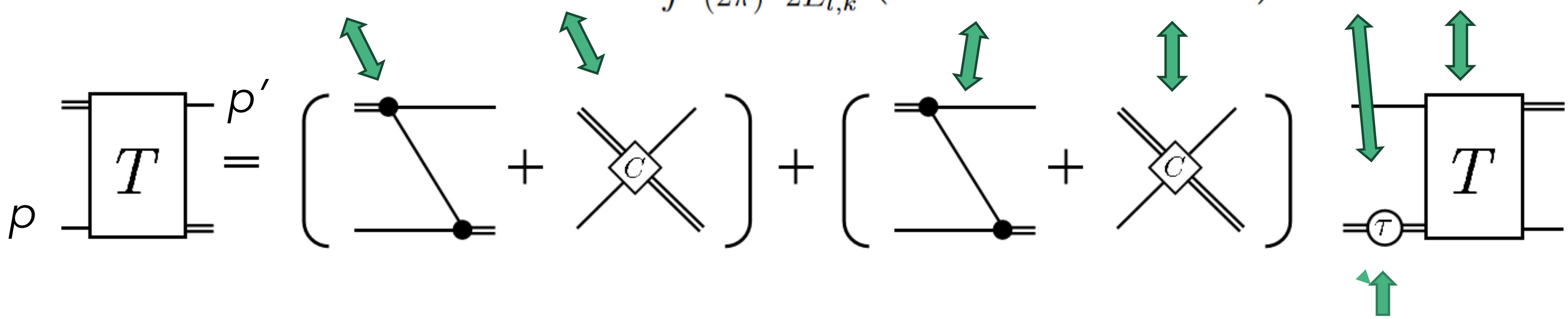


"Isobar" stands for two-body sub-amplitude which can be resonant or not; can be matched to CHPT expansion to one loop if desired. Isobars are re-parametrizations of full 2-body amplitudes [Bedaque] [Hammer]

Resulting scattering equation

- 4-dim BSE becomes 3-dim by putting spectator on-shell (choice)

$$\tilde{T}_{ji}(s,\boldsymbol{p}',\boldsymbol{p}) = \tilde{B}_{ji}(s,\boldsymbol{p}',\boldsymbol{p}) + \tilde{C}_{ji}(s,\boldsymbol{p}',\boldsymbol{p}) + \int \frac{\mathrm{d}^3 l}{(2\pi)^3\, 2E_{l,k}} \left(\tilde{B}_{jk}(s,\boldsymbol{p}',\boldsymbol{l}) + \tilde{C}_{jk}(s,\boldsymbol{p}',\boldsymbol{l})\right) \tilde{\tau}_k(\sigma_l)\, \tilde{T}_{ki}(s,\boldsymbol{l},\boldsymbol{p})$$



Exchange B:
- Complex
- Required by unitarity

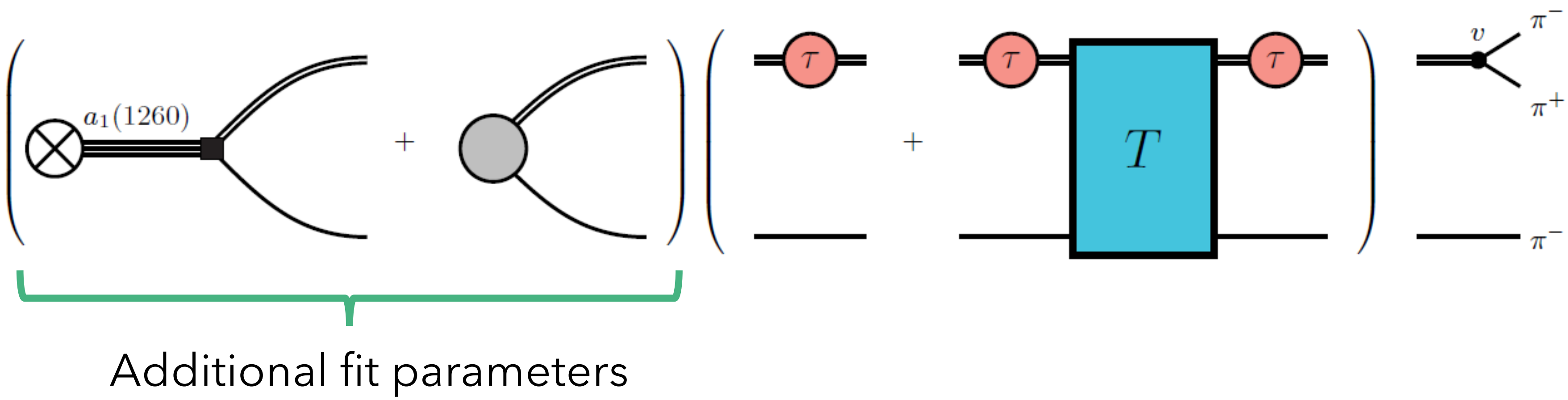
Contact term C:
- Does not destroy unitarity
- <u>Free parametrization: fit to data</u>

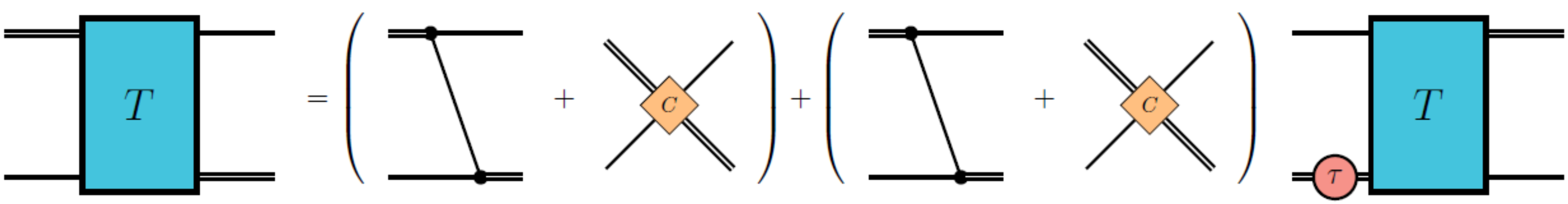
Isobar-spectator Green's functions

The a_1(1260) and its Dalitz plots

[Sadasivan 2020]

- Disconnected and connected decays for three-body untarity



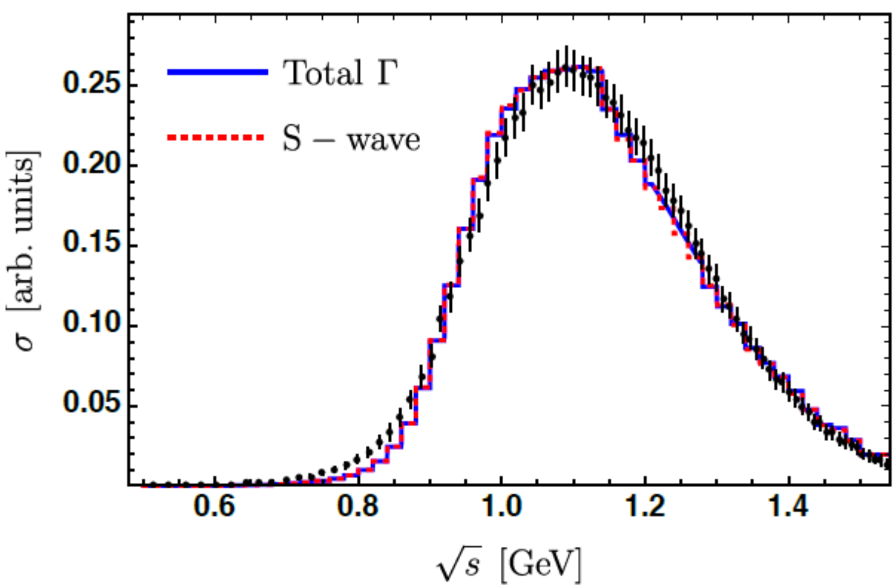


Fitting the lineshape & predicting Dalitz plots

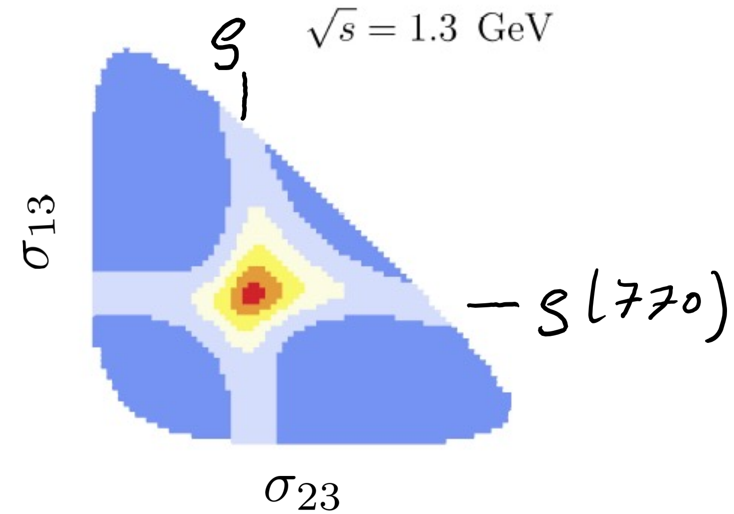
[Sadasivan 2020]

- One can have $\pi\rho$ in S- and D-wave coupled channels
- Fit contact terms to the lineshape from Experiment (ALEPH)

$$a_1 \to \pi^-\pi^-\pi^+ \text{ (symmetrize } \pi^-\text{'s!)}$$



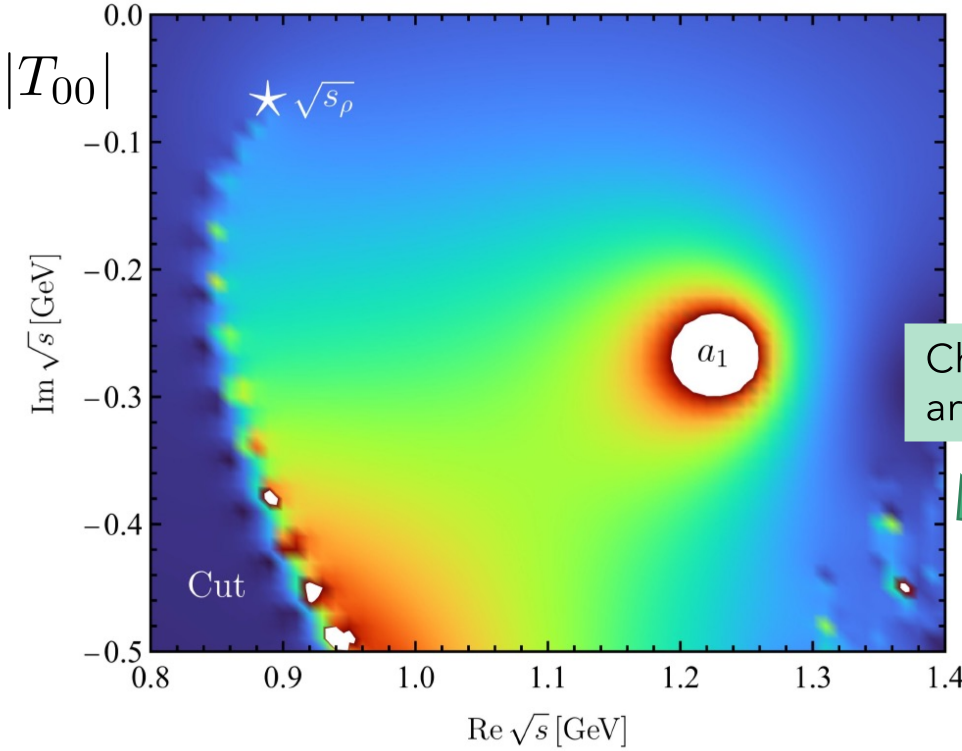
predict →

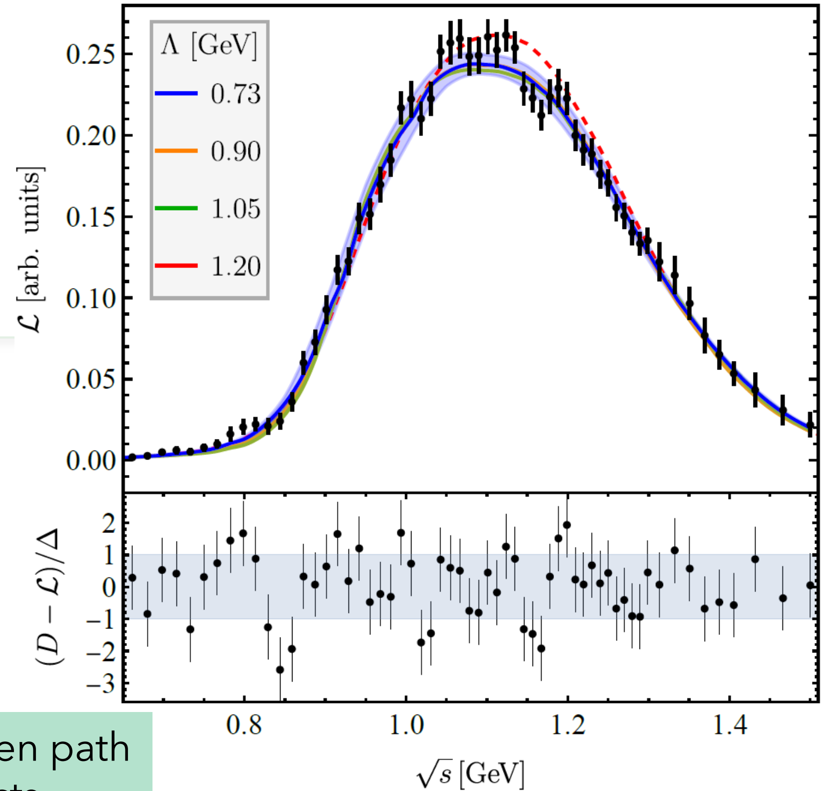


Where is the resonance pole in s? Pole positions & residues are reaction-independent characteristics of resonance mass, width, and branching ratios

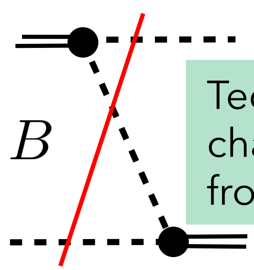
Result: Pole position

Technicalities analytic cont.: contour deformation — [Sadasivan (2021)] [Doering (2009)]



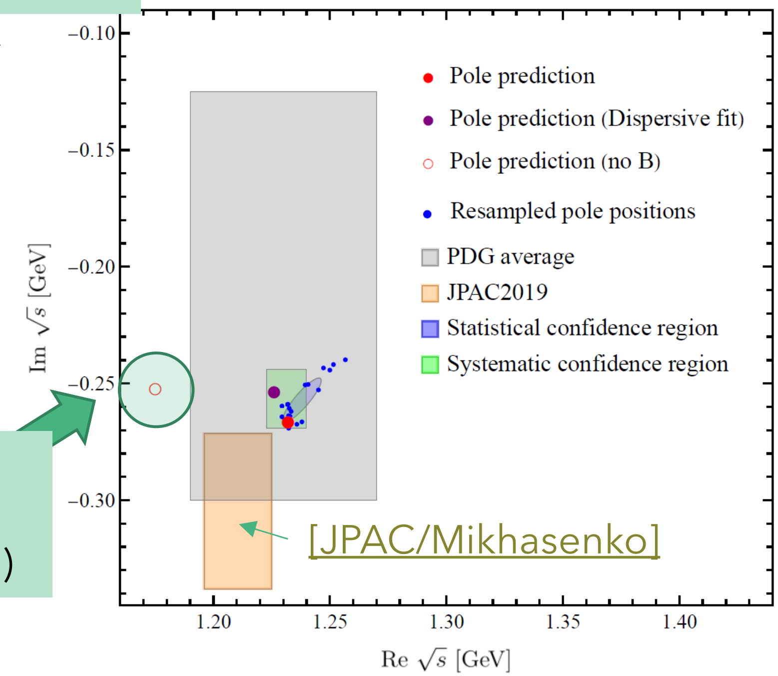


Chosen path artifacts



Technical challenges from 3B cuts

If the B-term is neglected + refit (unitarity violated)



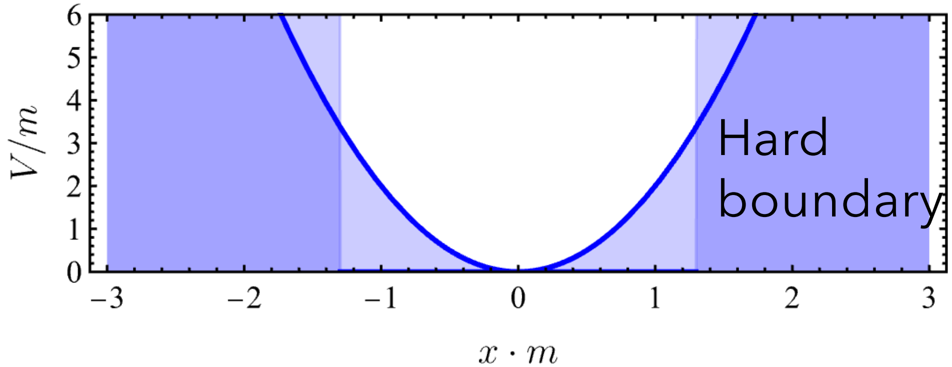
[JPAC/Mikhasenko]

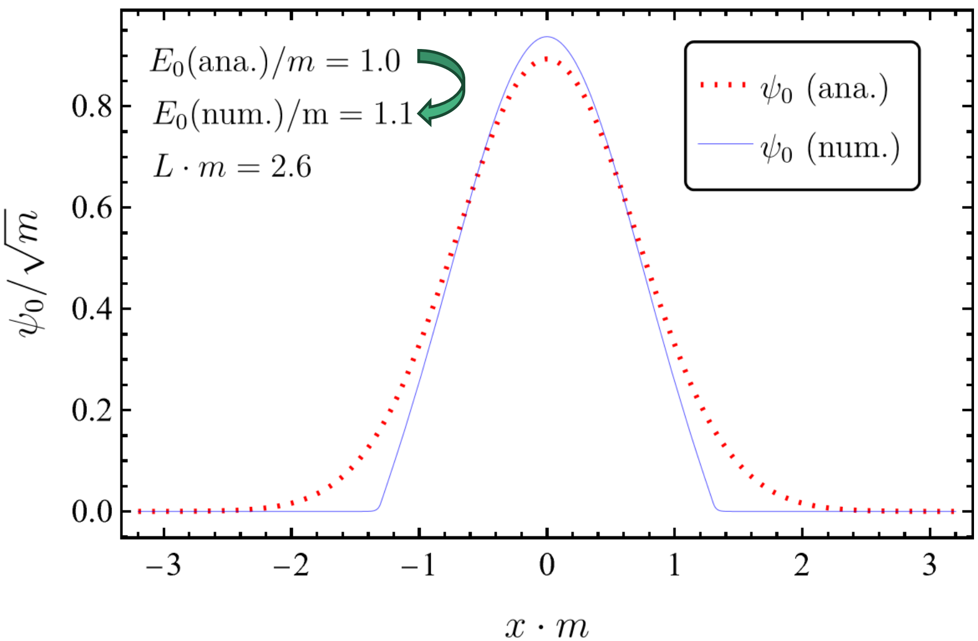
See also: S. M. Dawid [2303.04394 [nucl-th]]

Finite-Volume Effects

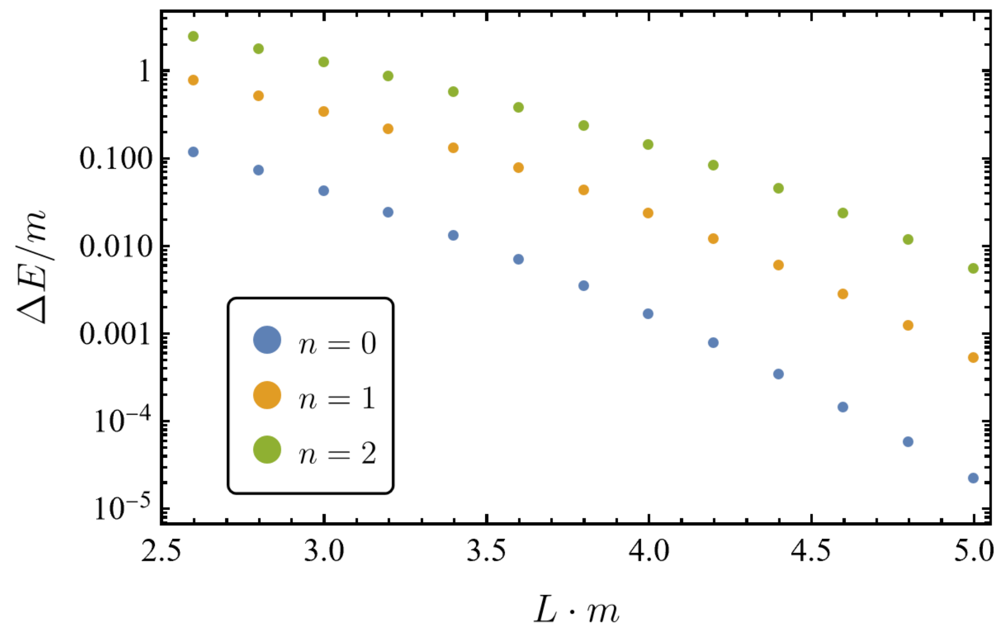
A wave function is squeezed into a finite volume

[https://blogs.gwu.edu/doring/]





Distortion of the energy spectrum as function of box size



Periodic Boundary Conditions

$$\Psi(\vec{x}) \stackrel{!}{=} \Psi(\vec{x} + \hat{\mathbf{e}}_i\, L) = \exp(i\, L\, q_i)\, \Psi(\vec{x})$$

$$q_i = \frac{2\pi}{L} n_i, \qquad n_i \in \mathbb{Z}$$

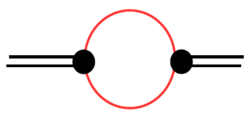
only discrete momenta allowed

Lattice QCD: Finite-volume unitarity (FVU)

Two-body unitarity

↓

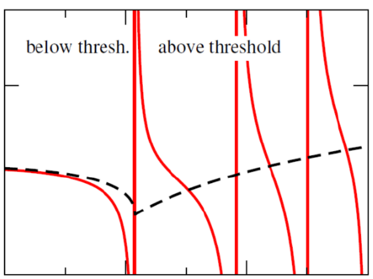
On-shell condition



$\int \to \sum$

↓

Imaginary parts



Infinite → Fin. Vol

↓

Power-law fin-vol. effects

↓

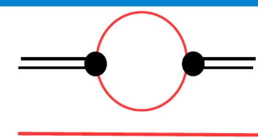
Lüscher

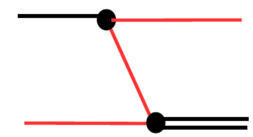
$$p\cot\delta(p) = -8\pi\sqrt{s}\left(\tilde{G}(E) - \mathrm{Re}\,G(E)\right)$$

Three-body unitarity

↓

On-shell condition





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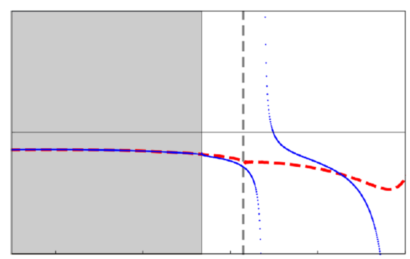
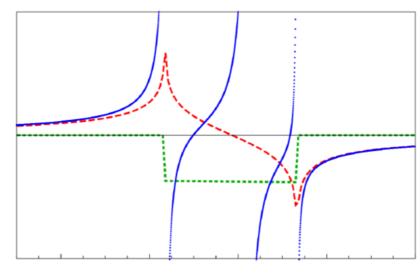
Imaginary parts





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Power-law fin-vol. effects

↓

Quantization Condition

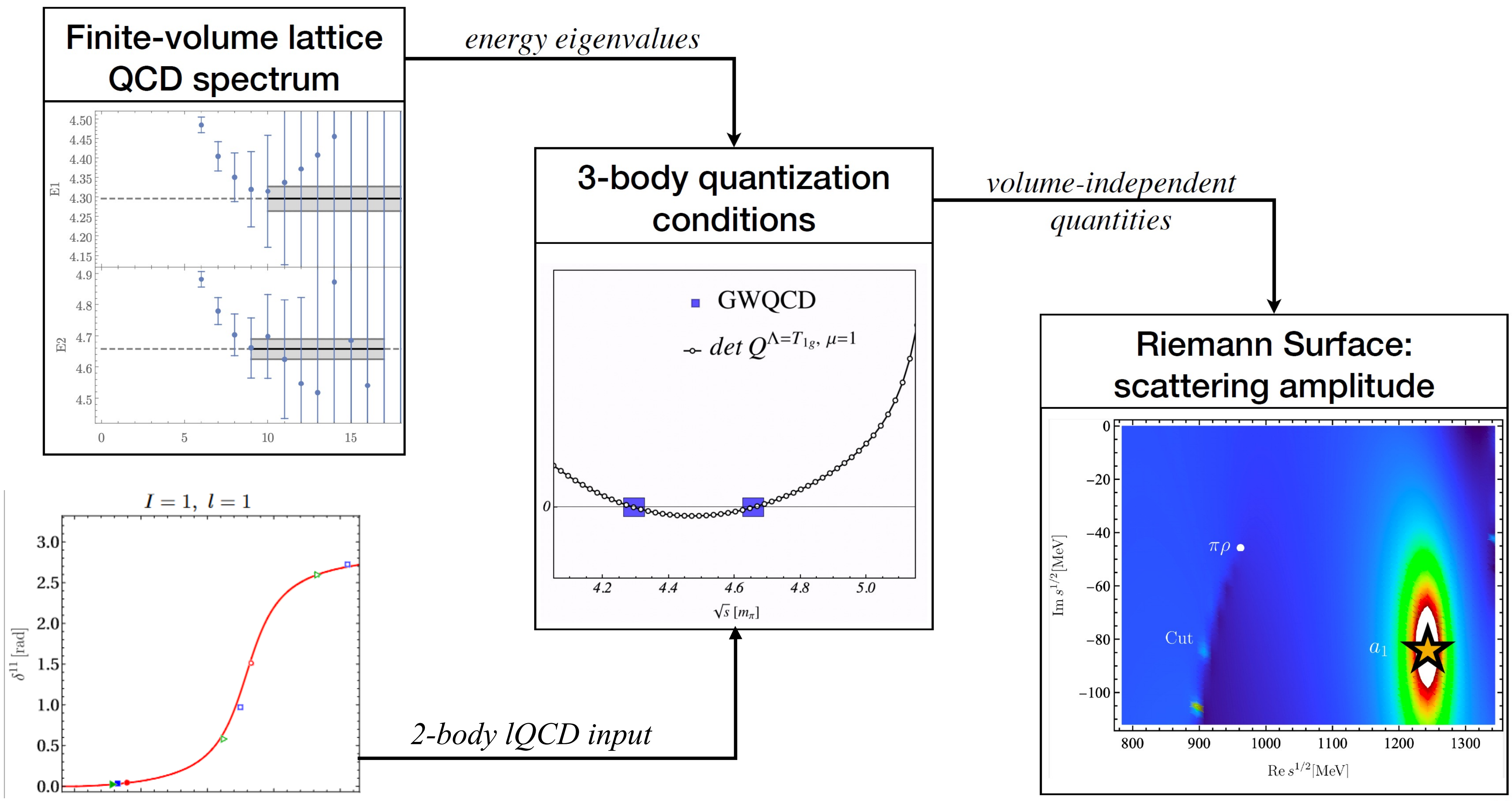
$$\mathrm{Det}\left(\mathbf{B}^{\Gamma \mathbf{ss}'}_{\mathbf{uu}'}(\mathbf{W}^2) + \frac{\mathbf{2E_s\,L^3}}{\vartheta(\mathbf{s})}\tau_{\mathbf{s}}(\mathbf{W}^2)^{-1}\delta_{\mathbf{ss}'}\delta_{\mathbf{uu}'}\right) = \mathbf{0}$$

[Review Mai et al.]

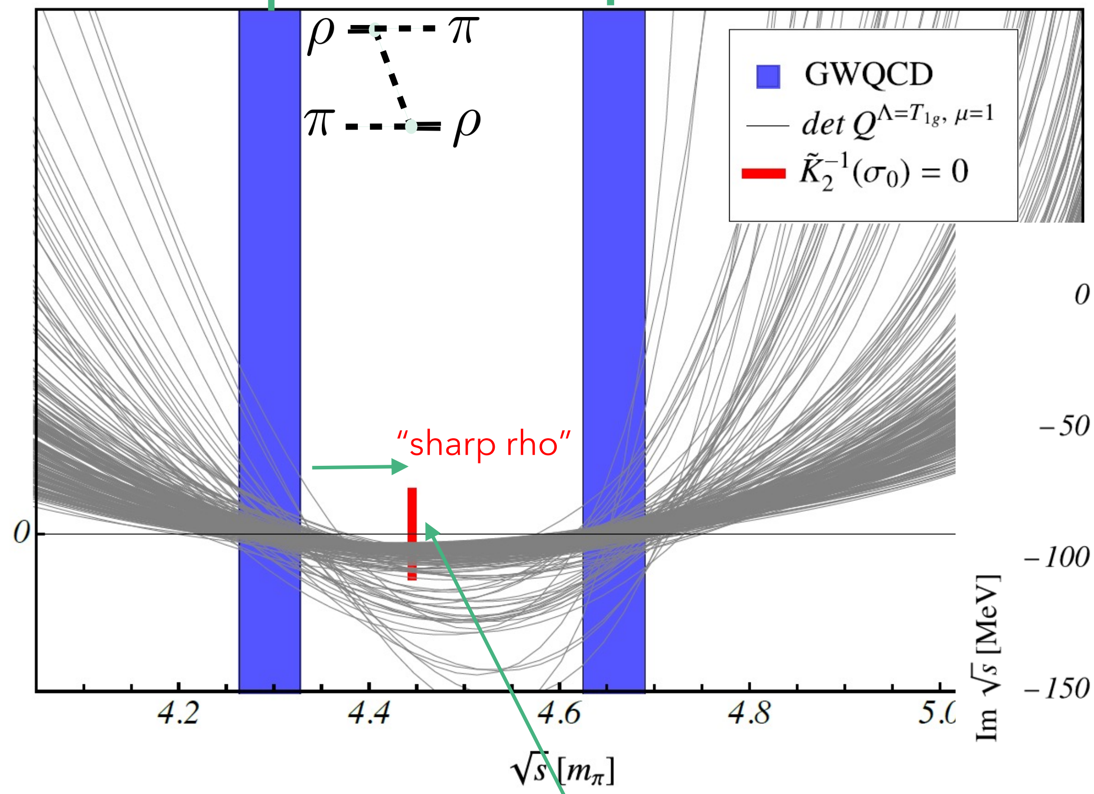
Extraction of $a_1(1260)$ from lQCD

[Mai/MD/GWQCD, PRL 2021]

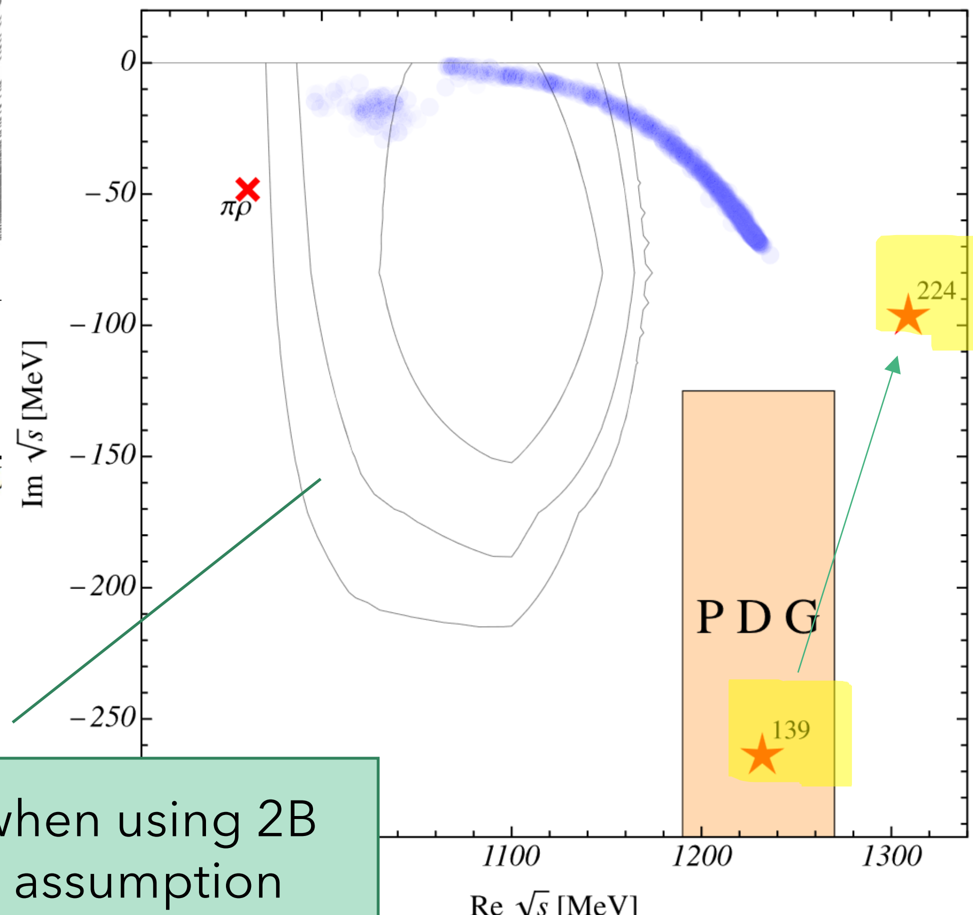
- First-ever three-body resonance from 1st principles (with explicit three-body dynamics).



a_1 results – Details



$$0 = \det\left[B(s) + C(s) - E_L\left(\tilde{K}_2^{-1}(s) - \Sigma_2^L(s)\right)\right]_{\substack{(\lambda'\lambda)\\(\boldsymbol{p}'\boldsymbol{p})}}$$

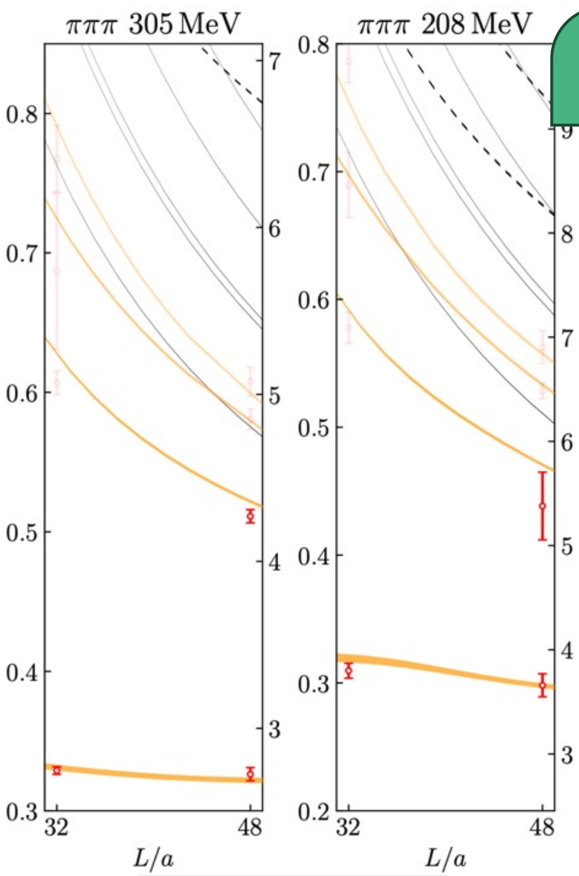


PDF for pole when using 2B Breit-Wigner assumption

$\omega \to \pi\pi\pi$ from Lattice QCD

[Yan, Mai, et al., PRL (2024)]

Lattice QCD
Nf = 2 + 1 Clover fermions
2/3 particle operators
2 pion masses, 2 volumes



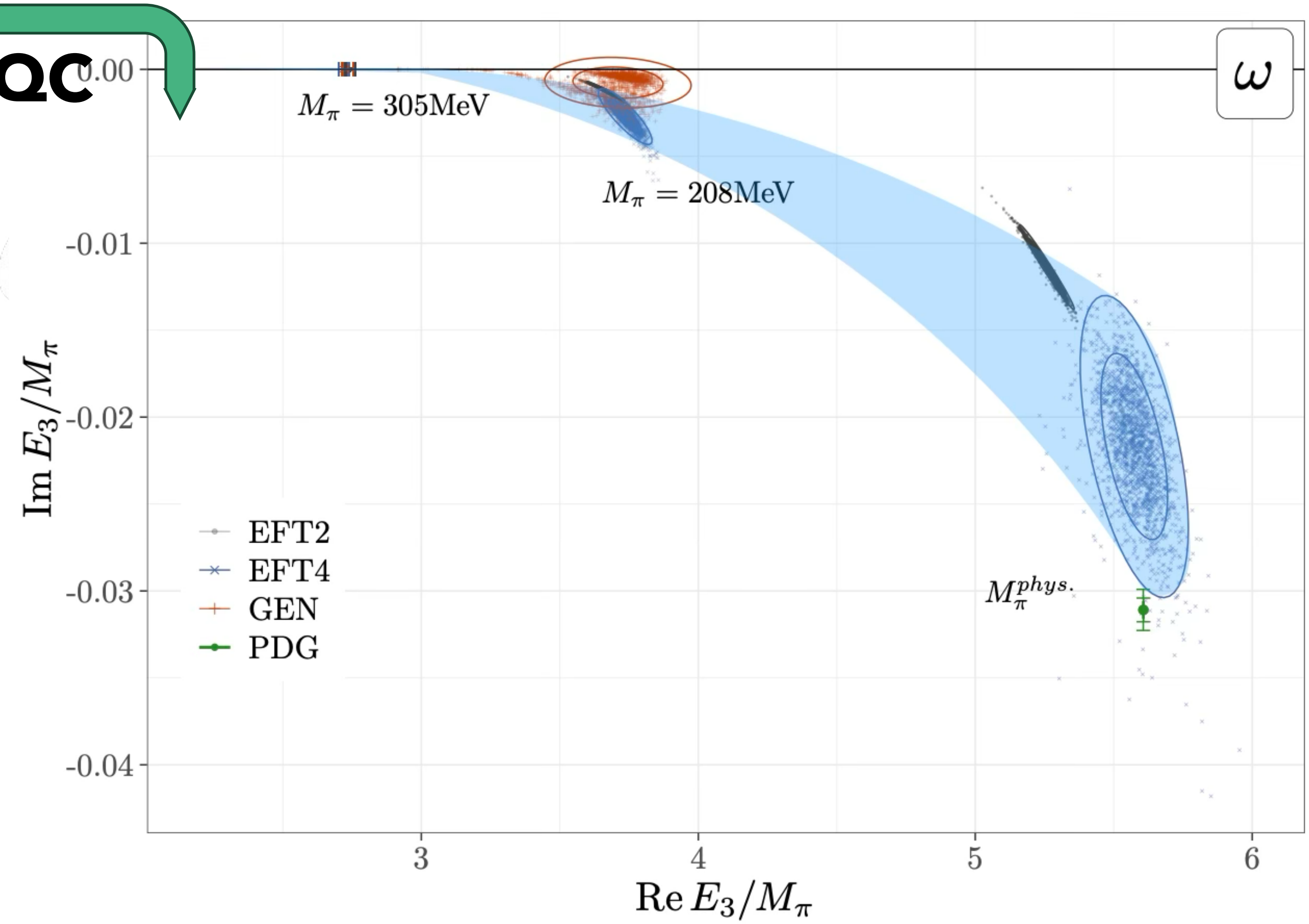
+ $\pi\pi$ levels

FVU

QC

Result

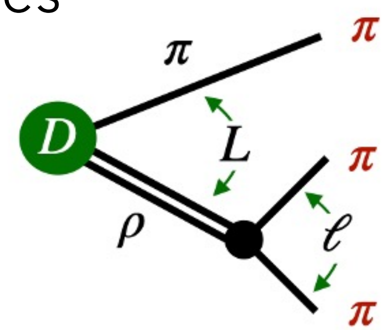
- Various EFT based ansatzes
- ω becomes a bound state at ~300 MeV
- at the physical point very close to the EXP value

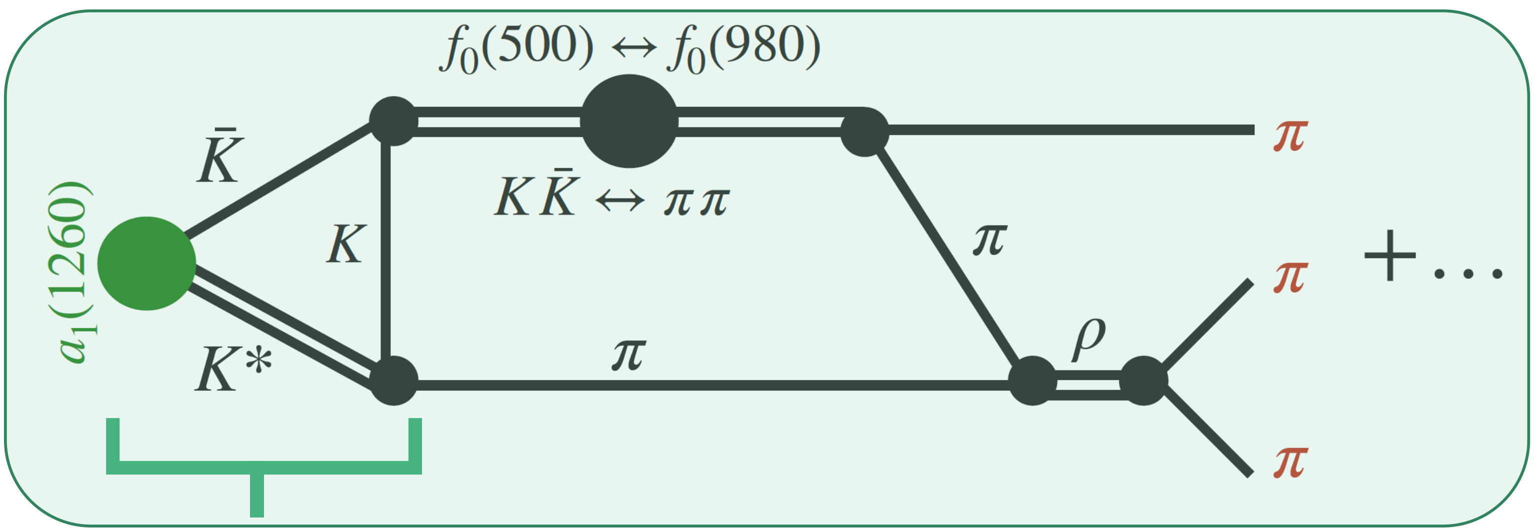


Coupled-channel, unitary amplitudes

[Feng, Gil, MD, Molina, Mai, Shastry, Szczepaniak, PRD '24]

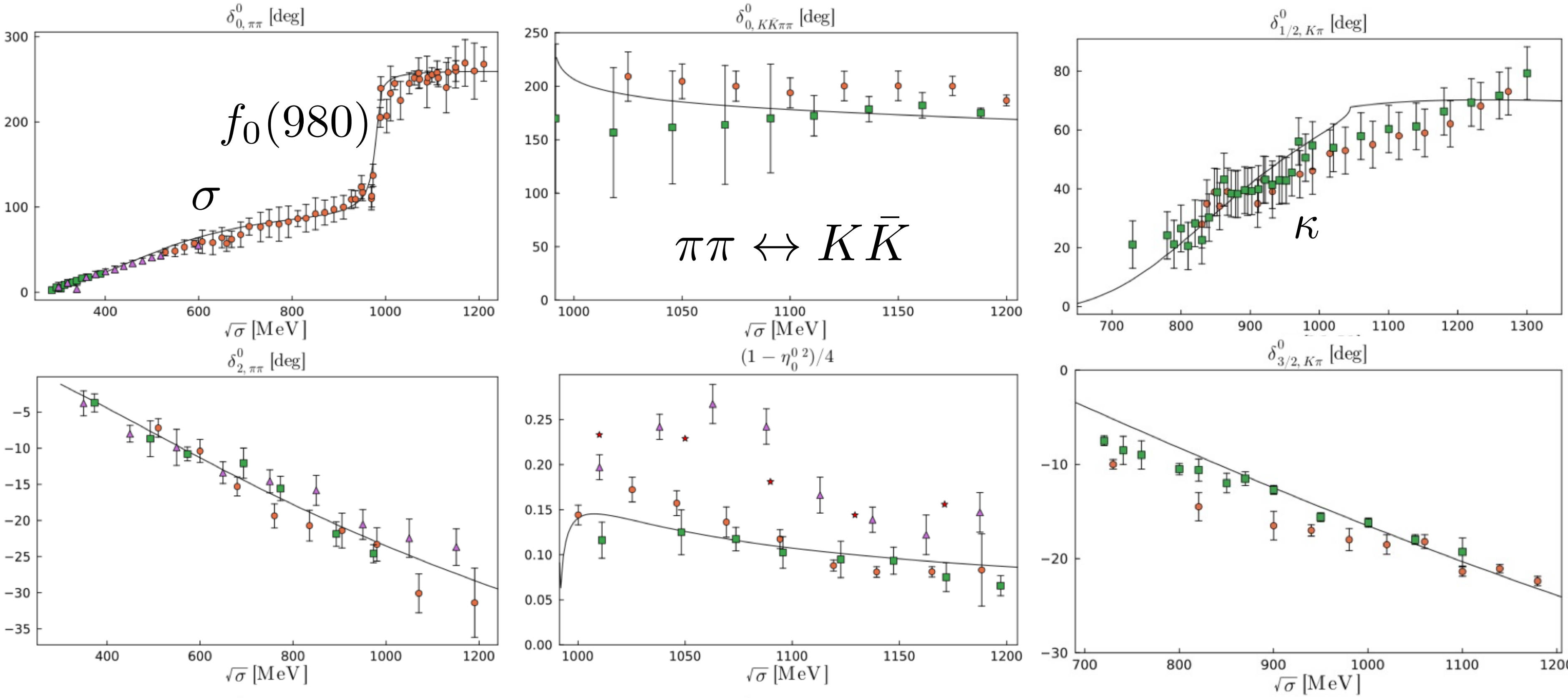
- Coupled-channel, coupled-partial wave amplitudes
- Unitarity manifest
- In-flight transitions of isobars: $\pi\pi \leftrightarrow K\bar{K}$
- All isospins: $I = 0,\ 1/2,\ 1, 3/2,\ 2$
- All subsystems up to P-wave, including $f_0(500),\ \rho,\ f_0(980), K^*,\ \kappa$
- Example:

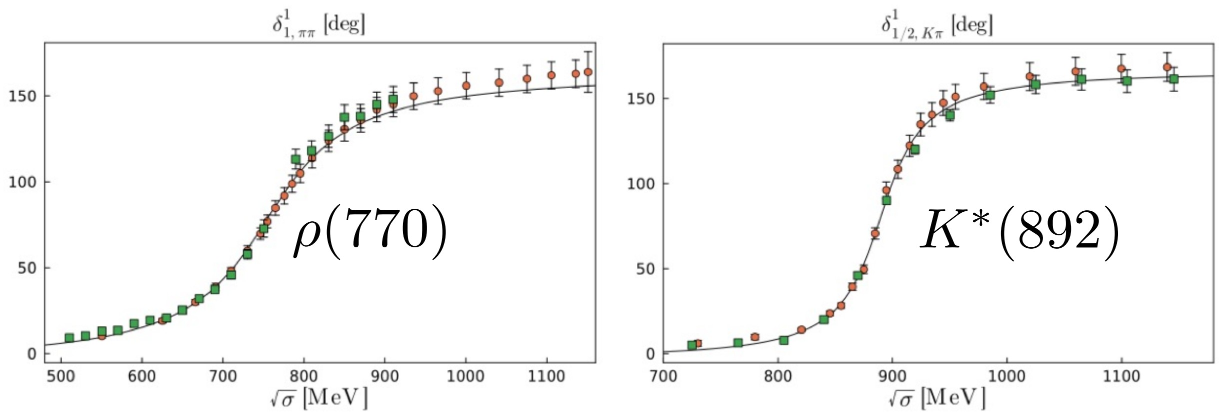




"Triangle"

Two-body input (including $\pi\pi \leftrightarrow K\bar{K}$)



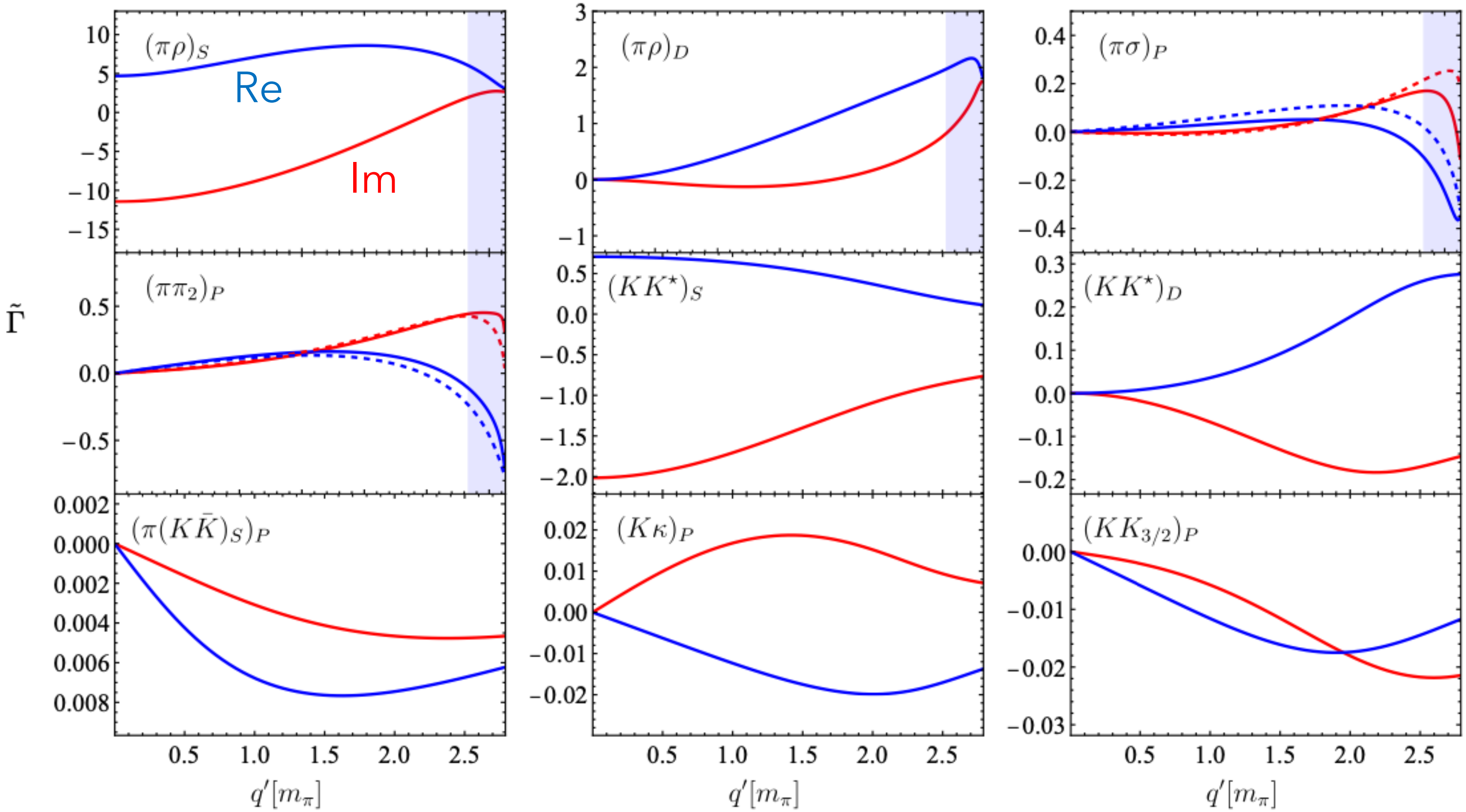


Notation:

$$\delta^{\ell}_{I,i},\ i \in \{\pi\pi, K\bar{K}, K\pi\}$$

Production amplitude 9-channel model

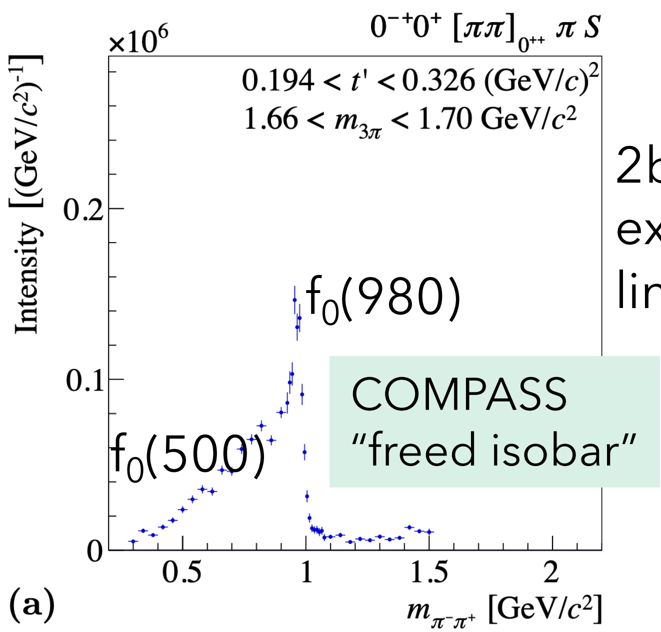
(Only the (non-trivial) rescattering piece) $3m_\pi < W < 2m_K + m_\pi$



Dashed lines: with **πρ** switched off (influence of coupled channels)

Future applications: Line-shape modifications

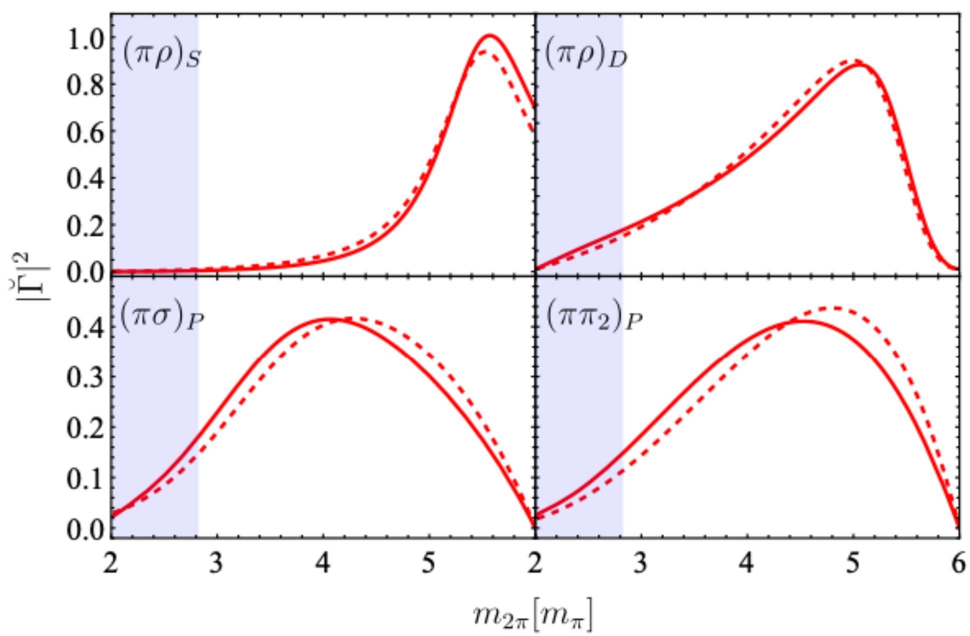
Lineshapes in the analysis of experimental data (COMPASS)



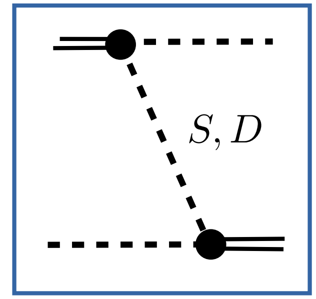
[Phys. Rev. D 95, 032004 (2017)]

2b lineshape extracted from 3b lineshape

Predicted line-shape modifications by three-body corrections and coupled-channels:



Summary

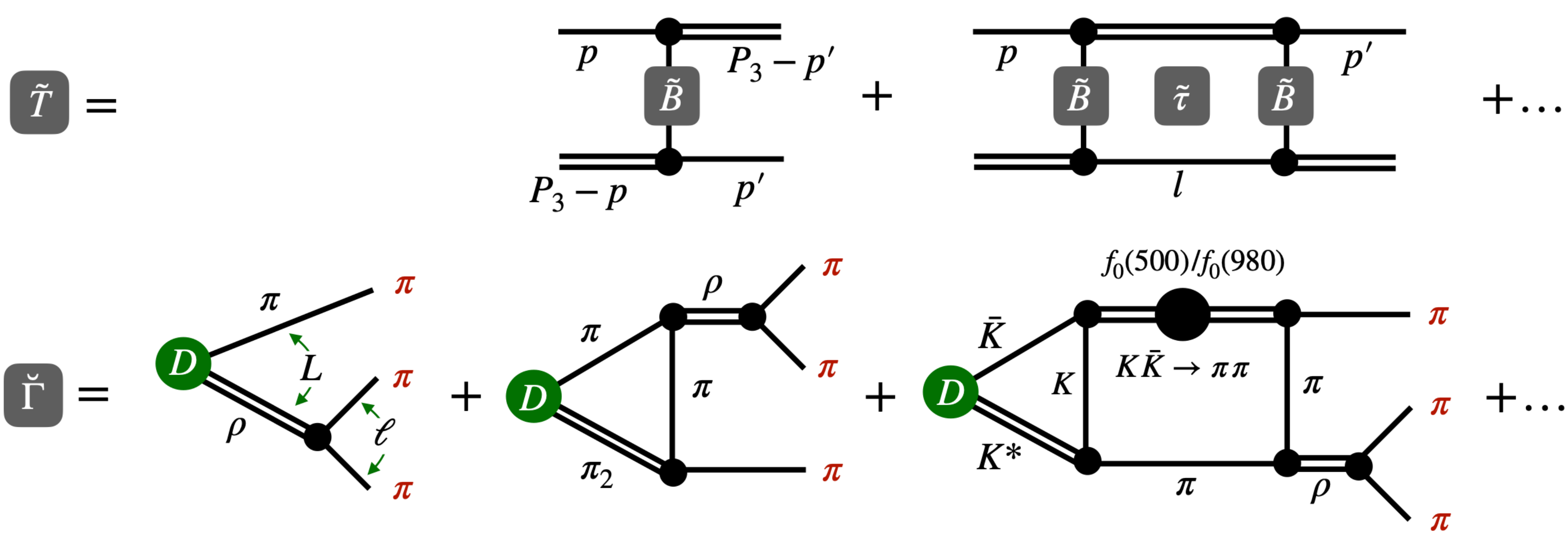


- Three-body systems beyond the tree-level isobar model:
  - Three-body effects in lineshapes and Dalitz plots
  - Resonance pole positions subject to 3-body modifications
  - Extension to 9 channels with systematic inclusion up to P-wave isobars
  - Future: data analysis, dynamic generation of resonances, triangle effects, nonzero net strangeness, baryons,…
- Lattice QCD determining the explicit dynamics of resonant three-body systems:
  - First determination of existence and properties of a three-body resonance – the $a_1(1260)$ - in coupled channels by FVU,
  - Recently extension to $\boldsymbol{\omega}$ and its three-pion dynamics

Spare slides

Channel space

Isobar (S_I, I_I)	$(1,1)$		$(1,{}^1\!/_2)$		$(0,0)$		$(0,2)$	$(0,{}^1\!/_2)$	$(0,{}^3\!/_2)$
HB basis (11 Ch.)	$\pi\rho_{\lambda=\pm1,0}$		$KK^*_{\lambda=\pm1,0}$		$\pi\sigma$	$\pi(K\bar{K})_S$	$\pi\pi_2$	$K\kappa$	$K(\pi K)_S$
JLS basis (9 Ch.)	$(\pi\rho)_S$	$(\pi\rho)_D$	$(KK^*)_S$	$(KK^*)_D$	$(\pi\sigma)_P$	$(\pi(K\bar{K})_S)_P$	$(\pi\pi_2)_S$	$(K\kappa)_S$	$(K(\pi K)_S)_P$

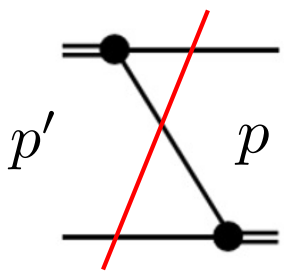


- Scattering matrix dimensions:
  Spectator momentum ⊗ JLS channels ⊗ isobar channels

How to solve the scattering equation

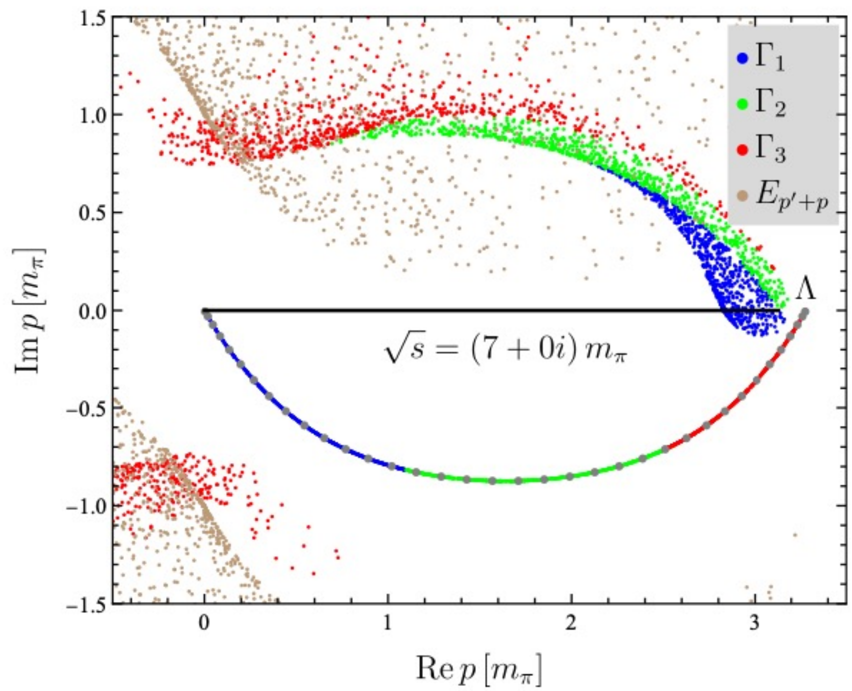
$$\tilde{T}_{ji}(s,p',p) = \tilde{B}_{ji}(s,p',p) + \tilde{C}_{ji}(s,p',p) + \int_0^\Lambda \frac{\mathrm{d}l\, l^2}{(2\pi)^3\, 2E_l} \left(\tilde{B}_{jk}(s,p',l) + \tilde{C}_{jk}(s,p',l)\right) \tilde{\tau}_k(\sigma_l)\, \tilde{T}_{kj}(s,l,p)$$

- **Three-body cuts**



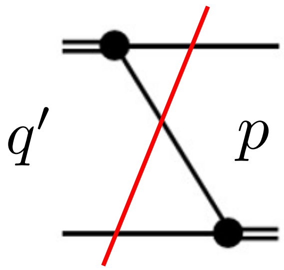
$$\tilde{B}_{ji}(s,\boldsymbol{p}',\boldsymbol{p}) = \frac{(\tilde{\boldsymbol{I}}_F)_{ji}\, v_j^*(p, P-p-p')v_i(p', P-p-p')}{2E_{p'+p}(\sqrt{s} - E_p - E_{p'} - E_{p'+p}) + i\epsilon}$$

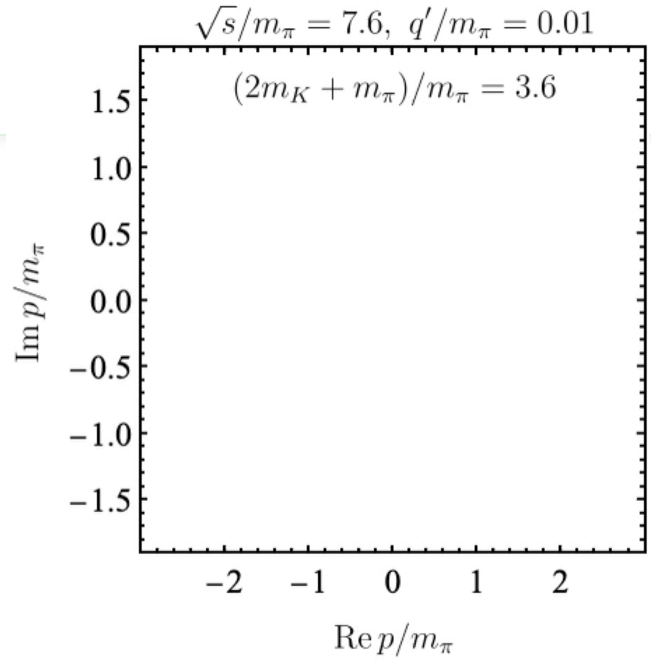
- Angle, energy dependent
- Depend also on p' and p
- Solve LSE for complex momenta on a deformed contour

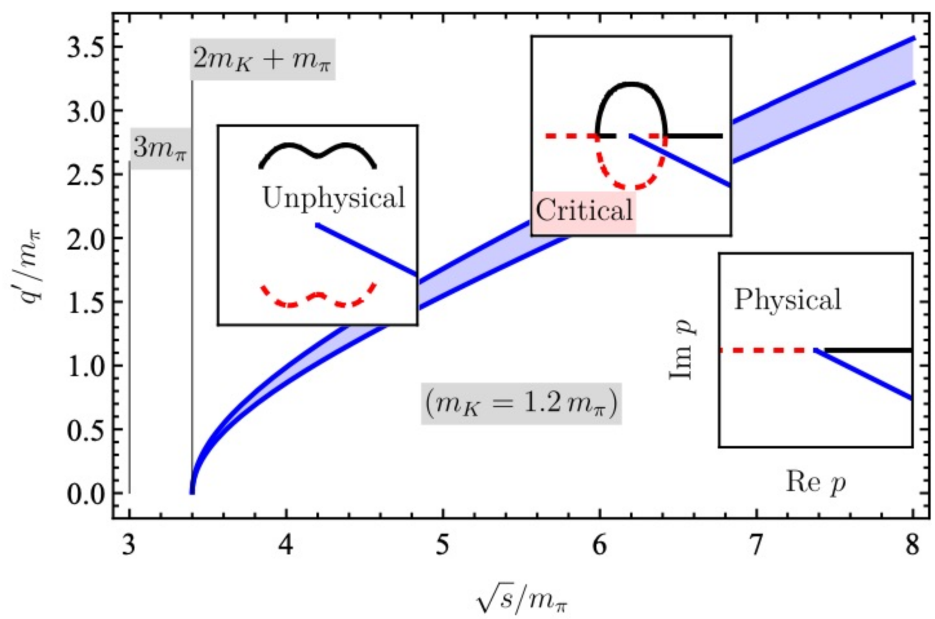


How to get the solution for real, physical momenta

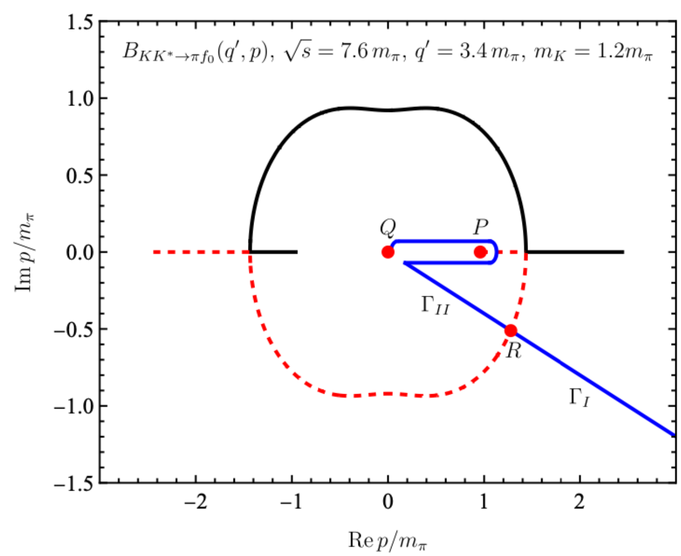
- No solution for real momenta in"critical region"







Solution 1: Contour deformation [Cahill & Sloane]



How to get the solution for real, physical momenta

Solution 2: Direct inversion [Ziegelmann et al.]

Production amplitude

$$\tilde{\Gamma}_j^T(s,q') = D_j(s,q') + \int_0^\Lambda \frac{dq\,q^2}{(2\pi)^3 2E_q}\,\tilde{B}ji(s,q',q)\,\tilde{\tau}_i(\sigma(q))\,\tilde{\Gamma}_i^T(s,q)$$

Ansatz

$$\tilde{\Gamma}^T(q) \approx \sum_{i=1}^N \tilde{\Gamma}^T(q_i)\,H_i(q)$$

with Lagrange polynomials

$$H_i(q) = \frac{\prod_{j\neq i}^N (q-q_j)}{\prod_{j\neq i}^N (q_i-q_j)}$$

Makes integral equation a matrix equation

$$\tilde{\Gamma}^T(q_j) = D(q_j) + A_{ji}\tilde{\Gamma}^T(q_i)$$

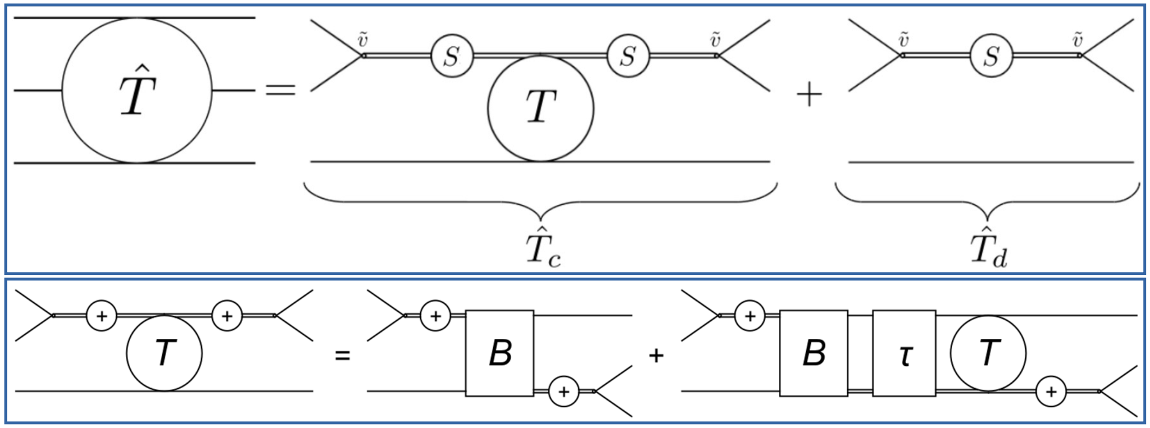
With singular integrals

$$A_{ji} = \int_0^\Lambda \frac{dq\,q^2}{(2\pi)^3 2E_q}\,\tilde{B}(q_j,q)\,\tilde{\tau}(\sigma(q))\,H_i(q)$$

… for which many established algorithms exist

Scattering amplitude

$3 \to 3$ scattering amplitude is a 3-dimensional integral equation



LS-type

- Imaginary parts of ***B*, *τ*** $^{-1}$ are fixed by **unitarity/matching**
- B, S are determined **consistently** through 8 different relations

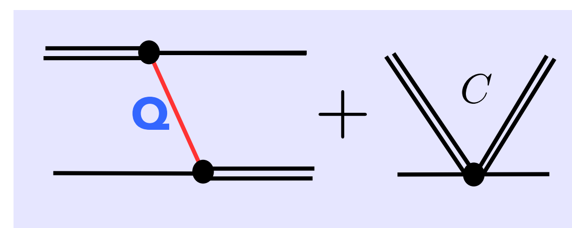
Matching → $$\mathrm{Disc}\, B(u) = 2\pi i \lambda^2 \frac{\delta\left(E_Q - \sqrt{m^2 + \mathbf{Q}^2}\right)}{2\sqrt{m^2 + \mathbf{Q}^2}}$$

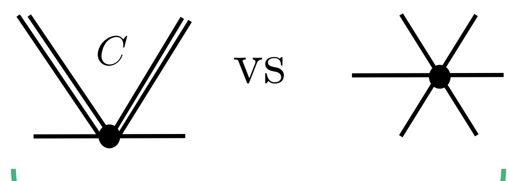
- un-subtracted dispersion relation

$$\langle q|B(s)|p\rangle = -\frac{\lambda^2}{2\sqrt{m^2+\mathbf{Q}^2}\left(E_Q - \sqrt{m^2+\mathbf{Q}^2} + i\epsilon\right)} + C$$

- one-**π** exchange in TOPT → *RESULT, NOT INPUT !*

- One <u>can</u> map to field theory but does not have to. Result is a-priori dispersive.

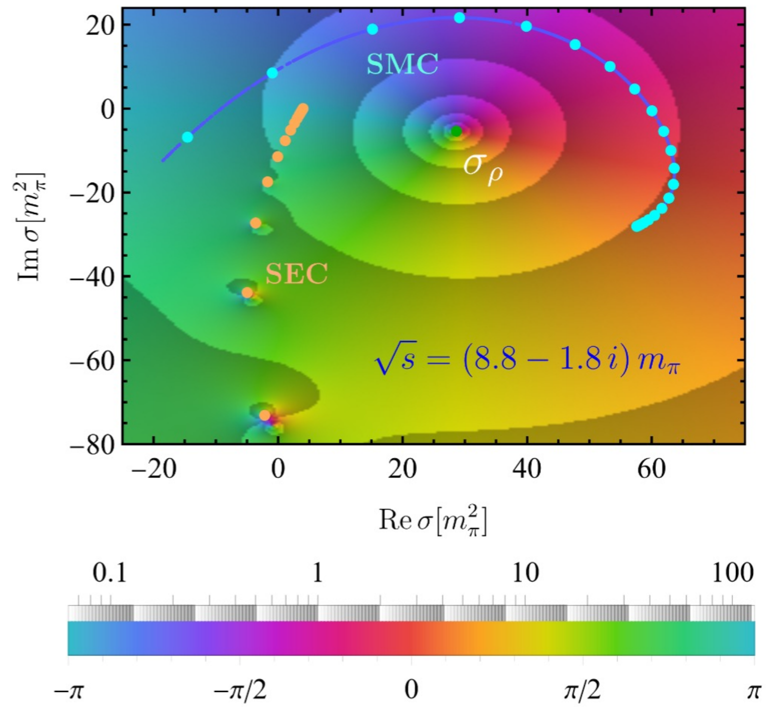




Add. Steps to map to theory might be needed [Brett (2021)]

Analytic cont. 3-body

[Sadasivan (2021)]
[Doering (2009)]



SMC

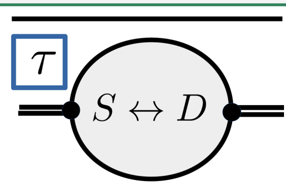
$$T^J_{LL'}(q_1,p) = \left(B^J_{LL'}(q_1,p) + C_{LL'}(q_1,p)\right) + \int_0^\Lambda \frac{\mathrm{d}l\, l^2}{(2\pi)^3 2E_l} \left(B^J_{LL''}(q_1,l) + C_{LL''}(q_1,l)\right) \tau(\sigma(l)) T^J_{L''L'}(l,p)$$

$$\tau^{-1}(\sigma) = K^{-1} - \Sigma\,,$$

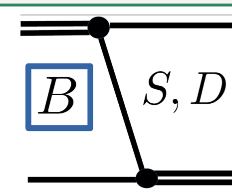
$$\Sigma = \int_0^\infty \frac{\mathrm{d}k\, k^2}{(2\pi)^3} \frac{1}{2E_k} \frac{\sigma^2}{\sigma'^2} \frac{\tilde{v}(k)^* \tilde{v}(k)}{\sigma - 4E_k^2 + i\epsilon}$$

$$B_{\lambda\lambda'}(\boldsymbol{p},\boldsymbol{p}') = \frac{v^*_\lambda(P-p-p',p)\, v_{\lambda'}(P-p-p',p')}{2E_{p'+p}\left(\sqrt{s} - E_p - E_{p'} - E_{p'+p} + i\epsilon\right)}$$

SEC



Singularities

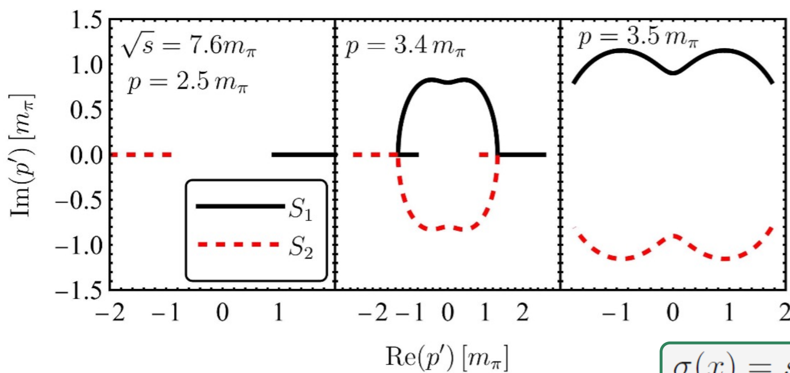


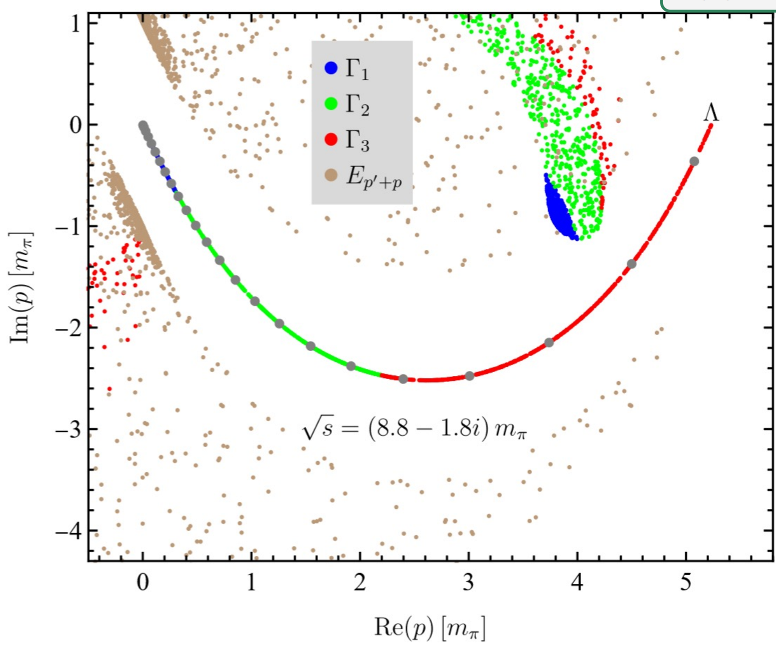
- Two contours (SMC and SEC)
- Deform both “adiabatically” to go to complex s
- Set of rules:
  - Contours cannot intersect with each others
  - Contours cannot intersect with (3-body) cuts
- Passing singularities left or right determines sheet

Analytic continuation 3-body (contd.)

- Three-body cuts

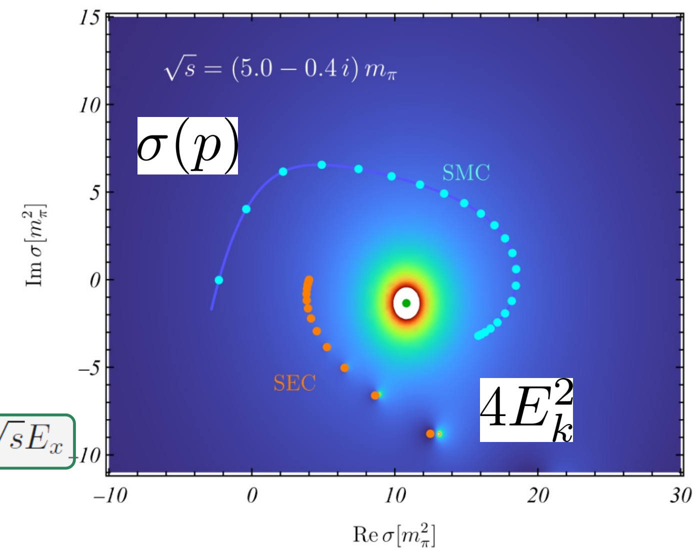
$$\sqrt{s} - E_p - E_{p'} - E_{p+p'} + i\epsilon = 0$$





$\sigma(x) = s + m_\pi^2 - 2\sqrt{s}E_x$

- Complex branch points



Integration limits at poles induce branch points