Resonances & Bound States

To investigate resonances, we must understand the analytic structure of the amplitude.
Let's continue with the $2 \to 2$ elastic scattering case. Recall the partial wave unitarity relation,

$$\mathrm{Im}\, \mathcal{M}_J(s) = \rho(s) |\mathcal{M}(s)|^2$$

Now, since $\rho(s) = 0$ below threshold on the real axis, $\rho(s) = \frac{\xi k^*}{8\pi E^*}$, $k^* = \frac{1}{2}\sqrt{s - 4m^2}$, then $\mathrm{Im}\, \mathcal{M}_J(s) = 0$ when $E^* = 2m$.

If a function $f(z)$ is real on a segment of the real axis, then $f(z^*) = f^*(z)$, this is the Schwartz reflection principle.

Causality implies that the amplitude for physical energies is the boundary value of some analytic function.

$$\boxed{\mathcal{M}_J^{(phys)}(s) = \lim_{\epsilon \to 0^+} \mathcal{M}_J(s + i\epsilon)}$$

The imaginary part of the amplitude is related to its <u>discontinuity</u>

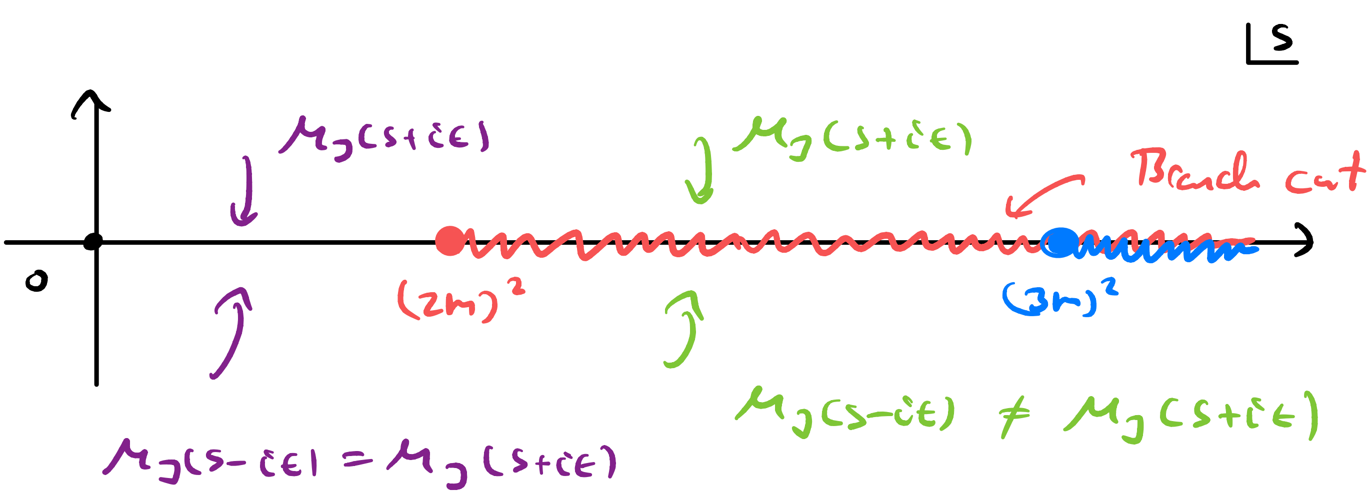
$$\text{Disc}\, f(x) \equiv \lim_{\epsilon \to 0^+} \left[f(x+i\epsilon) - f(x-i\epsilon) \right]$$

So,

$$\begin{aligned} 2i \,\text{Im}\, \mathcal{M}(s) &= \mathcal{M}(s) - \mathcal{M}^*(s) \\ &= \mathcal{M}(s+i\epsilon) - \mathcal{M}^*(s+i\epsilon) && \text{Analyticity} \\ &= \mathcal{M}(s+i\epsilon) - \mathcal{M}(s-i\epsilon) && \text{Schwartz} \\ &= \text{Disc}\, \mathcal{M}(s) \end{aligned}$$

Therefore, the unitarity condition gives us that the amplitude is discontinuous across the real s-axis,

$$\boxed{\text{Disc}\, \mathcal{M}_J(s) = 2i\rho(s) \, |\mathcal{M}_J(s)|^2}$$



Causality also gives that the amplitudes are analytic functions in the complex s-plane, except for unitarity cuts & pole singularities below threshold.

Bound States

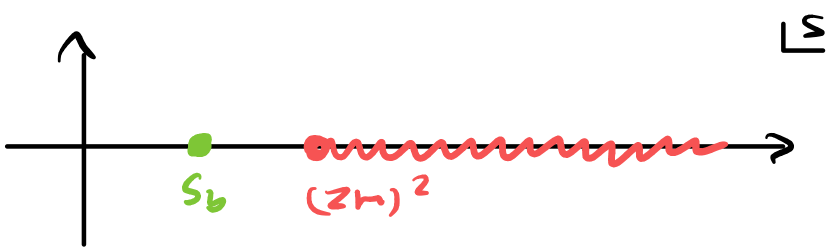
A bound state (stable state) is rigorously defined as a pole singularity on the real energy axis below threshold,

$$\boxed{\mathcal{M}(s) \sim -\frac{g_b^2}{s-s_b}}$$

coupling g_b (real)

mass $m_b = \sqrt{s_b}$ (real)

suppress J index



The binding energy is $B.E. = 2m - m_b$

The bound state has a spin J

Examples include the deuteron in np scattering.

Let's look at an example, consider S-wave scattering described by a leading order effective range, $a > 0$

$$\mathcal{M}(s) = \frac{8\pi\sqrt{s}}{\xi} \frac{1}{-\frac{1}{a} - i k^*}$$

pole singularity exists when $i k^* = -\frac{1}{a}$

Define binding momentum κ as $k^* = i\kappa$

$$\Rightarrow \quad \kappa = \frac{1}{a}$$

The bound state mass is

$$\boxed{\begin{aligned} m_b &= 2\sqrt{m^2 - \kappa^2} \\ &= 2\sqrt{m^2 - \frac{1}{a^2}} \end{aligned}} \qquad m_b < 2m \;!$$

The coupling is found by computing the residue at the pole,

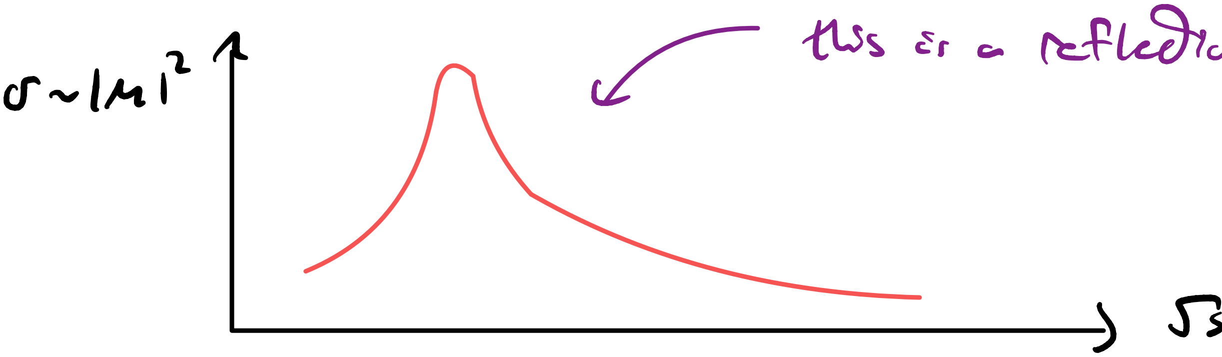
$$-g_b^2 = \oint \frac{ds}{2\pi i} \mathcal{M}(s)$$

$$\Rightarrow \quad \boxed{g_b = 8\sqrt{\frac{\pi m_b \kappa}{\xi}}} \qquad \text{(exercise)}$$

Resonances

What if we want to describe a cross-section with a classic resonance behavior?

this is a reflection of a pole



If we try to parametrize the K matrix by a single pole, e.g.,

$$K(s) = \frac{g_0^2 m_0^2}{m_0^2 - s}$$

here, g_0 & m_0 are just parameters

simple pole parametrization

so,

$$\mathcal{M}(s) = K(s) \frac{1}{1 - i\rho K(s)}$$

$$= \frac{g_0^2 m_0^2}{m_0^2 - s - i g_0^2 m_0^2 \rho}$$

This is an S-wave Breit-Wigner amplitude, describes an isolated, narrow-width peak where m_0 is approx. the peak location in $\sqrt{s}$

Let us massage it to a more familiar form for some, recall that $\rho = \frac{\xi\, k^*}{8\pi\sqrt{s}}$

$$\Rightarrow \quad \boxed{\mathcal{M} = \frac{g_0^2 m_0^2}{m_0^2 - s - i\sqrt{s}\,\Gamma(s)}}$$

where $\boxed{\Gamma(s) = \xi\, \frac{g_0^2 m_0^2\, k^*}{8\pi\, s}}$ energy-dependent width

m_0 & g_0 are <u>not</u> physical parameters. The only meaningful parameters are the pole positions.

What are the poles of our "Breit-Wigner"?

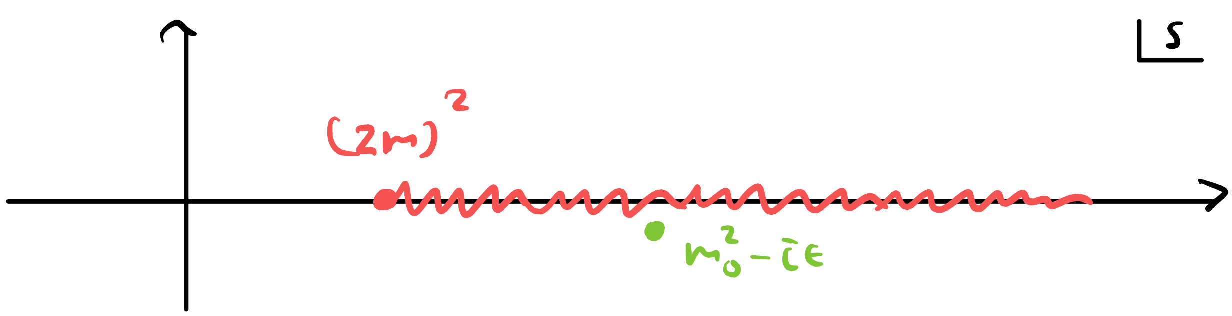
First, consider the narrow-width limit ($g_0 \to 0$)

then,
$$\mathcal{M}(s) = \frac{g_0^2 m_0^2}{m_0^2 - s - i\epsilon}$$

↪ $\epsilon \to 0$ as $g_0^2 \to 0$

pole at $\quad s = m_0^2 - i\epsilon$

$$\Rightarrow \sqrt{s} = \pm(m_0 - i\epsilon)$$



In the limits of $\epsilon \to 0$, the pole is a real pole, thus stable (bound state).

So, $\sqrt{s} = m_0 \equiv M$ actual mass

But, since we have scattering, $M > 2m$.
A real axis pole above threshold violates unitarity!

$$\mathrm{Im}\,\mathcal{M} \sim \delta(s - M^2)$$
$$\neq \rho |\mathcal{M}|^2 \sim \left(\frac{1}{s - M^2}\right)^2$$

$\Rightarrow$ NOT physically allowed.

What if we have a small g_0? (very narrow)

then,
$$\mathcal{M}(s) = \frac{g_0^2 m_0^2}{m_0^2 - s - i\sqrt{s}\,\Gamma(s)}$$

pole at $\quad m_0^2 - s - i\sqrt{s}\,\Gamma(s) = 0$

For vanishing g_0, $s = m_0^2$

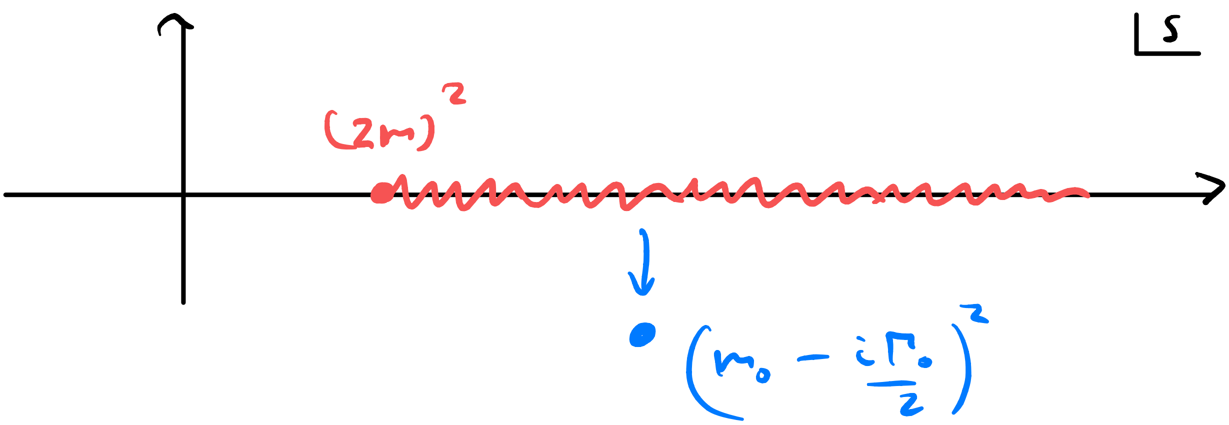
So, expand near $g_0 \to 0$ limit

$$s = m_0^2 - i\sqrt{s}\,\Gamma(s)$$

$$\simeq m_0^2 \mp i m_0 \Gamma(m_0^2)$$

$\Gamma(m_0^2) \equiv \Gamma_0$
a small constant

$$\simeq \left(m_0 \mp \frac{i\Gamma_0}{2}\right)^2 + \mathcal{O}(\Gamma_0^2)$$



Here, $M = m_0$ is resonance mass

$\Gamma = \Gamma_0$ is resonance width

$\tau = \frac{1}{\Gamma}$ is lifetime

In general, $M = M(m_0, g_0)$

$\Gamma = \Gamma(m_0, g_0)$

We define the resonance mass and width, in order to reproduce narrow width Breit-Wigner, as

$$\boxed{\sqrt{s_p} = M^2 - \frac{i\Gamma}{2}}$$

pole position s_p

An issue concerning analyticity complicates this definition. Causality imposes that the amplitude is analytic in the complex energy plane $\Rightarrow$ Cannot have pole singularities!

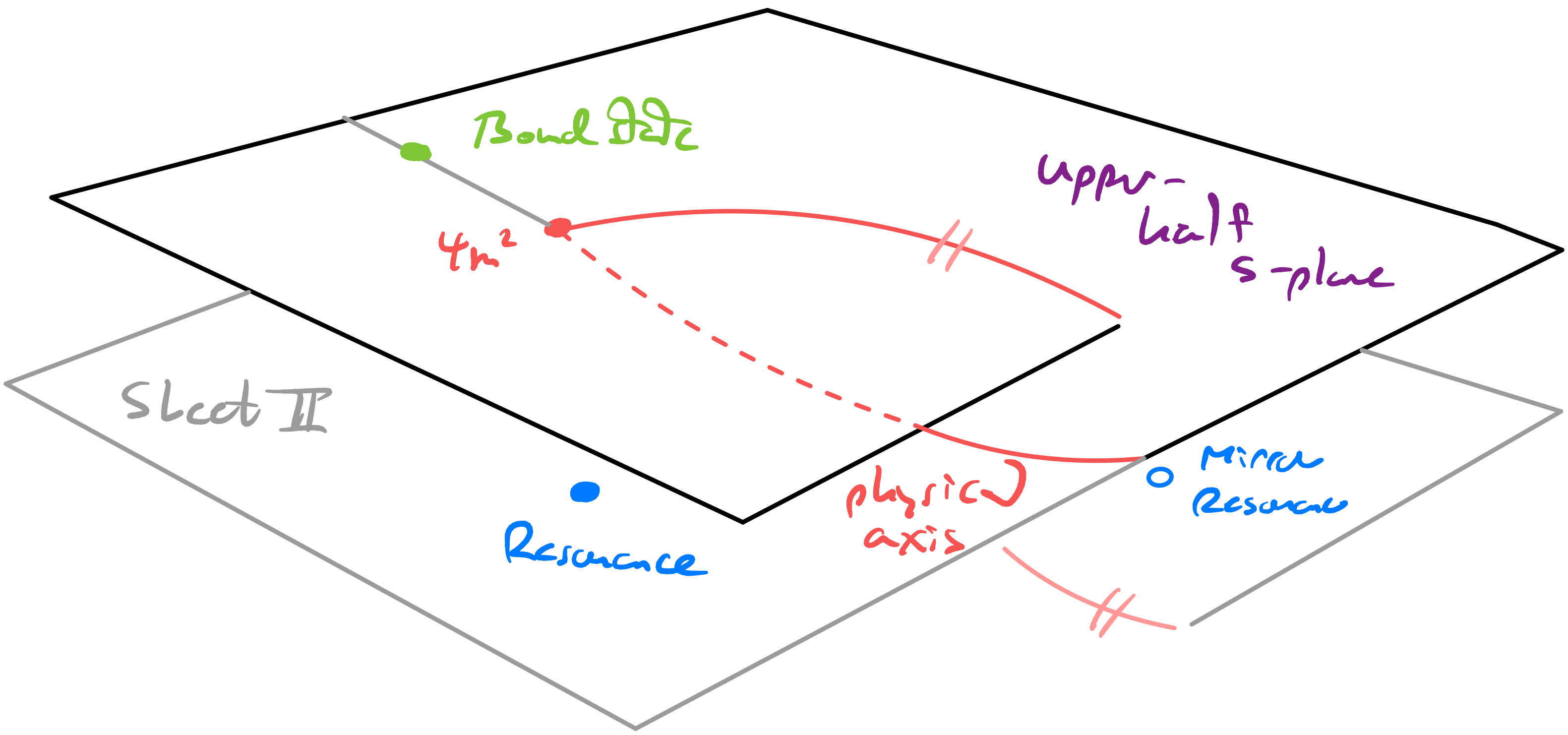
So, where are they? They are on the Second Riemann sheet!

Recall that the amplitude is discontinuous across the real axis for $s \geq (2m)^2$.

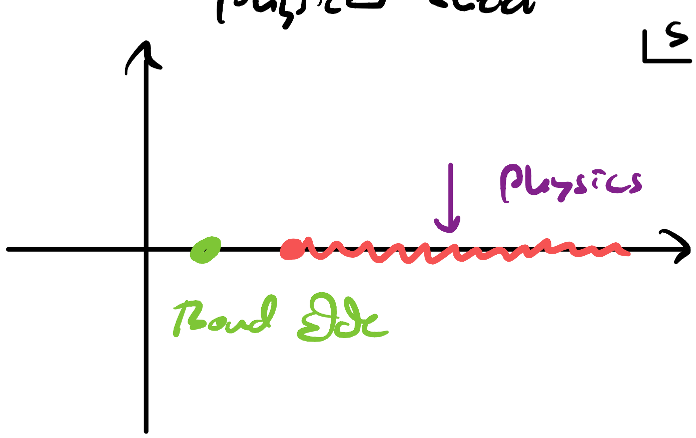
The branch cut is square-root-like,

(think of it being induced by phase space $p \sim \sqrt{s-4m^2}$)

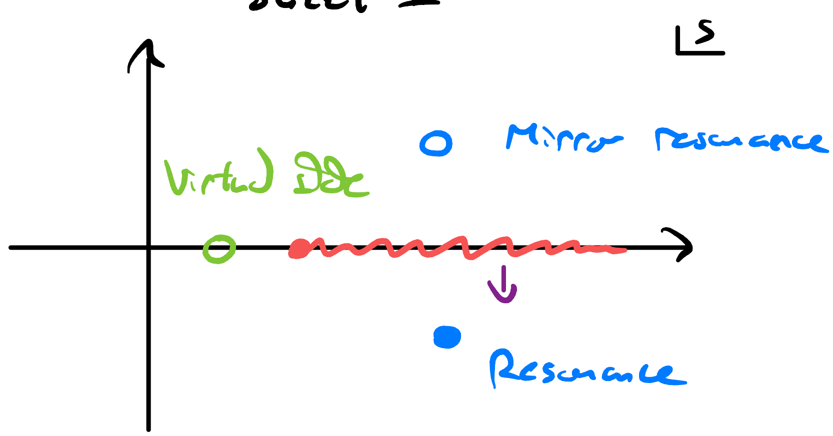
The resonance position is hiding underneath the branch cut on the 2nd sheet



Physical sheet



Sheet II



Constraining K-matrices

We have an analytic form for the amplitude, but we need to constrain K with external information. For Hadron physics, we have 3 options

- Experiment
  - Measure observables, e.g., $\sigma \sim |\mathcal{M}|^2$.
  - Constrain amplitude $\mathcal{M} = K \dfrac{1}{1-ipK}$
  - Parametrize K
  - Search for poles $\Rightarrow$ get hadron properties
- Theoretical model
  - Choose model or EFT, e.g., χ^{PT}, ...
  - Compute K, e.g. to some chiral order
  - Compare to data, fix parameters of theory
  - Search for poles $\Rightarrow$ get hadron properties
- Lattice QCD
  - Compute finite-volume spectrum E_n
  - Map E_n to K via Lüscher
  - Search for poles $\Rightarrow$ get hadron properties