Reaction Kinematics

Note that I will use the standard relativistic normalization,

$$\langle \vec{k} | \vec{p} \rangle = (2\pi)^3 2E_p \, \delta^{(3)}(\vec{k}-\vec{p})$$

Multiparticle states are defined as the direct product of states, but if they are identical, then we must take care of exchange symmetry,

e.g., for two identical particles,

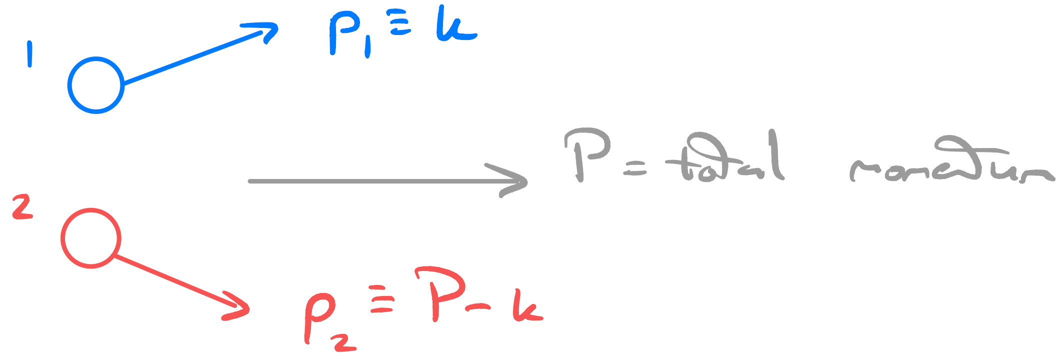
$$|\vec{p}, \vec{k}\rangle = \frac{1}{\sqrt{2!}}\Big(|\vec{p}\rangle \otimes |\vec{k}\rangle \pm |\vec{k}\rangle \otimes |\vec{p}\rangle \Big)$$

$\hookrightarrow$ + bosons
− fermions

So,

$$\langle \vec{p}'\vec{k}' | \vec{p}\,\vec{k} \rangle = (2\pi)^6 2E_p 2E_k \Big(\delta^{(3)}(\vec{k}-\vec{k}')\,\delta^{(3)}(\vec{p}-\vec{p}') \pm \delta^{(3)}(\vec{k}-\vec{p}')\,\delta^{(3)}(\vec{p}-\vec{k}') \Big)$$

Let us focus on two-body systems of identical particles with mass m.



Each particle satisfies the on-shell condition,

$$p_1^2 = k^2 = m^2 \quad , \quad p_2^2 = (P-k)^2 = m^2$$

It is useful to define the invariant mass of the two-body system, $\sqrt{s}$

$$\boxed{s = P^2 = (p_1+p_2)^2}$$

Mandelstam s

Since $P = (E, \vec{P})$, E = total energy,

$$\Rightarrow \quad s = E^2 - \vec{P}^2$$

For convenience, let us work in the center-of-momentum (CM) frame of the two-body system, defined by $\vec{P}^* = \vec{0}$

$$\Rightarrow \quad \vec{p}_1^{\,*} = -\vec{p}_2^{\,*} = \vec{k}^*$$

The total energy is $\boxed{E^* = \sqrt{s}}$

since $\quad E^* = E_1^* + E_2^* = 2E_1^* \quad (E_1^* = E_2^*)$

$$\Rightarrow \quad \boxed{E_1^* = E_2^* = \frac{\sqrt{s}}{2}}$$

Further, since $E_1 = \sqrt{m^2 + \vec{k}^{*2}}$, we find

$$\boxed{|\vec{k}^*| = \frac{1}{2}\sqrt{s - 4m^2}}$$

In the CM frame, the magnitude of the momentum is fixed by s.

An exercise is to show for $m_1 \neq m_2$,

$$E^*_{1,2} = \frac{\sqrt{s}}{2} \pm \frac{m_1^2 - m_2^2}{2\sqrt{s}}$$

$$|\vec{k}^*| \equiv |\vec{p}_1^{\,*}| = |\vec{p}_2^{\,*}| = \frac{1}{2\sqrt{s}} \lambda^{\frac{1}{2}}(s, m_1^2, m_2^2)$$

w/ Källen triangle function

$$\lambda(x,y,z) = x^2 + y^2 + z^2 - 2(xy + yz + zx)$$

For two-body states, often write as

$$|\vec{p}_1^{\,*}, \vec{p}_2^{\,*}\rangle \overset{\text{cm}}{\equiv} |E^*, \hat{k}^*\rangle$$

$\hookrightarrow$ $\hat{k}^* = (\theta_k^*, \varphi_k^*)$

orientation is unfixed

In a moving frame, $\vec{P} \neq \vec{0}$, $\quad k^* = k^*(E^*)$

$$|\vec{p}_1, \vec{p}_2\rangle \equiv |E^*, \vec{P}, \hat{k}^*\rangle$$

$\hookrightarrow$ $E^2 = E^{*2} + \vec{P}^2$

Probability Conservation – Unitarity

The probability for a reaction $\alpha \to \beta$ is

$$\text{Prob}(\alpha \to \beta) = |\langle\beta|\hat{S}|\alpha\rangle|^2$$

Assume properly normalized

The total probability for α to go to all final states β is

$$\sum_\beta \text{Prob}(\alpha \to \beta) = 1$$

$$\Rightarrow \quad 1 = \sum_\beta |\langle\beta|\hat{S}|\alpha\rangle|^2$$

$$= \sum_\beta \langle\alpha|\hat{S}^\dagger|\beta\rangle\langle\beta|\hat{S}|\alpha\rangle$$

$$= \langle\alpha|\hat{S}^\dagger S|\alpha\rangle$$

$$\Rightarrow \quad \boxed{\hat{S}^\dagger \hat{S} = \hat{1}}$$

The S-matrix is a unitary operator

Recall that

$$\hat{S} = \hat{1} + i\hat{T}$$

So, $\hat{S}^\dagger \hat{S} = \hat{1} \Rightarrow (\hat{1} - i\hat{T}^\dagger)(\hat{1} + i\hat{T}) = \hat{1}$

or,

$$\boxed{\hat{T} - \hat{T}^\dagger = i\hat{T}^\dagger \hat{T}}$$

This is the unitarity condition for the T-matrix. For $\alpha \to \beta$, we have for the amplitude

$$\boxed{\mathcal{M}_{\beta\alpha} - \mathcal{M}^*_{\alpha\beta} = i \sum\!\!\!\!\!\!\int_{\gamma} (2\pi)^4 \delta^{(4)}(P_\gamma - P_\alpha) \mathcal{M}^*_{\gamma\beta} \mathcal{M}_{\gamma\alpha}}$$

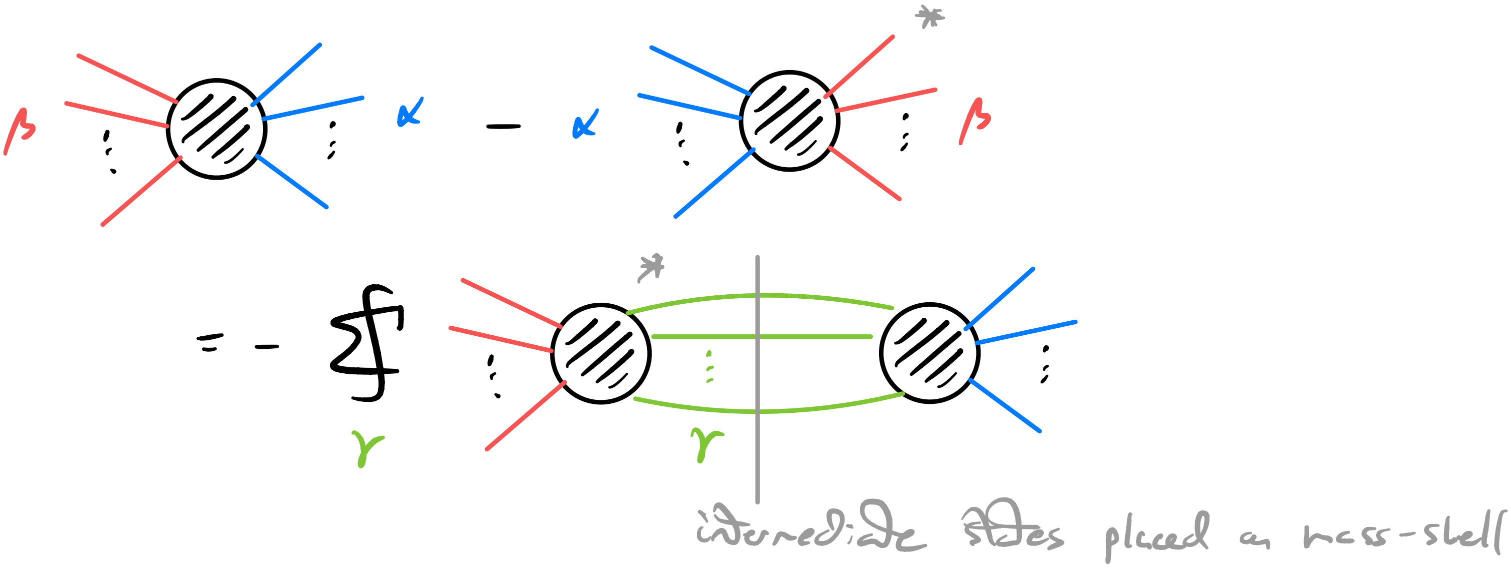
An exercise is to prove this

where we used the resolution of identity

$$\hat{1} = \sum\!\!\!\!\!\!\int_{\gamma} |\gamma\rangle\langle\gamma|$$

$\hookrightarrow$ sum over <u>all</u> scattering channels (an infinite number of terms)

Diagrammatically, the unitarity condition is



The CPT theorem relates $\alpha \to \beta$ to $\bar{\beta} \to \bar{\alpha}$

$$\mathcal{M}_{\beta\alpha} = \mathcal{M}_{\bar{\alpha}\bar{\beta}} \quad \text{by CPT}$$

If a system also exhibits CP symmetry, e.g., QCD, then

$$\mathcal{M}_{\beta\alpha} \underset{\text{CP}}{=} \mathcal{M}_{\bar{\beta}\bar{\alpha}} \underset{\text{CPT}}{=} \mathcal{M}_{\alpha\beta}$$

So, $\mathcal{M}_{\beta\alpha} - \mathcal{M}^*_{\alpha\beta} = \mathcal{M}_{\beta\alpha} - \mathcal{M}^*_{\beta\alpha} = 2i\,\mathrm{Im}\,\mathcal{M}_{\beta\alpha}$

$$\Rightarrow \boxed{2\,\mathrm{Im}\,\mathcal{M}_{\beta\alpha} = \sum\!\!\!\!\!\!\!\int_{\gamma} (2\pi)^4 \delta^{(4)}(P_\gamma - P_\alpha)\, \mathcal{M}^*_{\gamma\beta}\, \mathcal{M}_{\gamma\alpha}}$$

This is an extremely complicated non-linear integral equation for $\mathcal{M}_{\beta\alpha}$, which depends on its coupling to every other scattering amplitude (from which is allowed by symmetry).

However, by restricting the scope of the problem we wish to study, we can find a suitable $\mathcal{M}_{\beta\alpha}$ which satisfies unitarity.

Two-Body Elastic Scattering

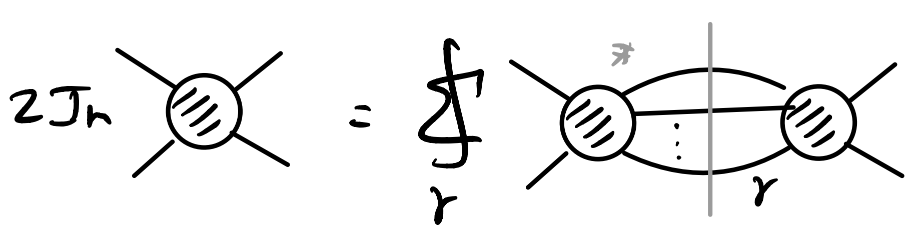
Lets consider elastic $2 \to 2$ scattering, e.g., $\pi^+\pi^+ \to \pi^+\pi^+$ scattering.

So,

$$|\alpha\rangle = |P, k\rangle$$

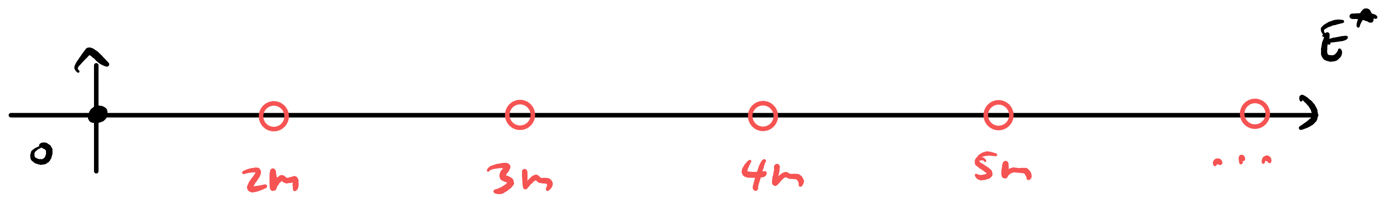
$$|\beta\rangle = |P', k'\rangle$$

So, unitarity is

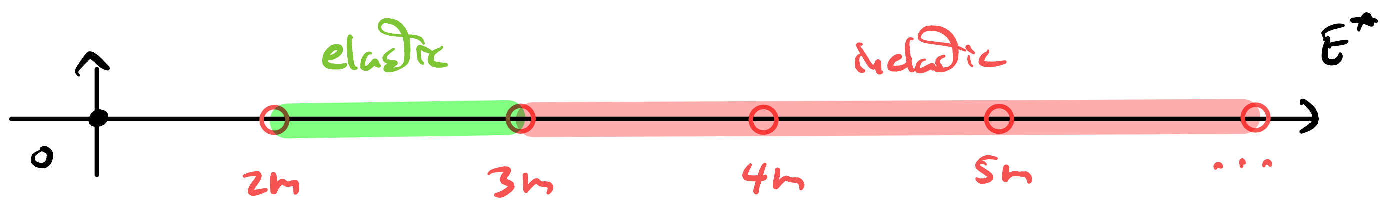


still a sum over an infinite number of states

For simplicity, let us consider only a single species of particle. Therefore, the production thresholds are, (in the two-body CM frame), $E^*_{th,n} = nm$



If we only care to describe the elastic kinematic region, i.e., $2m \le E^* < 3m$, then we effectively turn off all intermediate states except the two-body production,



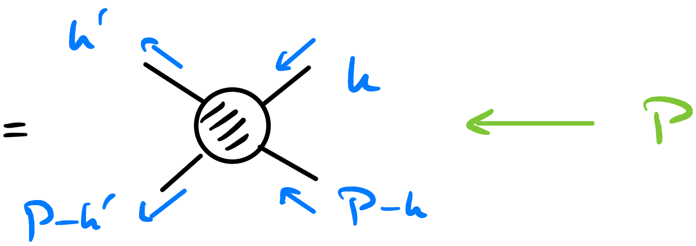
So,

$$\boxed{\begin{aligned} &2\,\mathrm{Im}\,\mathcal{M}(P,k',k) \\ &= \frac{1}{2!}\int\frac{d^3\vec{q}_1}{(2\pi)^3 2\omega_1}\int\frac{d^3\vec{q}_2}{(2\pi)^3 2\omega_2}(2\pi)^4\delta^{(4)}(P'-P)\,\mathcal{M}^*(P,q,k')\,\mathcal{M}(P,q,k)\end{aligned}}$$

symmetry factor

elastic unitarity

where $i\mathcal{M}(P,k',k) =$



& $\quad q_1 = q\,,\quad q_2 = P'-q\,,\quad \omega_1 = \sqrt{m^2+q^2}\,,\quad \omega_2 = \sqrt{m^2+(P'-k)^2}$

We can simplify this by exchanging $(\vec{q}_1,\vec{q}_2) \to (\vec{q},\vec{P}')$

$$\Rightarrow 2\,\mathrm{Im}\,\mathcal{M}(P,k',k)$$

$$= \frac{1}{2!}\int\frac{d^3\vec{q}}{(2\pi)^3 2\omega_1}\int\underbrace{\frac{d^3\vec{P}'}{(2\pi)^3 2\omega_2}(2\pi)^4\delta^{(4)}(P'-P)}_{\text{eliminates 3 integrals}}\,\mathcal{M}^*(P,q,k')\,\mathcal{M}(P,q,k)$$

$$= \frac{1}{2!}\,\frac{1}{(4\pi)^2}\int\frac{d^3\vec{q}}{\omega_1\omega_2}\,\delta(E'-E)\,\mathcal{M}^*(P,q,k')\,\mathcal{M}(P,q,k)$$

The last δ-function can be eliminated by going to the CM frame, $\vec{P}^* = \vec{0}$, & using spherical coordinates

$$\Rightarrow 2\,\mathrm{Im}\,\mathcal{M}(E^*, \hat{k}'^*, \hat{k}^*)$$
$$= \frac{1}{2!}\frac{1}{(4\pi)}\int \frac{d\hat{q}^*}{4\pi}$$
$$\times \int_0^\infty dq^* \frac{q^{*2}}{\omega_1^*\omega_2^*}\, \delta(E'^* - E^*)\, \mathcal{M}^*(E^*, \hat{q}^*, \hat{k}'^*)\mathcal{M}(E^*, \hat{q}^*, \hat{k}^*)$$

Now, $\delta(E'^* - E^*) = \dfrac{\omega_1^*\omega_2^*}{k^* E^*}\,\delta(q^* - k^*)$

$$\Rightarrow \mathrm{Im}\,\mathcal{M}(E^*, \hat{k}'^*, \hat{k}^*) = \underbrace{\frac{\xi\, k^*}{8\pi E^*}}_{\equiv \rho \text{ two-body phase space factor}} \int \frac{d\hat{q}^*}{4\pi}\mathcal{M}^*(E^*, \hat{q}^*, \hat{k}'^*)\mathcal{M}(E^*, \hat{q}^*, \hat{k}^*)$$

$\xi = \frac{1}{2!}$ symmetry factor

(if distinct, $\xi = 1$)

While simpler, the elastic unitarity condition is still a nonlinear integral equation. However, rotational symmetry will allow us to "solve" it.

Partial Wave Expansions

Under a rotation, amplitudes must transform appropriately. We can simplify this understanding by expanding the system into definite **partial waves**, that is states / amplitudes of definite angular momentum.

Since we assume the system is invariant under rotations, angular momentum is a good quantum number.

Let $|J, m_J\rangle$ be a state of definite angular momentum, with

$$\langle J' m_J' | J m_J \rangle = \delta_{J'J} \delta_{m_J' m_J}$$

&

$$\hat{1} = \sum_{J=0}^{\infty} \sum_{m_J=-J}^{J} |J m_J\rangle\langle J m_J|$$

So, two-body state in CM frame is then

$$|E^*, \hat{n}^*\rangle = \underbrace{\sqrt{4\pi}}_{\text{covariant normalization}} \sum_{J m_J} \underbrace{|E^*, J m_J\rangle}_{\text{two-body state of definite angular momentum}} Y^*_{J m_J}(\hat{n}^*)$$

Here,

$$Y_{Jm_J}(\hat{n}^*) = \langle \hat{n}^* | J m_J \rangle$$

are spherical harmonics.

So, for the 2→2 amplitude,

$$\boxed{\mathcal{M}(E^*, \hat{n}'^*, \hat{n}^*) = 4\pi \sum_{J', m_J'} \sum_{J, m_J'} Y_{J'm_J'}(\hat{n}'^*)\, \mathcal{M}^{J'J}_{m_J' m_J}(E^*)\, Y^*_{Jm_J}(\hat{n}^*)}$$

partial wave expansion

Partial wave amplitudes

Now, since total angular momentum is conserved,

$$\mathcal{M}^{J'J}_{m_J' m_J}(E^*) = \delta_{J'J}\, \delta_{m_J' m_J}\, \mathcal{M}_J(E^*)$$

independent of m_J, Wigner-Eckart theorem.

So,

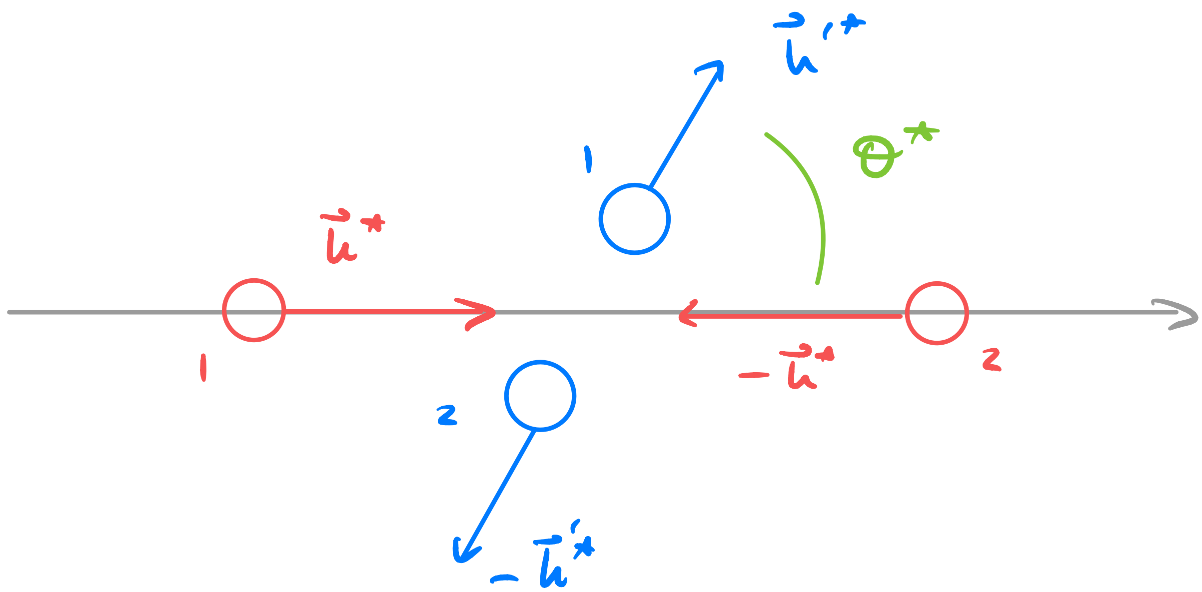
$$\mathcal{M}(E^*, \hat{n}'^*, \hat{n}^*) = 4\pi \sum_J \mathcal{M}_J(E^*) \sum_{m_J} Y_{Jm_J}(\hat{n}'^*)\, Y^*_{Jm_J}(\hat{n}^*)$$

Recall the spherical harmonic addition theorem,

$$\sum_{m_J} Y_{Jm_J}(\hat{n}'^*)\, Y^*_{Jm_J}(\hat{n}^*) = \frac{2J+1}{4\pi} P_J(\hat{n}'^* \cdot \hat{n}^*)$$

Legendre polynomials

The CM frame scattering angle, θ^*, is defined via $\cos\theta^* = \hat{k}'^* \cdot \hat{k}^*$



So,
$$\boxed{\mathcal{M}(E^*, \theta^*) = \sum_J (2J+1)\mathcal{M}_J(E^*)P_J(\cos\theta^*)}$$

We have traded the angular information for angular momentum, with the angular dependence being captured by known functions!

Partial wave analysis is useful since it diagonalizes the unitarity condition

$$\mathrm{Im}\,\mathcal{M}(E^*, \hat{k}'^*, \hat{k}^*) = \rho \int \frac{d\hat{q}^*}{4\pi} \mathcal{M}^*(E^*, \hat{q}^*, \hat{k}'^*)\, \mathcal{M}(E^*, \hat{q}^*, \hat{k}^*)$$

partial wave expand

using orthonormality of spherical harmonics,

$$\int d\hat{q}^*\, Y^*_{J' m_{J'}}(\hat{q}^*)\, Y_{J m_J}(\hat{q}^*) = \delta_{J'J}\, \delta_{m_{J'} m_J}$$

we find (exercise)

$$\boxed{\mathrm{Im}\,\mathcal{M}_J = \rho\, |\mathcal{M}_J|^2}$$

partial wave elastic unitarity condition

This is now an algebraic equation for $\mathcal{M}(s)$!

Analytic representations which satisfy the partial wave unitarity relation are called <u>on-shell representations</u>

For elastic scattering, at each energy E^*,
we need to describe two real numbers,

$$\mathcal{M}_J = \mathrm{Re}\,\mathcal{M}_J + i\,\mathrm{Im}\,\mathcal{M}_J$$

The unitarity condition relates $\mathrm{Im}\,\mathcal{M}$ to $\mathrm{Re}\,\mathcal{M}$.

Consider a polar representation,

$$\mathcal{M}_J(E^*) = |\mathcal{M}_J(E^*)|\, e^{i\delta_J(E^*)}$$

two real numbers at given E^*

But, unitarity condition,

$$\mathrm{Im}\,\mathcal{M}_J = |\mathcal{M}_J| \sin\delta_J = \rho\,|\mathcal{M}_J|^2$$

$$\Rightarrow \quad |\mathcal{M}_J| = \frac{\sin\delta_J}{\rho}$$

So,

$$\boxed{\mathcal{M}_J(E^*) = \frac{1}{\rho}\sin\delta_J(E^*)\, e^{i\delta_J(E^*)}}$$

Only 1 real function needed!
δ_J is called the <u>scattering phase shift</u>

The phase shift has the same interpretation as in NRQM scattering.

One can show (exercise)

$$\boxed{\mathcal{M}_J = \frac{1}{p}\,\frac{1}{\cot\delta_J - i}}$$

An alternative representation is the K-matrix form,

Consider

$$\operatorname{Im}\mathcal{M}_J = p\,|\mathcal{M}_J|^2$$

$$\Rightarrow \quad \frac{1}{|\mathcal{M}_J|^2}\operatorname{Im}\mathcal{M}_J = p \quad \Rightarrow \quad \operatorname{Im}\left(\frac{\mathcal{M}_J}{|\mathcal{M}_J|^2}\right) = p$$

Now, $\frac{z}{|z|^2} = \frac{1}{z^*}$, and $\operatorname{Im}\left(\frac{1}{z^*}\right) = -\operatorname{Im}(z^{-1})$

So, $$\boxed{\operatorname{Im}\mathcal{M}_J^{-1} = -p}$$

entirely known function

So,

$$\mathcal{M}_J^{-1} = \text{Real function} - ip$$
$$\equiv K^{-1} - ip$$

↳ K-matrix, real function

Inverting,

$$\boxed{\mathcal{M}_J = K_J \frac{1}{1 - ipK_J}}$$

this is non-perturbative!

Comparing to the phase shift,

$$K_J^{-1} = p \cot \delta_J$$

The K-matrix is a real function near the two-body scattering threshold. From scattering theory, we do not know the specific form. Some other input is needed, e.g., experiment, theory (QCD), ...

So, we can parametrize the function, and fit/constrain from some external data.

Threshold Behavior

The partial wave amplitudes contain kinematic singularities near threshold, $E^* \sim 2m$ ($k^* \to 0$).
The partial wave expansion,

$$\mathcal{M}(E^*, \hat{k}'^*, \hat{k}^*) = 4\pi \sum_J \mathcal{M}_J \sum_{m_J} Y_{Jm_J}(\hat{k}'^*)\, Y^*_{Jm_J}(\hat{k}^*)$$

As $k^* \to 0$, $\hat{k}^* \sim \hat{k}'^* \sim \frac{1}{k^*} \to \infty$

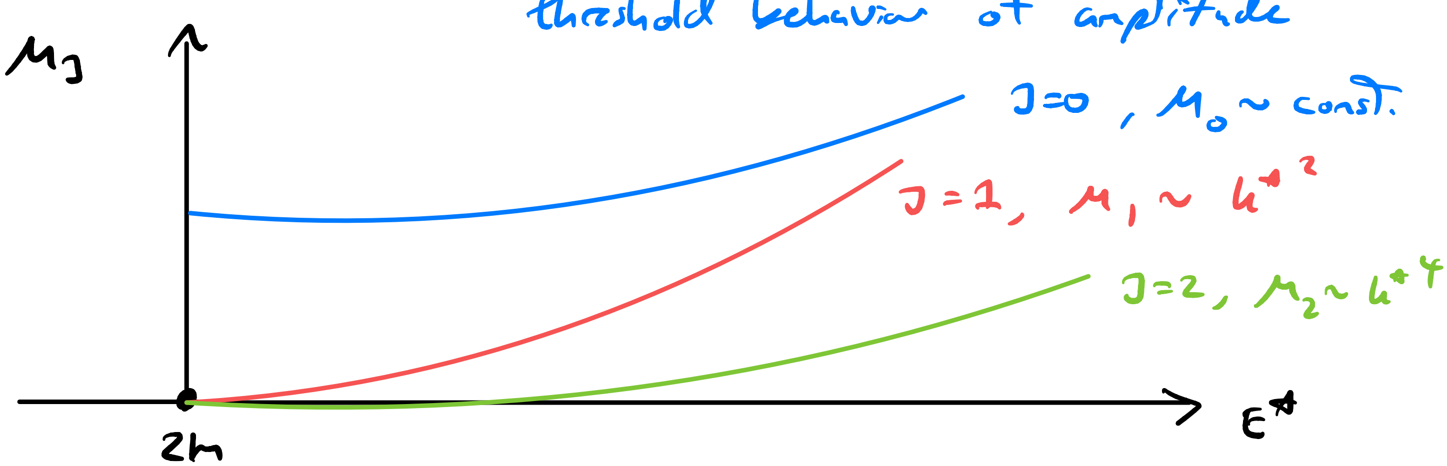
so, for fixed J,

$$\mathcal{M}(E^*, \hat{k}'^*, \hat{k}^*) \sim \mathcal{M}_J \frac{1}{(k^*)^{2J}}$$

To compensate the diverging behavior,

$$\boxed{\mathcal{M}_J \sim (k^*)^{2J} \quad \text{as} \quad k^* \to 0}$$

threshold behavior of amplitude



The threshold value for the S-wave scattering amplitude is related to the scattering length, a_0

$$\mathcal{M}_0 \sim a_0$$

$\hookrightarrow$ a measure of the "strength" of interaction

A common parametrization for low-energy scattering is the Effective Range expansion

$$\boxed{k^{*\,2J+1} \cot \delta_J = -\frac{1}{a_J} + \frac{1}{2} r_J k^{*2} + \mathcal{O}(k^{*4})}$$