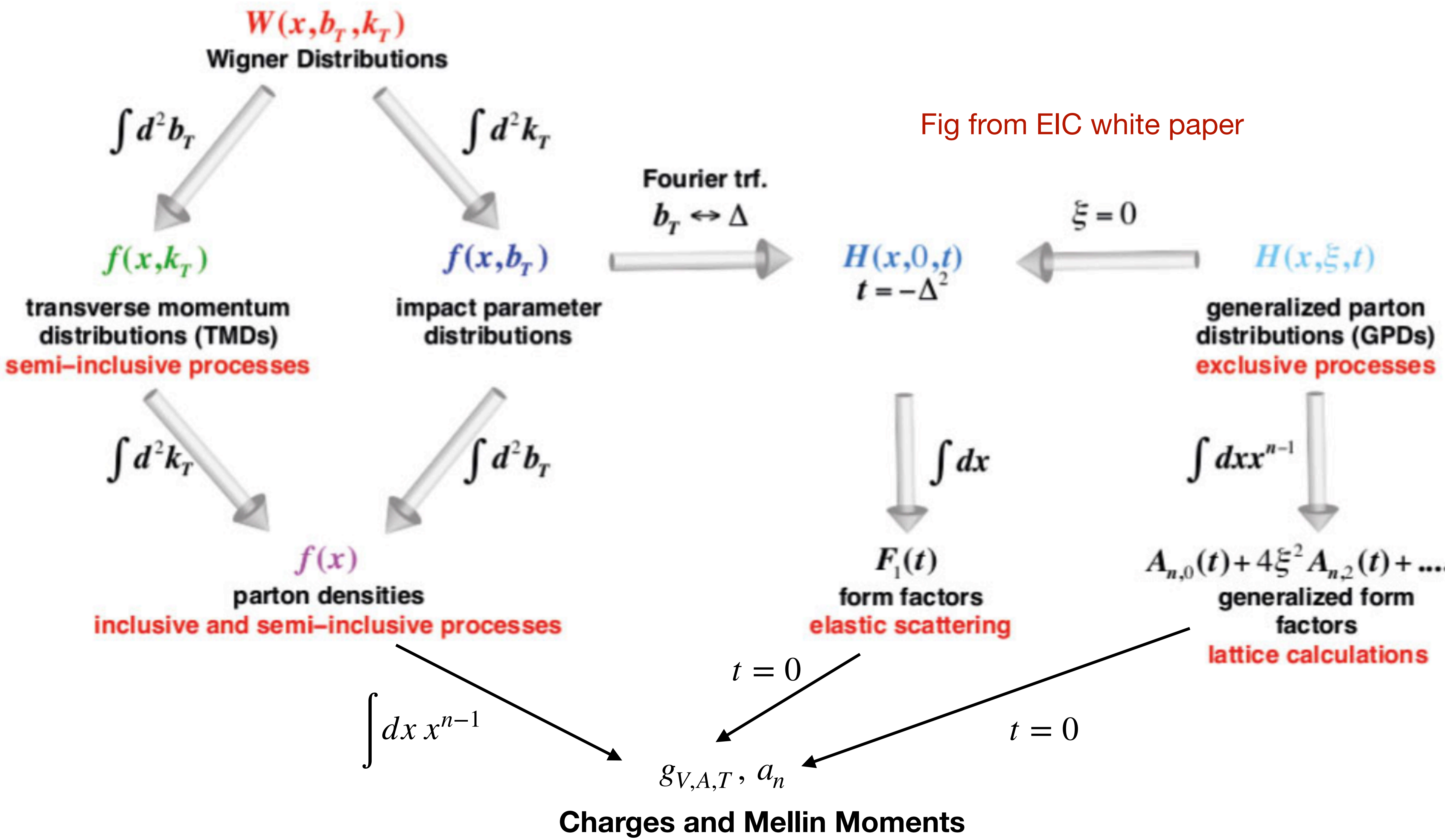
# Hadron Tomography from Lattice QCD

Joe Karpie

Jefferson Lab

# Overview of Objects in Hadron Structure

## Many ways to describe a hadron



# Parton Structure

For various flavors and spin combinations

Wigner Distribution/
Generalized Transverse Momentum Distribution (GTMD)

$F(x, b_T, k_T)$

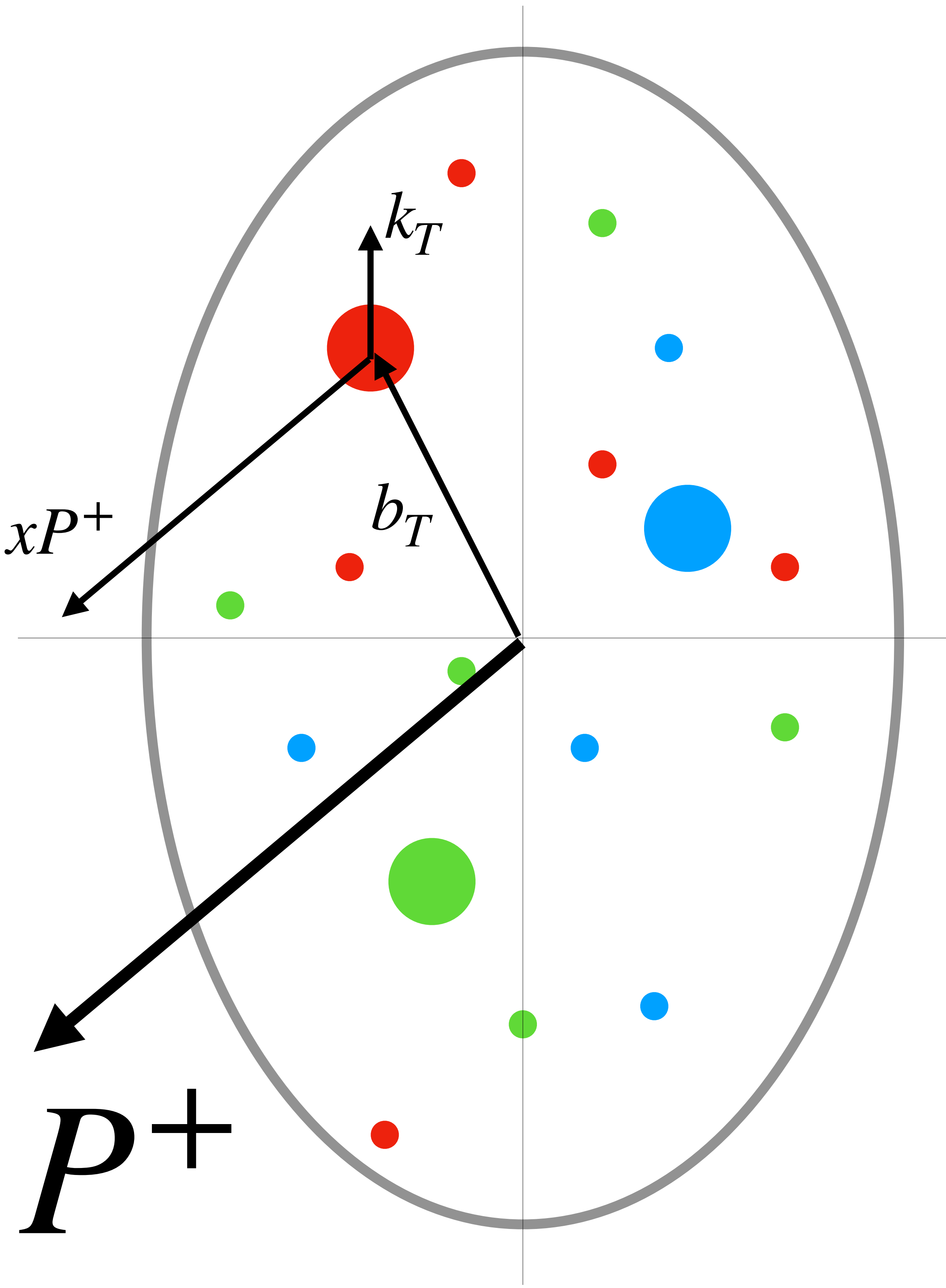
$\int d^2 b_t$ $\int d^2 k_t$

Transverse Momentum Distribution (TMD)
$f(x, k_T)$

Generalized Parton Distribution (GPD)
$f(x, b_T)$

$\int d^2 k_t$ $\int d^2 b_t$

Parton Distribution Function (PDF)
$f(x)$



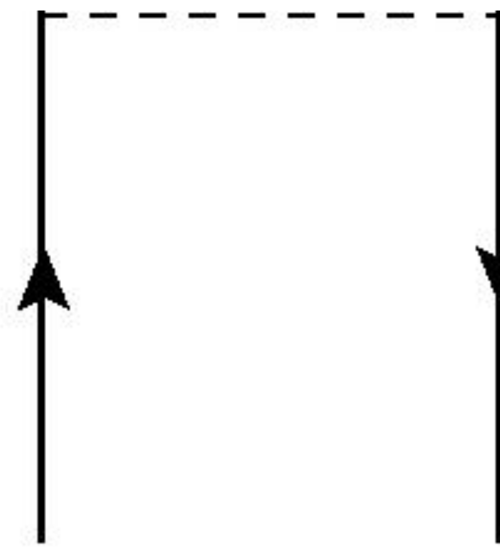
# Lattice Structure Review

- Matrix Elements from ratios of 3pt and 2pt functions at large Euclidean times
- Directly calculable matched the MS-bar scheme/scheme indep.
    - Charges
    - Form Factors
    - PDFs’ Mellin Moments
    - GPDs’ Mellin Moments
    - Ratios of TMDPDFs’ Mellin Moments
- Indirectly calculable from non-local operators needing factorization
    - PDFs and their moments
    - GPDs and their moments
    - TMDPDFs and the Collins-Soper Kernel

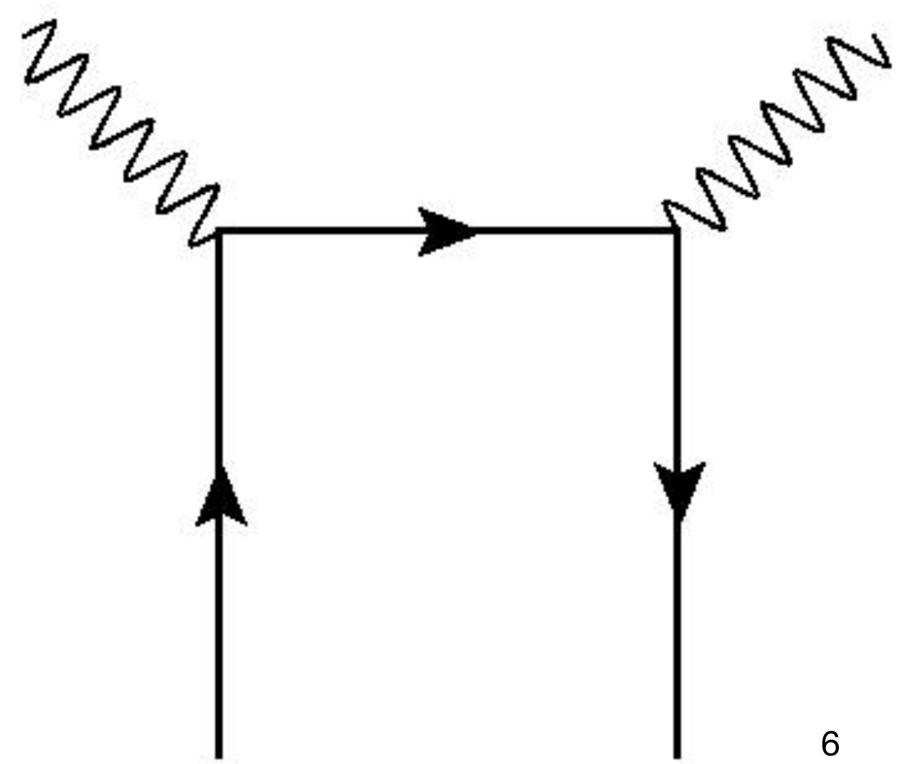
# Many non-local approaches

- **Wilson line operators**
  - LaMET X. Ji *Phys. Rev. Lett.* 110 (2013) 262002
  - Pseudo-PDF A. Radyushkin *Phys. Rev. D* 96 (2017) 3, 034025
- **Two current correlators**
  - Hadronic Tensor
    K.-F. Liu et al *Phys. Rev. Lett.* 72 1790 (1994)
    *Phys. Rev. D 62 (2000) 074501*
  - HOPE
    W. Detmold and C.-J. D. Lin, *Phys. Rev. D* 73 (2006) 014501
  - Short distance OPE
    V. Braun and D. Muller *Eur. Phys. J. C* 55 (2008) 349
  - OPE-without-OPE
    A. Chambers et al, *Phys. Rev. Lett.* 118 (2017) 242001
  - Good Lattice Cross Sections
    Y.-Q. Ma and J.-W. Qiu *Phys. Rev. Lett.* 120 (2018) 2, 022003

$$O_{WL}(x;z) = \bar{\psi}(x+z)\Gamma W(x+z;x)\psi(x)$$



$$O_{CC}(x,y) = J_{\Gamma}(x)J_{\Gamma'}(y)$$



# Wilson Line Matrix Elements

- In quasi-PDF/LaMET and pseudo-PDF/Short distance, separation and momentum swap roles

$$\nu = p \cdot z$$

  - Matrix element $M(p,z) = \langle p \,|\, \bar{\psi}(z)\gamma^\alpha W(z;0)\psi(0) \,|\, p\rangle$
  $= 2p^\alpha \mathscr{M}(\nu, z^2) + 2z^\alpha \mathscr{N}(\nu, z^2)$
  - Quasi-PDF: $\mathscr{M}(z, p_z) = \int_{-\infty}^{\infty} \frac{p_z dy}{2\pi} e^{iyp_z z}\, \tilde{q}(y, p_z^2)$ $\qquad z^2 \neq 0$
  - Large Momentum Effective Theory: X. Ji *Phys. Rev. Lett.* 110 (2013) 262002

$$\tilde{q}(y, p_z^2) = \int \frac{dx}{|x|} K\left(\frac{y}{x}, \frac{\mu^2}{(xp_z)^2}\right) q(x, \mu^2) + O\left(\frac{\Lambda_{\text{QCD}}^2}{(xp_z)^2}, \frac{\Lambda_{\text{QCD}}^2}{((1-x)p_z)^2}\right)$$

  - Pseudo-ITD:

$$\mathscr{M}(\nu, z^2) = \int dx\, C(x\nu, \mu^2 z^2) q(x, \mu^2) + O(\Lambda_{\text{QCD}}^2 z^2)$$

A. Radyushkin *Phys. Rev. D* 96 (2017) 3, 034025

# The Role of Separation and Momentum

- In quasi-PDF, pseudo-PDF, and Structure Functions, variables have common roles

**Scale:**

$p_z^2$ / $z^2$/$Q^2$

- Scale for factorization to PDF
- Scale in power expansion
- Keep away from $\Lambda_{\text{QCD}}^2$
- Technically only requires single value

**Dynamical variable:**

$z$ / $p_z$ , or $\nu = p \cdot z$ , $x_B$

- Variable describes non-perturbative dynamics
- Can take large or small value
- Want as many as are available
- Wider range improves the inverse problem

# Gluon Matrix Elements

- **General Matrix Element**

$$M^{\mu\alpha;\nu\beta}(z,p,s) = \langle p,s \,|\, \mathrm{Tr}\Big[F^{\mu\alpha}(z)W(z;0)F^{\nu\beta}(0)\Big] \,|\, p,s\rangle$$

  - Assume $z$ is along cardinal direction (eventually lattice axis)

- **Renormalization**

Z-Y. Li, Y-Q. Ma, J-W. Qiu.
*Phys. Rev. Lett.* 122 (2019) 6, 062002

  - Multiplicatively renormalizable
  - Depends on how many of $\mu, \nu, \rho, \sigma$ are in $z$ direction.

- Matrix element has complicated Lorentz decomposition in terms of $p^\mu, z^\mu, s^\mu$
  - Need to isolate amplitudes with leading twist contributions

# Spin Averaged matrix element

- Spin averaged combination $\mathcal{M}(\nu, z^3) = \frac{1}{2p_0^2}\left[M_{ti;it} + M_{ij;ij}\right]$
  - Gives **one** amplitude with leading twist contribution

$i, j = x, y$

I. Balitsky, W. Morris, A. Radyushkin Phys Rev D 105 (2022) 1, 014008
T. Khan et al (HadStruc) Phys. Rev. D 104 (2021) 9, 094516

# Lorentz decomposition

I. Balitsky, W. Morris, A. Radyushkin
PLB 808 (2020) 135621

- Many indices leads to complicated decomposition

Only term to contribute to light cone distribution's definition

$$M_{+i;i+}(\nu, z^2 = 0)$$

$$
\begin{aligned}
&M_{\mu\alpha;\lambda\beta}(z,p) = \\
&\left(g_{\mu\lambda}p_\alpha p_\beta - g_{\mu\beta}p_\alpha p_\lambda - g_{\alpha\lambda}p_\mu p_\beta + g_{\alpha\beta}p_\mu p_\lambda\right)\mathcal{M}_{pp} \\
&+\left(g_{\mu\lambda}z_\alpha z_\beta - g_{\mu\beta}z_\alpha z_\lambda - g_{\alpha\lambda}z_\mu z_\beta + g_{\alpha\beta}z_\mu z_\lambda\right)\mathcal{M}_{zz} \\
&+\left(g_{\mu\lambda}z_\alpha p_\beta - g_{\mu\beta}z_\alpha p_\lambda - g_{\alpha\lambda}z_\mu p_\beta + g_{\alpha\beta}z_\mu p_\lambda\right)\mathcal{M}_{zp} \\
&+\left(g_{\mu\lambda}p_\alpha z_\beta - g_{\mu\beta}p_\alpha z_\lambda - g_{\alpha\lambda}p_\mu z_\beta + g_{\alpha\beta}p_\mu z_\lambda\right)\mathcal{M}_{pz} \\
&+\left(p_\mu z_\alpha - p_\alpha z_\mu\right)\left(p_\lambda z_\beta - p_\beta z_\lambda\right)\mathcal{M}_{ppzz} \\
&+\left(g_{\mu\lambda}g_{\alpha\beta} - g_{\mu\beta}g_{\alpha\lambda}\right)\mathcal{M}_{gg} \,,
\end{aligned}
$$

- Isolate $\mathscr{M}_{pp}$ by choosing $M_{ti;it} + M_{ij;ji}$ where $i, j$ are summing over transverse $x, y$ only

# Spin Averaged matrix element

- Spin averaged combination $\mathscr{M}(\nu, z^2) = \frac{1}{2p_0^2}\left[M_{ti;it} + M_{ij;ij}\right]$

  $i, j = x, y$

  - Gives **one** amplitude with leading twist contribution

I. Balitsky, W. Morris, A. Radyushkin Phys Rev D 105 (2022) 1, 014008
T. Khan et al (HadStruc) Phys. Rev. D 104 (2021) 9, 094516

- Use **ratio** with finite continuum limit

$$\mathfrak{M}(\nu, z^2) = \frac{\mathscr{M}(p, z)\mathscr{M}(0,0)}{\mathscr{M}(p,0)\mathscr{M}(0,z)}$$

# Spin Averaged matrix element

- Spin averaged combination $\mathscr{M}(\nu, z^3) = \frac{1}{2p_0^2}\left[M_{ti;it} + M_{ij;ij}\right]$
  - Gives **one** amplitude with leading twist contribution

$i, j = x, y$

I. Balitsky, W. Morris, A. Radyushkin Phys Rev D 105 (2022) 1, 014008
T. Khan et al (HadStruc) Phys. Rev. D 104 (2021) 9, 094516

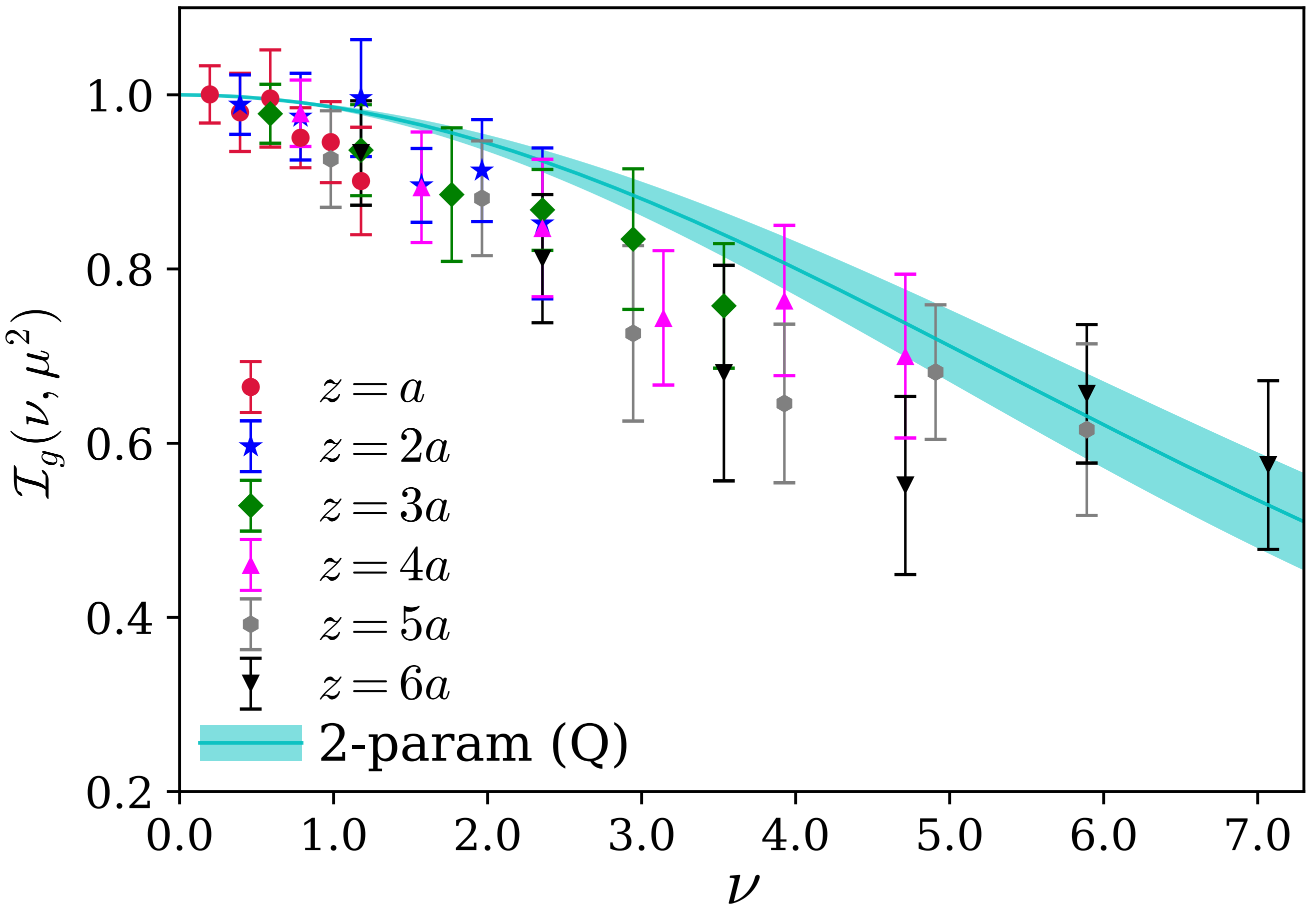
- Use **ratio** with finite continuum limit

$$\mathfrak{M}(\nu, z^2) = \frac{\mathscr{M}(p, z)\mathscr{M}(0,0)}{\mathscr{M}(p,0)\mathscr{M}(0,z)}$$

- Relation to **gluon** and **quark singlet** ITD

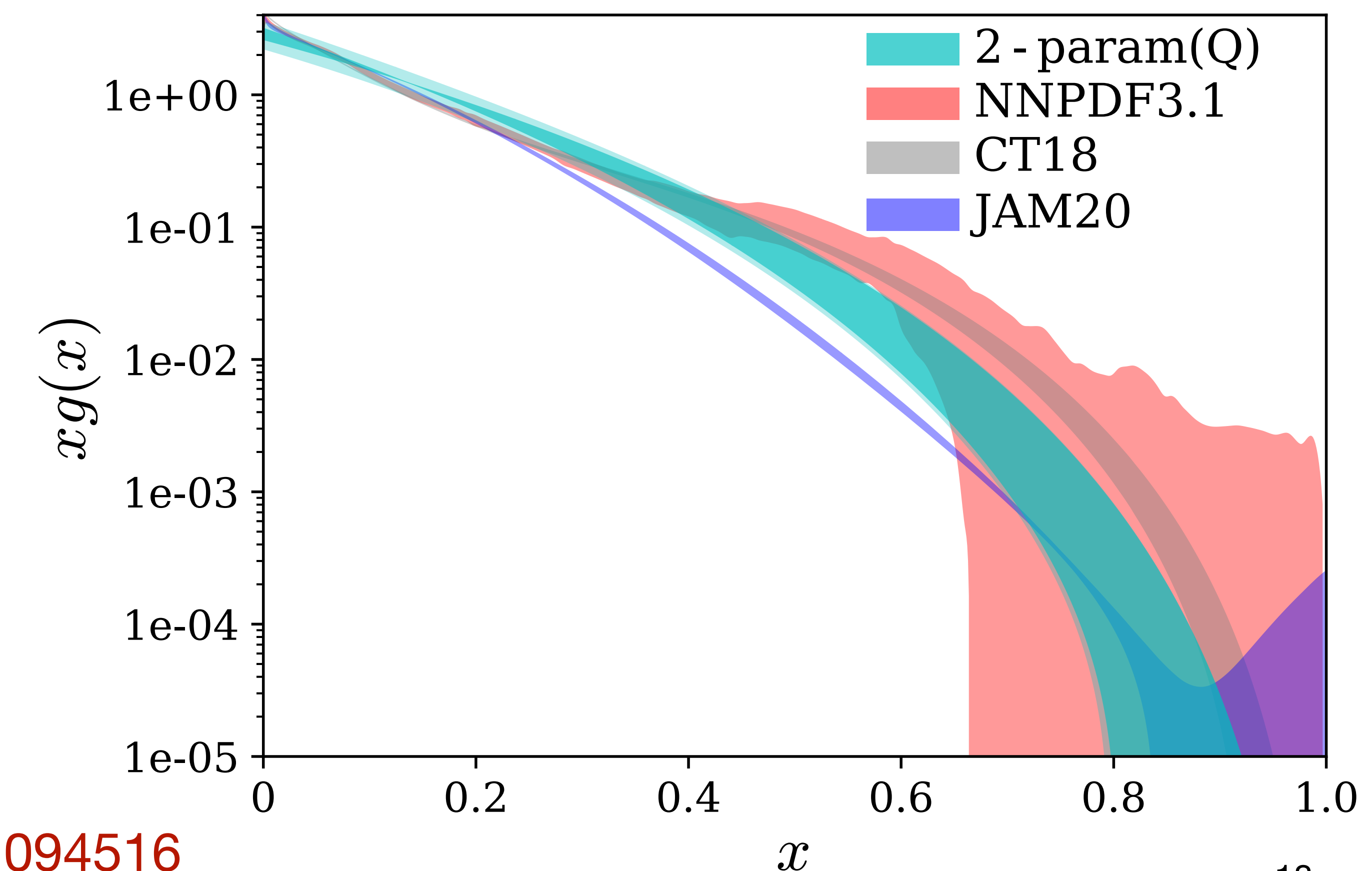
$$\langle x \rangle_g \mathfrak{M}(\nu, z^2) = \int_0^1 C^{gg}(u, \mu^2 z^2) I_g(u\nu, \mu^2) + C^{qg}(u, \mu^2 z^2) I_s(u\nu, \mu^2)$$

# Unpolarized Gluon PDF




$a = 0.094$ fm
$m_\pi = 358$ MeV

- ITD fit to cosine transform of
  $xg(x) = x^a(1-x)^b/B(a+1,b+1)$
- Qualitative agreement with global analysis

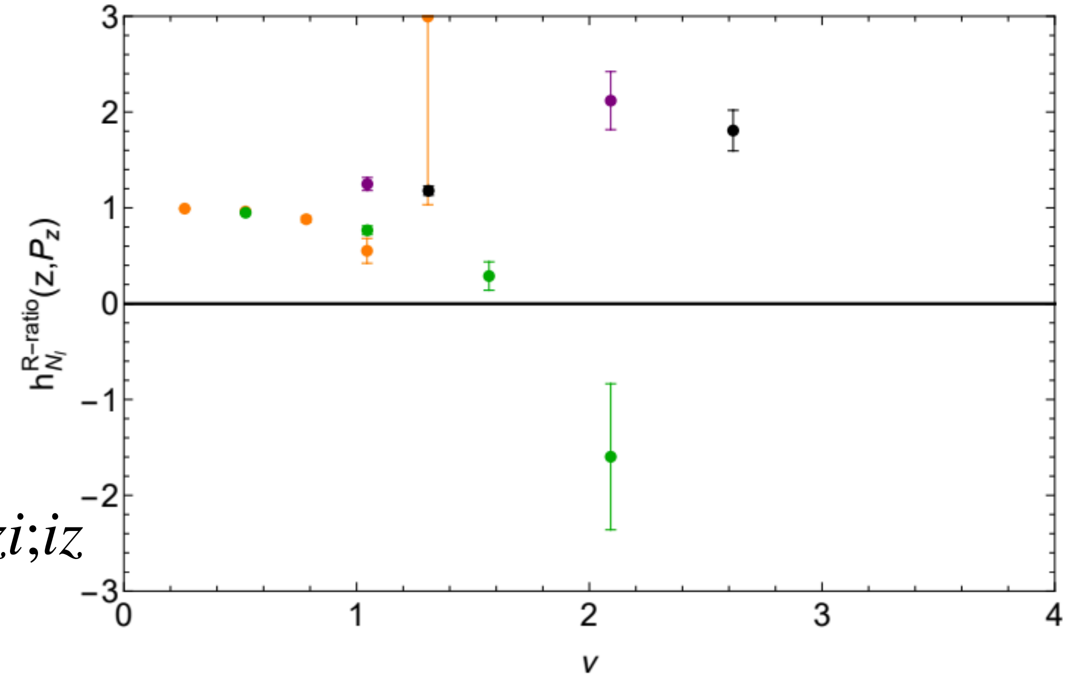



T. Khan et al (HadStruc) Phys. Rev. D 104 (2021) 9, 094516

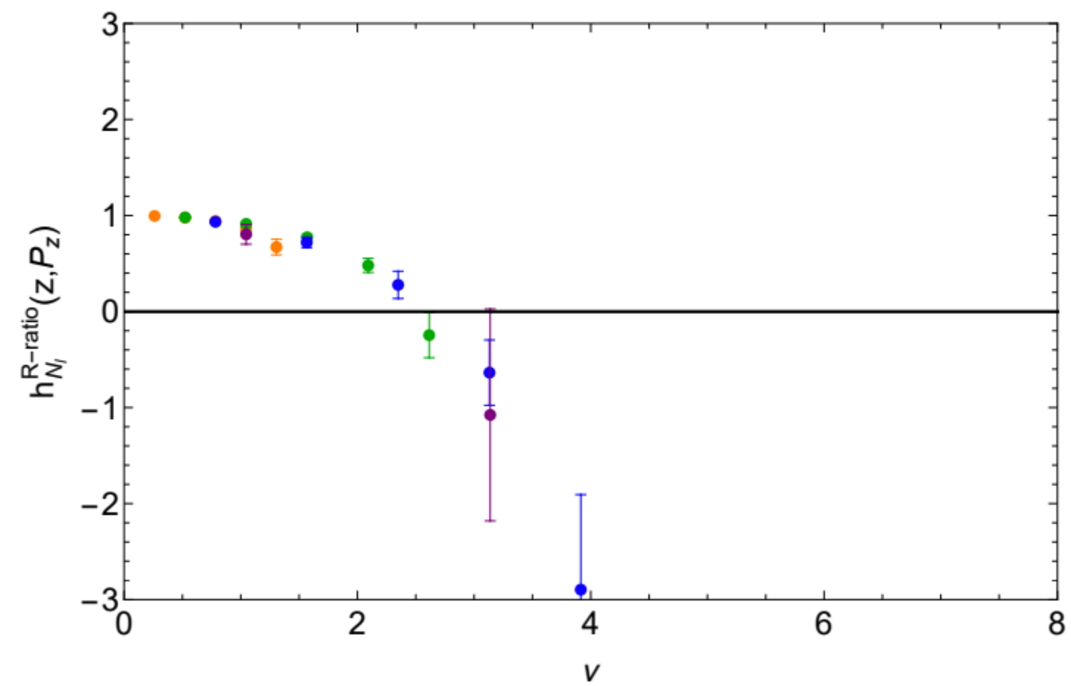
# What about other operators?

- All multiplicatively renormalizable
- In limits all approach light cone
- Clearly have different levels of contamination and signal
- Wilson3 smearing + Ratio Renorm.

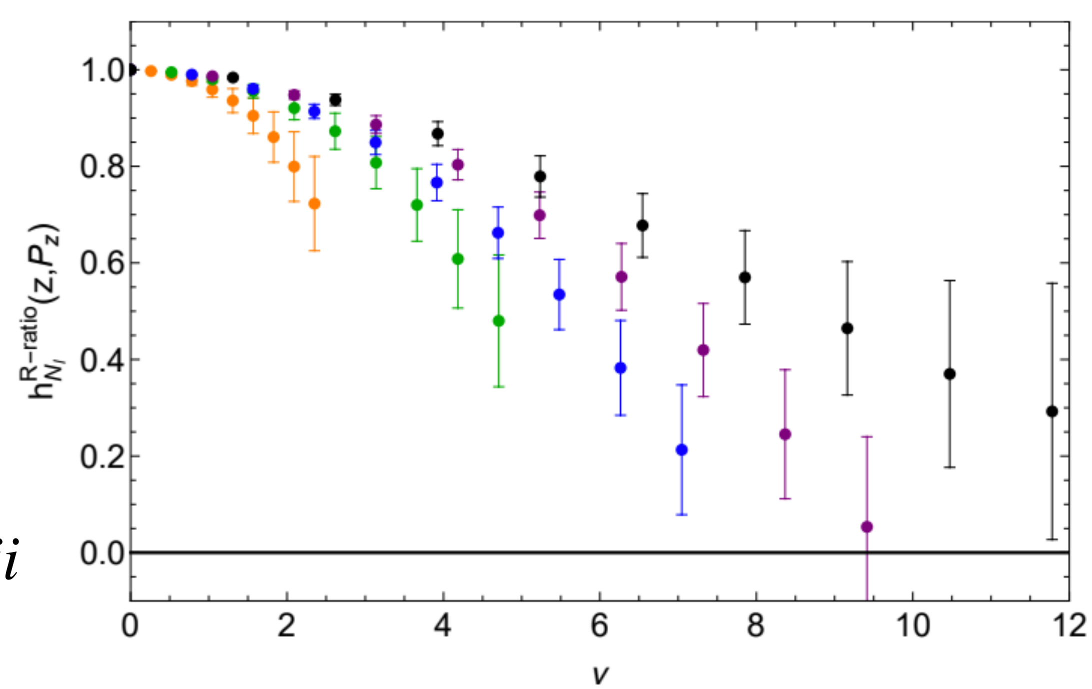
W. Good, K. Hasan, H.W. Lin 2409.02750




$M_{zi;iz}$




$$M_{z\mu;\mu z} = M_{zi;iz} + M_{zt;tz}$$




$M_{ti;it} + M_{ij;ji}$

# Helicity Gluon matrix element

I. Balitsky, W. Morris, A. Radyushkin JHEP 02 (2022) 193
C. Egerer et al (HadStruc) arXiv:2207.08733

- Helicity Gluon Matrix Element:

$$\widetilde{M}_{\mu\alpha;\nu\beta}(z,p,s) = \frac{1}{2}\epsilon_{\nu\beta\rho\sigma}M_{\mu\alpha;\rho\sigma} = \langle p,s\,|\,\mathrm{Tr}\big[F^{\mu\alpha}(z)W(z;0)\,\widetilde{F}^{\nu\beta}(0)\big]\,|p,s\rangle$$

- Useful Combination $\widetilde{\mathscr{M}}(z,p) = \big[\widetilde{M}_{ti;it} + \widetilde{M}_{ij;ij}\big]$
    - Gives **two** amplitudes, one has no leading twist contribution

# Lorentz decomposition

$$\widetilde{M}^{(2)}_{\mu\alpha;\lambda\beta}(z,p) = (sz)\big(g_{\mu\lambda}p_\alpha p_\beta - g_{\mu\beta}p_\alpha p_\lambda - g_{\alpha\lambda}p_\mu p_\beta + g_{\alpha\beta}p_\mu p_\lambda\big)\widetilde{\mathcal{M}}_{pp}$$
$$+(sz)\big(g_{\mu\lambda}z_\alpha z_\beta - g_{\mu\beta}z_\alpha z_\lambda - g_{\alpha\lambda}z_\mu z_\beta + g_{\alpha\beta}z_\mu z_\lambda\big)\widetilde{\mathcal{M}}_{zz}$$
$$+(sz)\big(g_{\mu\lambda}z_\alpha p_\beta - g_{\mu\beta}z_\alpha p_\lambda - g_{\alpha\lambda}z_\mu p_\beta + g_{\alpha\beta}z_\mu p_\lambda\big)\widetilde{\mathcal{M}}_{zp}$$
$$+(sz)\big(g_{\mu\lambda}p_\alpha z_\beta - g_{\mu\beta}p_\alpha z_\lambda - g_{\alpha\lambda}p_\mu z_\beta + g_{\alpha\beta}p_\mu z_\lambda\big)\widetilde{\mathcal{M}}_{pz}$$
$$+(sz)\big(p_\mu z_\alpha - p_\alpha z_\mu\big)\big(p_\lambda z_\beta - p_\beta z_\lambda\big)\widetilde{\mathcal{M}}_{ppzz}$$
$$+(sz)\big(g_{\mu\lambda}g_{\alpha\beta} - g_{\mu\beta}g_{\alpha\lambda}\big)\widetilde{\mathcal{M}}_{gg}$$

I. Balitsky, W. Morris, A. Radyushkin
JHEP 02 (2022) 193

$\tilde{M}_{+i;i+}(\nu, z^2 = 0)$

**Multiple terms contribute to light cone distribution's definition**

$$\widetilde{M}^{(1)}_{\mu\alpha;\lambda\beta}(z,p) = \Big(g_{\mu\lambda}s_\alpha p_\beta - g_{\mu\beta}s_\alpha p_\lambda - g_{\alpha\lambda}s_\mu p_\beta + g_{\alpha\beta}s_\mu p_\lambda\Big)\widetilde{\mathcal{M}}_{sp}$$
$$+\Big(g_{\mu\lambda}p_\alpha s_\beta - g_{\mu\beta}p_\alpha s_\lambda - g_{\alpha\lambda}p_\mu s_\beta + g_{\alpha\beta}p_\mu s_\lambda\Big)\widetilde{\mathcal{M}}_{ps}$$
$$+\Big(g_{\mu\lambda}s_\alpha z_\beta - g_{\mu\beta}s_\alpha z_\lambda - g_{\alpha\lambda}s_\mu z_\beta + g_{\alpha\beta}s_\mu z_\lambda\Big)\widetilde{\mathcal{M}}_{sz}$$
$$+\Big(g_{\mu\lambda}z_\alpha s_\beta - g_{\mu\beta}z_\alpha s_\lambda - g_{\alpha\lambda}z_\mu s_\beta + g_{\alpha\beta}z_\mu s_\lambda\Big)\widetilde{\mathcal{M}}_{zs}$$
$$+(p_\mu s_\alpha - p_\alpha s_\mu)(p_\lambda z_\beta - p_\beta z_\lambda)\widetilde{\mathcal{M}}_{pspz}$$
$$+(p_\mu z_\alpha - p_\alpha z_\mu)(p_\lambda s_\beta - p_\beta s_\lambda)\widetilde{\mathcal{M}}_{pzps}$$
$$+(s_\mu z_\alpha - s_\alpha z_\mu)(p_\lambda z_\beta - p_\beta z_\lambda)\widetilde{\mathcal{M}}_{szpz}$$
$$+(p_\mu z_\alpha - p_\alpha z_\mu)(s_\lambda z_\beta - s_\beta z_\lambda)\widetilde{\mathcal{M}}_{pzsz}$$

Want: $M_{\Delta g}(\nu, z^2) = \big[\widetilde{\mathcal{M}}^{(+)}_{sp} - \nu\widetilde{\mathcal{M}}_{pp}\big]$

Can get: $\widetilde{\mathcal{M}}(z,p) = \big[\widetilde{M}_{ti;it} + \widetilde{M}_{ij;ij}\big]$

$$= M_{\Delta g} - \frac{m^2 z^2}{\nu}\widetilde{\mathcal{M}}_{pp}$$

$$= M_{\Delta g} - \frac{m^2}{p_z^2}\nu\widetilde{\mathcal{M}}_{pp}$$

# Helicity Gluon matrix element

I. Balitsky, W. Morris, A. Radyushkin JHEP 02 (2022) 193
C. Egerer et al (HadStruc) arXiv:2207.08733

- Helicity Gluon Matrix Element:

$$\widetilde{M}_{\mu\alpha;\nu\beta}(z,p,s) = \frac{1}{2}\epsilon_{\nu\beta\rho\sigma}M_{\mu\alpha;\rho\sigma} = \langle p,s\,|\,\mathrm{Tr}\left[F^{\mu\alpha}(z)W(z;0)\,\widetilde{F}^{\nu\beta}(0)\right]|\,p,s\rangle$$

- Useful Combination $\widetilde{\mathcal{M}}(z,p) = \left[\widetilde{M}_{ti;it} + \widetilde{M}_{ij;ij}\right]$
    - Gives **two** amplitudes, one has no leading twist contribution
- Use ratio with finite continuum limit

$$\widetilde{\mathfrak{M}}(\nu,z^2) = i\frac{\left[\widetilde{\mathcal{M}}(z,p)/p_z p_0\right]/Z_L(z/a)}{\mathcal{M}(0,z^2)/m^2}$$

# Helicity Gluon matrix element

I. Balitsky, W. Morris, A. Radyushkin JHEP 02 (2022) 193
C. Egerer et al (HadStruc) arXiv:2207.08733

- Helicity Gluon Matrix Element:

$$\widetilde{M}_{\mu\alpha;\nu\beta}(z,p,s) = \frac{1}{2}\epsilon_{\nu\beta\rho\sigma}M_{\mu\alpha;\rho\sigma} = \langle p,s|\,\mathrm{Tr}\big[F^{\mu\alpha}(z)W(z;0)\,\widetilde{F}^{\nu\beta}(0)\big]\,|p,s\rangle$$

- Useful Combination $\widetilde{\mathscr{M}}(z,p) = \big[\widetilde{M}_{ti;it} + \widetilde{M}_{ij;ij}\big]$
  - Gives **two** amplitudes, one has no leading twist contribution
- Use ratio with finite continuum limit

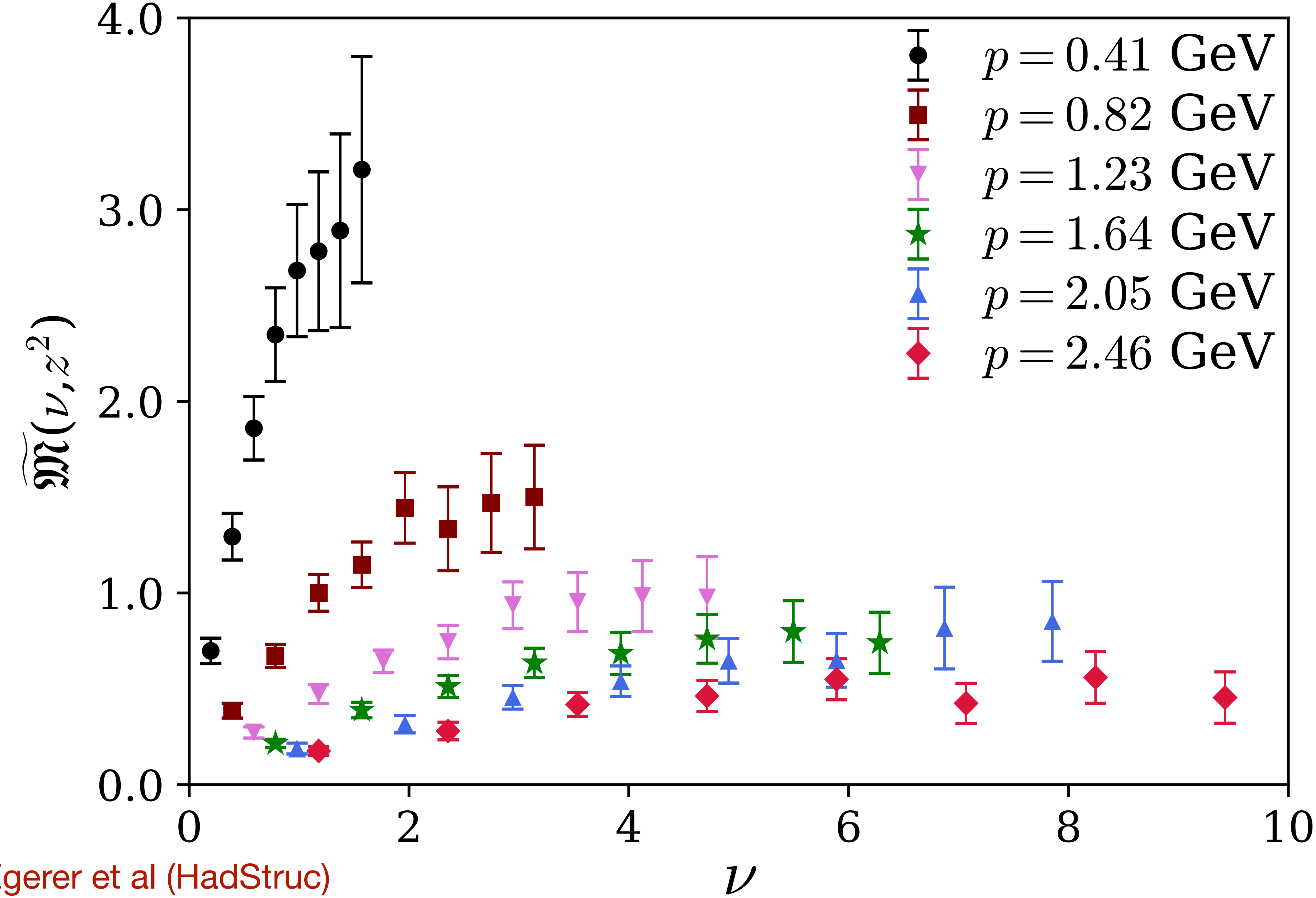
$$\widetilde{\mathfrak{M}}(\nu,z^2) = i\frac{\big[\widetilde{\mathscr{M}}(z,p)/p_z p_0\big]/Z_L(z/a)}{\mathscr{M}(0,z^2)/m^2}$$

- Relation to gluon and quark singlet ITD

$$\langle x\rangle_g \widetilde{\mathfrak{M}}(\nu,z^2) = \int_0^1 \widetilde{C}^{gg}(u,\mu^2z^2)\widetilde{I}_g(u\nu,\mu^2) + \widetilde{C}^{qg}(u,\mu^2z^2)\widetilde{I}_s(u\nu,\mu^2)$$

# Helicity Gluon Matrix Element

- Large contamination from $\frac{m^2}{p_z^2}\nu\widetilde{\mathscr{M}}_{pp}$ will need to be removed



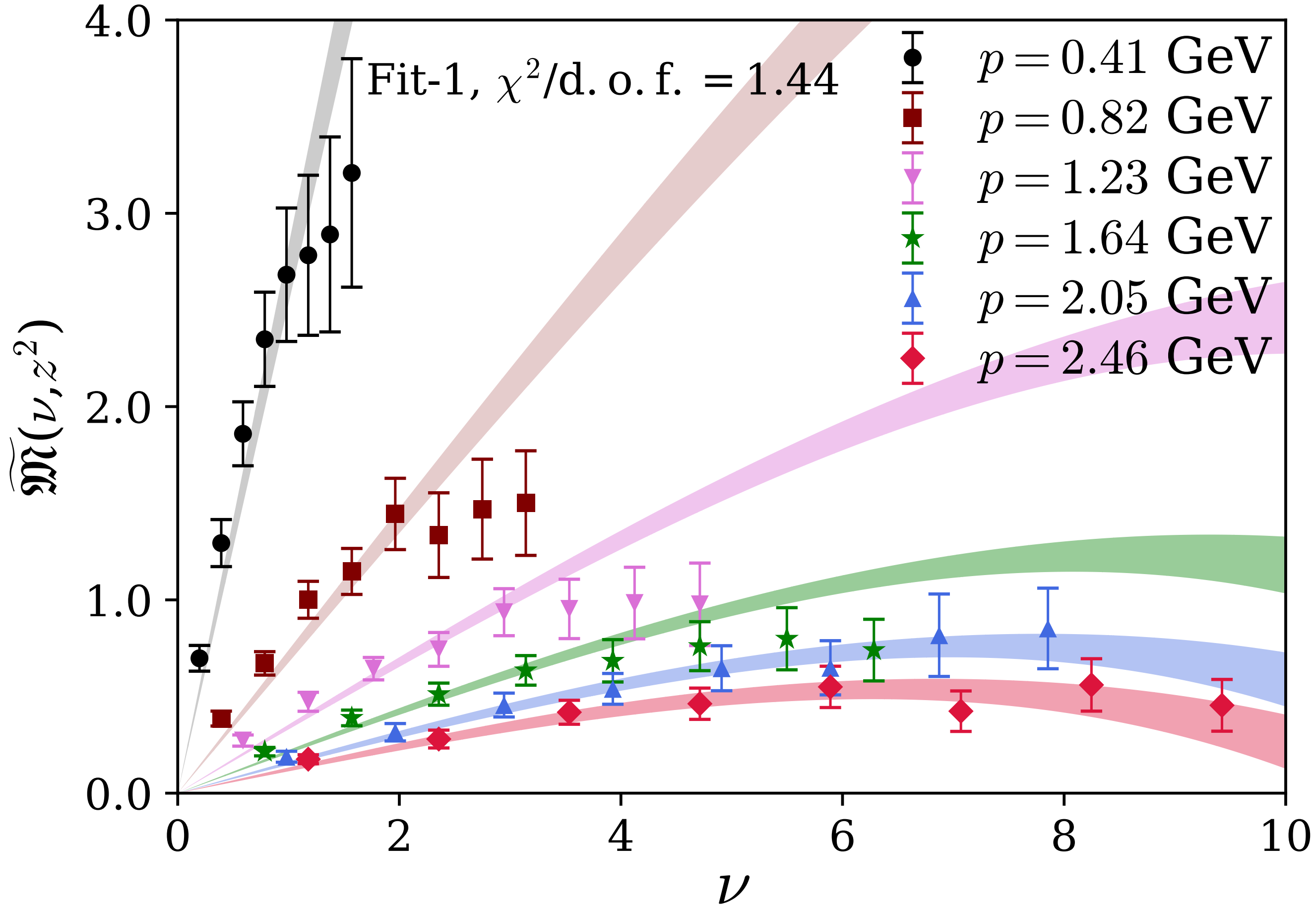

$a = 0.094$ fm

$m_\pi = 358$ MeV

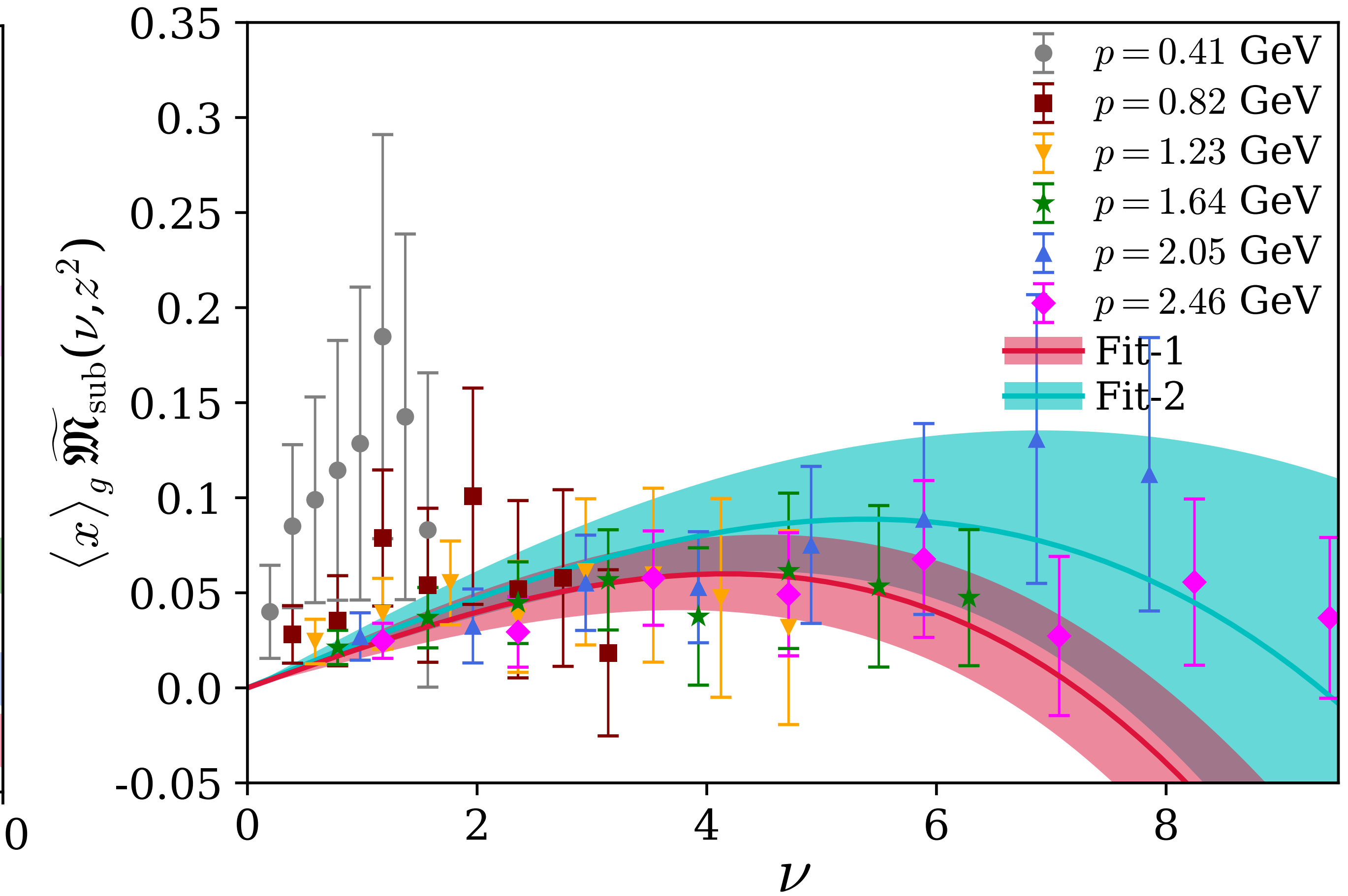
C. Egerer et al (HadStruc)
arXiv: 2207.08733

# Correcting Helicity Gluon Results

- Model both terms




- Subtract rest frame



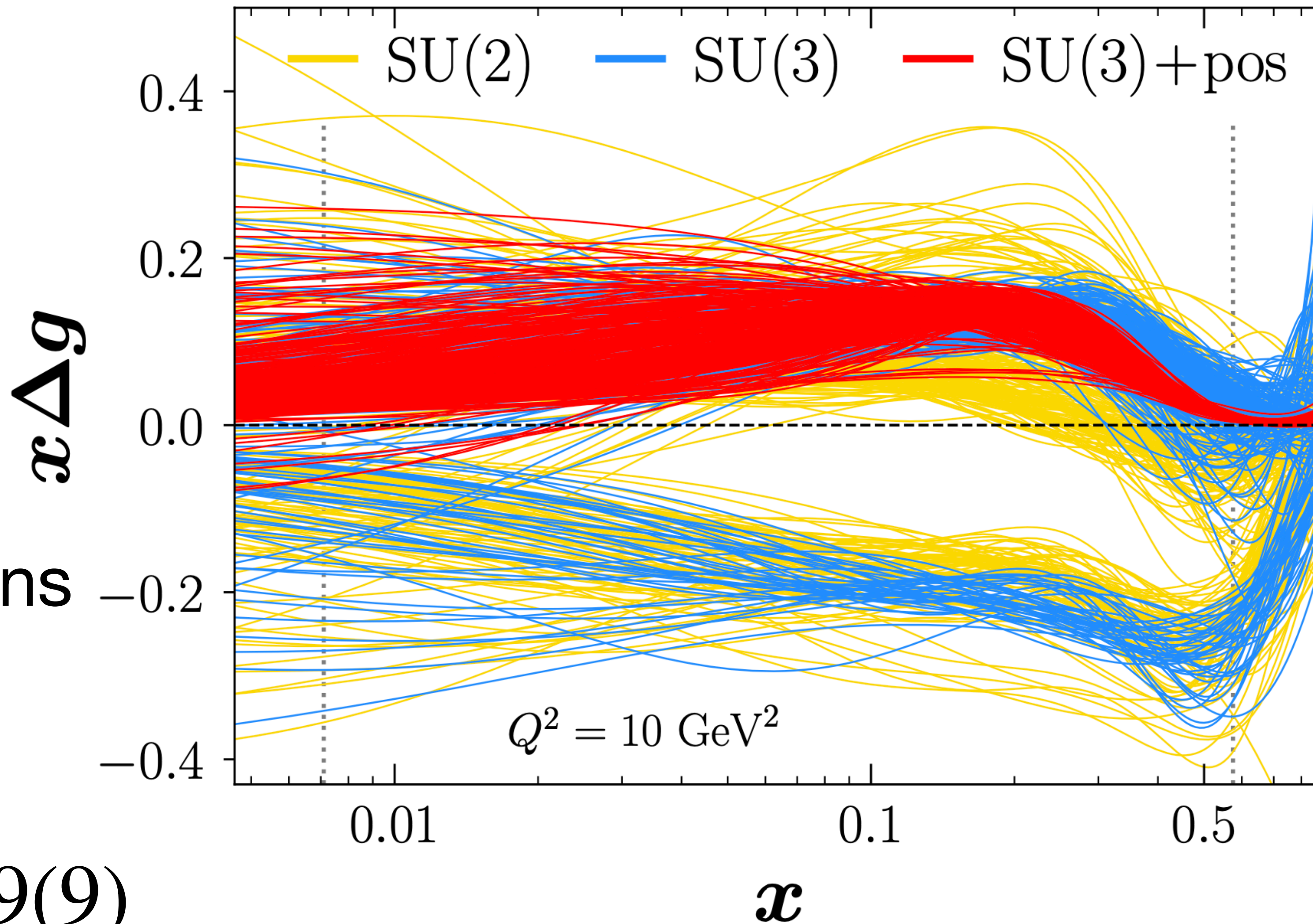

$a = 0.094$ fm

$m_\pi = 358$ MeV

C. Egerer et al (HadStruc) arXiv: 2207.08733

# Spinning gluons

Y. Zhou et al (JAM) Phys. Rev. D 105, 074022 (2022)




- Positivity removed from JAM helicity gluon PDF

$$g_\uparrow(x), g_\downarrow(x) > 0 \rightarrow |\Delta g| \leq g(x)$$

  - Reveals new band of solutions

- With constraint: $\Delta G = 0.39(9)$

R. Jaffe and A. Manohar, Nucl. Phys. B 337, 509 (1990)

- Without constraint: $\Delta G = 0.3(5)$

$$J = \frac{1}{2}\Delta\Sigma + L_q + L_G + \Delta G$$

- Lattice: $\Delta G = 0.251(47)(16)$

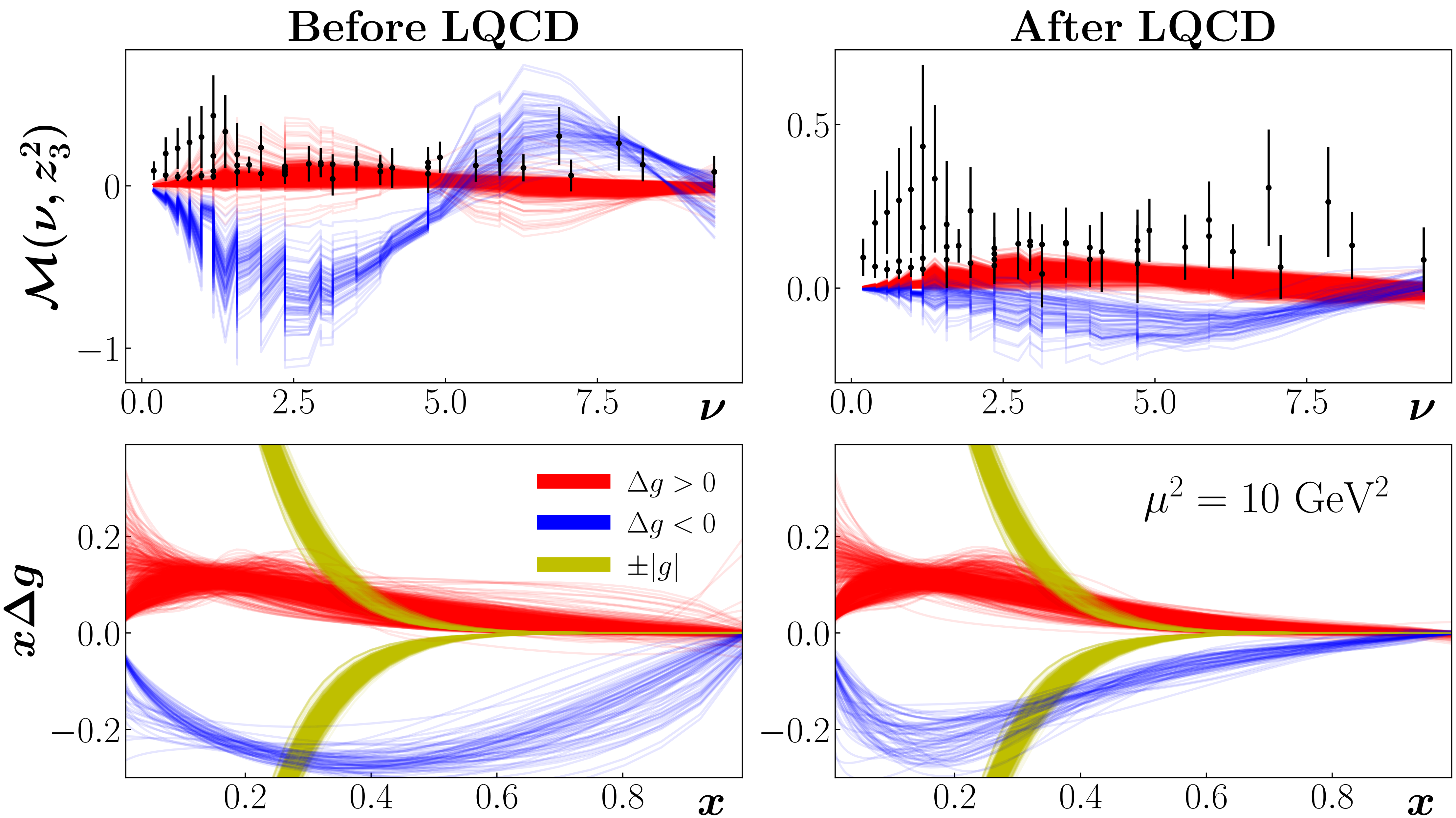
$$\Delta G = \int dx\, \Delta g(x)$$

Y-B. Yang et al ( $\chi$-QCD ) Phys. Rev. Lett. 118, 102001 (2017)
K-F. Liu arXiv: 2112.08416

# Spinning gluons

Y. Zhou et al Phys. Rev. D 105, 074022 (2022)
C. Egerer et al (HadStruc) arXiv:2207.08733

Can lattice data affect phenomenological polarized gluon analysis?



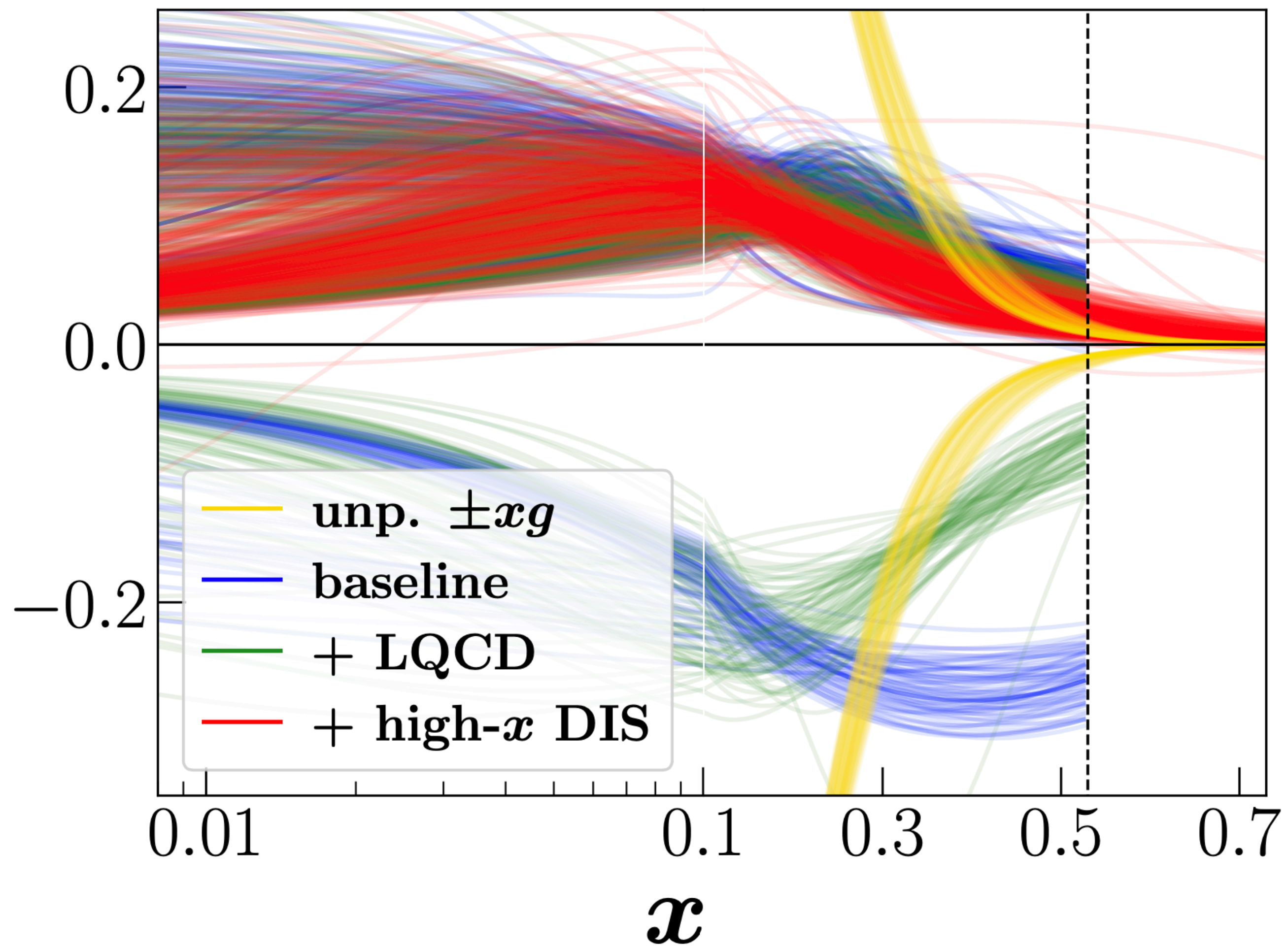

- The positive and negative solutions without positivity constraints
- Only positive band consistent with lattice data, but is too noisy to constrain.

$$\Delta G = \int d\nu\, I_g(\nu)$$

# Resolution of the helicity sign

- Rejection of negative helicity gluon PDF requires
  - RHIC Spin Asymmetries
    - Linear and quadratic in $\Delta g$
  - Lattice QCD matrix element
    - Linear in $\Delta g$
  - JLab high-x DIS from relaxing cuts on Final state mass
    - Linear in $\Delta g$
    - $W^2 > 10\,\mathrm{GeV}^2 \to W^2 > 4\,\mathrm{GeV}^2$

N.T. Hunt-Smith et al arXiv:2403.08117



# Pause for Gluon review

- Gluons are important contributions to spin physics
- Gluon matrix elements are disconnected and very noisy
- Identifying correct Lorentz invariant structures is critical to approaching the light cone

# Generalized Parton Distributions

- Generalized Ioffe time distributions

$$z^2 = 0$$

$$I^\mu(p', p, z = z^-, \mu^2) = \langle p' | \bar{q}\left(-\frac{z^-}{2}\right)\gamma^\mu W\left(-\frac{z^-}{2}, \frac{z^-}{2}\right) q\left(\frac{z^-}{2}\right) | p \rangle_{\mu^2}$$

**Ioffe Time**

$$\nu = \frac{p + p'}{2} \cdot z = P \cdot z$$

**Momentum Transfer**

**squared**

$$t = (p' - p)^2 = q^2$$

**Skewness**

$$\xi = \frac{q \cdot z}{P \cdot z}$$

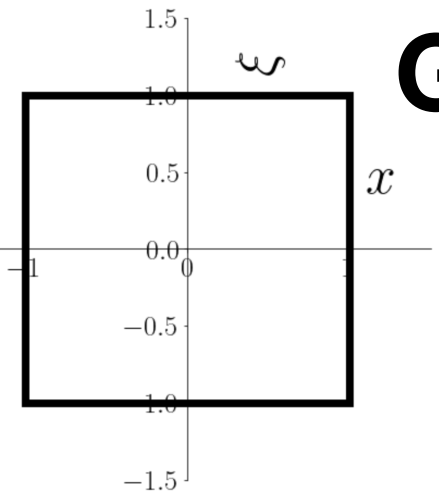
- Generalized Parton Distributions

$$\sigma^{\mu a} = \sigma^{\mu\nu} a_\nu$$

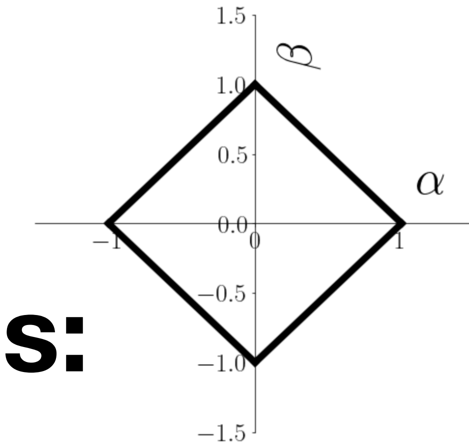
$$\int \frac{d\nu}{2\pi} e^{-i\nu x} z_\mu I^\mu(\nu, t, \xi, \mu^2) = H(x, t, \xi)\, \bar{u}' \not{z} u + E(x, t, \xi)\, \bar{u}' \frac{i\sigma^{zq}}{2m} u$$

X. Ji PRL 78 (1997) 610-613

# Two Faced Distributions




**GPDs:** ⟷ **Radon Transform** ⟷ **DDs:**



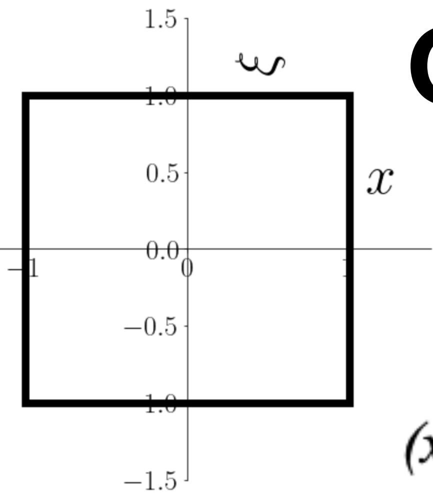

$$f(x,\xi) = \int_{-1}^{1} d\beta \int_{-1+|\beta|}^{1-|\beta|} d\alpha\, \delta(x-\beta-\xi\alpha)\tilde{f}(\alpha,\beta)$$

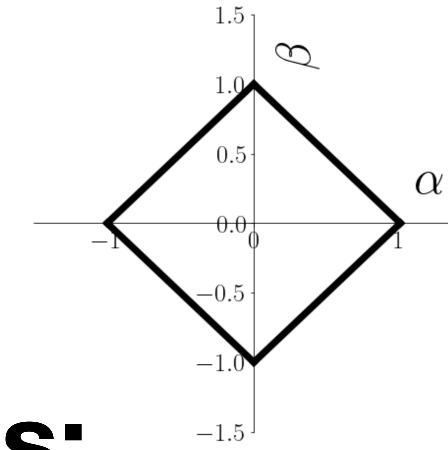
- Interpretation: average/change in parton momentum fraction
- Mellin moments give Form Factors and Angular Momentum decomposition
- Complex interrelation of variables
  - ERBL/DGLAP regions and polynomiality

Statue of Janus Bifrons (Wikipedia)

- Interpretation: Hybridize PDFs/DAs
- $\beta$ acts like PDF $x$
- $\alpha$ acts like DA $x$
- GPD evolution and polynomiality arise naturally from parameterized DD

# Two Faced Distributions

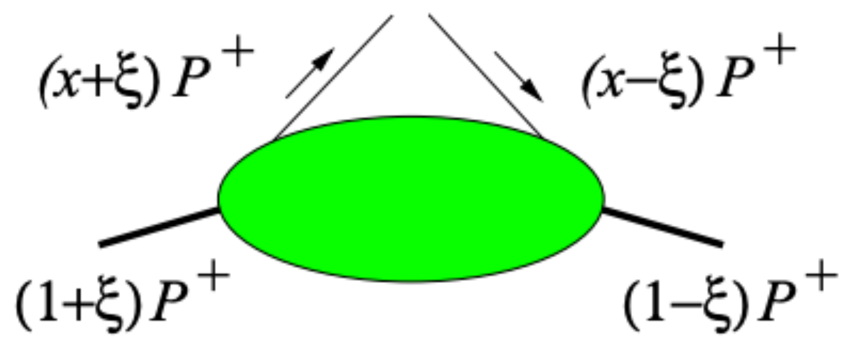


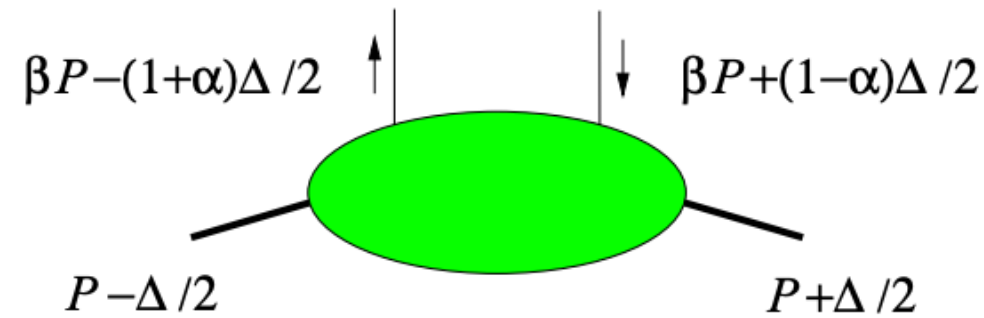




**GPDs:** ⟷ **DDs:**

Radon Transform

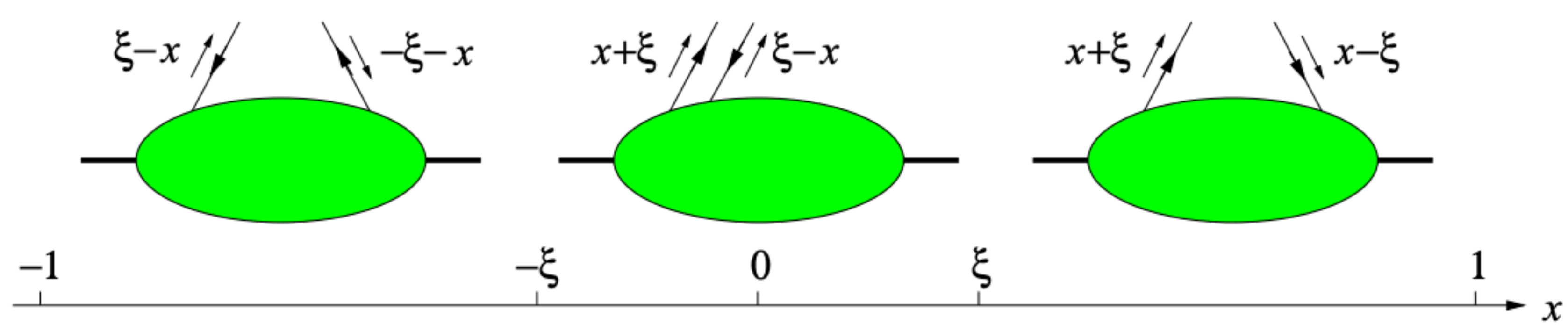
$$f(x,\xi) = \int_{-1}^{1} d\beta \int_{-1+|\beta|}^{1-|\beta|} d\alpha\, \delta(x - \beta - \xi\alpha)\tilde{f}(\alpha,\beta)$$




Figs from ``Generalized Parton Distributions”
M. Diehl arXiv:0307382

- Interpretations depend on kinematics
- $\xi$ is an externally measured variable




**anti-quark** **$q\bar{q}$ correlation** **quark**

- Break momenta into flow through operator (s-channel) and flow out of operators (t-channel)
- Sy

# Polynomiality of Mellin moments

- Polynomiality of moments $$\int dx x^n H(x,\xi;t) = \sum_{k=0,even}^{n} \xi^k A_{n,k}(t)$$

- Double Distribution is even in $\alpha$

$$\int dx x^n H(x,\xi;t) = \int_{-1}^{1} d\beta \int_{-1+|\beta|}^{1-|\beta|} d\alpha\, (\beta - \xi\alpha)^n h(\alpha,\beta;t)$$

- Ioffe-time Fourier Transform of $\delta(x-\beta-\xi\alpha)e^{i\nu x}$

$$A_1(\nu,\xi\nu;t) = \int_{-1}^{1} d\beta e^{i\nu\beta} \int_{-1+|\beta|}^{1-|\beta|} d\alpha \cos(\alpha\xi\nu) h(\alpha,\beta;t)$$

# Symmetries of the lattice

## Continuum rotation vs Lattice rotation

Continuous symmetry $O(4)$

Infinite number of Irreducible Representations (irreps) labeled by integers/half integers called spin

Spin is conserved since different irreps don't mix

Discrete and Finite symmetry $H(4)$

Hypercube symmetry group has 192 Elements with 13 irreps

Each irrep has contributions from many, but not all, spins

# Gravitational FFs

D. Hackett, D. Pefkou, P. Shanahan PRL 132, 251904 (2024)

- Energy Momentum tensor

$$\hat{T}_g^{\mu\nu} = 2\mathrm{Tr}\left[-F^{\mu\alpha}F_{\alpha}^{\nu} + \frac{1}{4}g^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta}\right],$$

$$\hat{T}_q^{\mu\nu} = \sum_f [i\bar{\psi}_f D^{\{\mu}\gamma^{\nu\}}\psi_f],$$

- Lorentz Decomposition

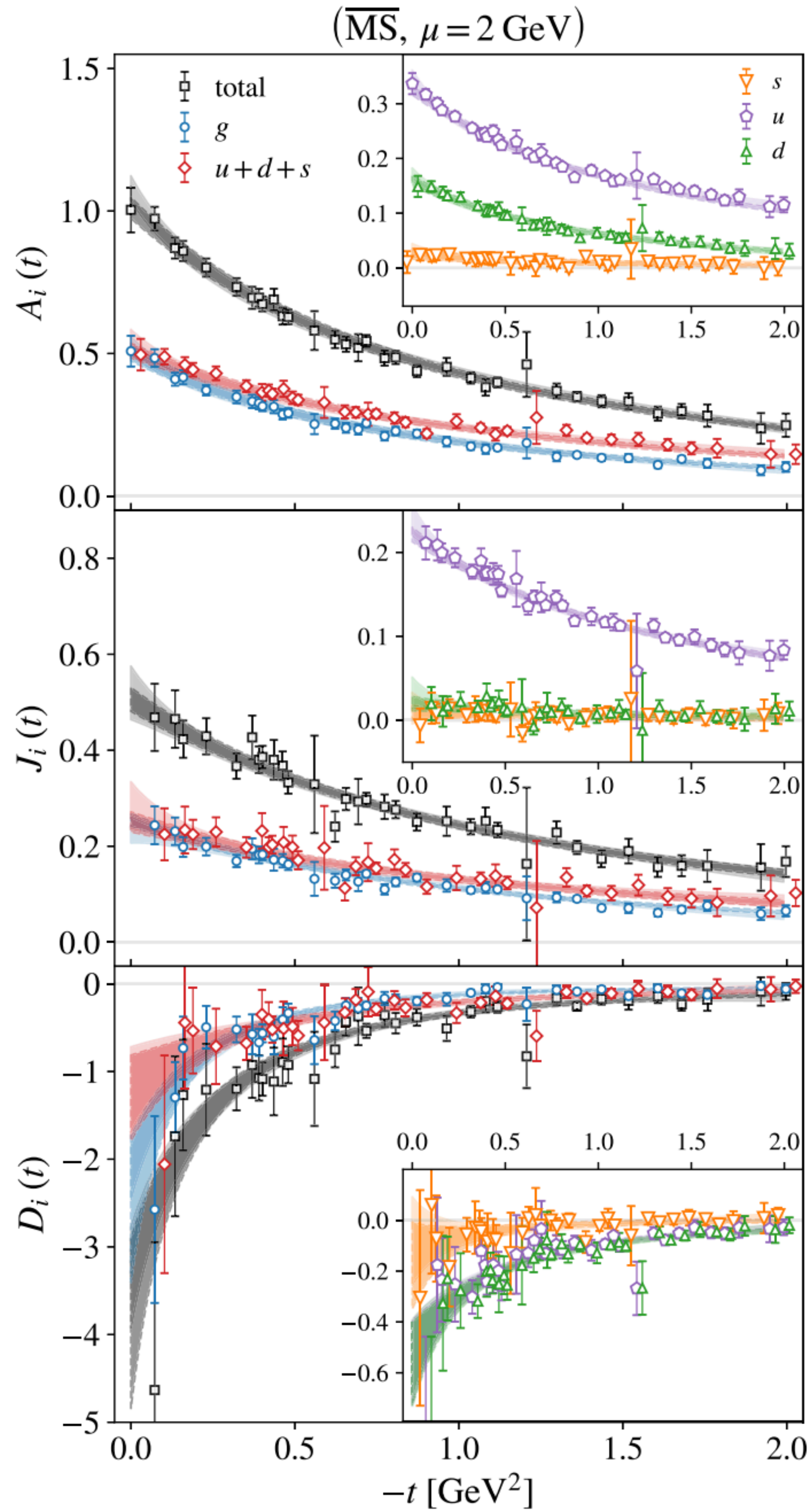
$$\langle N(\mathbf{p}', s')|\hat{T}^{\mu\nu}|N(\mathbf{p}, s)\rangle = \frac{1}{m}\bar{u}(\mathbf{p}', s')\Big[P^{\mu}P^{\nu}A(t) + iP^{\{\mu}\sigma^{\nu\}\rho}\Delta_{\rho}J(t) + \frac{1}{4}(\Delta^{\mu}\Delta^{\nu} - g^{\mu\nu}\Delta^2)D(t)\Big]u(\mathbf{p}, s),$$

- Gives Mellin moments of GPDs

$$A(t) = \int dx x H(x, \xi = 0, t)$$

$$D(t) = \int dx x D(x, t)$$

$$J(t) = \int dx x [H + E](x, \xi = 0, t)$$



# Gravitational FFs

$$F_1(t) = \int d^3b e^{-iq\cdot b}\tilde{F}_1(b)$$

D. Hackett, D. Pefkou, P. Shanahan PRL 132, 251904 (2024)

$$\langle r^2\rangle_{\text{charge}} = \frac{\int d^3b\, b^2 \tilde{F}_1(b)}{\int d^3b\, \tilde{F}_1(b)} = \frac{F_1'(t)}{F_1(0)}$$

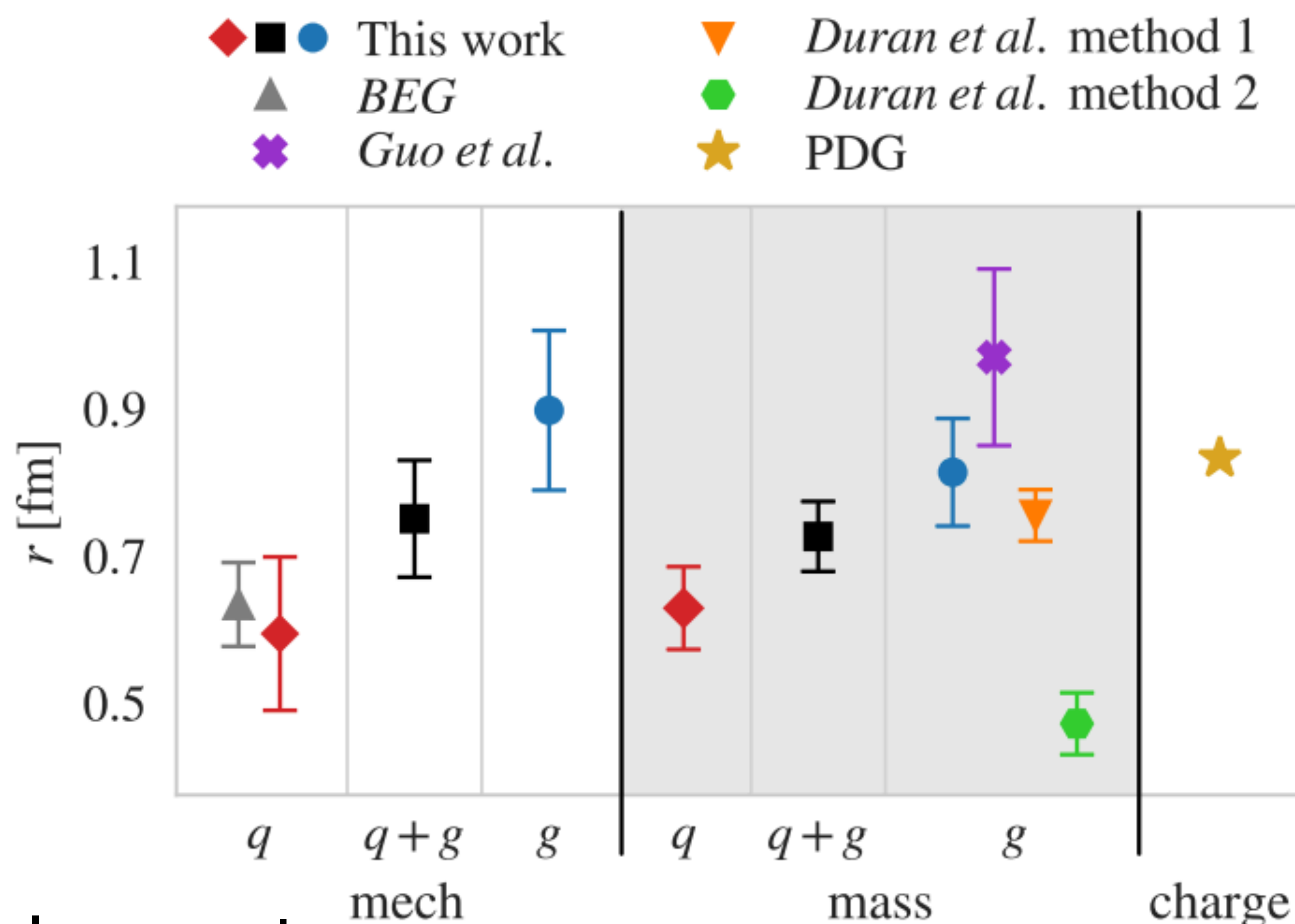
- Energy Momentum tensor

$$\hat{T}_g^{\mu\nu} = 2\text{Tr}\left[-F^{\mu\alpha}F^{\nu}_{\alpha} + \frac{1}{4}g^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta}\right],$$

$$\hat{T}_q^{\mu\nu} = \sum_f [i\bar{\psi}_f D^{\{\mu}\gamma^{\nu\}}\psi_f],$$

- Lorentz Decomposition

$$\langle N(\mathbf{p}', s')|\hat{T}^{\mu\nu}|N(\mathbf{p}, s)\rangle = \frac{1}{m}\bar{u}(\mathbf{p}', s')\left[P^\mu P^\nu A(t) + iP^{\{\mu}\sigma^{\nu\}\rho}\Delta_\rho J(t) + \frac{1}{4}(\Delta^\mu\Delta^\nu - g^{\mu\nu}\Delta^2)D(t)\right]u(\mathbf{p}, s),$$



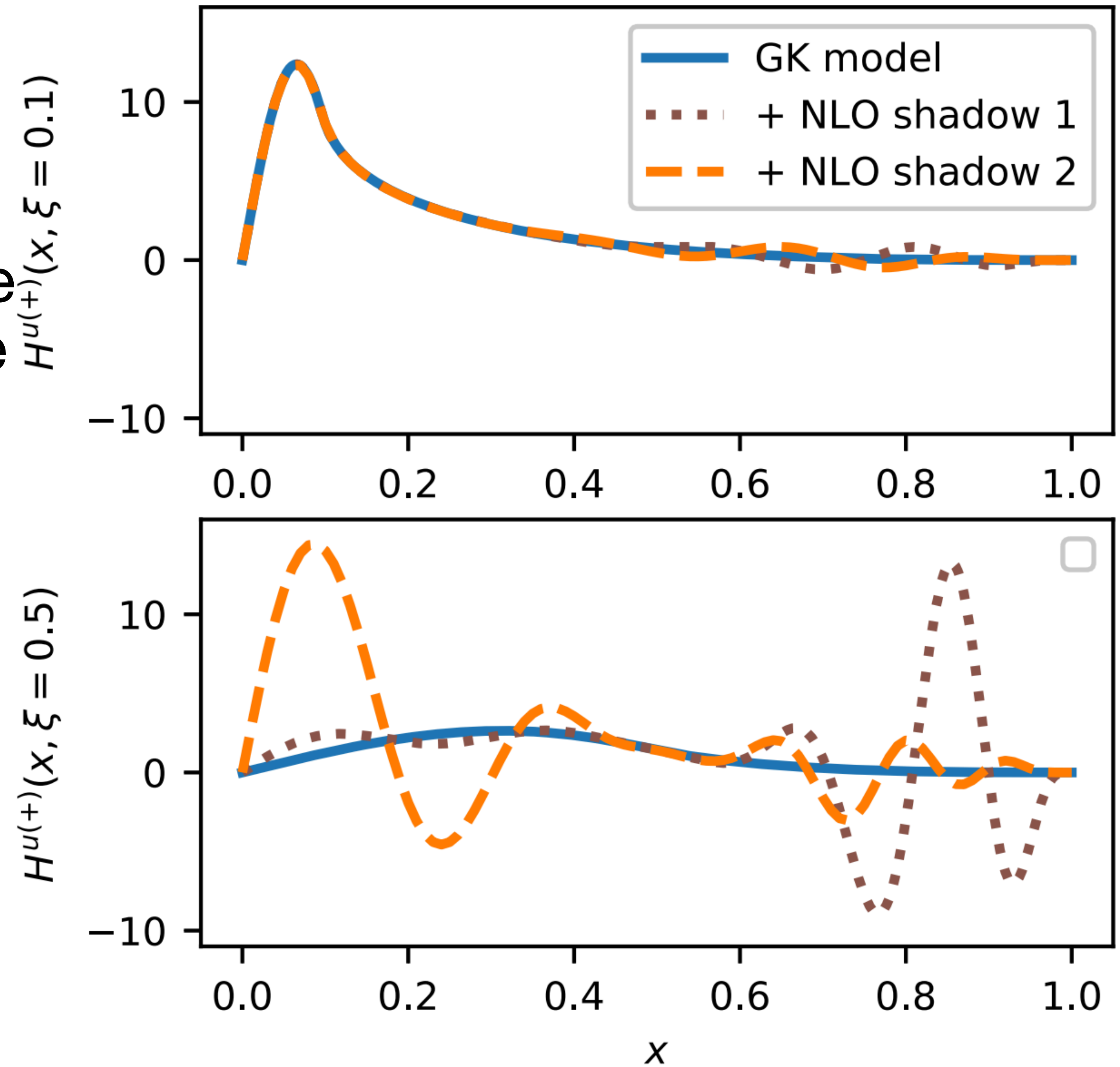

- Relation to different radii from slopes at $t = 0$

$$\epsilon(t) = m\left[A(t) - \frac{t}{4m^2}\big(D(t) + A(t) - 2J(t)\big)\right] = \int d^3b e^{-iq\cdot b}\tilde{\epsilon}(b) \quad \langle r^2\rangle_{\text{mass}} = \frac{\int d^3b\, b^2\tilde{\epsilon}(b)}{\int d^3b\, \tilde{\epsilon}(b)} = \frac{\epsilon_1'(t)}{\epsilon(0)}$$

# GPDs and their shadows

$$\mathscr{H}(\xi, t) = \int_{-1}^{1} \frac{dx}{2\xi} T^q(\frac{x}{\xi}, \mu^2, Q^2) H^{q(+)}(x, \xi, \mu^2)$$

- DVCS access Compton Form Factor
- Model dependence in GDPs from DVCS alone
- "Shadow GPDs" added to the "true" GPD would not change DVCS cross sections
- Lattice will potentially lack shadow GPDs
  - Improved control of kinematics




Shadow GPDs: Phys Rev D 103 (2021) 11, 114019

# Lorentz off the Lightcone (GPDs)

- Generalized pseudo-ITD

$$M^\mu(p',p,z) = \langle p' | \bar{q}\left(-\frac{z}{2}\right)\gamma^\mu W\left(-\frac{z}{2},\frac{z}{2}\right)q\left(\frac{z}{2}\right)|p\rangle$$

$$\mathfrak{M}^\mu(p',p,z) = \sum_{i=1}^{8} K^\mu(p',p,z)A_i(\nu,\xi,t)$$

S. Bhattacharya et al PRD 106 (2022) 11, 114512

**Gordon Identity means decomposition is not unique**

# Lorentz Decomposition

S. Bhattacharya et al PRD 106 (2022) 11, 114512

$$F^\mu(z,P,\Delta) = \bar{u}(p_f,\lambda')\left[\frac{P^\mu}{m}A_1 + mz^\mu A_2 + \frac{\Delta^\mu}{m}A_3 + im\sigma^{\mu z}A_4 + \frac{i\sigma^{\mu\Delta}}{m}A_5 + \frac{P^\mu i\sigma^{z\Delta}}{m}A_6 + mz^\mu i\sigma^{z\Delta}A_7 + \frac{\Delta^\mu i\sigma^{z\Delta}}{m}A_8\right]u(p_i,\lambda)$$

$$\bar{u}(p')\gamma^\mu u(p) = \frac{P^\mu}{m}\bar{u}(p')u(p) + \frac{i}{2m}\bar{u}(p')\sigma^{\mu\Delta}u(p)$$

**Gordon Identity means decomposition is not unique**

$\Delta = q$

$\langle\langle\Gamma\rangle\rangle = \bar{u}(p_f)\Gamma u(p_i)$

H. Dutrieux et al (HadStruc) JHEP 08 (2024) 162

$$\begin{aligned}\mathcal{M}^\mu(p_f,p_i,z) = {}& \langle\langle\gamma^\mu\rangle\rangle\mathcal{A}_1\left(\nu,\xi,t,z^2\right) + z^\mu\langle\langle\mathbb{1}\rangle\rangle\mathcal{A}_2\left(\nu,\xi,t,z^2\right) + i\langle\langle\sigma^{\mu z}\rangle\rangle\mathcal{A}_3\left(\nu,\xi,t,z^2\right) \\ & + \frac{i}{2m}\langle\langle\sigma^{\mu q}\rangle\rangle\mathcal{A}_4\left(\nu,\xi,t,z^2\right) + \frac{q^\mu}{2m}\langle\langle\mathbb{1}\rangle\rangle\mathcal{A}_5\left(\nu,\xi,t,z^2\right) \\ & + \frac{i}{2m}\langle\langle\sigma^{zq}\rangle\rangle\left[P^\mu\mathcal{A}_6\left(\nu,\xi,t,z^2\right) + q^\mu\mathcal{A}_7\left(\nu,\xi,t,z^2\right) + z^\mu\mathcal{A}_8\left(\nu,\xi,t,z^2\right)\right].\end{aligned} \tag{2.8}$$

# Lorentz Decomposition

S. Bhattacharya et al PRD 106 (2022) 11, 114512

$$F^\mu(z,P,\Delta) = \bar{u}(p_f,\lambda')\Big[\frac{P^\mu}{m}A_1 + mz^\mu A_2 + \frac{\Delta^\mu}{m}A_3 + im\sigma^{\mu z}A_4 + \frac{i\sigma^{\mu\Delta}}{m}A_5 + \frac{P^\mu i\sigma^{z\Delta}}{m}A_6 + mz^\mu i\sigma^{z\Delta}A_7 + \frac{\Delta^\mu i\sigma^{z\Delta}}{m}A_8\Big]u(p_i,\lambda)$$

**$A_5$ will give $H+E$**

$$\bar{u}(p')\gamma^\mu u(p) = \frac{P^\mu}{m}\bar{u}(p')u(p) + \frac{i}{2m}\bar{u}(p')\sigma^{\mu\Delta}u(p)$$

**Gordon Identity means decomposition is not unique**

$\Delta = q$

$\langle\langle\Gamma\rangle\rangle = \bar{u}(p_f)\Gamma u(p_i)$

**$A_4$ will give $E$**

H. Dutrieux et al (HadStruc) JHEP 08 (2024) 162

$$\mathcal{M}^\mu(p_f,p_i,z) = \langle\langle\gamma^\mu\rangle\rangle\mathcal{A}_1(\nu,\xi,t,z^2) + z^\mu\langle\langle\mathbb{1}\rangle\rangle\mathcal{A}_2(\nu,\xi,t,z^2) + i\langle\langle\sigma^{\mu z}\rangle\rangle\mathcal{A}_3(\nu,\xi,t,z^2) + \frac{i}{2m}\langle\langle\sigma^{\mu q}\rangle\rangle\mathcal{A}_4(\nu,\xi,t,z^2) + \frac{q^\mu}{2m}\langle\langle\mathbb{1}\rangle\rangle\mathcal{A}_5(\nu,\xi,t,z^2) + \frac{i}{2m}\langle\langle\sigma^{zq}\rangle\rangle\left[P^\mu\mathcal{A}_6(\nu,\xi,t,z^2) + q^\mu\mathcal{A}_7(\nu,\xi,t,z^2) + z^\mu\mathcal{A}_8(\nu,\xi,t,z^2)\right]. \quad (2.8)$$

# Double Distribution Representation

A. Radyushkin PRD 59 (1999) 014030; JHEP 03 (2023) 086

- On light cone, isolate symmetric-traceless by contracting with $z$

$$z^\mu M^\mu(\nu, z^2=0) = \bar{u}' \not{z} u H_{DD} + \frac{i}{2m}\bar{u}'\sigma^{zq} u E_{DD} + z\cdot q \frac{\bar{u}' u}{M} D$$

$$H_{DD}(\nu, \xi, t, z^2=0) = \int d\alpha d\beta e^{-i\beta\nu - i\alpha\xi\nu} h(\alpha, \beta, t)$$

# Double Distribution Representation

## Twist 3 contamination from new DDs

A. Radyushkin JHEP 03 (2023) 086

- On light cone, isolate symmetric-traceless by contracting with $z$

$$z^\mu M^\mu(\nu, z^2=0) = \bar{u}' \not{z} u H_{DD} + \frac{i}{2m} \bar{u}' \sigma^{zq} u E_{DD} + z \cdot q \frac{\bar{u}' u}{M} D$$

$$H_{DD}(\nu, \xi, t, z^2=0) = \int d\alpha d\beta e^{-i\beta\nu - i\alpha\xi\nu} h(\alpha, \beta, t)$$

- Off light cone, identity longitudinal and transverse DDs

$r = -q$

$$\begin{aligned}
M^\lambda &= (\bar{u}'\gamma^\lambda u) H_{DD} - \frac{1}{2M}(\bar{u}' i\sigma^{\lambda r} u) E_{DD} + r^\lambda \frac{\bar{u}' u}{M} D \\
&+ \left[r^\lambda (\mathcal{P}z) - \mathcal{P}^\lambda (rz)\right] \frac{\bar{u}' u}{M} Y \\
&- \frac{1}{M}\left[(\bar{u}' i\sigma^{zr} u)\mathcal{P}^\lambda - (\bar{u}' i\sigma^{\lambda r} u)(\mathcal{P}z)\right] X_1 \\
&- \frac{1}{M}\left[(\bar{u}' i\sigma^{zr} u) r^\lambda - (\bar{u}' i\sigma^{\lambda r} u)(rz)\right] X_2 \\
&+ (\bar{u}' i\sigma^{\lambda z} u) M X_3 \\
&+ i(\bar{u}' u) M z^\lambda Z_1 - (\bar{u}' i\sigma^{zr} u) M z^\lambda Z_2 \ .
\end{aligned}$$

- Original terms (first line)
- 0 when contracted with $z^\mu$ (Y, $X_1$, $X_2$, $X_3$ terms)
- $z^2$ when contracted with $z^\mu$ ($Z_1$, $Z_2$ terms)

# Double Distribution Representation

## Twist 3 contamination from new DDs

A. Radyushkin JHEP 03 (2023) 086

- On light cone, isolate symmetric-traceless by contracting with $z$

$$z^\mu M^\mu(\nu, z^2=0) = \bar{u}' \not{z} u H_{DD} + \frac{i}{2m}\bar{u}'\sigma^{zq} u E_{DD} + z\cdot q \frac{\bar{u}'u}{M} D$$

$$H_{DD}(\nu,\xi,t,z^2=0) = \int d\alpha d\beta e^{-i\beta\nu - i\alpha\xi\nu} h(\alpha,\beta,t)$$

- Off light cone, identity longitudinal and transverse DDs

$r = -q$

$$M^\lambda = (\bar{u}'\gamma^\lambda u)H_{DD} - \frac{1}{2M}(\bar{u}' i\sigma^{\lambda r} u)E_{DD} + r^\lambda \frac{\bar{u}'u}{M} D$$
$$+ \left[r^\lambda(\mathcal{P}z) - \mathcal{P}^\lambda(rz)\right]\frac{\bar{u}'u}{M} Y$$
$$- \frac{1}{M}\left[(\bar{u}' i\sigma^{zr} u)\mathcal{P}^\lambda - (\bar{u}' i\sigma^{\lambda r} u)(\mathcal{P}z)\right] X_1$$
$$- \frac{1}{M}\left[(\bar{u}' i\sigma^{zr} u) r^\lambda - (\bar{u}' i\sigma^{\lambda r} u)(rz)\right] X_2$$
$$+ (\bar{u}' i\sigma^{\lambda z} u) M X_3$$
$$+ i(\bar{u}'u) M z^\lambda Z_1 - (\bar{u}' i\sigma^{zr} u) M z^\lambda Z_2 \ .$$

$$\bar{u}'\gamma^\mu u = \frac{P^\mu}{m}\bar{u}'u + \frac{i}{2m}\bar{u}\sigma^{\mu\Delta}u$$

**Gordon Identity to spread $Y$ contamination**

**Isolate unique structures**

$$M^\lambda = (\bar{u}'\gamma^\lambda u)\left[H_{DD} - (rz)Y\right] + r^\lambda \frac{\bar{u}'u}{M}\left[D + (\mathcal{P}z)Y\right]$$
$$- \frac{1}{2M}(\bar{u}' i\sigma^{\lambda r} u)\left[E_{DD} + (rz)Y\right]$$
$$- \frac{1}{M}\left[(\bar{u}' i\sigma^{zr} u)\mathcal{P}^\lambda - (\bar{u}' i\sigma^{\lambda r} u)(\mathcal{P}z)\right] X_1$$
$$- \frac{1}{M}\left[(\bar{u}' i\sigma^{zr} u) r^\lambda - (\bar{u}' i\sigma^{\lambda r} u)(rz)\right] X_2$$
$$+ (\bar{u}' i\sigma^{\lambda z} u) M X_3$$
$$+ i(\bar{u}'u) M z^\lambda Z_1 - (\bar{u}' i\sigma^{zr} u) M z^\lambda Z_2 \ , \qquad (4.1$$

# Double Distribution Representation

A. Radyushkin JHEP 03 (2023) 086

Off light cone, identity longitudinal and transverse DDs

$$
\begin{aligned}
M^\lambda = & (\bar{u}'\gamma^\lambda u)\,[H_{DD} - (rz)Y] + r^\lambda \frac{\bar{u}'u}{M}[D + (\mathcal{P}z)Y] \\
& - \frac{1}{2M}(\bar{u}' i\sigma^{\lambda r} u)[E_{DD} + (rz)Y] \\
& - \frac{1}{M}\left[(\bar{u}' i\sigma^{zr} u)\mathcal{P}^\lambda + (\bar{u}' i\sigma^{\lambda r} u)(\mathcal{P}z)\right] X_1 \\
& - \frac{1}{M}\left[(\bar{u}' i\sigma^{zr} u) r^\lambda + (\bar{u}' i\sigma^{\lambda r} u)(rz)\right] X_2 \\
& + (\bar{u}' i\sigma^{\lambda z} u) M X_3 \\
& + i(\bar{u}'u) M z^\lambda Z_1 - (\bar{u}' i\sigma^{zr} u) M z^\lambda Z_2 \;, \qquad (4.16)
\end{aligned}
$$

Want to find combinations which will cancel the transverse degrees of freedom

$$
\begin{aligned}
\mathcal{M}^\mu(p_f, p_i, z) = & \langle\langle \gamma^\mu \rangle\rangle \mathcal{A}_1\left(\nu, \xi, t, z^2\right) + z^\mu \langle\langle \mathbb{1} \rangle\rangle \mathcal{A}_2\left(\nu, \xi, t, z^2\right) + i\langle\langle \sigma^{\mu z} \rangle\rangle \mathcal{A}_3\left(\nu, \xi, t, z^2\right) \\
& + \frac{i}{2m}\langle\langle \sigma^{\mu q} \rangle\rangle \mathcal{A}_4\left(\nu, \xi, t, z^2\right) + \frac{q^\mu}{2m}\langle\langle \mathbb{1} \rangle\rangle \mathcal{A}_5\left(\nu, \xi, t, z^2\right) \qquad (2.8) \\
& + \frac{i}{2m}\langle\langle \sigma^{zq} \rangle\rangle \left[P^\mu \mathcal{A}_6\left(\nu, \xi, t, z^2\right) + q^\mu \mathcal{A}_7\left(\nu, \xi, t, z^2\right) + z^\mu \mathcal{A}_8\left(\nu, \xi, t, z^2\right)\right] .
\end{aligned}
$$

# Lorentz off the Lightcone (GPDs)

- Generalized pseudo-ITD

$$M^{\mu}(p',p,z) = \langle p' | \bar{q}\big(-\frac{z}{2}\big)\gamma^{\mu}W\big(-\frac{z}{2},\frac{z}{2}\big)q\big(\frac{z}{2}\big)|p\rangle$$

$$\mathfrak{M}^{\mu}(p',p,z) = \sum_{i=1}^{8} K^{\mu}(p',p,z)A_i(\nu,\xi,t)$$

S. Bhattacharya et al PRD 106 (2022) 11, 114512

- Leading twist amplitudes in Ioffe time space

H. Dutrieux et al (HadStruc) JHEP 08 (2024) 16

$$H(\nu,\xi,t) = \int dx e^{i\nu x} H(x,\xi,t) = \int d\alpha d\beta e^{i\nu(\beta+\alpha\xi)}[H_{DD}(\beta,\alpha,t) + \delta(\beta)D(\alpha)]$$

$$= \lim_{z^2\to 0} A_1(\nu,\xi,t,z^2) - \xi A_5(\nu,\xi,t,z^2)$$

$$E(\nu,\xi,t) = \int dx e^{i\nu x} E(x,\xi,t) = \int d\alpha d\beta e^{i\nu(\beta+\alpha\xi)}[E_{DD}(\beta,\alpha,t) - \delta(\beta)D(\alpha)]$$

$$= \lim_{z^2\to 0} A_4(\nu,\xi,t,z^2) + \nu A_6(\nu,\xi,t,z^2) - 2\nu\xi A_7(\nu,\xi,t,z^2) + \xi A_5(\nu,\xi,t,z^2)$$

# Isolation of Amplitudes: SVD

$$\mathfrak{M}^\mu(p', p, z) = \sum_{i=1}^{8} K^\mu(p', p, z) A_i(\nu, \xi, t, z^2)$$

Calculate for fixed $\nu, \xi, t, z^2$ and vary initial/final spin and $\mu$ to build matrix equation

$$\mathfrak{M} = KA$$

Pseudo-inverse solution $\tilde{A}$ gives minimum of $\chi^2 = |K\tilde{A} - \mathfrak{M}|^2$

$$\tilde{A} = K^+ \mathfrak{M}$$

Penrose, Math Proc. CPS 52, 17-19 (1956)

https://numpy.org/doc/stable/reference/generated/numpy.linalg.pinv.html

**Already used for Lattice Form Factor calculations**

2 non-degenerate spin combinations and 4 values of $\mu$

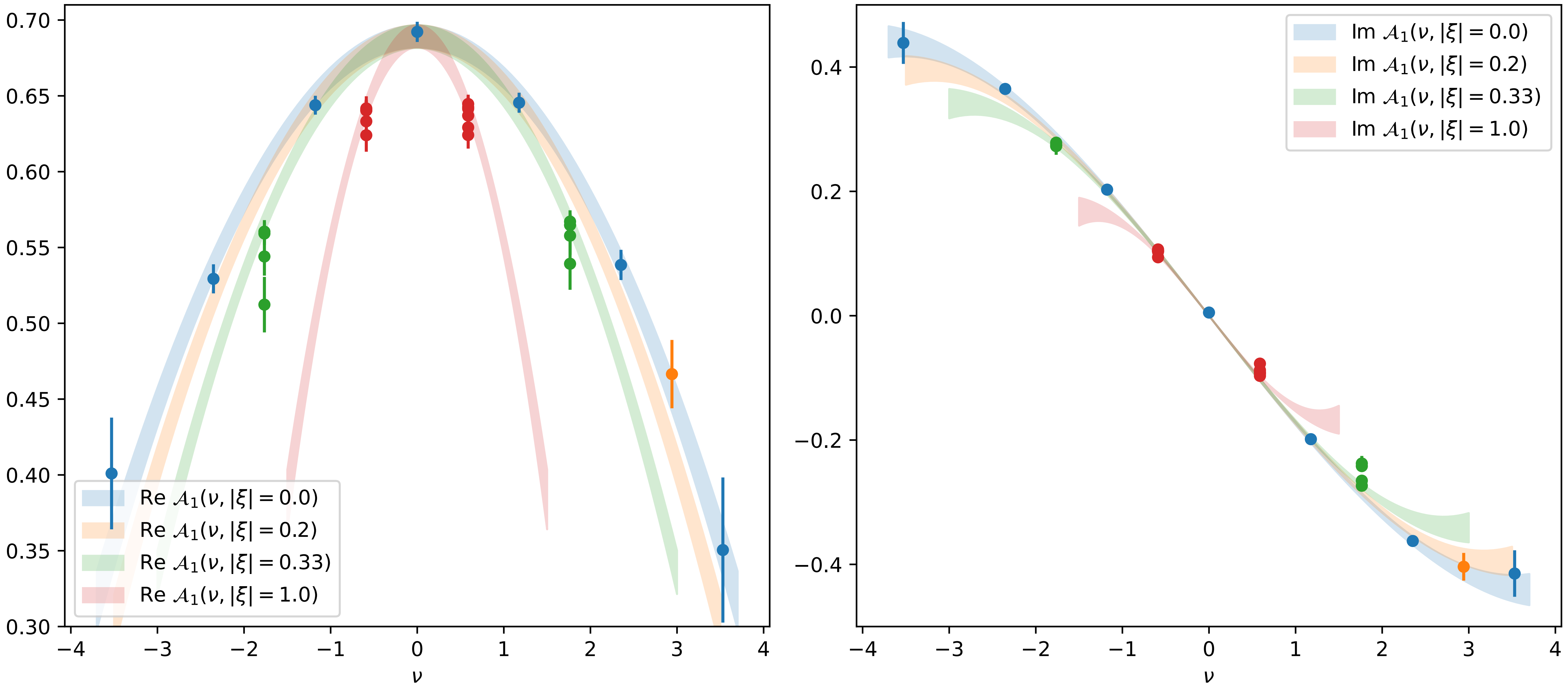
8 unique equations can be formed for 8 unknowns

# Fits to Moments

V. Braun and D. Muller *Eur. Phys. J. C* 55 (2008) 349

## Mellin moments are coeffs of polynomials in $\nu, \xi\nu$

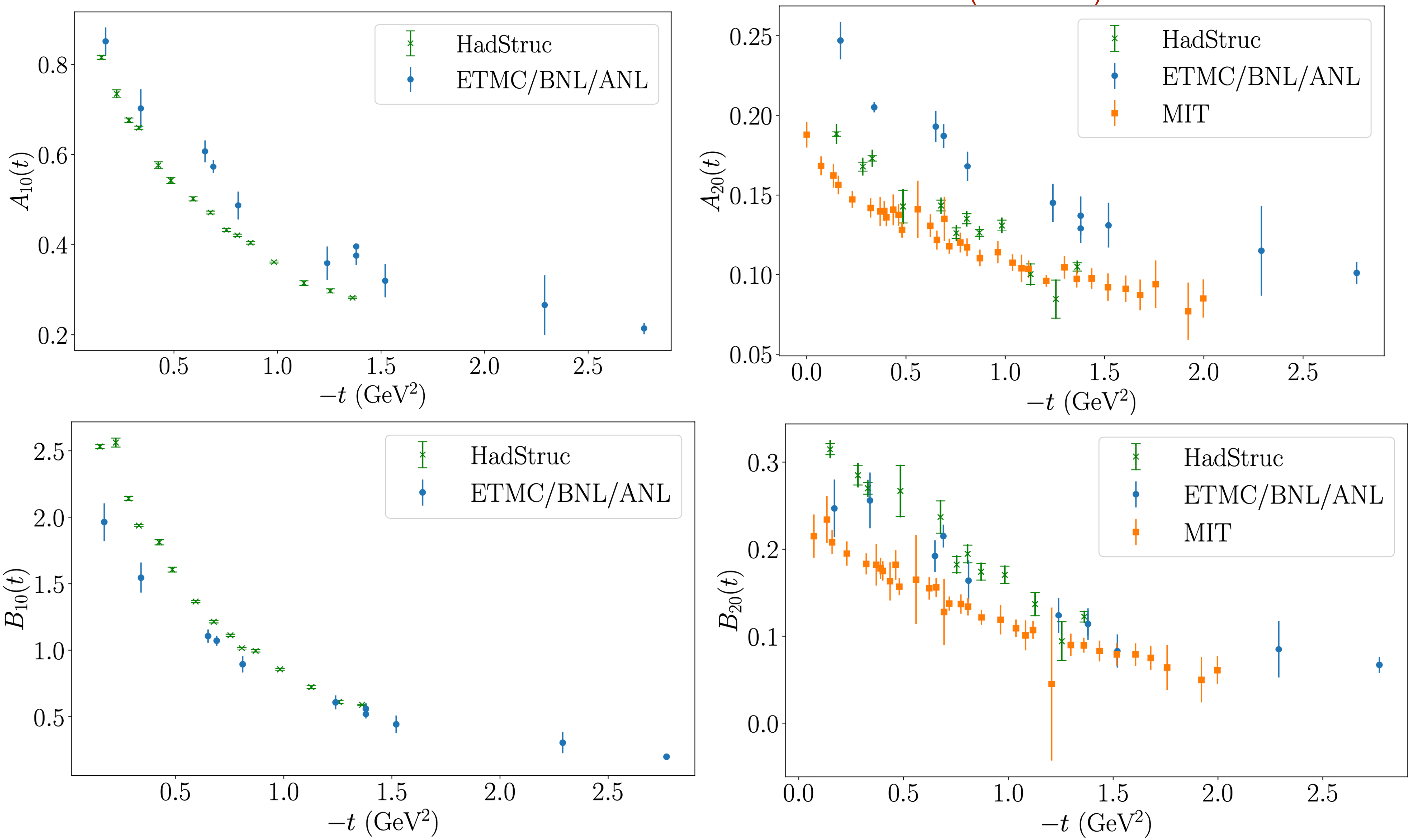
$$A_1(\nu, \xi\nu, t) = F_1(t) - i\nu A_{20}(t) - \frac{\nu^2}{2}\left(A_{30}(t) + \xi^2 A_{32}(t)\right) + i\frac{\nu^3}{6}\left(A_{40}(t) + \xi^2 A_{42}(t)\right) + \ldots$$




H. Dutrieux et al (HadStruc) JHEP 08 (2024) 162

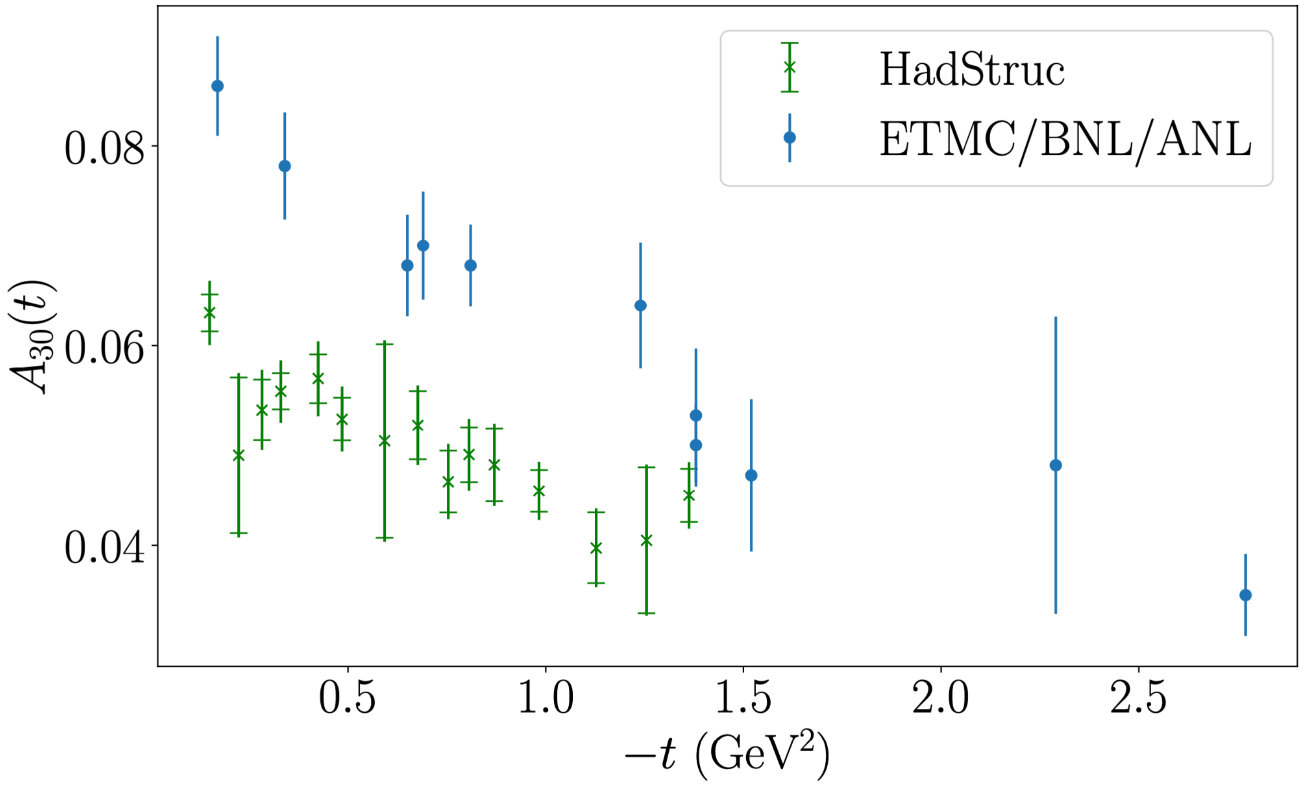
# Moments of $H$ and $E$

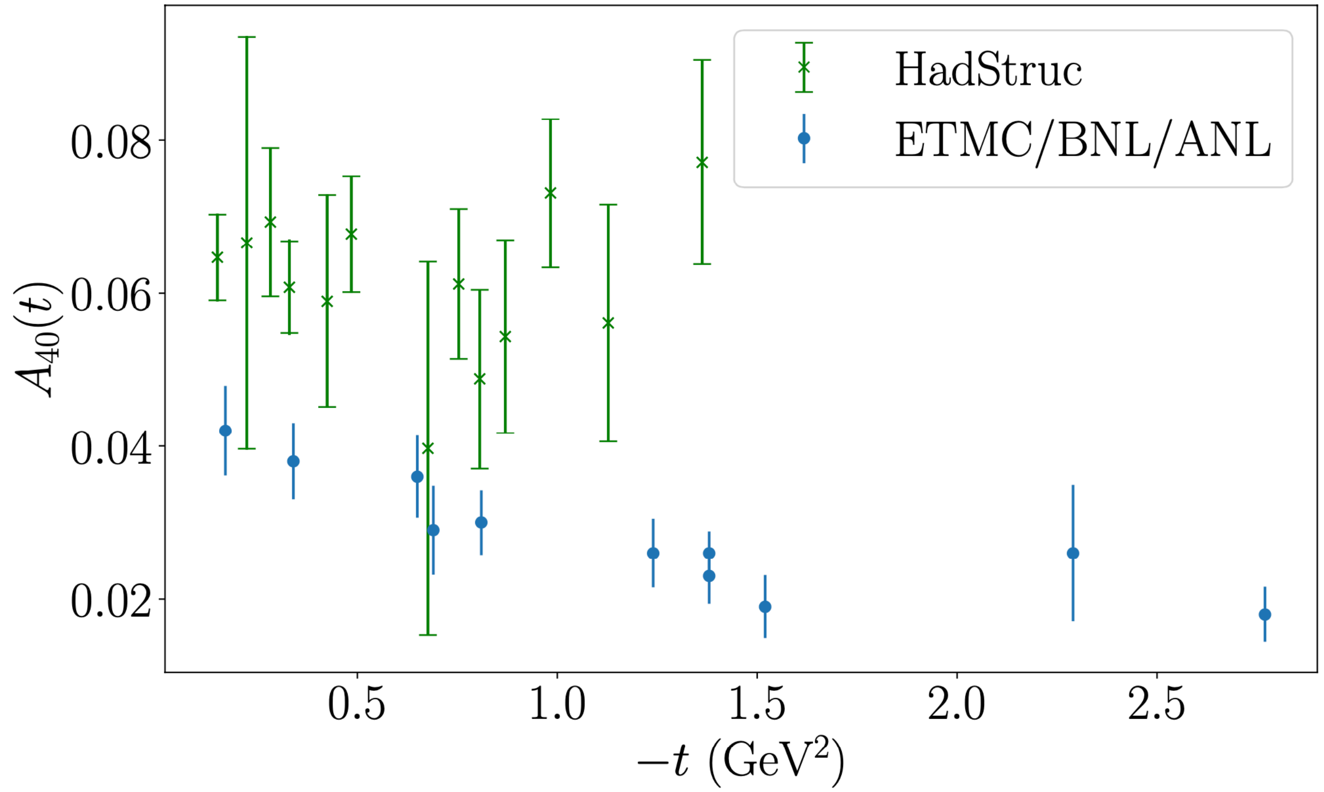
D. Hackett, D. Pefkou, P. Shanahan (MIT) arXiv:2301.08484
S. Bhattacharya et al (ETMC/BNL/ANL) arXiv:2305.11117
H. Dutrieux et al (HadStruc) arXiv:2405.10304

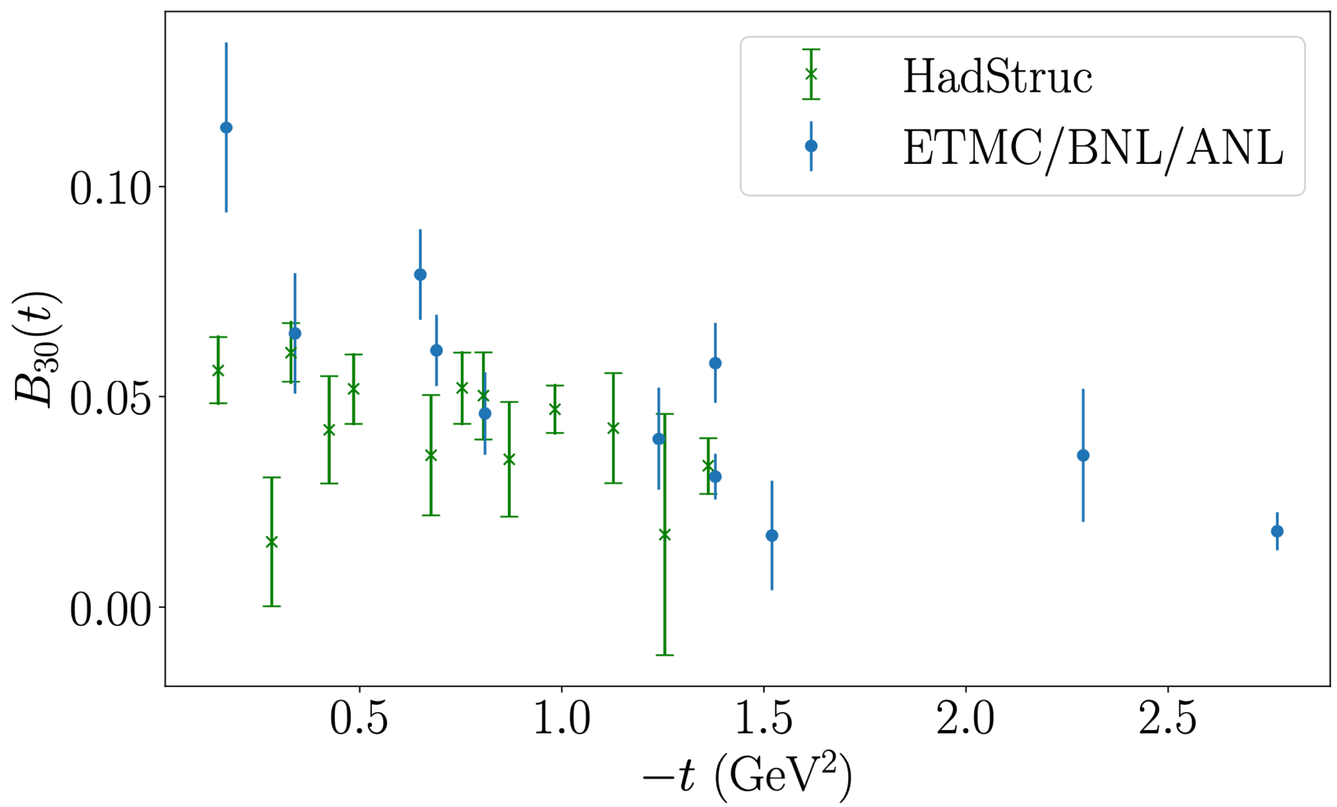


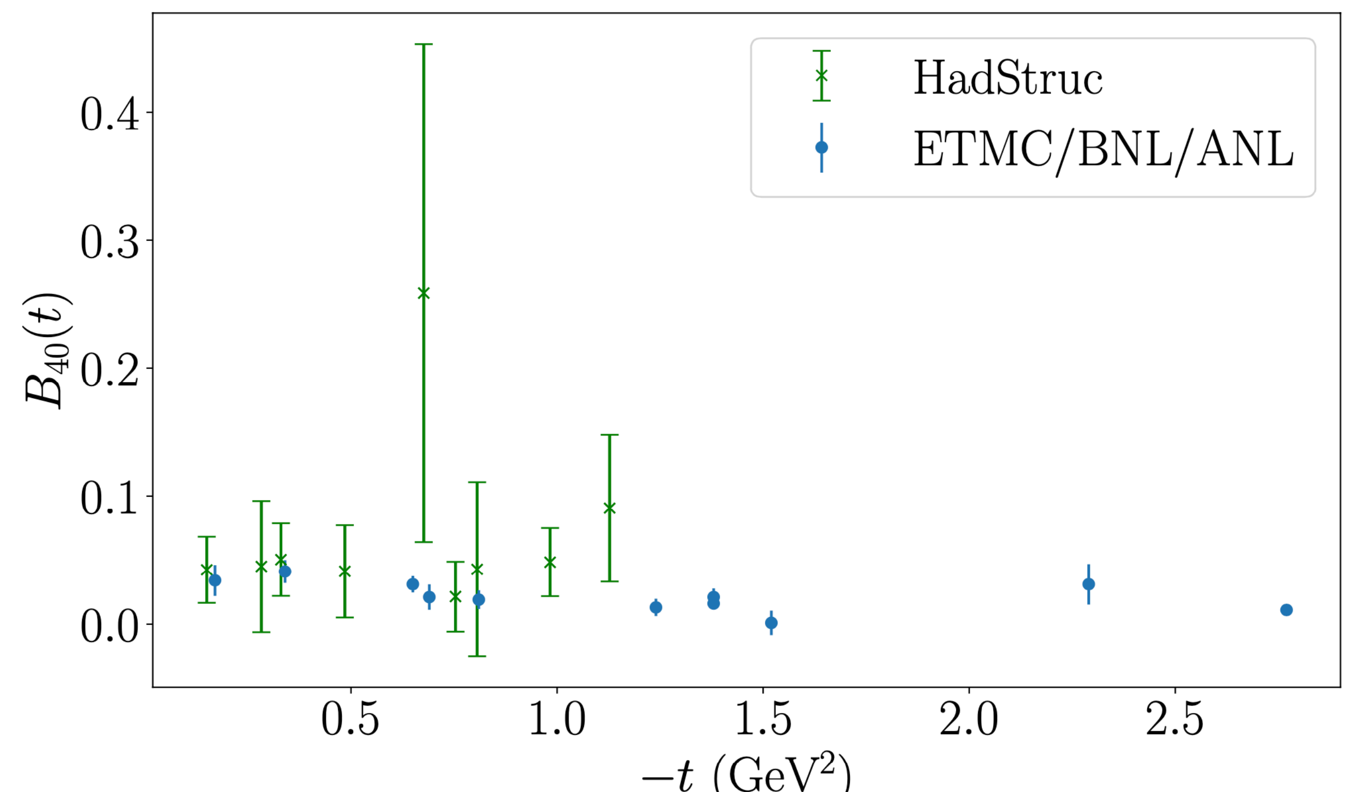
# Moments of $H$ and $E$

S. Bhattacharya et al (ETMC/BNL/ANL) arXiv:2305.11117
H. Dutrieux et al (HadStruc) arXiv:2405.10304



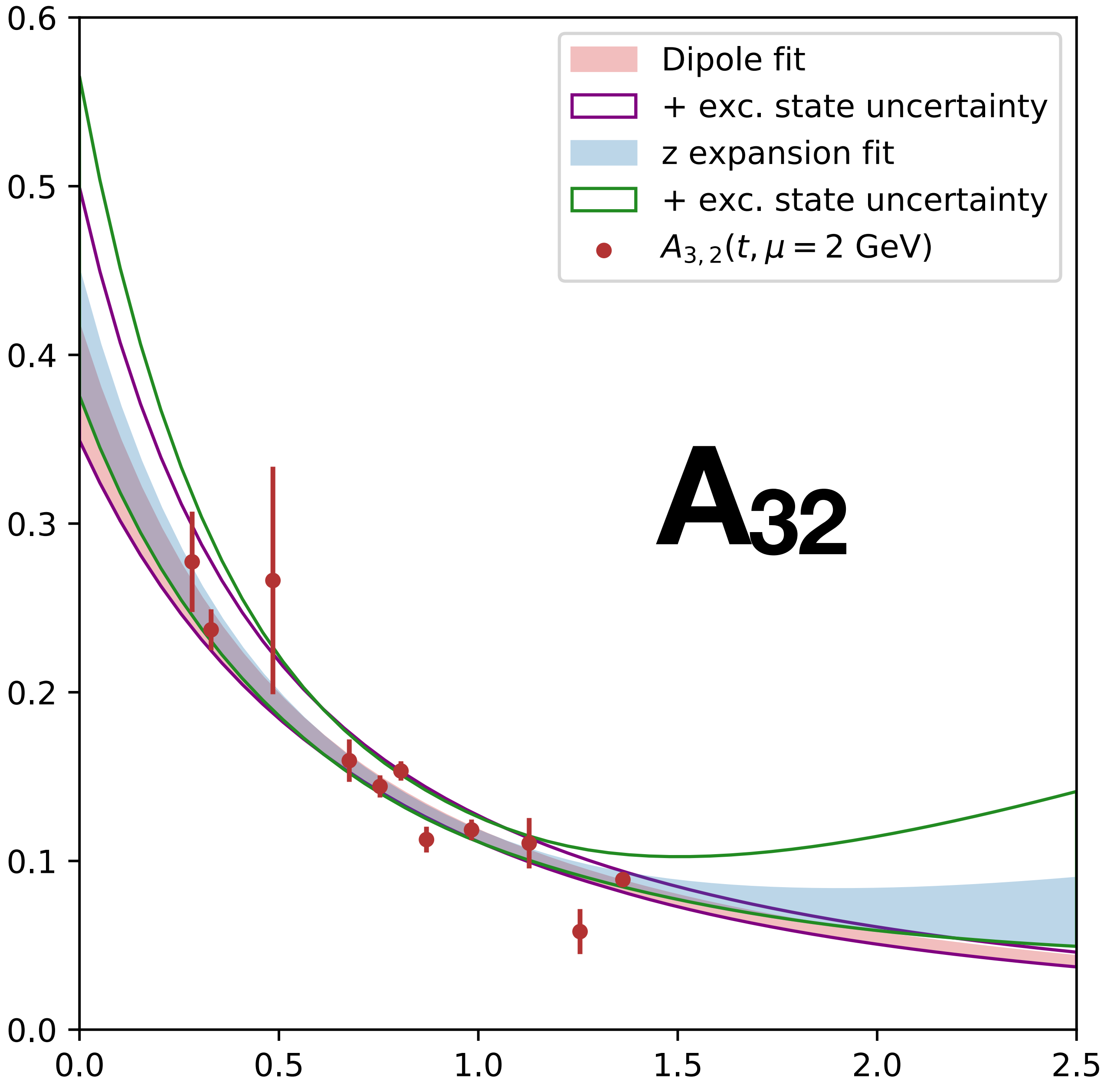



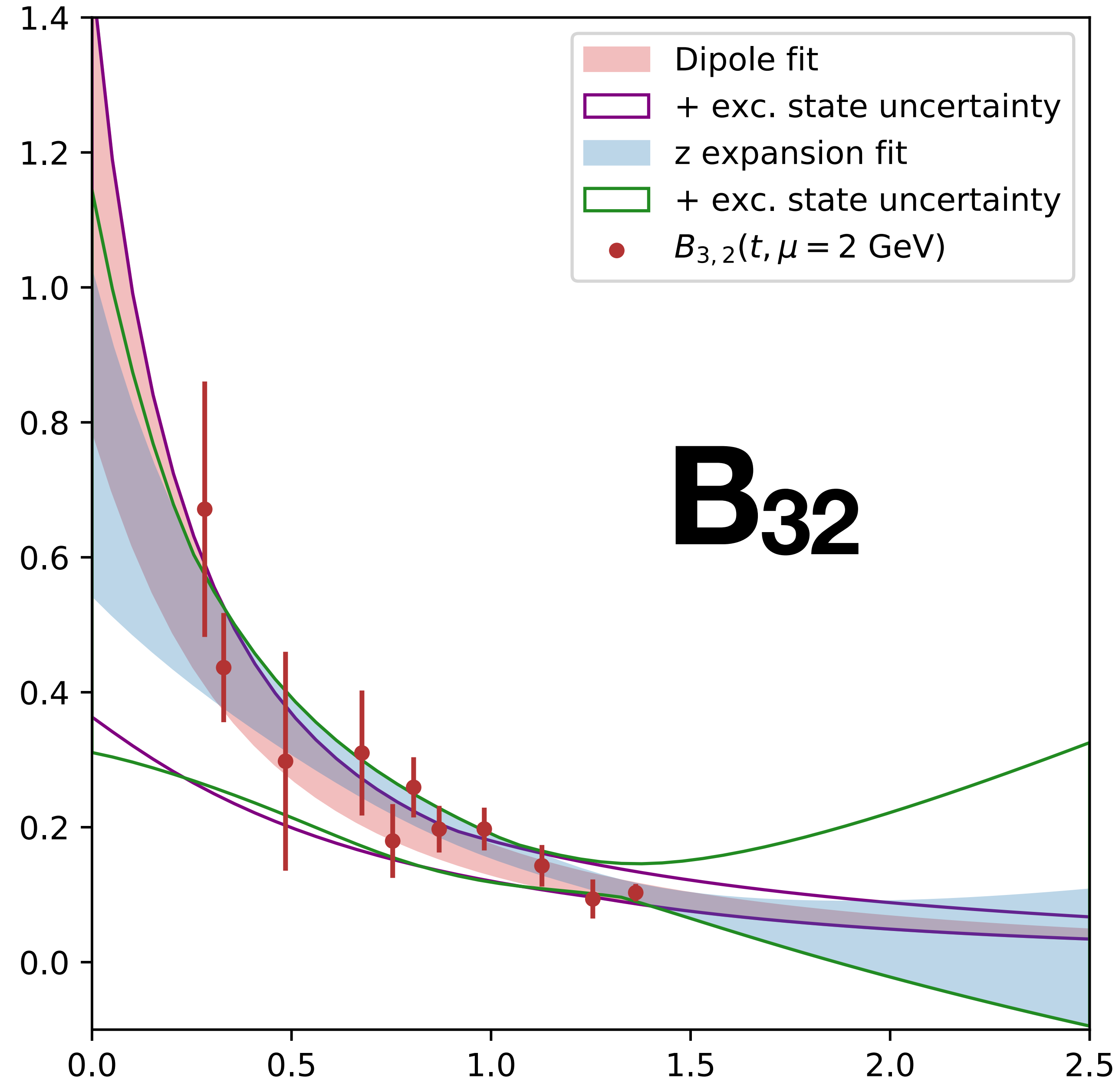



# Skewness dependence of moments

H. Dutrieux et al (HadStruc) arXiv:2405.10304

$$A_1(\nu, \xi\nu, t) = F_1(t) - i\nu A_{20}(t) - \frac{\nu^2}{2}\left(A_{30}(t) + \xi^2 A_{32}(t)\right) + i\frac{\nu^3}{6}\left(A_{40}(t) + \xi^2 A_{42}(t)\right) + \ldots$$

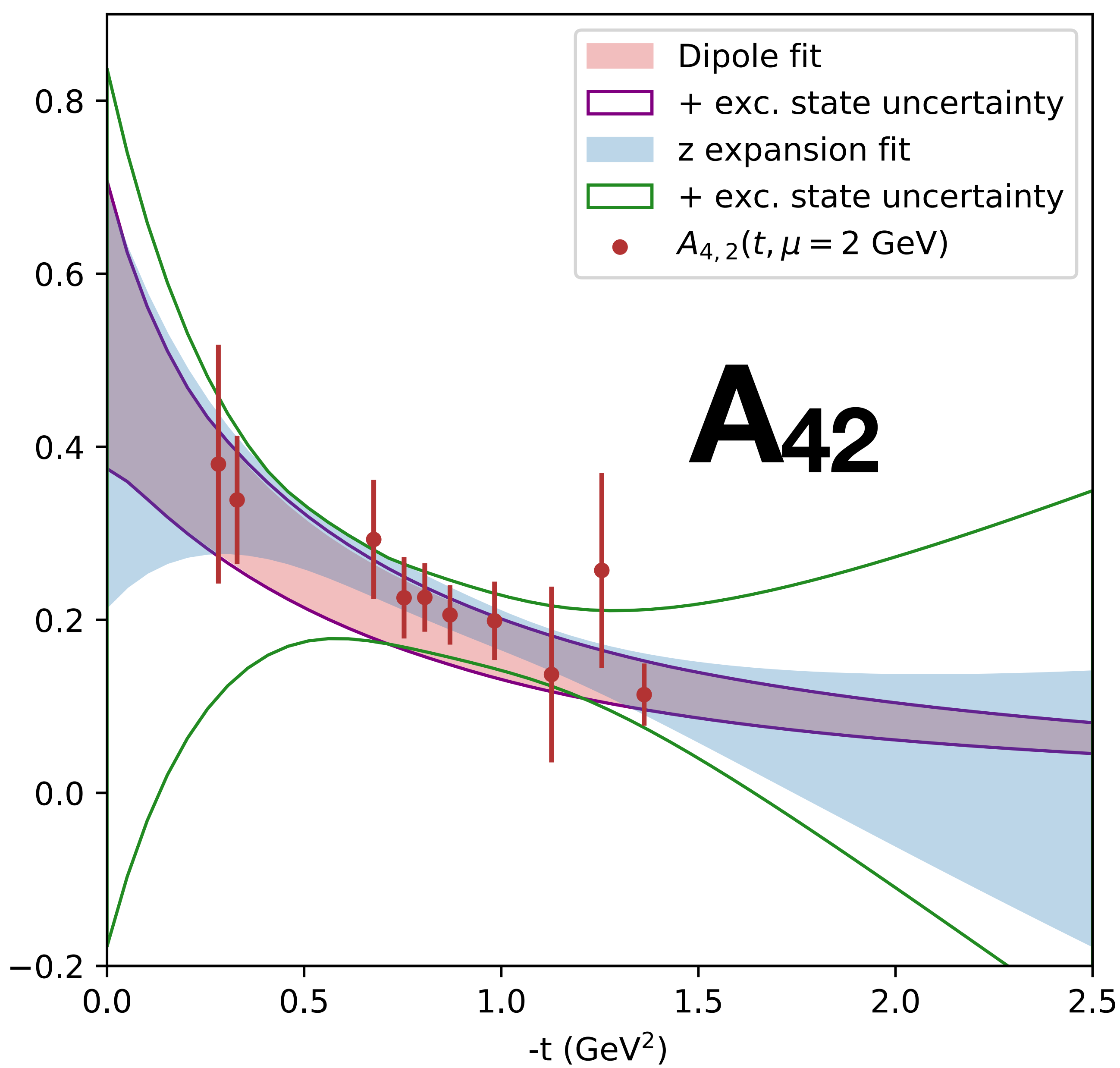


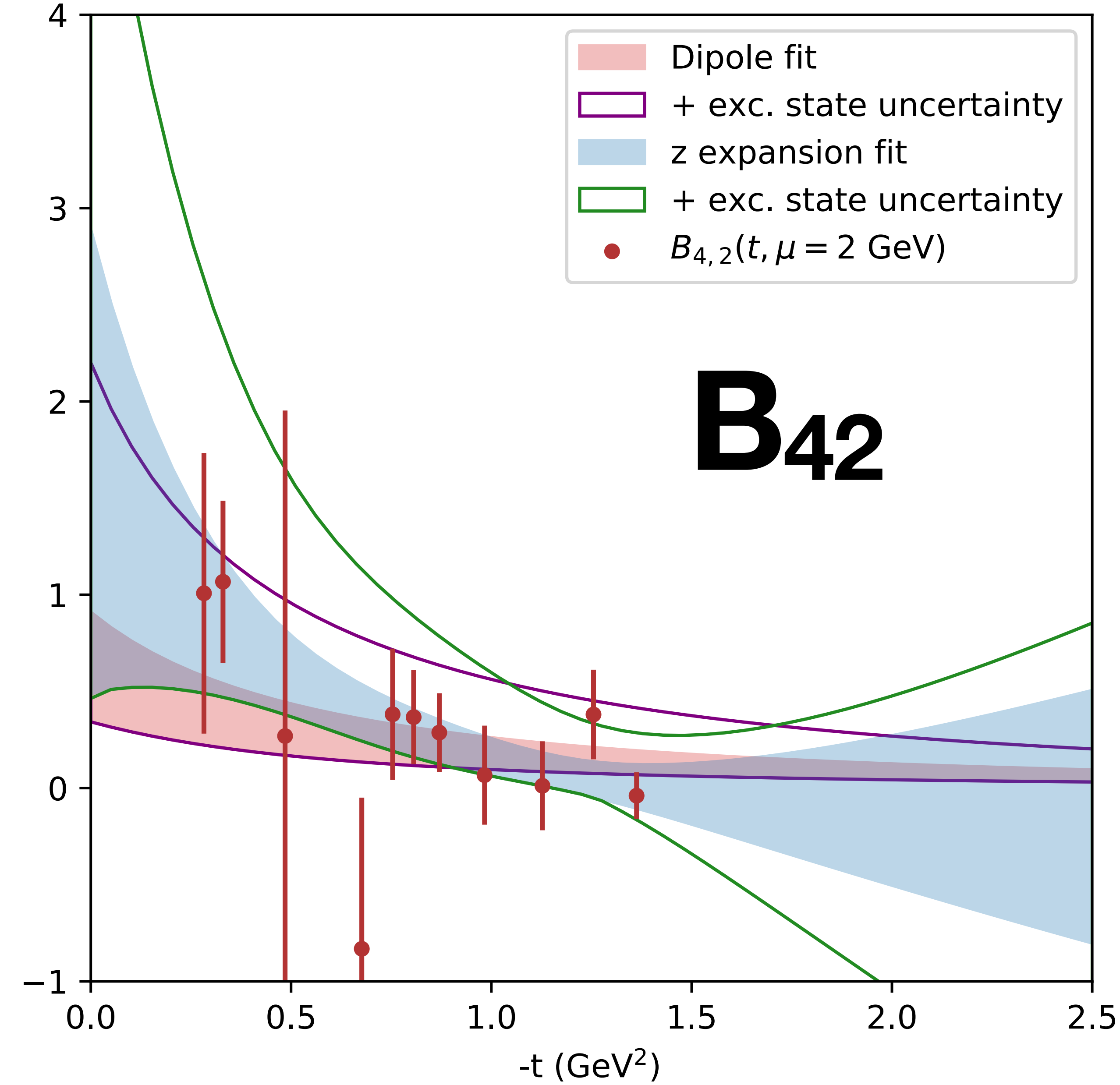



# Skewness dependence of moments

H. Dutrieux et al (HadStruc) arXiv:2405.10304

$$A_1(\nu, \xi\nu, t) = F_1(t) - i\nu A_{20}(t) - \frac{\nu^2}{2}\left(A_{30}(t) + \xi^2 A_{32}(t)\right) + i\frac{\nu^3}{6}\left(A_{40}(t) + \xi^2 A_{42}(t)\right) + \ldots$$



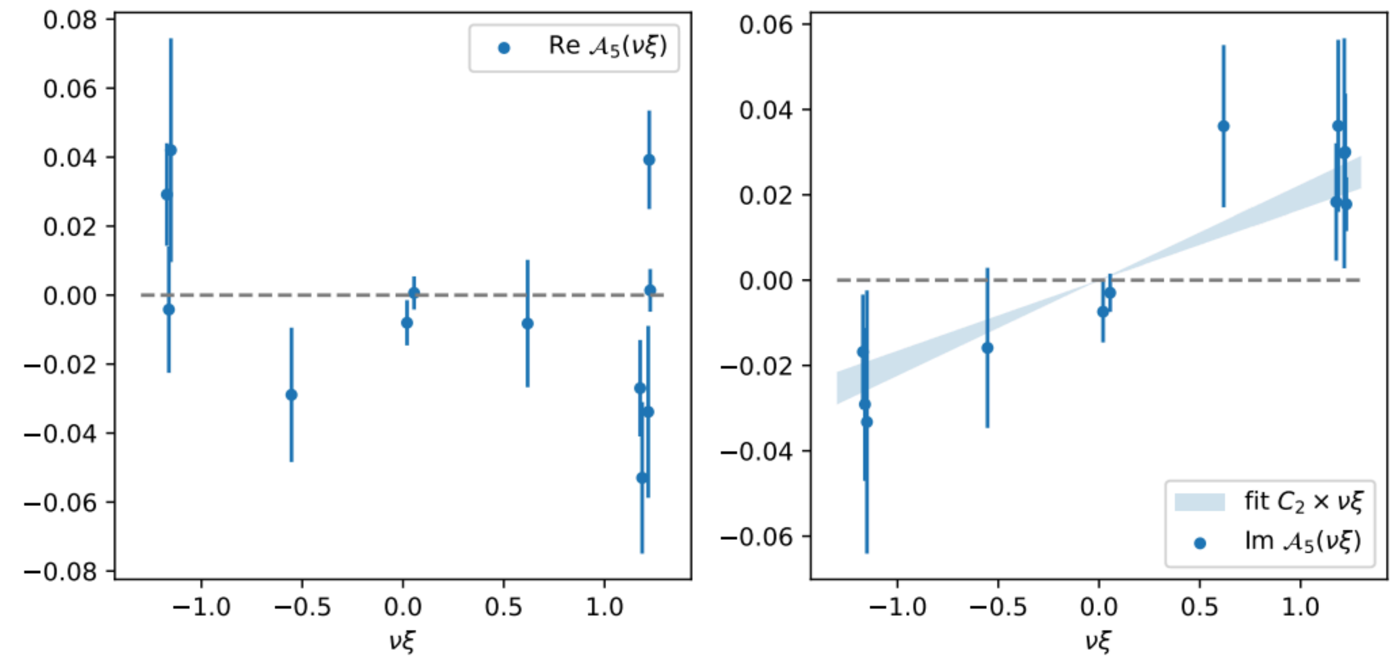
# Moment of iso-vector $D$ term

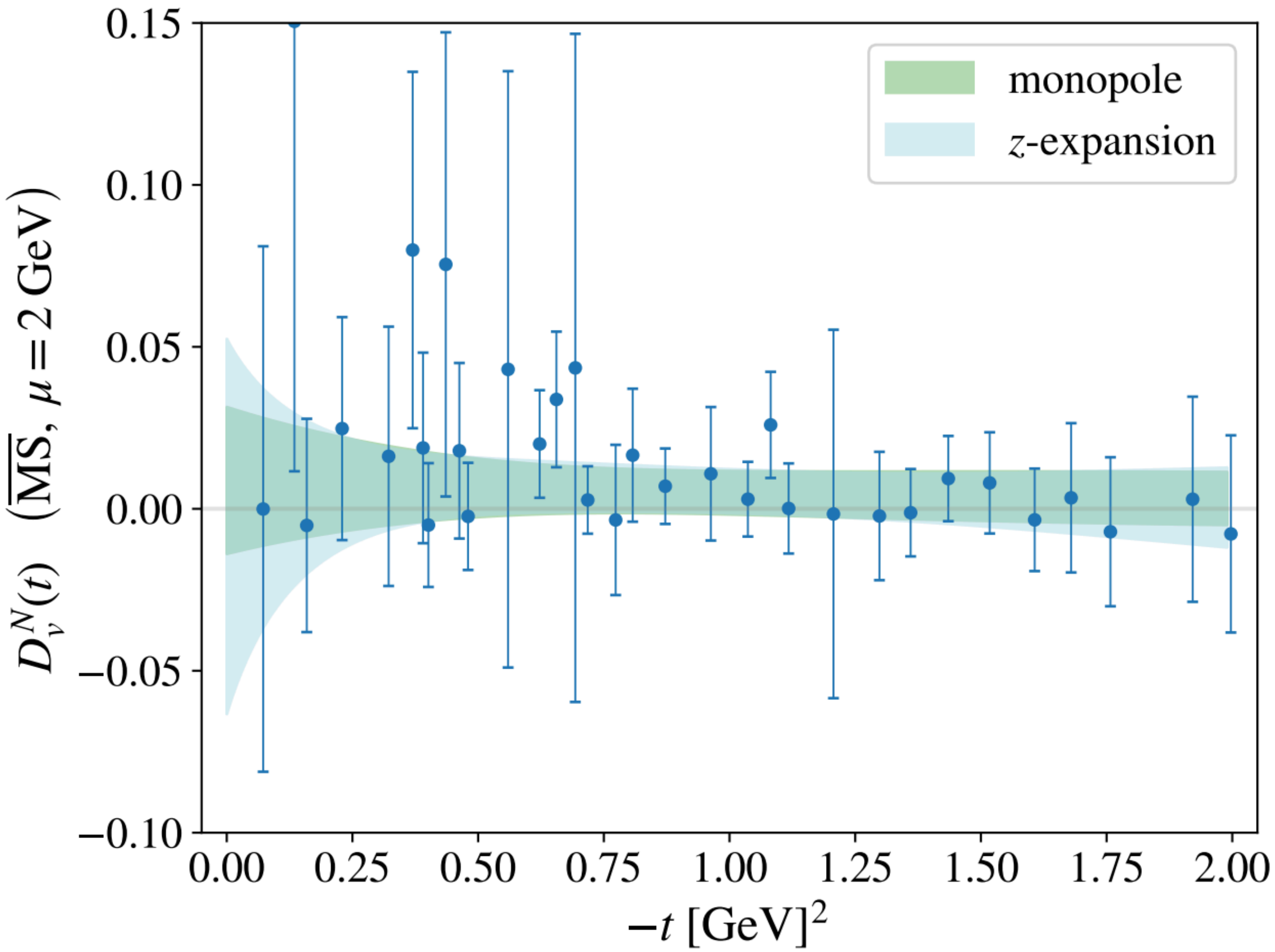
$$\xi\nu = q \cdot z$$

$$A_5(\xi\nu) = D + 2\nu Y = i\xi\nu \int d\alpha\, \alpha D(\alpha) + \ldots$$

- $\nu = 0$ removes $Y$ DD
- $D$ is odd function of $\alpha$

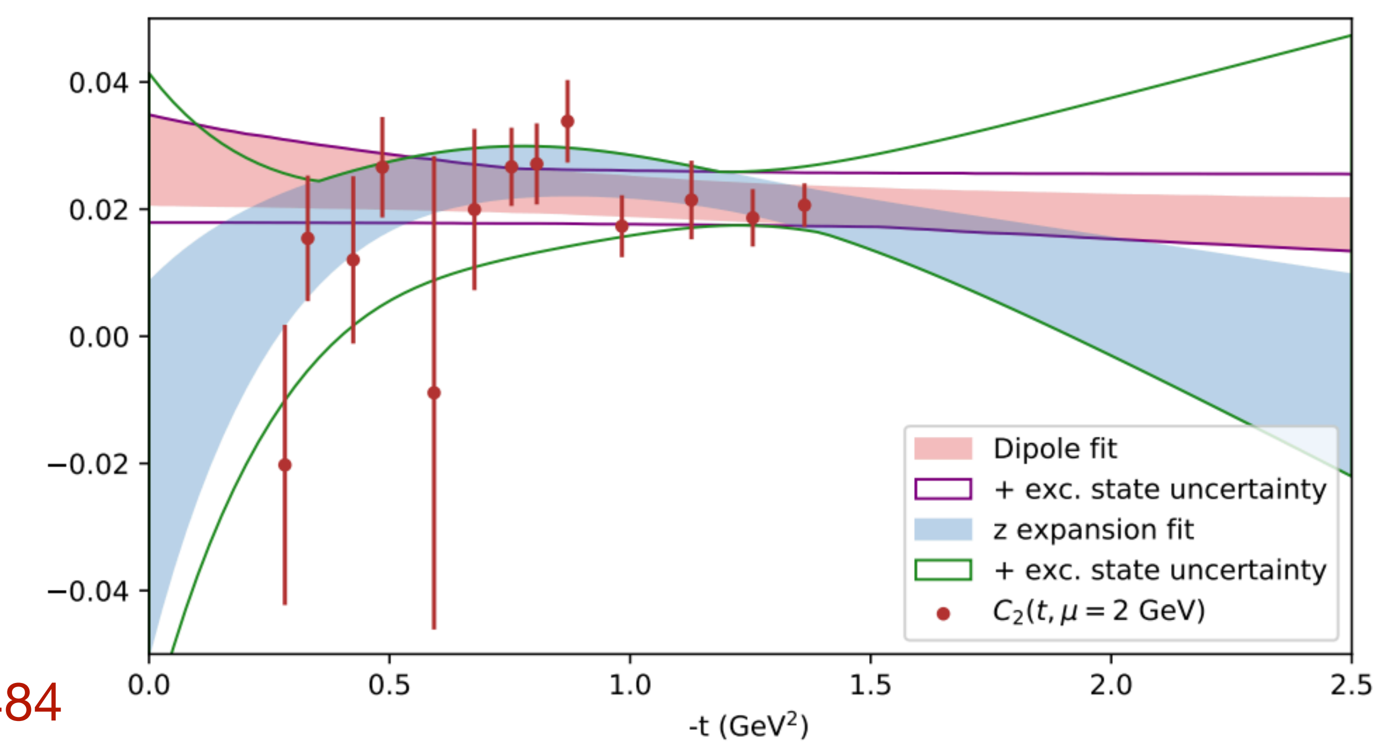
**Highly Correlated Data have more information that plots imply**




D. Hackett, D. Pefkou, P. Shanahan (MIT) arXiv:2301.08484

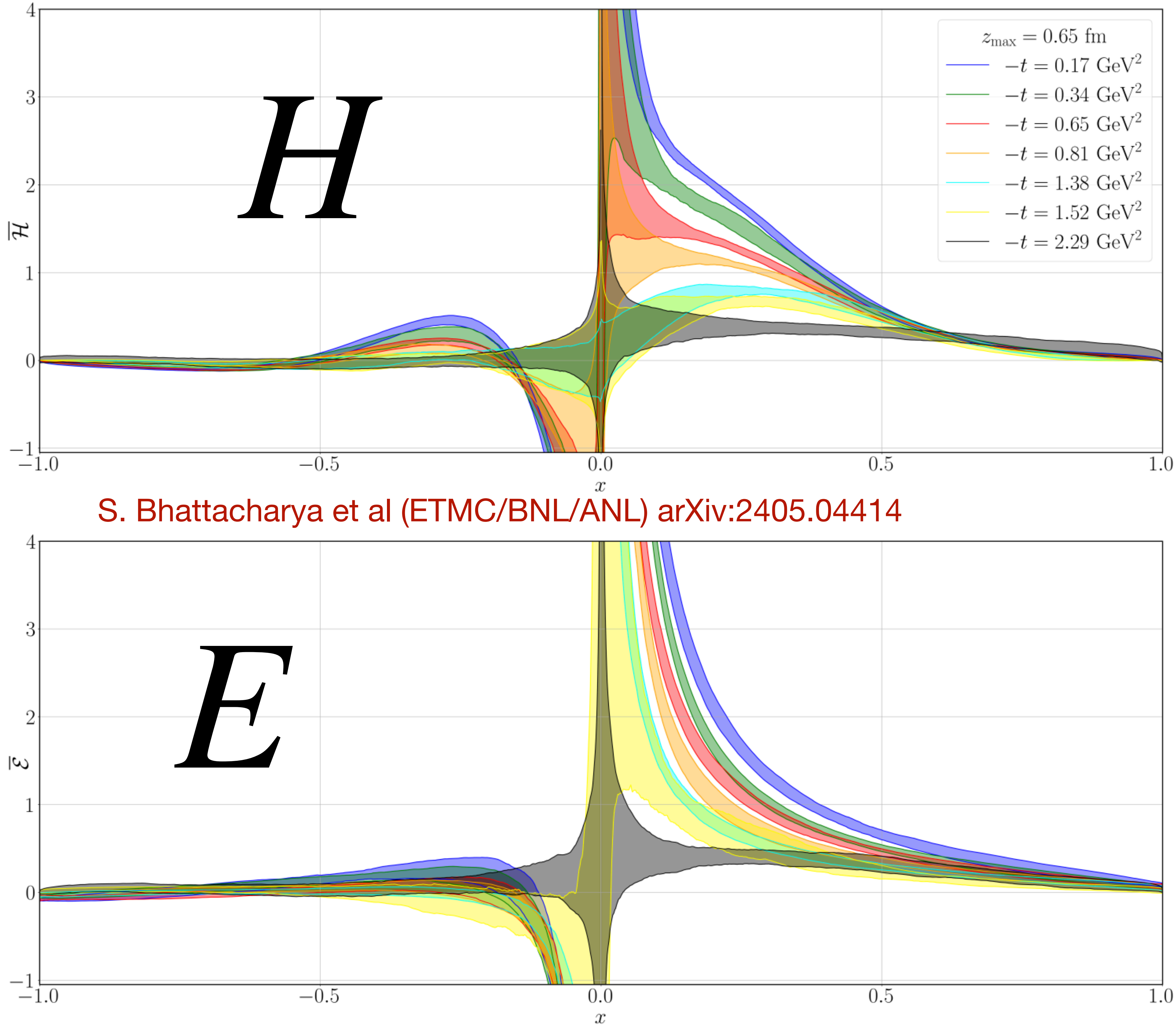
H. Dutrieux et al (HadStruc) arXiv:2405.10304



# $x$ Dependence of GPDs

- Results from follow up pseudo-PDF fit from $z \leq 0.65$ fm
- Invert by fits
- As $-t$ increases the GPD flattens

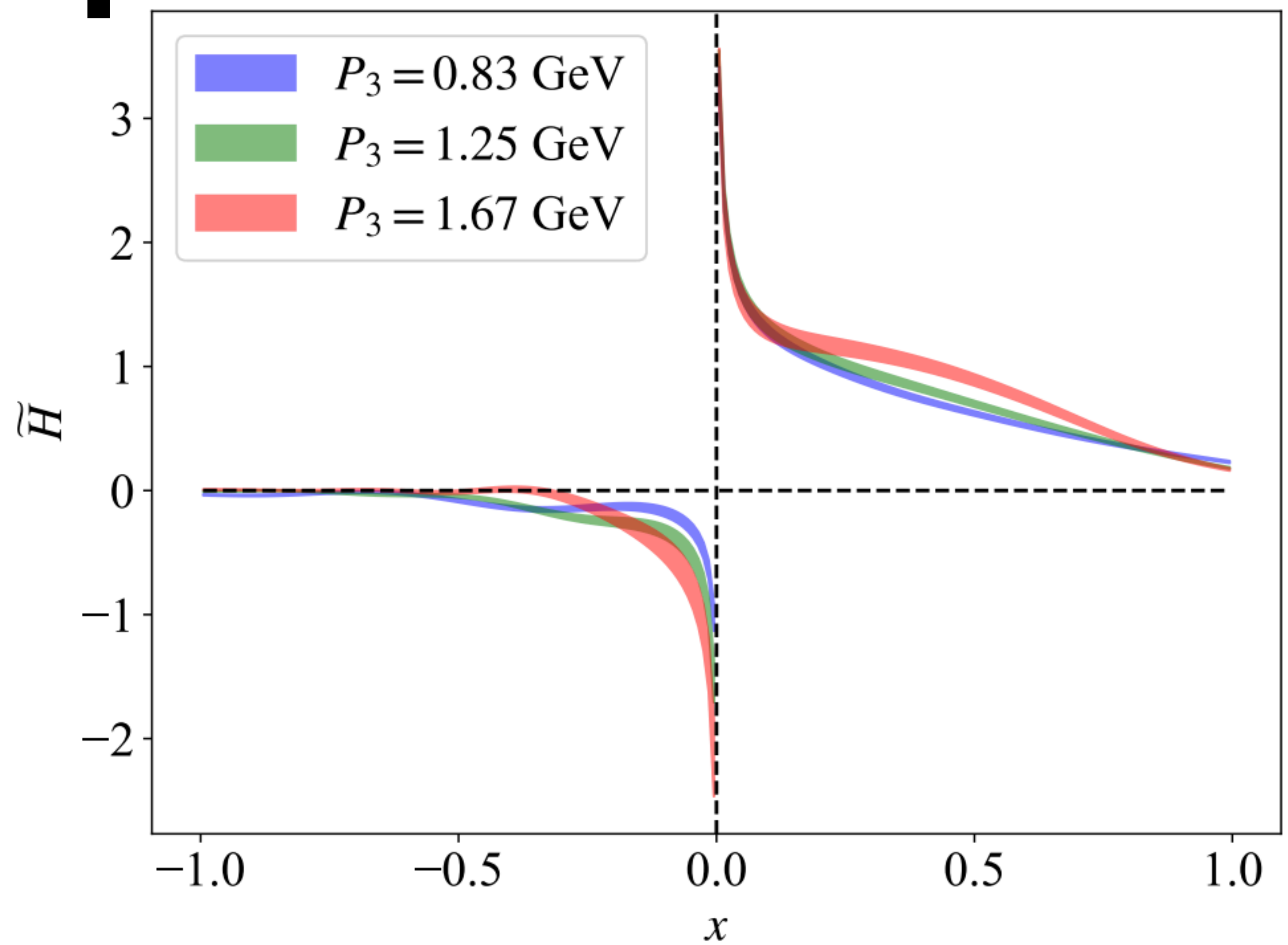
$$\xi = 0$$

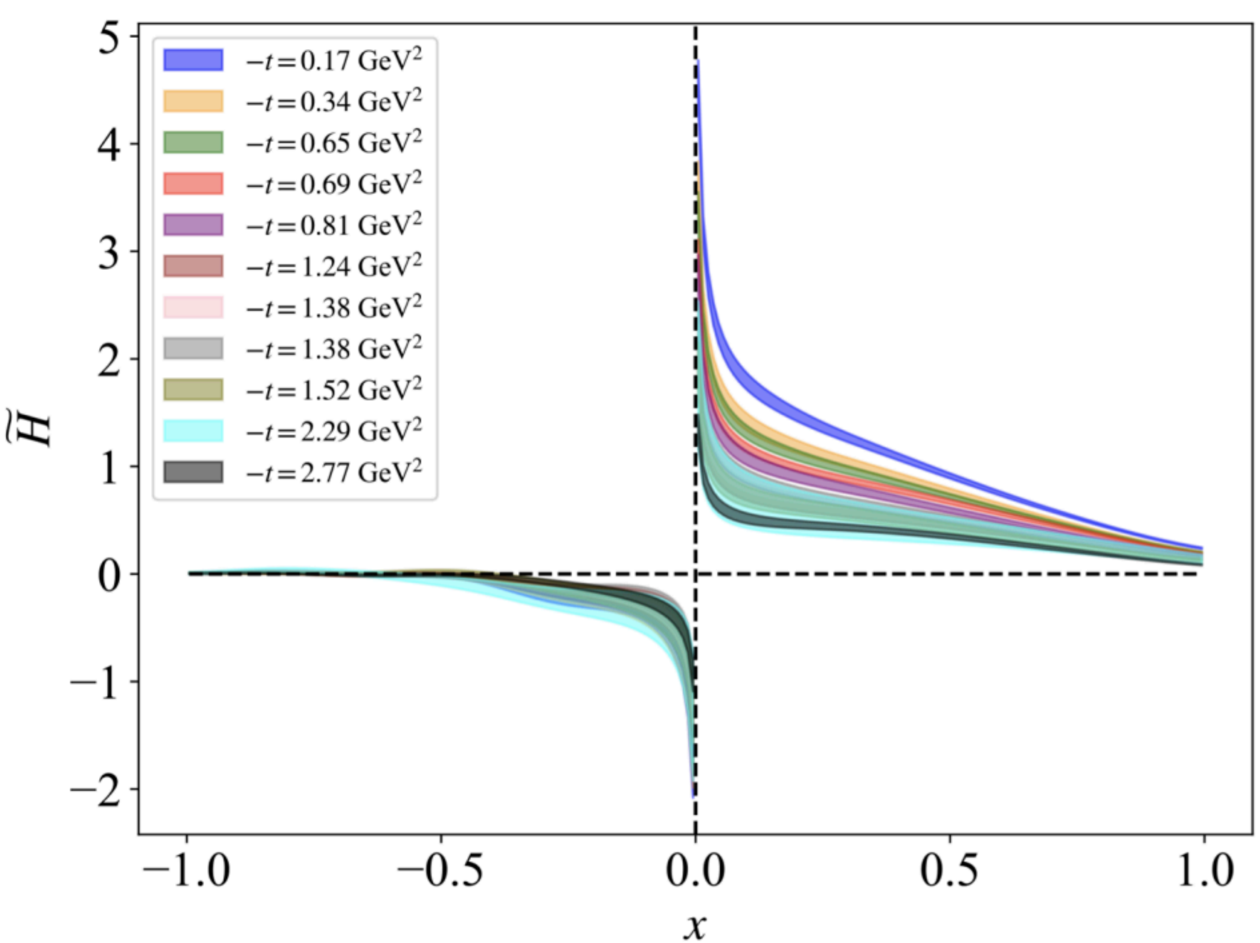



S. Bhattacharya et al (ETMC/BNL/ANL) arXiv:2405.04414

# $x$ Dependence of polarized GPDs

- Results from polarized quasi-GPD analysis
- Invert with Backus Gilbert
- As $-t$ increases the GPD flattens




$$\xi = 0$$

$$\tilde{H}$$

S. Bhattacharya et al (ETMC/BNL/ANL) arXiv:2310.13114

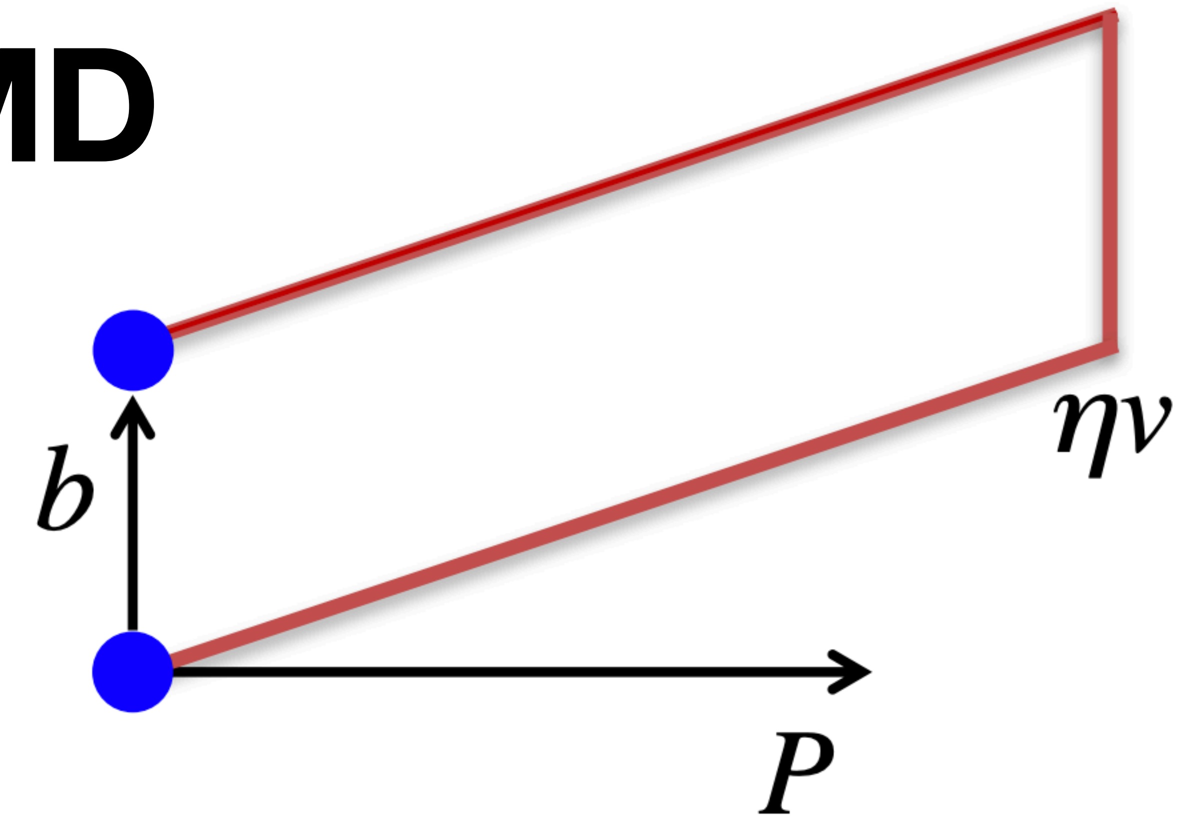
# GPDs summary

- Direct Calculations of flavor dependence of FFs and GFFs

- Indirect Moments from the pseudo-ITD $\nu$ Taylor expansion

- Indirect inverse to obtain (un)polarized GPDs which show proper drops in $-t$

# Moments of the TMD




- Correlator with staple shape

$$\Phi^{[\Gamma]}(x, \boldsymbol{k}_T, P, S, \ldots)$$

$$= \int \frac{d^2 \boldsymbol{b}_\mathrm{T}}{(2\pi)^2} \int \frac{d(b}{2\pi.} \Phi^{[\Gamma]}(x, \boldsymbol{k}_T, P, S, \ldots)$$

$$= \int \frac{d^2 \boldsymbol{b}_\mathrm{T}}{(2\pi)^2} \int \frac{d(b\cdot P)}{2\pi P^+} e^{ix(b\cdot P) - i\boldsymbol{b}_\mathrm{T}\cdot \boldsymbol{k}_\mathrm{T}} \; \widetilde{\Phi}^{[\Gamma]}_{\mathrm{subtr.}}\Big|_{b^+=0}$$

- Unsubtracted has renormalization and soft factor
    - Completely mulitplicative!

$$\widetilde{\Phi}^{[\Gamma]}_{\mathrm{subtr.}}(b, P, S, \ldots)$$

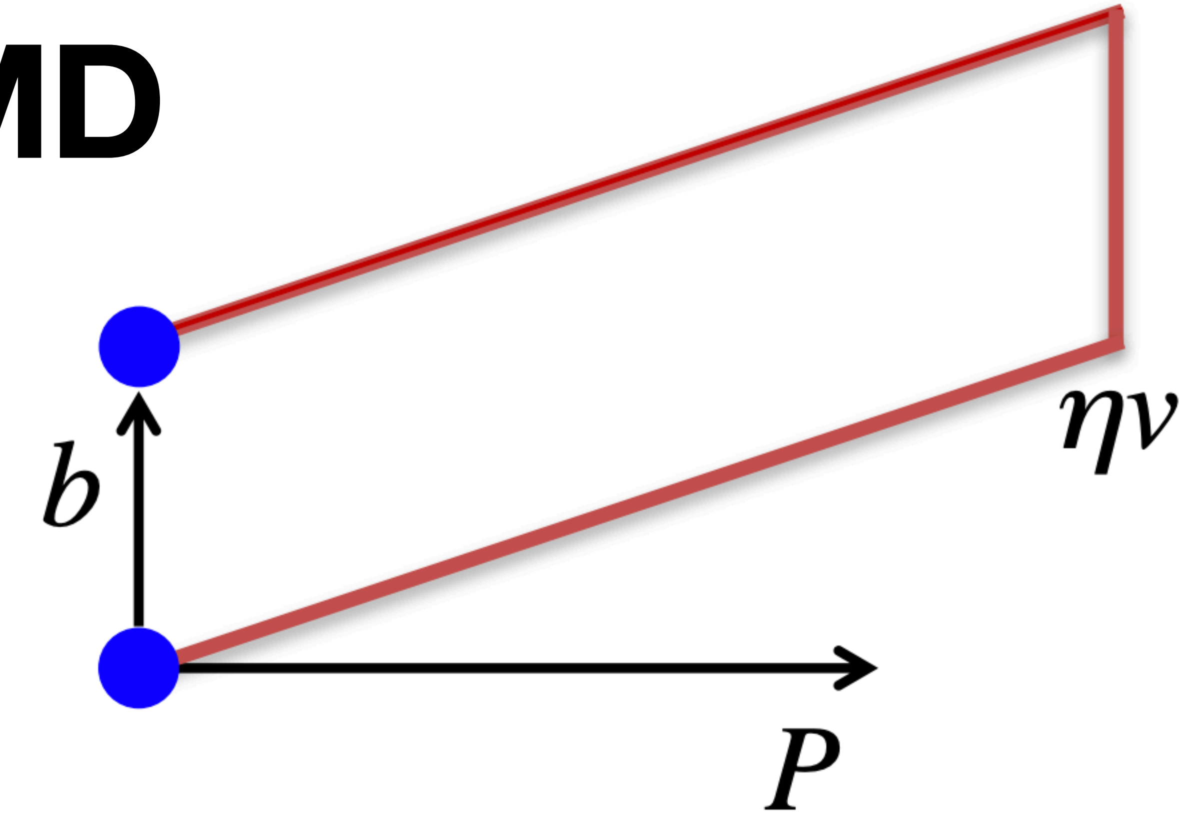
$$= \widetilde{\Phi}^{[\Gamma]}_{\mathrm{unsubtr.}}(b, P, S, \ldots) \cdot \mathcal{S} \cdot Z_{\mathrm{TMD}} \cdot Z_2$$

- Light cone limit obtained by

$$\hat{\zeta} = \frac{v \cdot P}{\sqrt{|v^2||P^2|}} \to \infty$$

B. Yoon et al PRD 96 (2017) 9, 094508

# Moments of the TMD




- Correlator with staple shape

$$\Phi^{[\Gamma]}(x, \boldsymbol{k}_T, P, S, \ldots)$$
$$= \int \frac{d^2\boldsymbol{b}_\mathrm{T}}{(2\pi)^2} \int \frac{d(b}{2\pi.} \, \Phi^{[\Gamma]}(x, \boldsymbol{k}_T, P, S, \ldots)$$
$$= \int \frac{d^2\boldsymbol{b}_\mathrm{T}}{(2\pi)^2} \int \frac{d(b\cdot P)}{2\pi P^+} e^{ix(b\cdot P) - i\boldsymbol{b}_\mathrm{T}\cdot\boldsymbol{k}_\mathrm{T}} \; \widetilde{\Phi}^{[\Gamma]}_{\text{subtr.}}\Big|_{b^+=0}$$

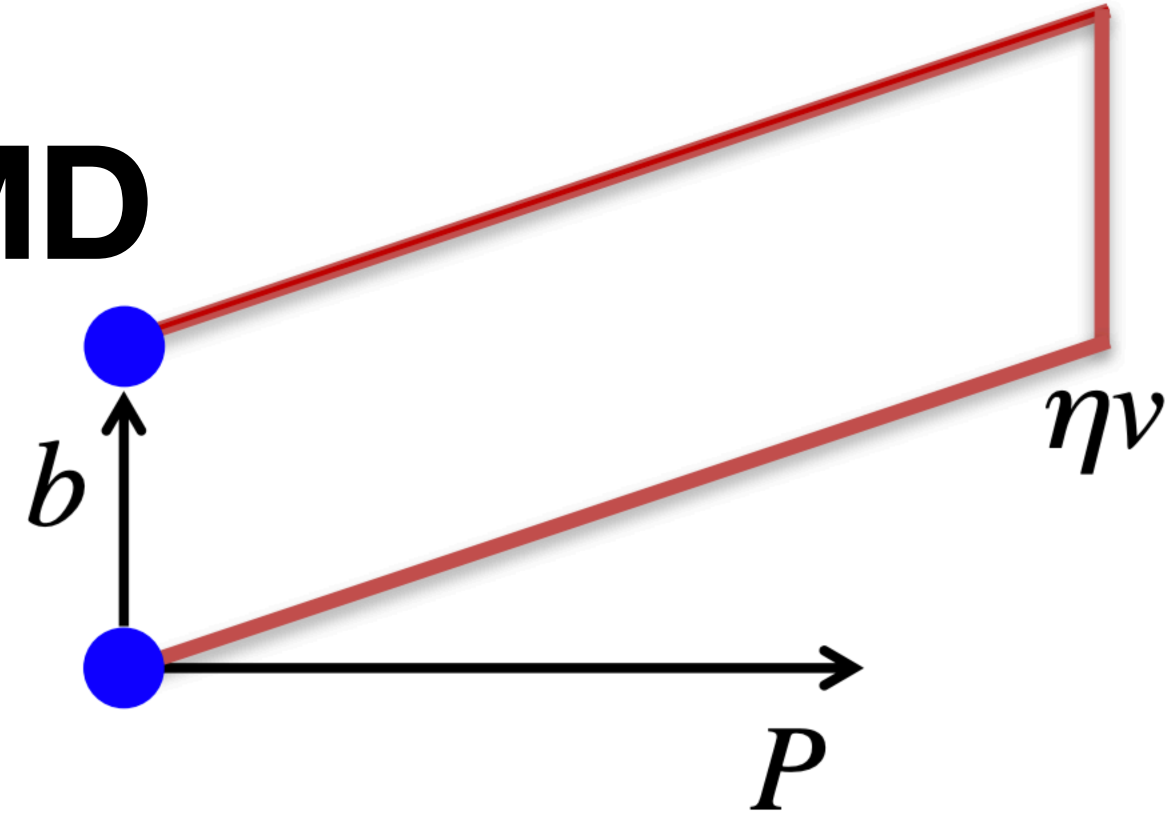
- Lorentz Decomp

$$\Phi^{[\gamma^+]} = f_1 - \frac{\epsilon_{ij}\boldsymbol{k}_i\boldsymbol{S}_j}{m_N} f_{1T}^\perp$$
$$\Phi^{[\gamma^+\gamma^5]} = \Lambda g_1 + \frac{\boldsymbol{k}_\mathrm{T}\cdot\boldsymbol{S}_\mathrm{T}}{m_N} g_{1T}$$
$$\Phi^{[i\sigma^{i+}\gamma^5]} = \boldsymbol{S}_i h_1 + \frac{(2\boldsymbol{k}_i\boldsymbol{k}_j - \boldsymbol{k}_\mathrm{T}^2\delta_{ij})\boldsymbol{S}_j}{2m_N^2} h_{1T}^\perp + \frac{\Lambda\boldsymbol{k}_i}{m_N} h_{1L}^\perp + \frac{\epsilon_{ij}\boldsymbol{k}_j}{m_N} h_1^\perp$$

$$\frac{1}{2P^+}\widetilde{\Phi}^{[\gamma^+]}_{\text{unsubtr.}} = \widetilde{A}_{2B} + im_N\epsilon_{ij}\boldsymbol{b}_i\boldsymbol{S}_j\widetilde{A}_{12B} \quad (8)$$
$$\frac{1}{2P^+}\widetilde{\Phi}^{[\gamma^+\gamma^5]}_{\text{unsubtr.}} = -\Lambda\widetilde{A}_{6B} + i[(b\cdot P)\Lambda - m_N(\boldsymbol{b}_\mathrm{T}\cdot\boldsymbol{S}_\mathrm{T})]\widetilde{A}_{7B} \quad (9)$$
$$\frac{1}{2P^+}\widetilde{\Phi}^{[i\sigma^{i+}\gamma^5]}_{\text{unsubtr.}} = im_N\epsilon_{ij}\boldsymbol{b}_j\widetilde{A}_{4B} - \boldsymbol{S}_i\widetilde{A}_{9B} - im_N\Lambda\boldsymbol{b}_i\widetilde{A}_{10B} + m_N[(b\cdot P)\Lambda - m_N(\boldsymbol{b}_\mathrm{T}\cdot\boldsymbol{S}_\mathrm{T})]\boldsymbol{b}_i\widetilde{A}_{11B} \quad (10)$$

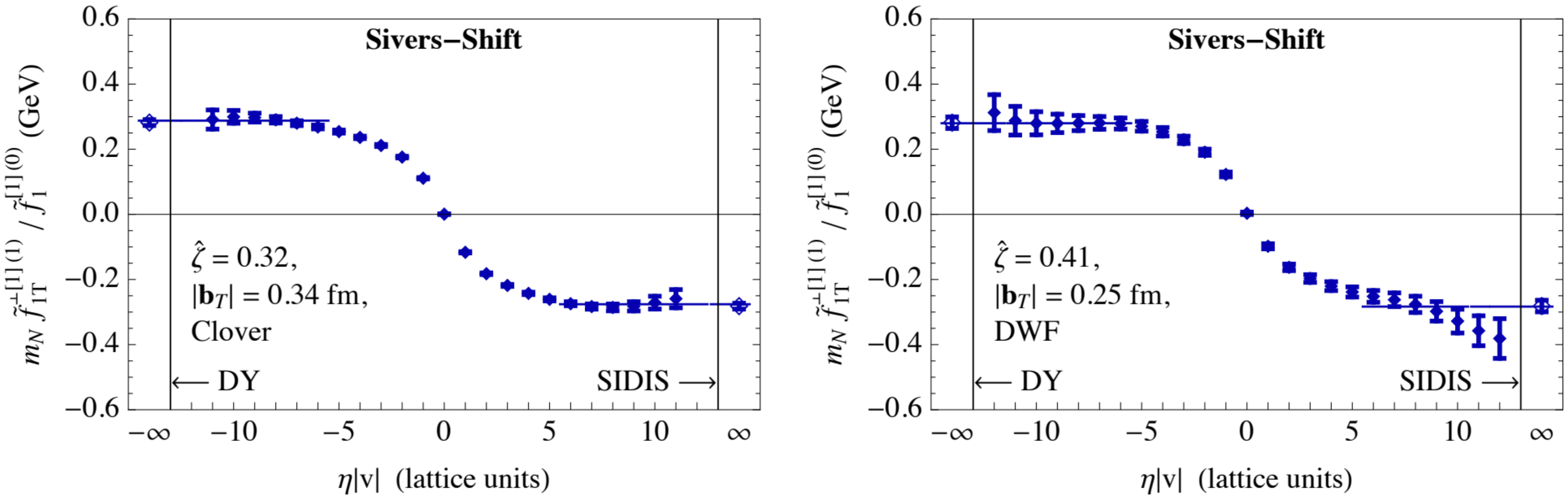
B. Yoon et al PRD 96 (2017) 9, 094508

# Moments of the TMD



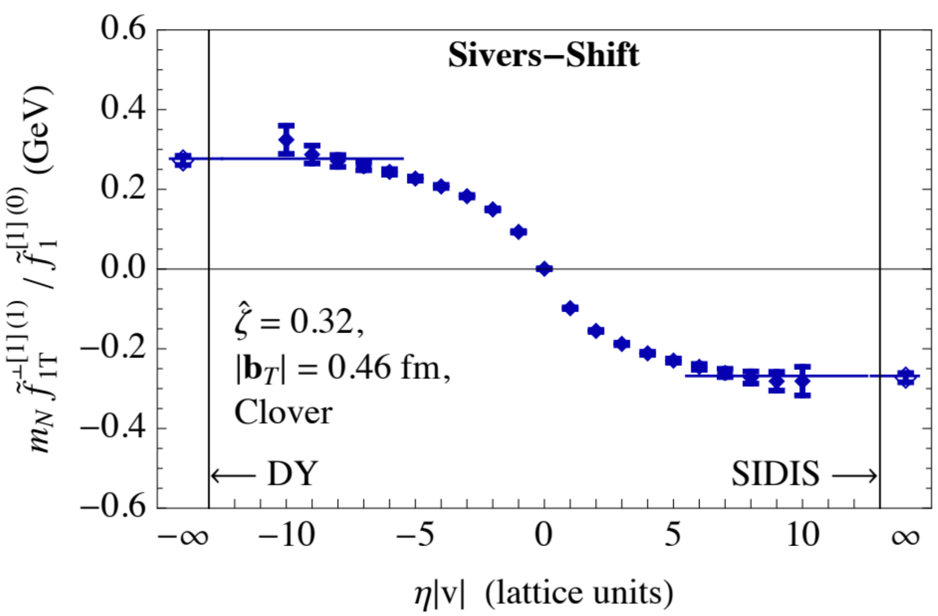

- Moments are taken from derivatives

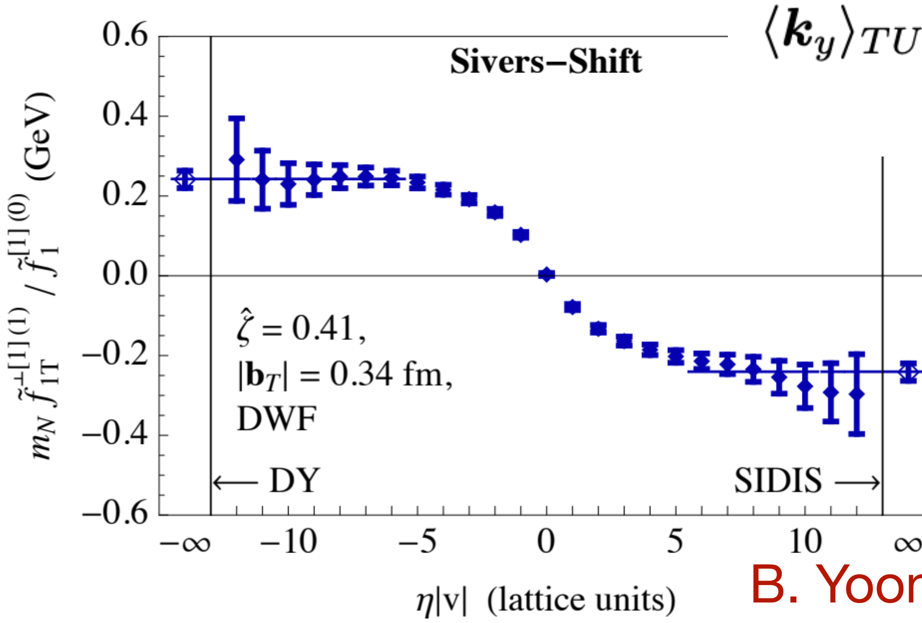
$$\tilde{f}^{[m](n)}(\boldsymbol{b}_\mathrm{T}^2, \ldots) \equiv n! \left(-\frac{2}{m_N^2}\partial_{\boldsymbol{b}_\mathrm{T}^2}\right)^n \int_{-1}^{1} dx\, x^{m-1} \cdot \qquad (19)$$
$$\cdot \int d^2\boldsymbol{k}_\mathrm{T}\, e^{i\boldsymbol{b}_\mathrm{T}\cdot\boldsymbol{k}_\mathrm{T}} f(x, \boldsymbol{k}_\mathrm{T}^2, \ldots) \;.$$




- Transversely polarized nucleon
- Unpolarized quarks
- How are the quarks moving in the final perpendicular direction?

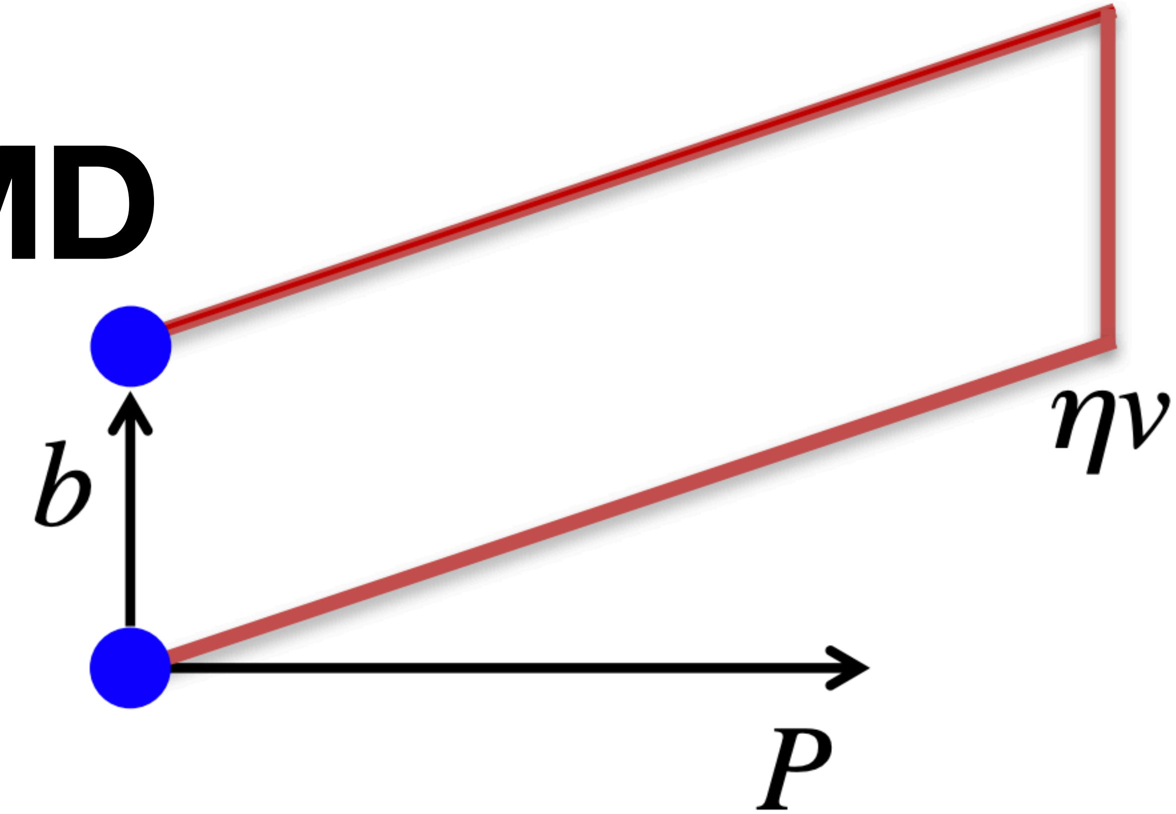
$$\langle \boldsymbol{k}_y \rangle_{TU}(\boldsymbol{b}_\mathrm{T}^2; \ldots) \equiv m_N \frac{\tilde{f}_{1T}^{\perp[1](1)}(\boldsymbol{b}_\mathrm{T}^2; \ldots)}{\tilde{f}_1^{[1](0)}(\boldsymbol{b}_\mathrm{T}^2; \ldots)} ,$$




B. Yoon et al PRD 96 (2017) 9, 094508

# Moments of the TMD

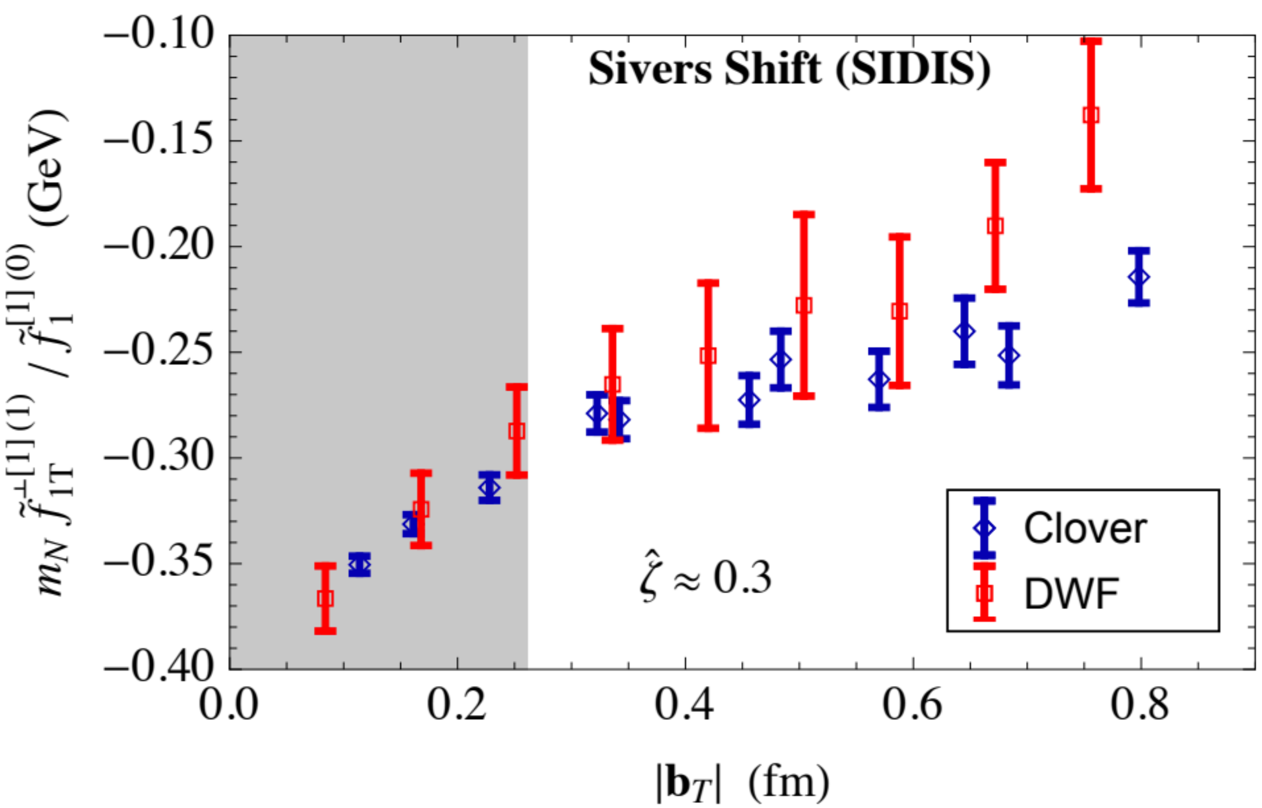


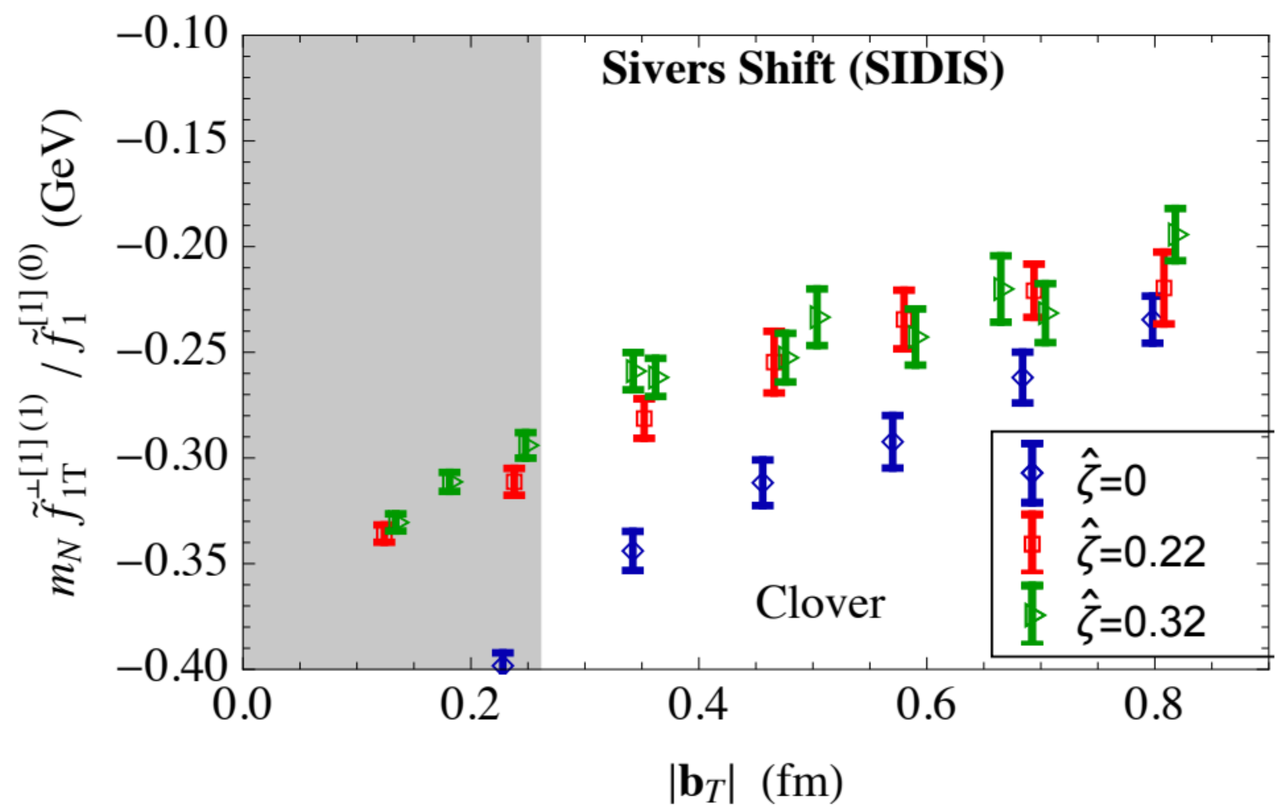

- Moments are taken from derivatives

$$\tilde{f}^{[m](n)}(\boldsymbol{b}_{\mathrm{T}}^2, \ldots) \equiv n! \left( -\frac{2}{m_N^2} \partial_{\boldsymbol{b}_{\mathrm{T}}^2} \right)^n \int_{-1}^{1} dx\, x^{m-1} \cdot \quad (19)$$

$$\cdot \int d^2\boldsymbol{k}_{\mathrm{T}}\, e^{i\boldsymbol{b}_{\mathrm{T}} \cdot \boldsymbol{k}_{\mathrm{T}}} f(x, \boldsymbol{k}_{\mathrm{T}}^2, \ldots) \, .$$

- Checking action and $\hat{\zeta} \to \infty$ limit




B. Yoon et al PRD 96 (2017) 9, 094508

# Generalized Transverse Momentum Moments

M. Engelhardt PRD 95 (2017) 9,094505

## Interpolating the angular momentum decompositions

**What shape for link $U$?**

- Generalized TMD

$$W^{U}_{\Lambda'\Lambda}(x,\Delta_T,k_T) = \frac{1}{2}\int \frac{dz^- d^2 z_T}{(2\pi)^3} e^{i(xP^+z^- - k_T\cdot z_T)} \left.\frac{\langle p',\Lambda'|\overline{\psi}(-z/2)\gamma^+ U\psi(z/2)|p,\Lambda\rangle}{\mathcal{S}[U]}\right|_{z^+=0}$$

$$W^{U}_{\Lambda'\Lambda} = \frac{1}{2m}\bar{u}(p',\Lambda')\left[F_{11} + \frac{i\sigma^{i+}k_T^i}{P^+}F_{12} + \frac{i\sigma^{i+}\Delta_T^i}{P^+}F_{13} + \frac{i\sigma^{ij}k_T^i\Delta_T^j}{m^2}F_{14}\right]u(p,\Lambda)$$

- Ji definition GPDs: Straight link
- Jaffe Manohar GTMDs: Staple link

**Conveniently Renormalization constants and Soft Factor $S[U]$ are multiplicative**

$$L_3^U = \int dx \int d^2 b_T \int d^2 k_T \ (b_T \times k_T)_3 \mathcal{W}^U(x,b_T,k_T) \ = -\int dx \int d^2 k_T \left.\frac{k_T^2}{m^2}F_{14}\right|_{\Delta_T=0}$$

$$L_3^U = \frac{1}{2P^+}\epsilon_{ij}\frac{\partial}{\partial z_{T,i}}\ \frac{\partial}{\partial \Delta_{T,j}}\left.\frac{\langle p',S'=\vec{e}_3|\overline{\psi}(-z/2)\gamma^+ U\psi(z/2)|p,S=\vec{e}_3\rangle}{\mathcal{S}(z_T^2)}\right|_{z^+=z^-=0\,,\ \Delta_T=0\,,\ z_T\to 0}$$

# Generalized Transverse Momentum Moments

M. Engelhardt PRD 95 (2017) 9,094505

## Interpolating the angular momentum decompositions

**What shape for link $U$?**

- Generalized TMD

$$W^{U}_{\Lambda'\Lambda}(x,\Delta_T,k_T) = \frac{1}{2}\int \frac{dz^- d^2 z_T}{(2\pi)^3} e^{i(xP^+z^- - k_T\cdot z_T)} \left.\frac{\langle p',\Lambda'|\overline{\psi}(-z/2)\gamma^+ U\psi(z/2)|p,\Lambda\rangle}{\mathcal{S}[U]}\right|_{z^+=0}$$

$$W^{U}_{\Lambda'\Lambda} = \frac{1}{2m}\bar{u}(p',\Lambda')\left[F_{11} + \frac{i\sigma^{i+}k_T^i}{P^+}F_{12} + \frac{i\sigma^{i+}\Delta_T^i}{P^+}F_{13} + \frac{i\sigma^{ij}k_T^i\Delta_T^j}{m^2}F_{14}\right]u(p,\Lambda)$$

- Ji definition GPDs: Straight link
- Jaffe Manohar GTMDs: Staple link

**Conveniently Renormalization constants and Soft Factor $S[U]$ are multiplicative**

- Use ratio with unpolarized TMD moment with sum rule to cancel renormalization and soft factor

$$f_1 = F_{11}|_{\Delta_T=0} = W^{U}_{++}|_{\Delta_T=0}$$

$$n = \int dx \int d^2k_T\, f_1 = \frac{1}{2P^+}\left.\frac{\langle p',S'=\vec{e}_3|\overline{\psi}(-z/2)\gamma^+ U\psi(z/2)|p,S=\vec{e}_3\rangle}{\mathcal{S}(z_T^2)}\right|_{z^+=z^-=0,\Delta_T=0,z_T\to 0}$$

# Generalized Transverse Momentum Moments

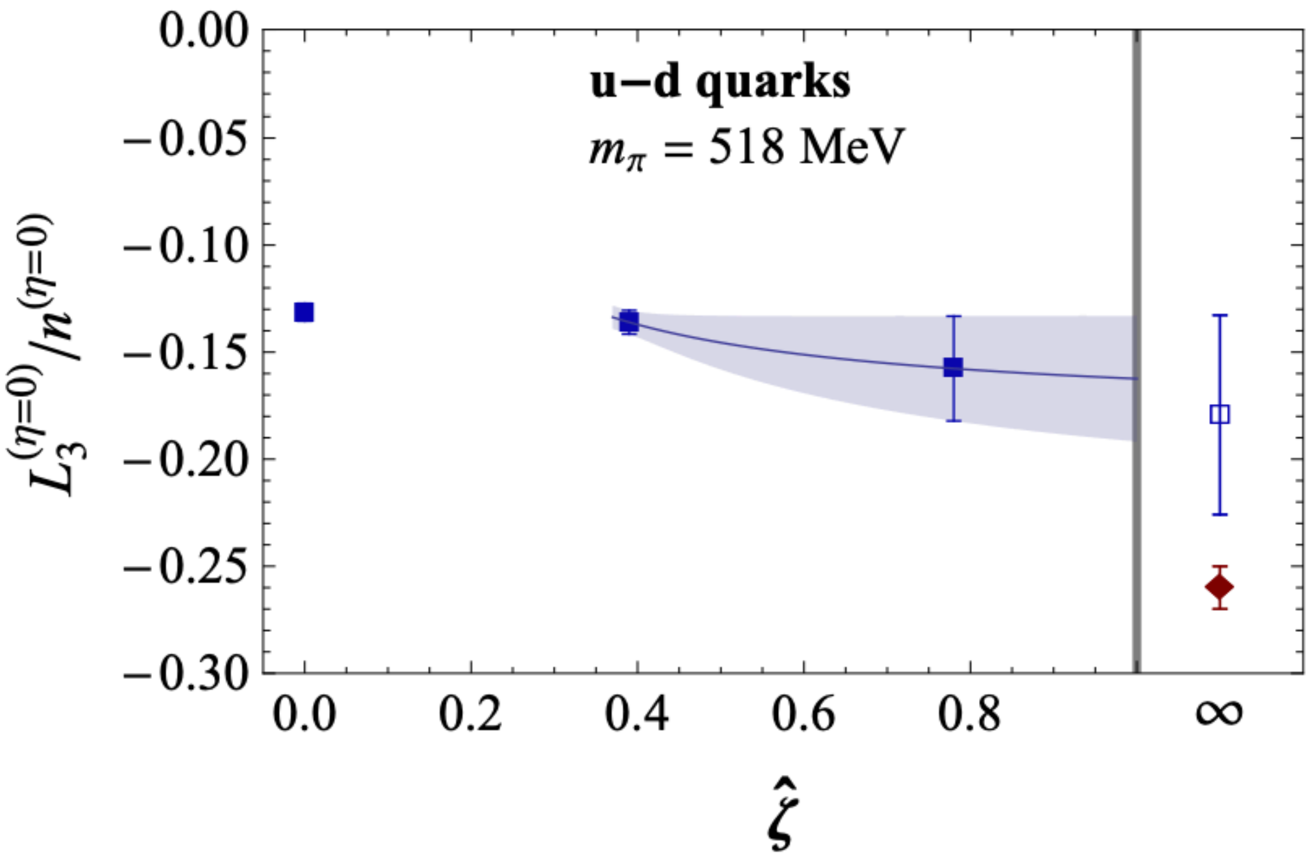
M. Engelhardt PRD 95 (2017) 9,094505

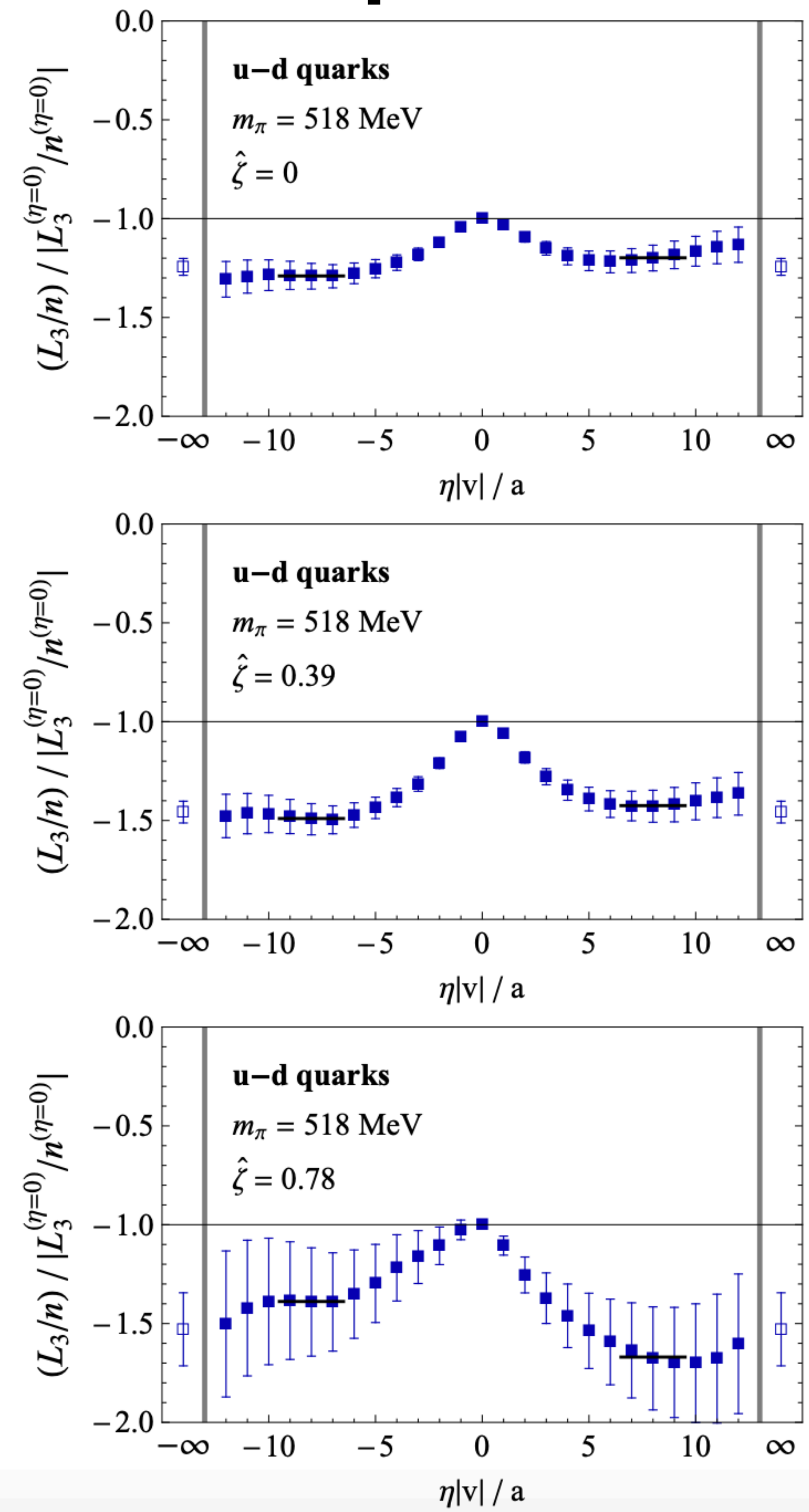
## Interpolating the angular momentum decompositions

- Ji definition GPDs: Straight link
- Jaffe Manohar GTMDs: Staple link

$$\hat{\zeta} = \frac{v \cdot P}{\sqrt{|v^2||P^2|}} \to \infty$$

- Comparison of direct moment for Ji definition with large rapidity extrapolation



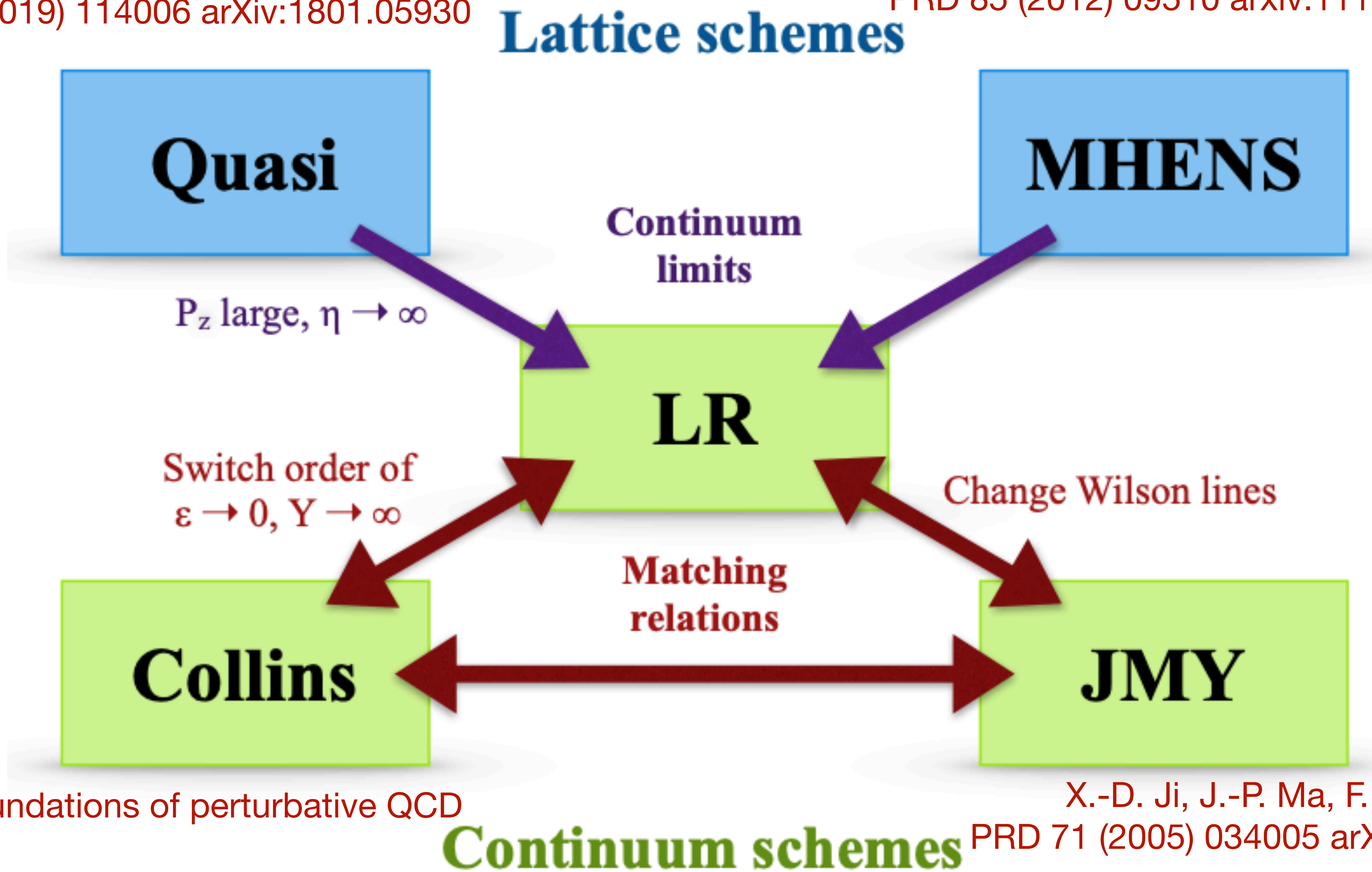



# Different TMDPDF Schemes

## Large Rapidity: One Scheme to unite them all

X.-D. Ji, L.-C. Jin, F. Yuan, J.-H. Zhang, Y. Zhao
PRD 99 (2019) 114006 arXiv:1801.05930

Musch-Hagler-Engelhardt-Negele-Schafer
PRD 85 (2012) 09510 arxiv:1111.4249

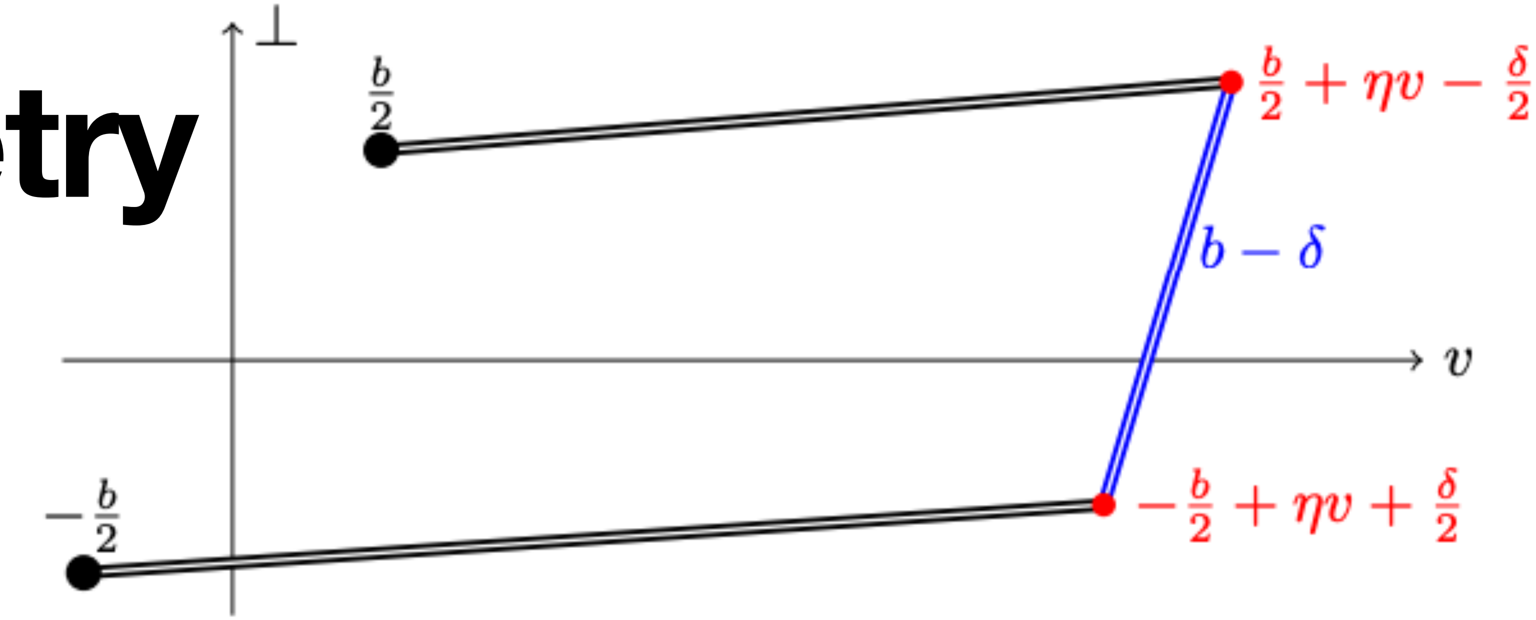



J.Collins, Foundations of perturbative QCD

X.-D. Ji, J.-P. Ma, F. Yuan
PRD 71 (2005) 034005 arXiv:0404183

Fig 1 of M. Ebert, S. Schindler, I. Stewart, Y. Zhao 2201.08401

# Staple Geometry




- Staple given by 3 vectors

Fig/Tab 2 of M. Ebert, S. Schindler, I. Stewart, Y. Zhao 2201.08401

| | Collins / LR | JMY | Quasi | MHENS |
|---|---|---|---|---|
| $b^\mu$ | $(0, b^-, b_\perp)$ | $(0, b^-, b_\perp)$ | $(0, b_T^x, b_T^y, \tilde{b}^z)$ | $(0, b_T^x, b_T^y, \tilde{b}^z)$ |
| $v^\mu$ | $(-e^{2y_B}, 1, 0_\perp)$ | $(v^- e^{2y'_B}, v^-, 0_\perp)$ | $(0, 0, 0, -1)$ | $(0, v^x, v^y, v^z)$ |
| $\delta^\mu$ | $(0, b^-, 0_\perp)$ | $(0, b^-, 0_\perp)$ | $(0, 0, 0, \tilde{b}^z)$ | $(0, 0, 0_\perp)$ |
| $P^\mu$ | $\frac{m_h}{\sqrt{2}}(e^{y_P}, e^{\;y_P}, 0_\perp)$ | $\frac{m_h}{\sqrt{2}}(e^{y_P}, e^{\;y_P}, 0_\perp)$ | $m_h(\cosh y_{\tilde{P}}, 0, 0, \sinh y_{\tilde{P}})$ | $m_h\left(\cosh y_P, \frac{P^x}{m_h}, \frac{P^y}{m_h}, \sinh y_P\right)$ |
| $b^2$ | $-b_T^2$ | $-b_T^2$ | $-b_T^2 - (\tilde{b}^z)^2$ | $-b_T^2 - (\tilde{b}^z)^2$ |
| $(\eta v)^2$ | $-2\eta^2 e^{2y_B}$ | $2\eta^2 (v^-)^2 e^{2y'_B}$ | $-\tilde{\eta}^2$ | $-\eta^2 \vec{v}^2$ |
| $P \cdot b$ | $\frac{m_h}{\sqrt{2}} b^- e^{y_P}$ | $\frac{m_h}{\sqrt{2}} b^- e^{y_P}$ | $-m_h \tilde{b}^z \sinh y_{\tilde{P}}$ | $m_h \sinh y_P \tilde{b}^z + P^x b_T^x + P^y b_T^y$ |
| $\frac{b \cdot (\eta v)}{\sqrt{\lvert(\eta v)^2 b^2\rvert}}$ | $-\frac{b^- e^{y_B}}{\sqrt{2}\, b_T}\,\mathrm{sgn}(\eta)$ | $\frac{b^- e^{y'_B}}{\sqrt{2}\, b_T}\,\mathrm{sgn}(\eta)$ | $\frac{\tilde{b}^z}{\sqrt{(\tilde{b}^z)^2 + b_T^2}}\,\mathrm{sgn}(\eta)$ | $\frac{b_T^x v^x + b_T^y v^y + \tilde{b}^z v^z}{\sqrt{v_T^2 + (v^z)^2}\sqrt{b_T^2 + (\tilde{b}^z)^2}}$ |
| $\frac{P \cdot (\eta v)}{\sqrt{P^2 \lvert\eta v\rvert^2}}$ | $\sinh(y_P - y_B)\,\mathrm{sgn}(\eta)$ | $\cosh(y_P - y'_B)\,\mathrm{sgn}(\eta)$ | $\sinh y_{\tilde{P}}\,\mathrm{sgn}(\eta)$ | $\frac{P^x v^x + P^y v^y + m_h v^z \sinh y_P}{\sqrt{v_T^2 + (v^z)^2}\sqrt{m_h^2 + P_x^2 + P_y^2}}$ |
| $\frac{\delta^2}{b^2}$ | $0$ | $0$ | $\frac{(\tilde{b}^z)^2}{b_T^2 + (\tilde{b}^z)^2}$ | $0$ |
| $\frac{b \cdot \delta}{b^2}$ | $0$ | $0$ | $\frac{(\tilde{b}^z)^2}{b_T^2 + (\tilde{b}^z)^2}$ | $0$ |
| $\frac{P \cdot \delta}{P \cdot b}$ | $1$ | $1$ | $1$ | $0$ |
| $\frac{\delta \cdot (\eta v)}{b \cdot (\eta v)}$ | $1$ | $1$ | $1$ | $0$ |
| $P^2$ | $m_h^2$ | $m_h^2$ | $m_h^2$ | $m_h^2$ |

- Lorentz invariants in the different definitions can be related to each other

- Appropriate limits must be taken for correspondence

- $\tilde{b}^z \to 0$ sets $\frac{\delta^2}{b^2} = 0$

# Stapleless Gauge Fixed quasi-TMDs

X. Gao, Wei-Liu, Y. Zhao PRD 109 (2024) 9, 094506
Y. Zhao arXiv:2311.01391

- Wilson lines contain many complications such as power divergences in $z/a$

- Fix to a gauge and take operator with out Wilson line

$$O^\mu(b) = \bar{q}(-\frac{b}{2})\gamma^\mu q(\frac{b}{2})$$

- Under infinite boost the Coulomb gauge $\overrightarrow{\nabla} \cdot A = 0$ corresponds to the light-cone gauge $A^+ = 0$

- Requires gauge fixing: Introduces new $O(a)$ errors and complications from Gribov copies (copies of equivalent yet different configurations some same HMC sample with the identical gauge fixing)

# TMDWF and the CS Kernel

- TMDPDF operator has evolution with a rapidity scale via the CS Kernel

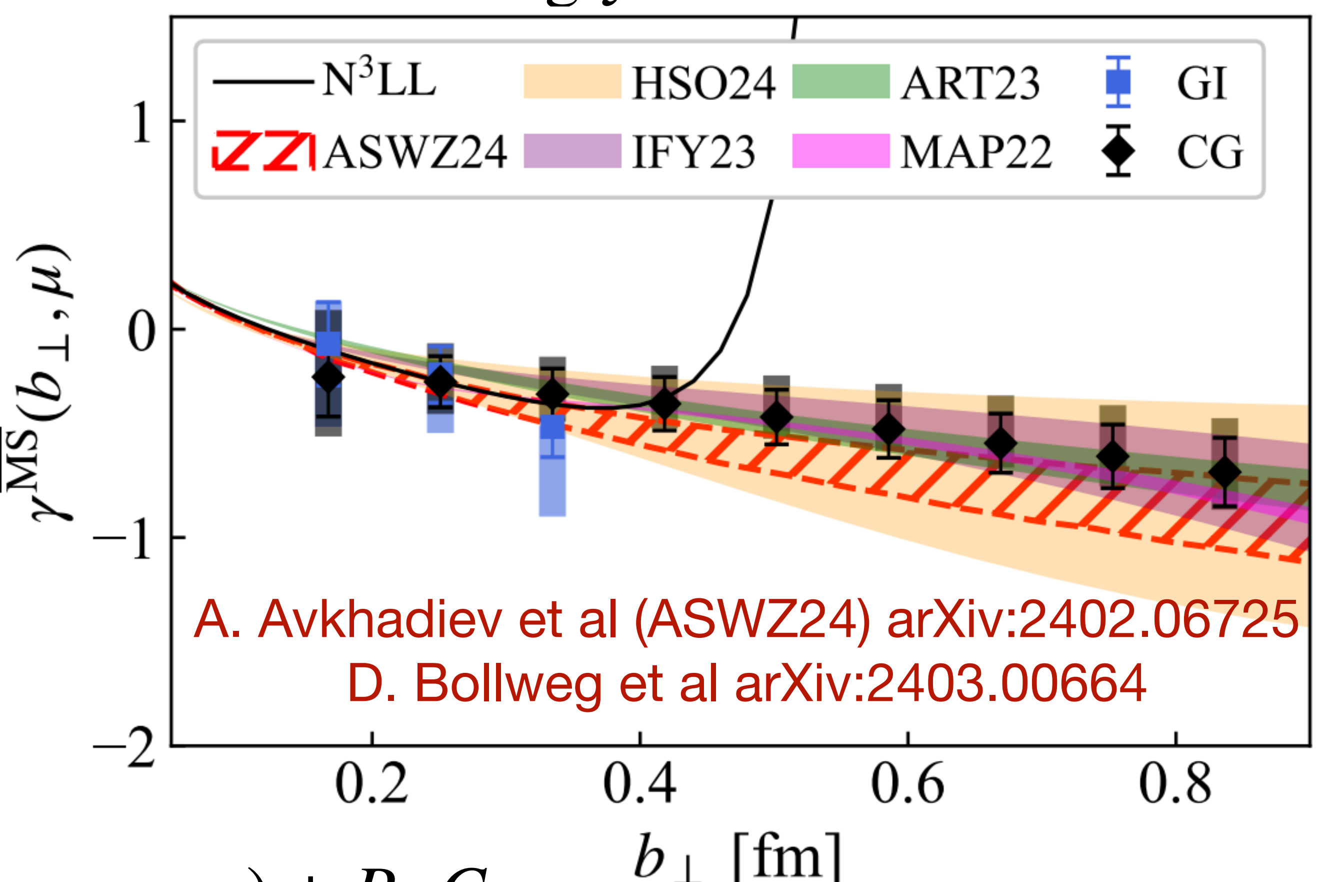
- Really it's an operator dependent evolution of how the stap

- CS Kernel is perturbatively calculable only at low $b_T$

- Use LaMET to relate $p_z \to \zeta$

- Finite difference approximation using the TMDWF

$$\gamma(b_T, \mu) = 2\frac{d}{d\log\zeta}\log f^{\mathrm{TMD}}(x, b_T, \mu, \zeta)$$

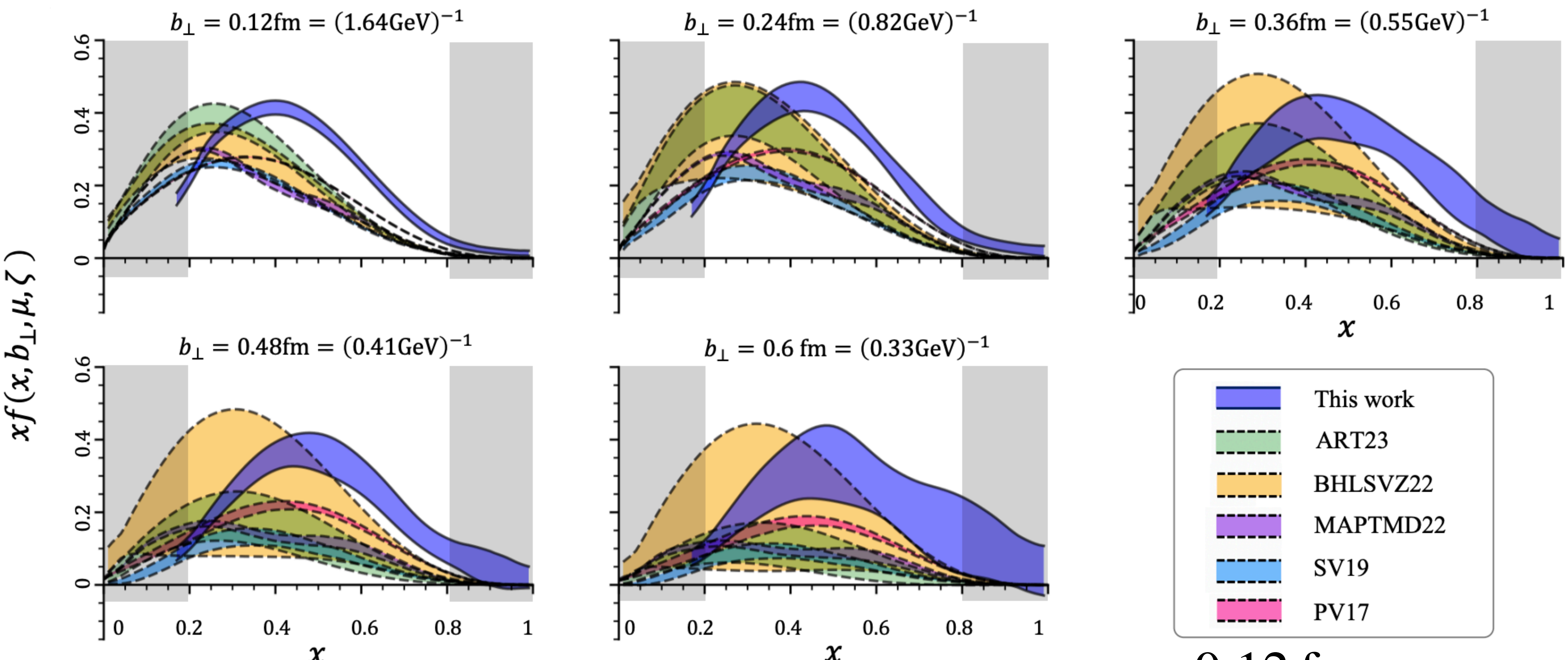



$$\langle 0 | O^\mu(b_\parallel, b_T) | p \rangle = \int dx e^{ixp\cdot z}\phi(x, b_T, p, \mu)$$

$$\gamma(b_T, \mu) = \frac{1}{\log\frac{p_2}{p_1}}\log\frac{\phi(x, b_T, p_2, \mu)}{\phi(x, b_T, p_1, \mu)} + \delta\gamma^{\overline{\mathrm{MS}}}(x, \mu, p_1, p_2) + P.C.$$

# $x$ dependence of TMDPDF

- TMDPDF from two ensembles extrapolated to physical pion mass
- Different momenta give consistent results within unshaded region

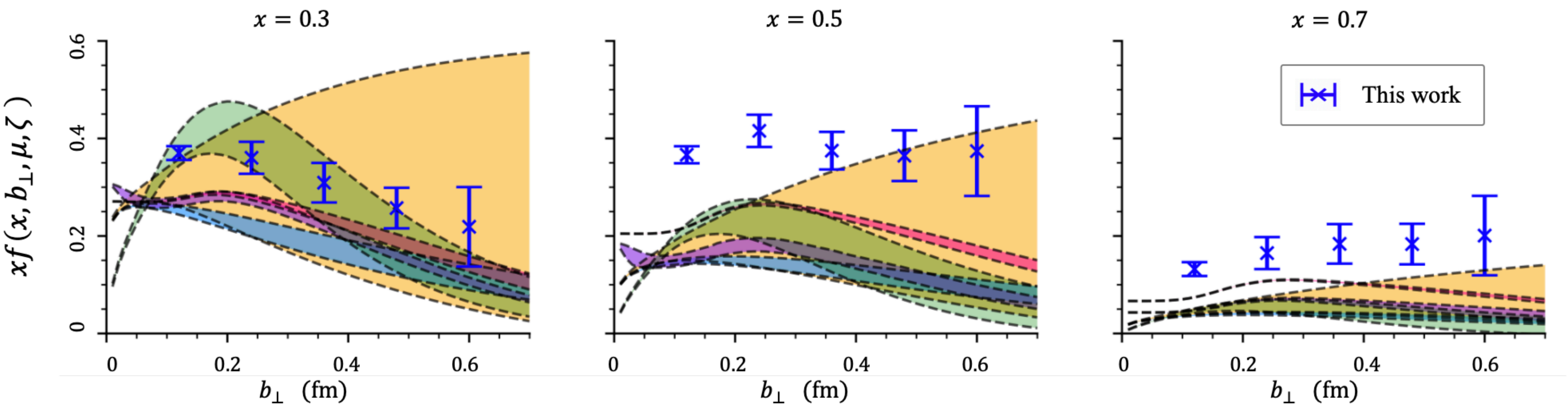



$a = 0.12\ \text{fm}$

J-C He et al (LPC) arXiv:221.02340

# $x$ dependence of TMDPDF

- TMDPDF from two ensembles extrapolated to physical pion mass
- Different momenta give consistent results within unshaded region




J-C He et al (LPC) arXiv:221.02340

$a = 0.12\,\mathrm{fm}$

# TMD Summary

- Important part of the TMDs are non-perturbative large $b_T$ behavior accessible from lattice calculations

- Direct calculations capable of understanding orbital angular momentum of quarks and reproduce sign shifts

- Quasi-PDF methods used to determine CS Kernel, Soft factor (didn't show you), and TMDPDF

# Lattice QCD and Structure

- Lattice QCD is a difficult but powerful method for QCD
  - Gain access to hadrons in terms of quarks and gluons
  - Lose analytic control of results and gain statistical uncertainty
- Vast computing resources are needed
  - Optimized codes for efficient algorithms are required to have best cost/benefit
- Entering era where high quality results of complicated hadron structure observables are in reach