Hadron Structure from Lattice QCD

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Lattice QCD reminder

Path integral is beginning of QFTs

- Want to describe hadrons in a theory of quarks and gluons

Gauge Action

$$S_{\text{QCD}}[A^\mu, \psi, \bar{\psi}] = \int d^4x \ \sum_f \bar{\psi}_i^f(x) \left[i\not{D}_{ij} - m_f \delta_{ij} \right] \psi_j^f(x) - \frac{1}{4} F^a_{\mu\nu}(x) F_a^{\mu\nu}(x)$$

Dirac Matrix

$$D^\mu_{ij} = \partial^\mu \delta_{ij} + g A^\mu_a t^a_{ij}$$

- Feynman Path Integral for vacuum expectation values

$$\langle O(t_1)\ldots O(t_N) \rangle_{conn} = \frac{\int d[A^\mu] d[\psi] d[\bar{\psi}] \ O(t_1)\ldots O(t_N) \, e^{iS_{\text{QCD}}[A^\mu, \psi, \bar{\psi}]}}{\int d[A^\mu] d[\psi] d[\bar{\psi}] \ e^{iS_{\text{QCD}}[A^\mu, \psi, \bar{\psi}]}}$$

Minkowski

- $O(t_1)$ is any function of the fields at a fixed time
  - Maybe it can create/annihilate a nucleon or pion or it represents a series of effective Weak or EM interactions

Lattice QCD reminder

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Dirac Matrix

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$$\langle O(t_1) \ldots O(t_N) \rangle_{conn} = \frac{\int d[A^\mu] d[\psi] d[\bar{\psi}] \; O(t_1) \ldots O(t_N) \, e^{-S^E_{\text{QCD}}[A^\mu, \psi, \bar{\psi}]}}{\int d[A^\mu] d[\psi] d[\bar{\psi}] \; e^{-S^E_{\text{QCD}}[A^\mu, \psi, \bar{\psi}]}}$$

Euclidean

- $O(t_1)$ is any function of the fields at a fixed time
    - Maybe it can create/annihilate a nucleon or pion or it represents a series of effective Weak or EM interactions

- I will always assume products of operators in VEV will have explicit and fixed times to simplify time ordering and Wick rotations.

Markov Chain Monte Carlo Recap

A very large dimension integral to do

- For Numerical Evaluation, we start in Euclidean time and Wick rotate to Minkowski at later step **(IF POSSIBLE!)**
- We want to evaluate a lattice regulated path integral for any action at any coupling

$$\langle O \rangle = \int d[\phi]\, O(\phi) \frac{e^{-S_E[\phi]}}{Z}$$

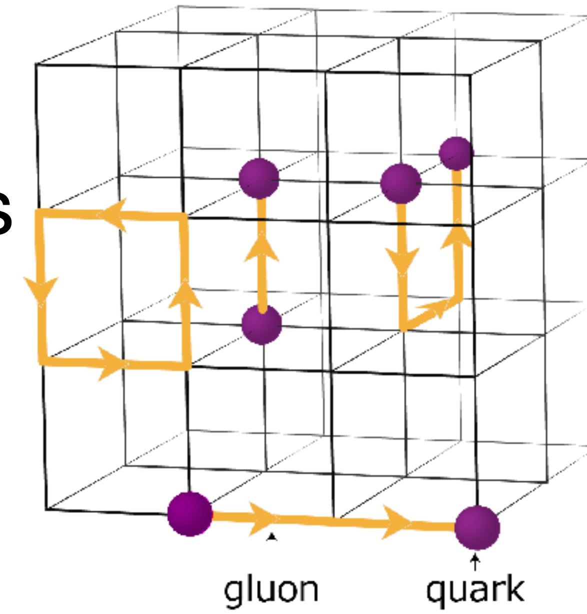
- Sample variables (the fields on lattice sites) $\{\phi_x\}$ with probability $P[\phi] = e^{-S_E[\phi]}/Z$

- Apply measurement to the samples and average to approximate path integral

$$\langle O \rangle = Z^{-1} \int d[\phi]\, O(\phi) e^{-S_E[\phi]} \approx N^{-1} \sum_i^N O(\phi_i)$$

Discretizing the Path Integral

Free Scalar fields as random Gaussian variables



- Feynman Path Integral for vacuum expectation values

$$\langle O_1 \ldots O_N \rangle^E_{conn} = \frac{\int d[\phi]\ O_1 \ldots O_N\, e^{-S^E[\phi]}}{\int d[\phi]\ e^{-S^E[\phi]}} \qquad d[\phi(x)] \to \prod_x d\phi_x$$

- Philosophy: After discretizing infinite field of operators to finite grid, replace operators with actual numbers and matrices.
- Generate random values of field with probability $P[\phi] = \frac{1}{Z}\exp[-S_E[\phi]]$
- Free Scalar Lagrangian $L_E(\phi) = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{m^2}{2}\phi^2$

Allow field to only be on a 4-d grid with spacing a

Gaussian Random Number for each grid point

$$S_E(\phi) = a^4 \sum_{x\in\Lambda} \frac{1}{2}\sum_{\mu=1}^{4}\left(\frac{\phi(x+a\hat{\mu}) - \phi(x-a\hat{\mu})}{2a}\right)^2 + \frac{m^2}{2}\phi^2(x) = \frac{a^4}{2}\sum_{x,y\in\Lambda}\phi_x M_{xy}\phi_y$$

- Average $O_1 \ldots O_N$ as function of variables from each random sample

For quarks in QCD, M has dimension $V \times N_c \times N_s \sim (48^3 \times 96) \times 3 \times 4 \approx 127M$

2 point functions in Euclidean time

Times are important to fix for translation to Minkowski space

- What is Euclidean time dependence of correlator

$$\langle O(T)O(0)\rangle^E_{conn} = \frac{\int d[\phi]\ O(T)O(0)\, e^{-S^E[\phi]}}{\int d[\phi]\ e^{-S^E[\phi]}}$$

O is any operator of interest from a fixed timeslice.

Could be $O(t) = \phi(\vec{0}, t)$ **or** $O(t) = \sum_{\vec{x}} e^{i\vec{p}\cdot\vec{x}}\phi(\vec{x}, t)$ **or** $O(t) = \sum_{\vec{x},\vec{y}} e^{i\vec{p}\cdot(\vec{x}-\vec{y})}\phi(\vec{x}, t)\phi(\vec{y}, t)$

- Insert complete set of energy Eigen states (sum in finite volume)

$$\langle O(T)O(0)\rangle^E_{conn} = \langle\Omega|O(T)O(0)|\Omega\rangle = \sum_n \frac{1}{2E_n}\langle\Omega|O(T)|n\rangle\langle n|O(0)|\Omega\rangle = \sum_n |Z_n|^2 e^{-E_nT}$$

$$\sum_n \frac{1}{2E_n}|n\rangle\langle n|$$

$$O(T) = e^{HT}O(0)e^{-HT} \qquad Z_n = \frac{1}{\sqrt{2E_n}}\langle\Omega|O(0)|n\rangle$$

Time translation in Euclidean spacetime

$$H|n\rangle = E_n|n\rangle \qquad H|\Omega\rangle = 0|\Omega\rangle \qquad O(T) = e^{iHT}O(0)e^{-iHT}$$

Time translation in Minkowski spacetime

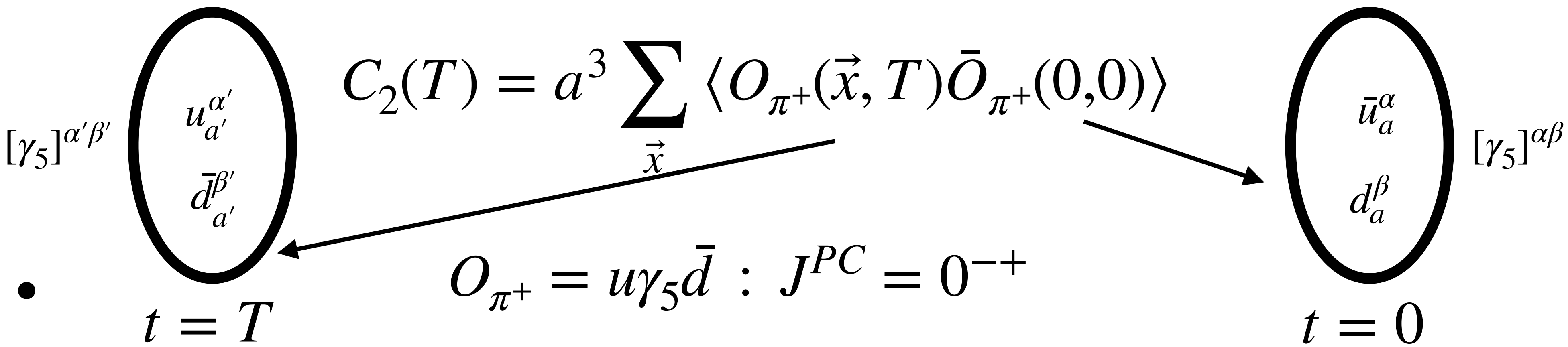
- Low Energy spectrum dominates the large Euclidean time limit

Wick's theorem makes graphs

Just like in PT, but only done for quarks

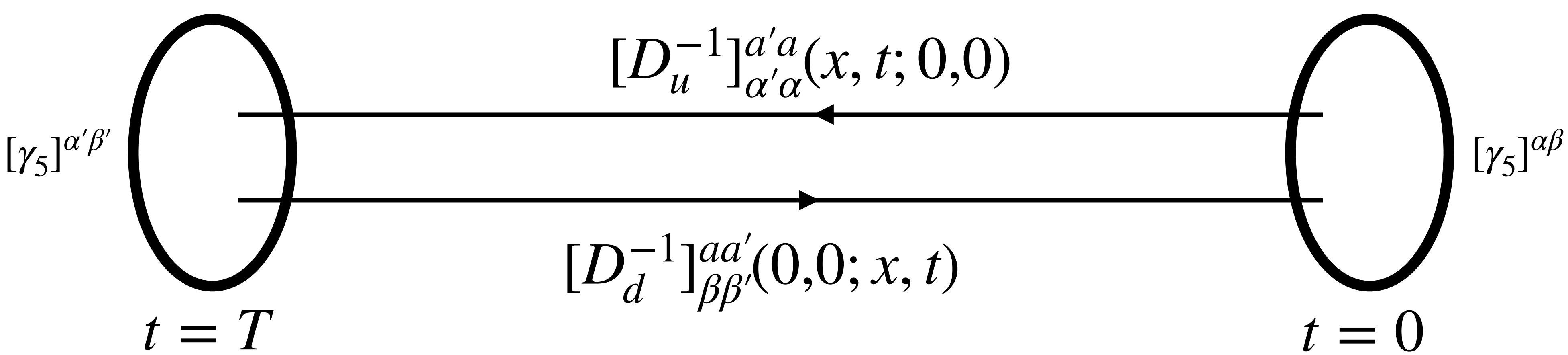
$$\langle O(t_1)\ldots O(t_N)\rangle_{conn} = \frac{\int d[A^\mu]\prod_q [d[q]d[\bar{q}]]\ O(t_1)\ldots O(t_N)\, e^{-S_g^E[A^\mu]-\sum_q \bar{q}D_q q}}{\int d[A^\mu]\prod_q [d[q]d[\bar{q}]]\ e^{-S^E_{\rm QCD}}}$$

- Interpolator field $O_h(t)$ has quantum numbers of desired hadron



Any operator with right flavor and J^{PC} will do

- Wick's Theorem contracts spin-color-space matrices



Propagators are inverse Dirac matrix are functions of gauge links and contain info on quark/gluon interactions and quark loops from determinant

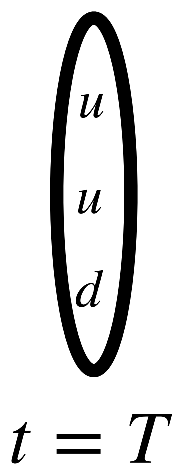
- Simply for light quarks $C_2 = \langle \mathrm{Tr}\left[D^{-1}(t;0)\gamma_5 D^{-1}(0;t)\gamma_5\right]\rangle$

Trace spin and color

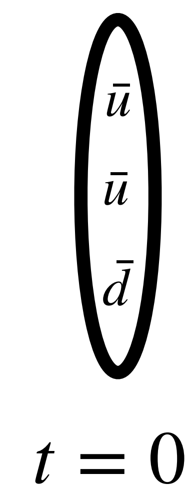
Wick's theorem makes graphs

Just like in PT, but these include all gluon interactions

- Interpolator field $O_h(t)$ has quantum numbers of desired hadron

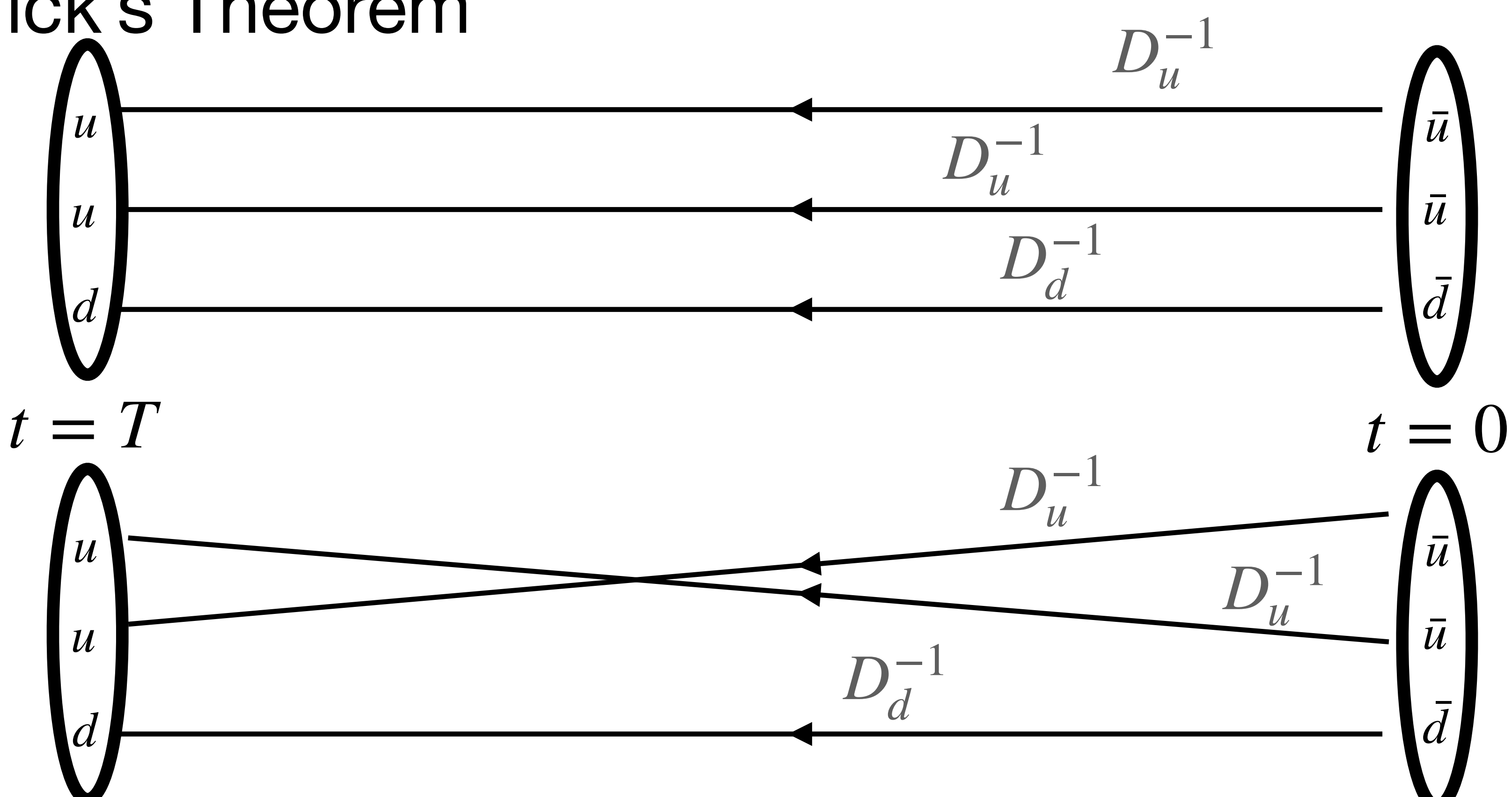


Interpolators can define specific spin and color combinations to make a Nucleon



Any operator with right flavor and J^{PC} will do

- Wick's Theorem



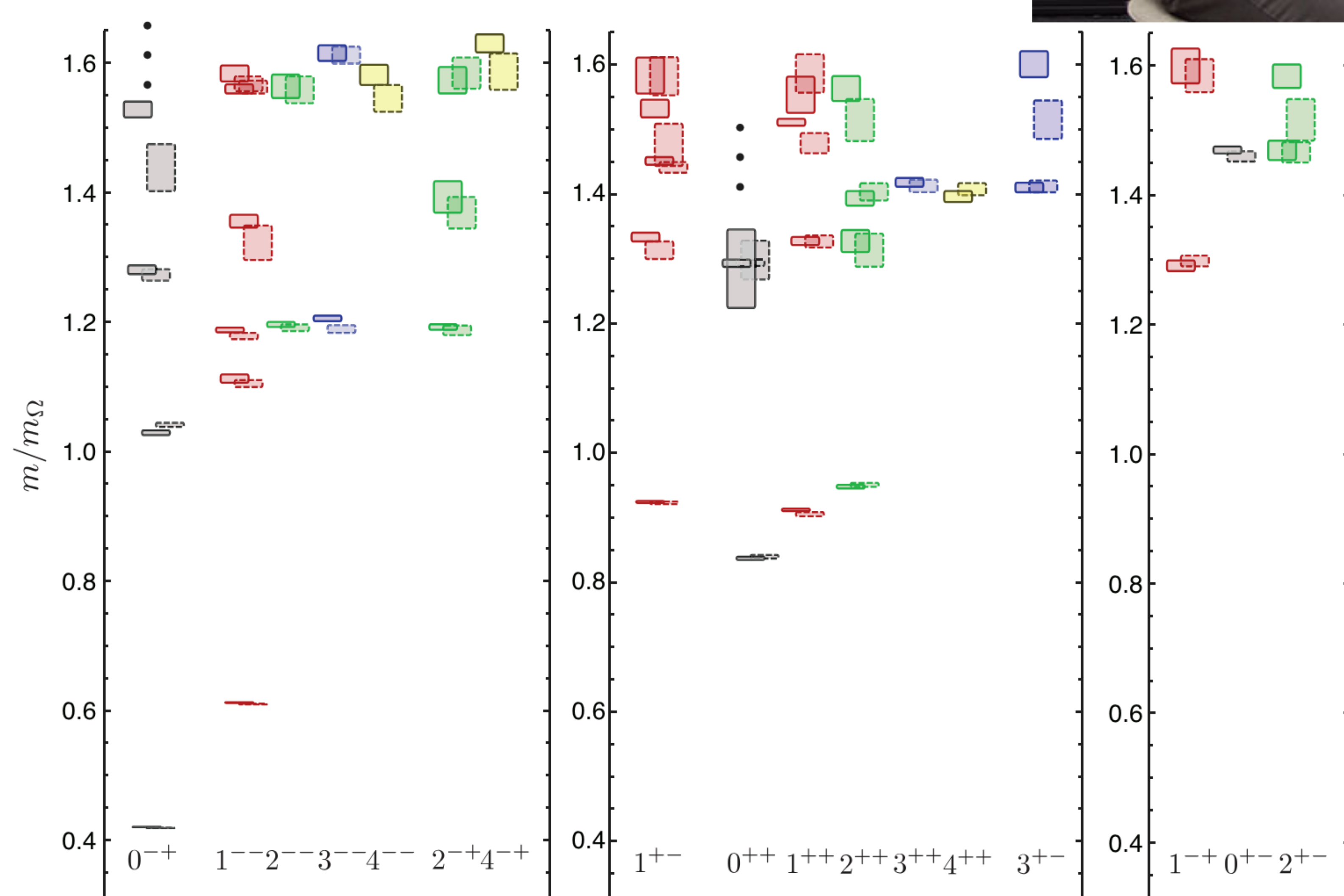
Propagators are inverse Dirac matrix are functions of gauge links and contain info on quark/gluon interactions and quark loops from determinant

Hadron Spectrum

HadSpec Collaboration

$$C^{ij}(T) = \langle O^i(T)\bar{O}^j(0)\rangle = \sum_n Z_n^i Z_n^{*j} e^{-E_n T}$$

- Studying correlation matrix access higher states with GEVP
- PRD 82 (2010) 034508



Spin, Parity,
Charge Conjugation

J^{PC}

Matrix elements of hadrons

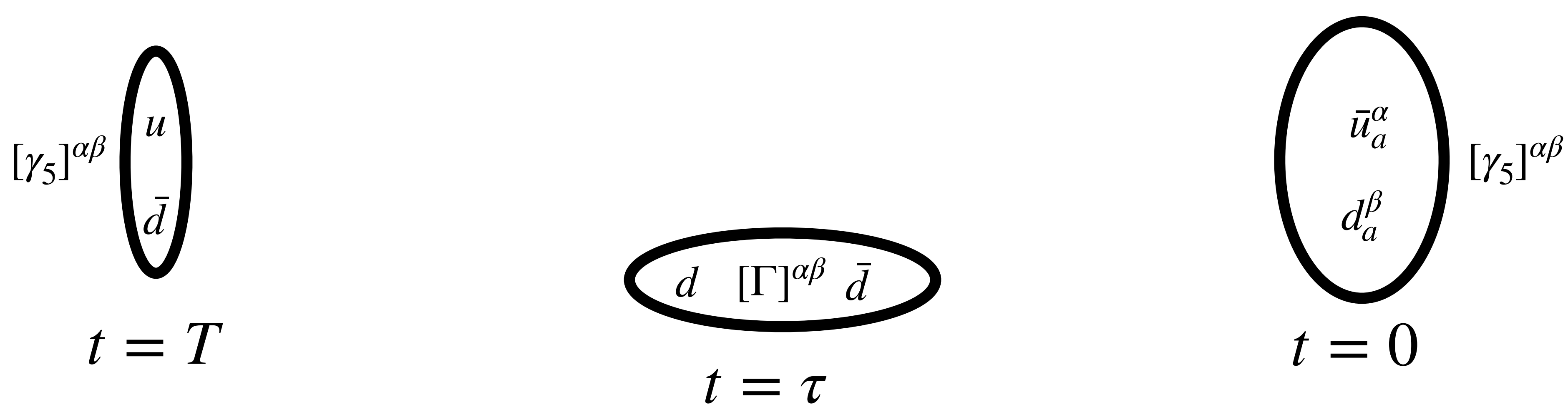
- 3 operators: $\langle O(T)\, J(\tau)\, \bar{O}(0)\rangle = \sum_{n,m} Z_n Z_m^* e^{-E_n(T-\tau)-E_m\tau}\langle n\,|\,J\,|\,m\rangle$
- Expand with complete set of states

$$O(T) = e^{HT}O(0)e^{-HT} \qquad Z_n = \frac{1}{\sqrt{2E_n}}\langle\Omega\,|\,O(0)\,|\,n\rangle$$

Time translation in Euclidean space

- Wick contractions: Connect quarks in all possible ways

$$J(\tau) = \sum_{\vec{x}} [\bar{d}\Gamma d](\vec{x},\tau)$$



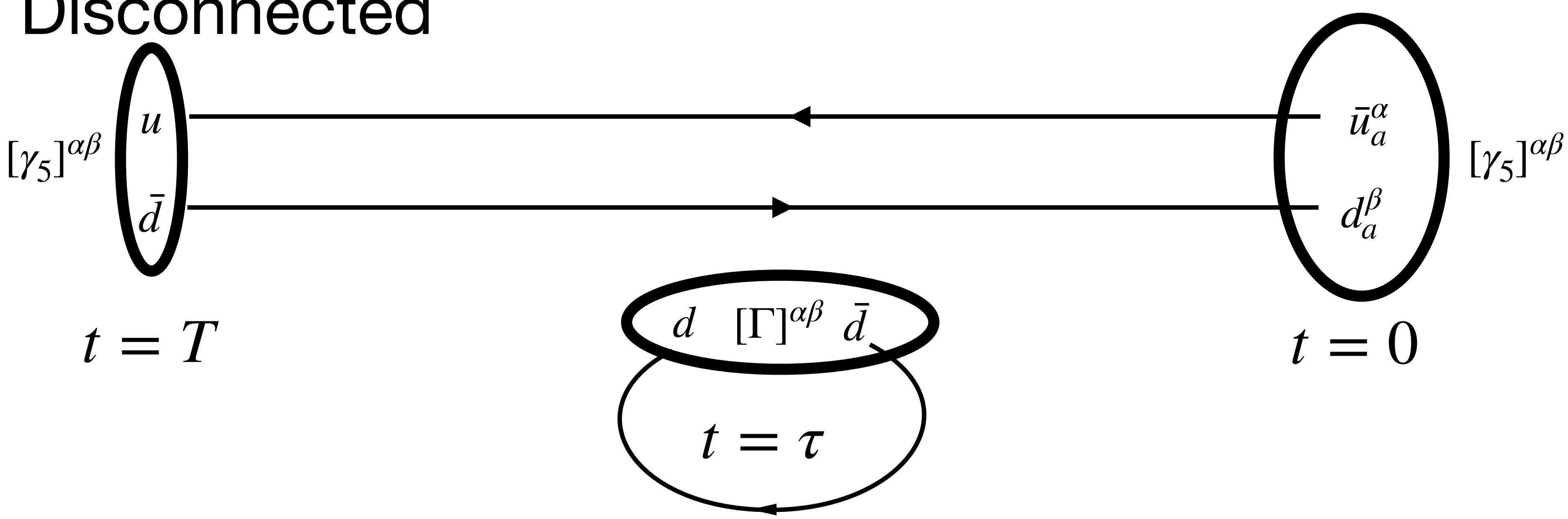
Matrix elements of hadrons

THESE ARE NOT FEYNMAN DIAGRAMS WHERE DISCONNECTED DIAGRAMS ARE 0.

$$\langle O(t_1)\ldots O(t_N)\rangle_{conn} = \frac{\int d[A^\mu]d[\psi]d[\bar\psi]\;\; O(t_1)\ldots O(t_N)\, e^{iS_{\mathrm{QCD}}[A^\mu,\psi,\bar\psi]}}{\int d[A^\mu]d[\psi]d[\bar\psi]\;\; e^{iS_{\mathrm{QCD}}[A^\mu,\psi,\bar\psi]}}$$

- 3 operators: $\langle O(T)\, J(\tau)\, \bar O(0)\rangle = \sum_{n,m} Z_n Z_m^* e^{-E_n(T-\tau)-E_m\tau}\langle n\,|\,J\,|\,m\rangle$

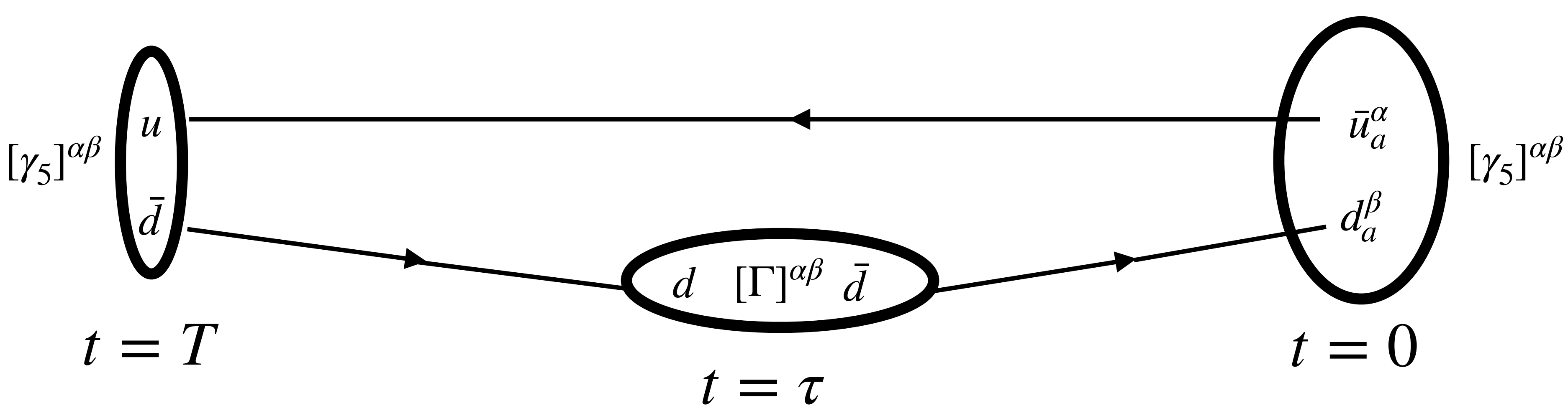
Disconnected



Disconnected diagrams are noisier. Avoid with iso-vector $u - d$ quarks

$$A(\tau) = \sum_{\vec x} [\bar d \Gamma d](\vec x, \tau)$$

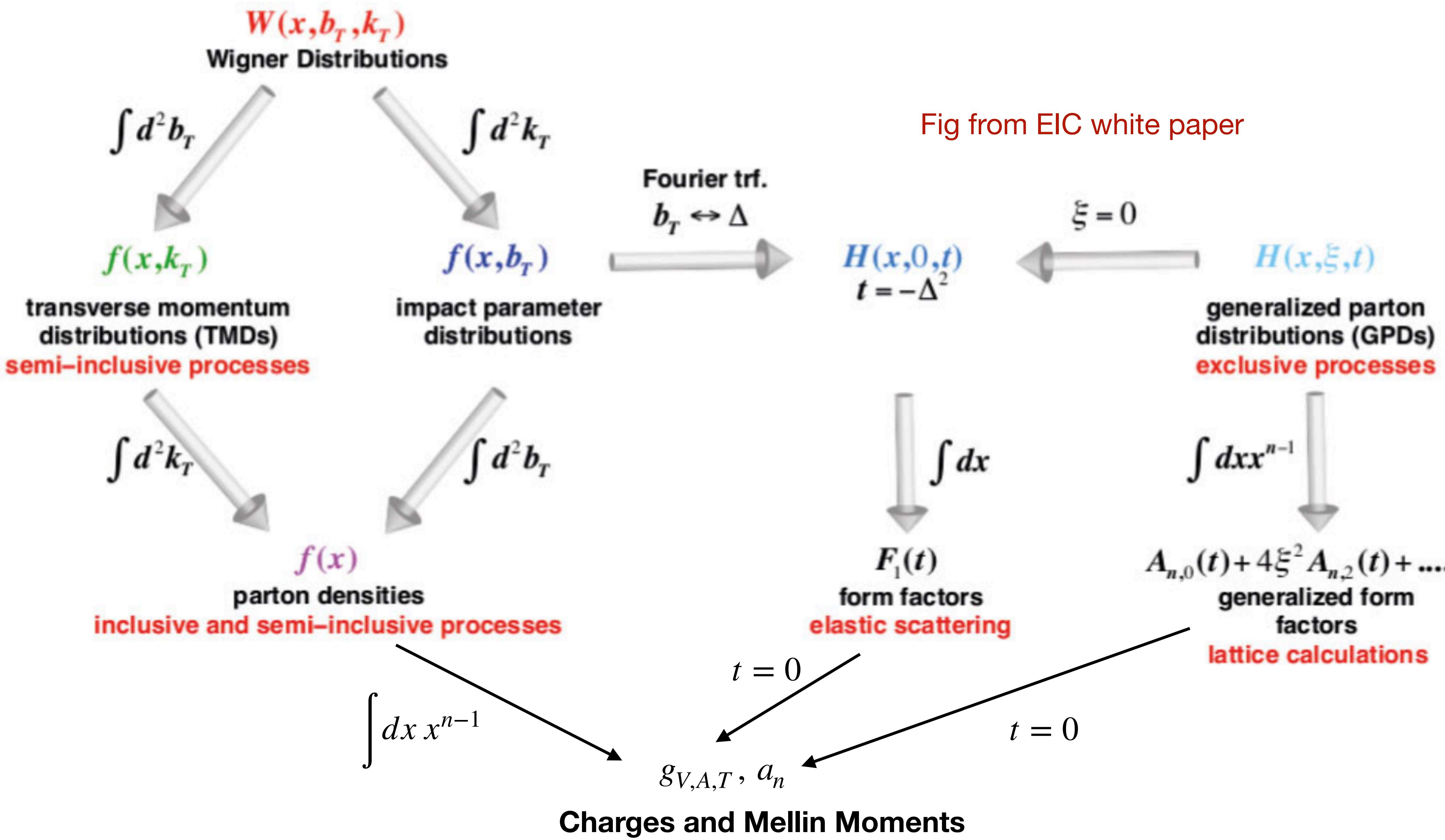
Connected



Overview of Objects in Hadron Structure

Many ways to describe a hadron

Fig from EIC white paper



Lattice Structure Overview

- Matrix Elements from ratios of 3pt and 2pt functions at large Euclidean times
- **Directly calculable from local operators** matched the MS-bar scheme / scheme independent ratios
    - Charges
    - Form Factors
    - PDFs' Mellin Moments
    - GPDs' Mellin Moments
    - Ratios of (G)TMDPDFs' Mellin Moments
- **Indirectly calculable from non-local operators** after a factorization
    - PDFs
    - GPDs
    - TMDPDFs and the Collins-Soper Kernel

Catches of a lattice calculation

All systematics are improvable, but at what cost?

- Finite lattice spacing $a \sim 0.045 - 0.1$ fm

  Polynomial of a to model

- Finite volumes $L \sim 3 - 5$ fm and $m_\pi L \sim 4 - 6$

  Single hadron: Exponential decay in $m_\pi L$ to model
  Multi-hadron: Polynomial in L^{-1} which Luscher method removes

- Heavy quarks / pions $m_\pi \sim 140 - 600$ MeV

  Chiral PT gives polynomials and logs of m_π to model

- Excited state control $\Delta \sim 140 - 500$ MeV

  Use larger T and do better fits
  Variational can separate lowest states

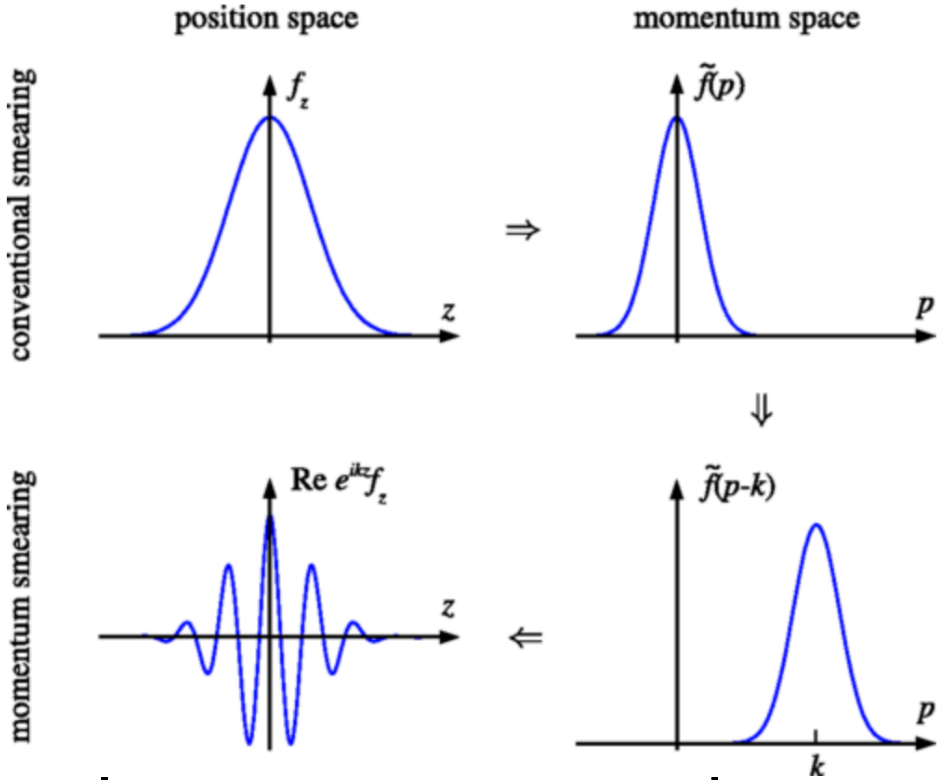
- Statistics **Always there**

  Beg the DOE for bigger computer

Difficulty Reaching High Momentum

$$\langle O(T)\, J(\tau)\, \bar{O}(0)\rangle = \sum_{n,m} Z_n Z_m^* e^{-E_n(T-\tau)-E_m\tau}\langle n\,|\,J\,|\,m\rangle \qquad Z_n = \langle \Omega\,|\,O(0)\,|\,n\rangle$$

- **Smearing interpolating operator for high overlap and signal**
  - Momentum smearing
    G. Bali et al Phys. Rev. D 93 (2016) 9, 094515
  - Distillation smears the operators
    M. Peardon, et al, Phys. Rev. D 80 (2009) 054506
    C. Egerer et al Phys. Rev. D 103 (2021) 3, 034502



- **Excited state energy gap shrinks**
  - Larger times needed for ground state
  - (Summed) GEVP techniques can remove lowest states and suppress remaining
    J. Bulava, M. Donnellan, R. Sommer JHEP 01 (2012) 140
- **Exponentially suppressed signal-to-noise ratio**
  - Lanczos approach to separate noise and signal modes of transfer matrix
    M. Wagman 2406.20009
    D. Hackett and M. Wagman 2407.21777

Local Charges

$$\langle O(T)\, J(\tau)\, \bar{O}(0)\rangle = \sum_{n,m} Z_n Z_m^* e^{-E_n(T-\tau)-E_m\tau}\langle n|J|m\rangle$$

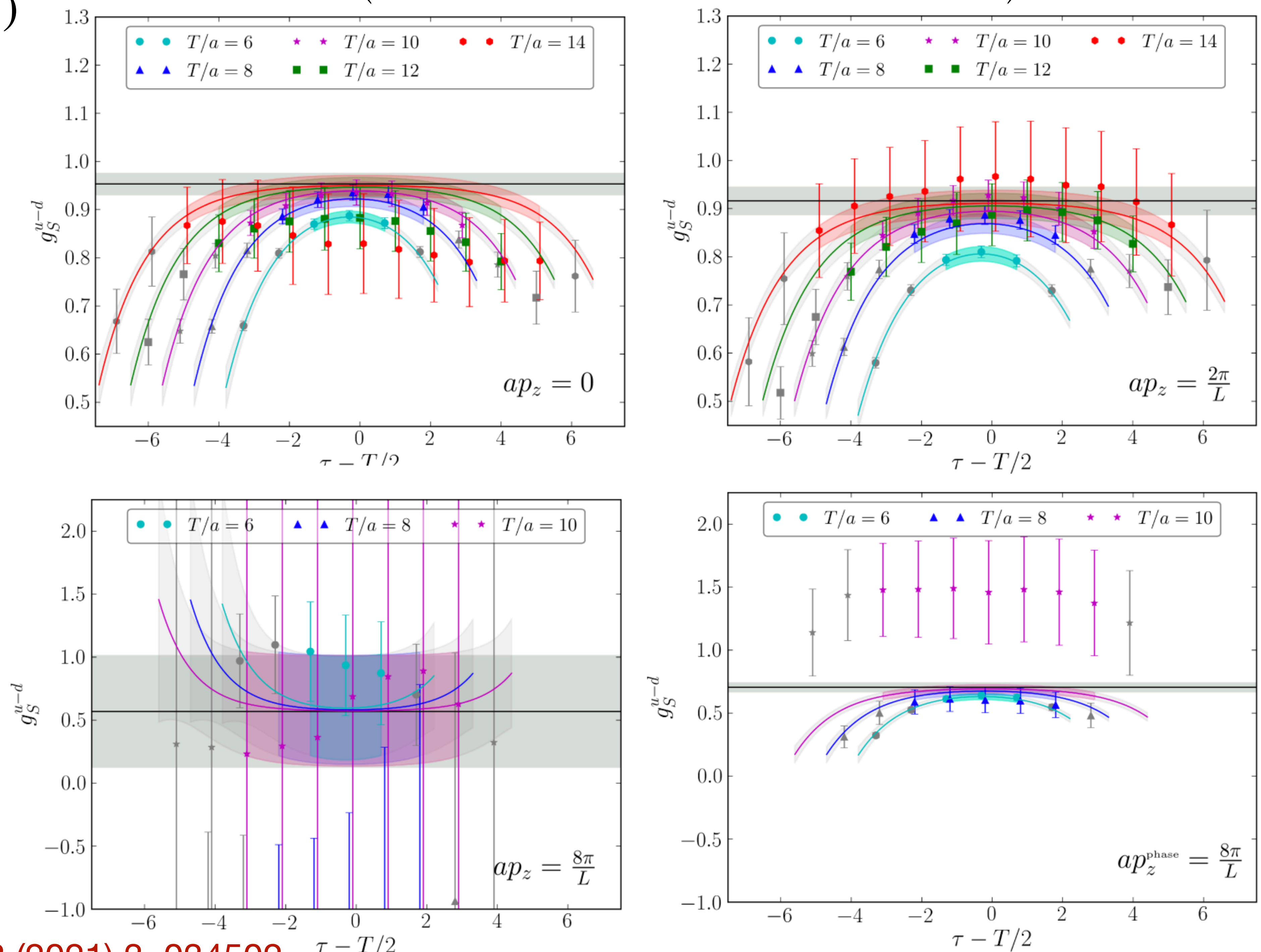
$$\langle O(T)\bar{O}(0)\rangle = \sum_{n} |Z_n|^2 e^{-E_nT}$$

$$J^{\Gamma}_{qq'} = \bar{q}\Gamma q'$$

$$\Gamma = 1$$

Simple for testing, but still important for structure

- Ratios of 3pt and 2pt give series of matrix elements $\dfrac{C_3(T,\tau)}{C_2(T)} = \langle 0|J|0\rangle + O\left(e^{-(E_1-E_0)T}, e^{-(E_1-E_0)(T-\tau)}, e^{-(E_1-E_0)\tau}\right)$
- Near $\tau = T \,\mathrm{or}\, 0$ excited states give curvature
- Model plateau and leading excited states

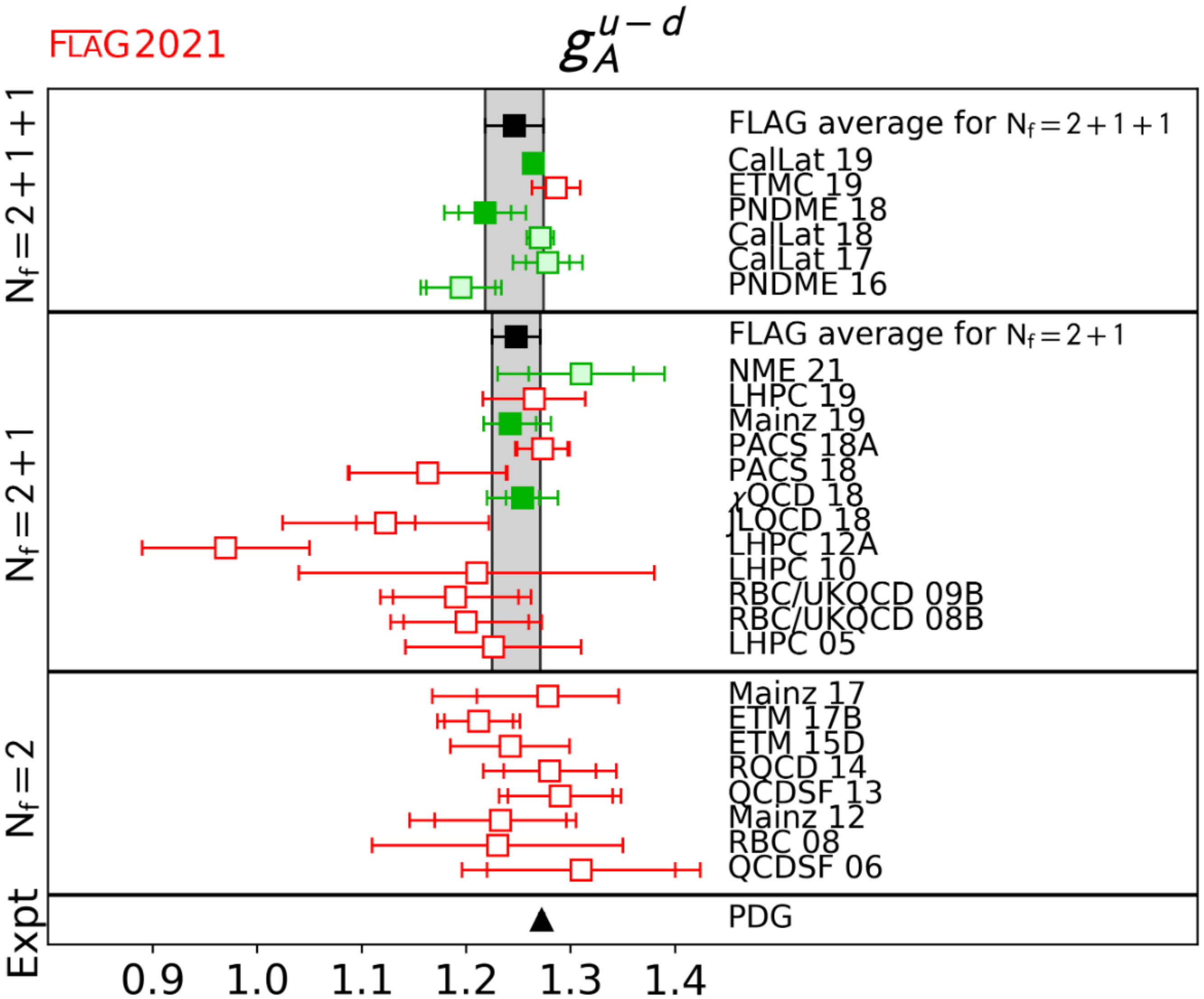


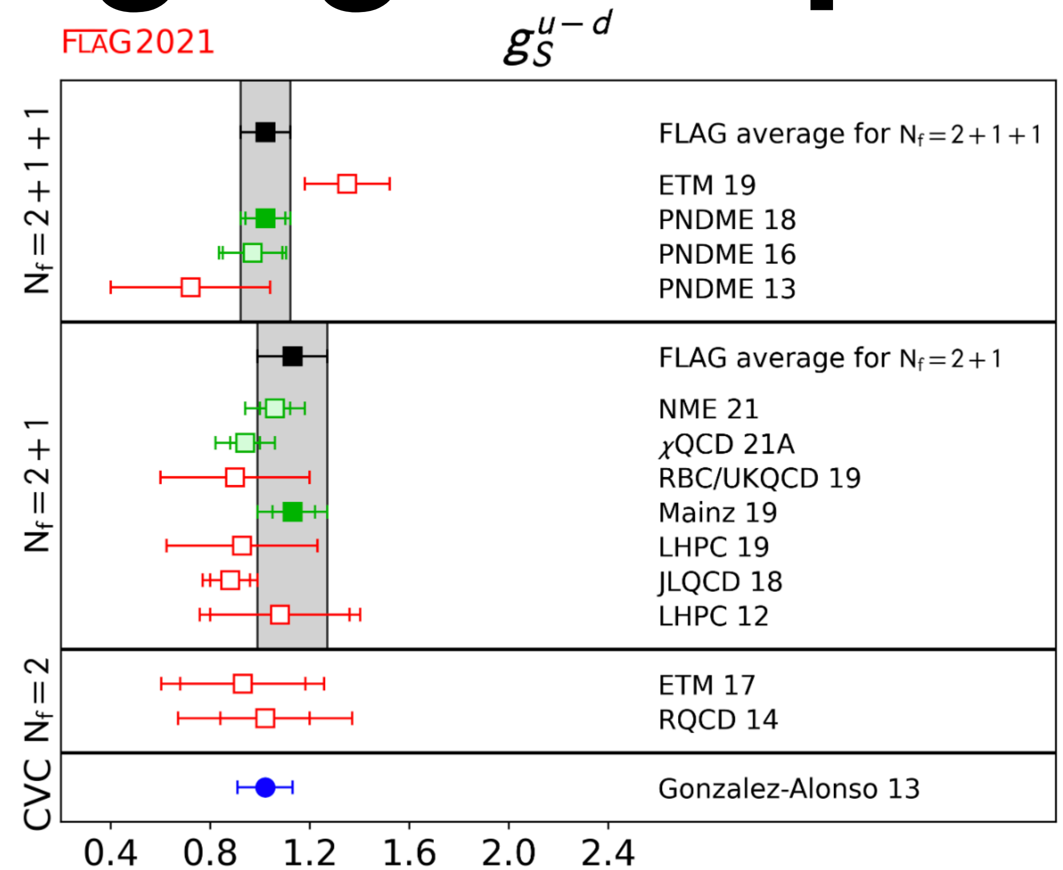
C. Egerer et al Phys. Rev. D 103 (2021) 3, 034502

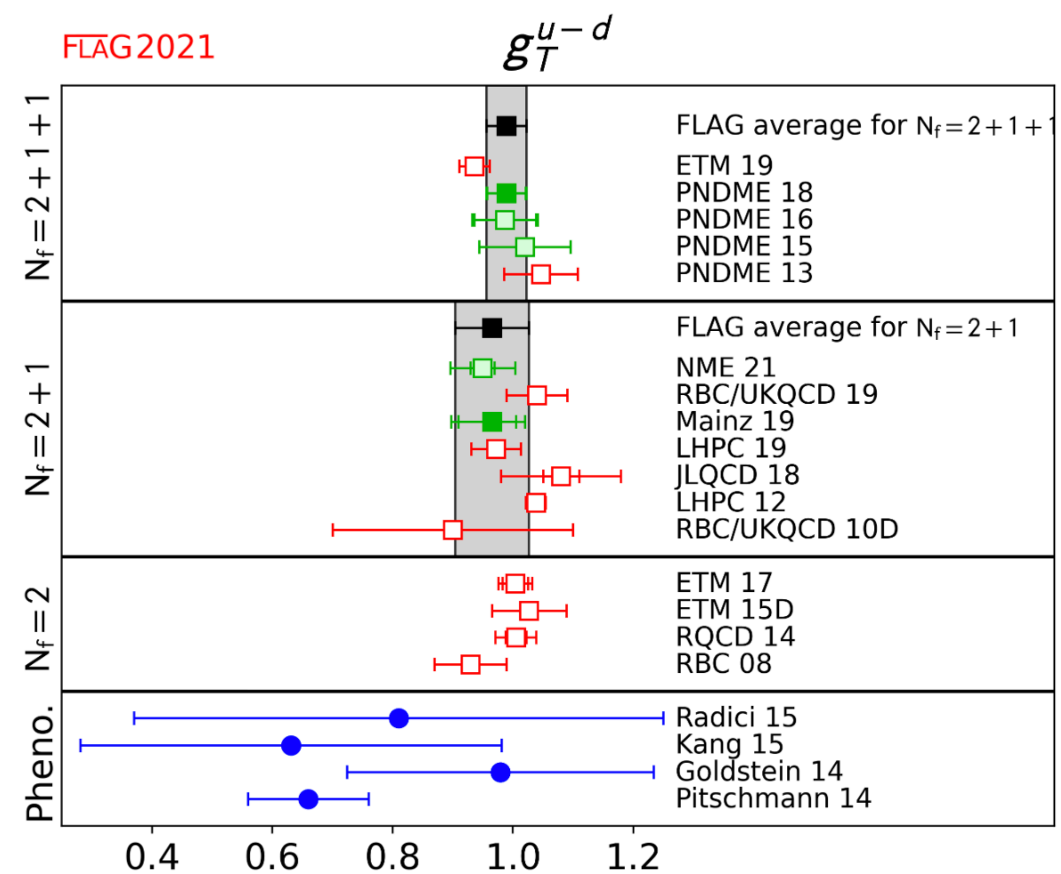
Flavour Lattice Averaging Group

http://flag.unibe.ch/2021/Nucleon%20matrix%20elements

- FLAG Review 2021 (hopefully 2024 will appear)
- Green means continuum, pion mass, infinite volume and excited states all under some control
- Connected only $u - d$







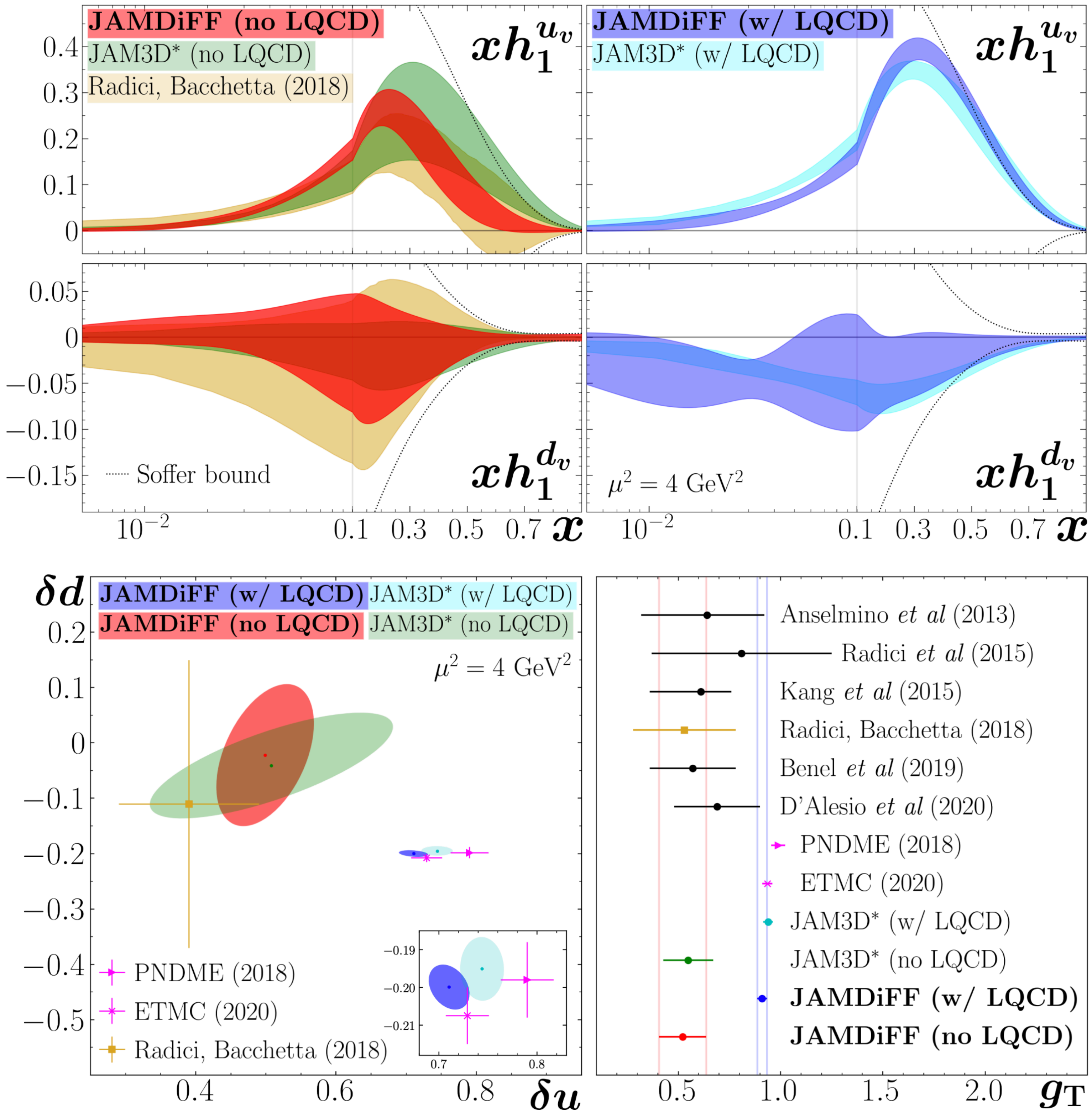
Use of charges in global fits

C. Cocuzza et al JAM Collab PRL 132 091901 (2024)

- Tensor Charge and Transversity PDF

$$g_T^{u-d} = \int dx\, h_1^{u-d}(x)$$

- Initially appear to have tension
- Adding Lattice QCD charges to analysis removes tension and improves precision



Form Factors

$$\langle p'| J^\mu(q = p' - p)|p\rangle = \bar{u}_N(p')\big(\gamma^\mu F_1(t) + \frac{i\sigma^{\mu q}}{2m}F_2(t)\big)u_N(p)$$

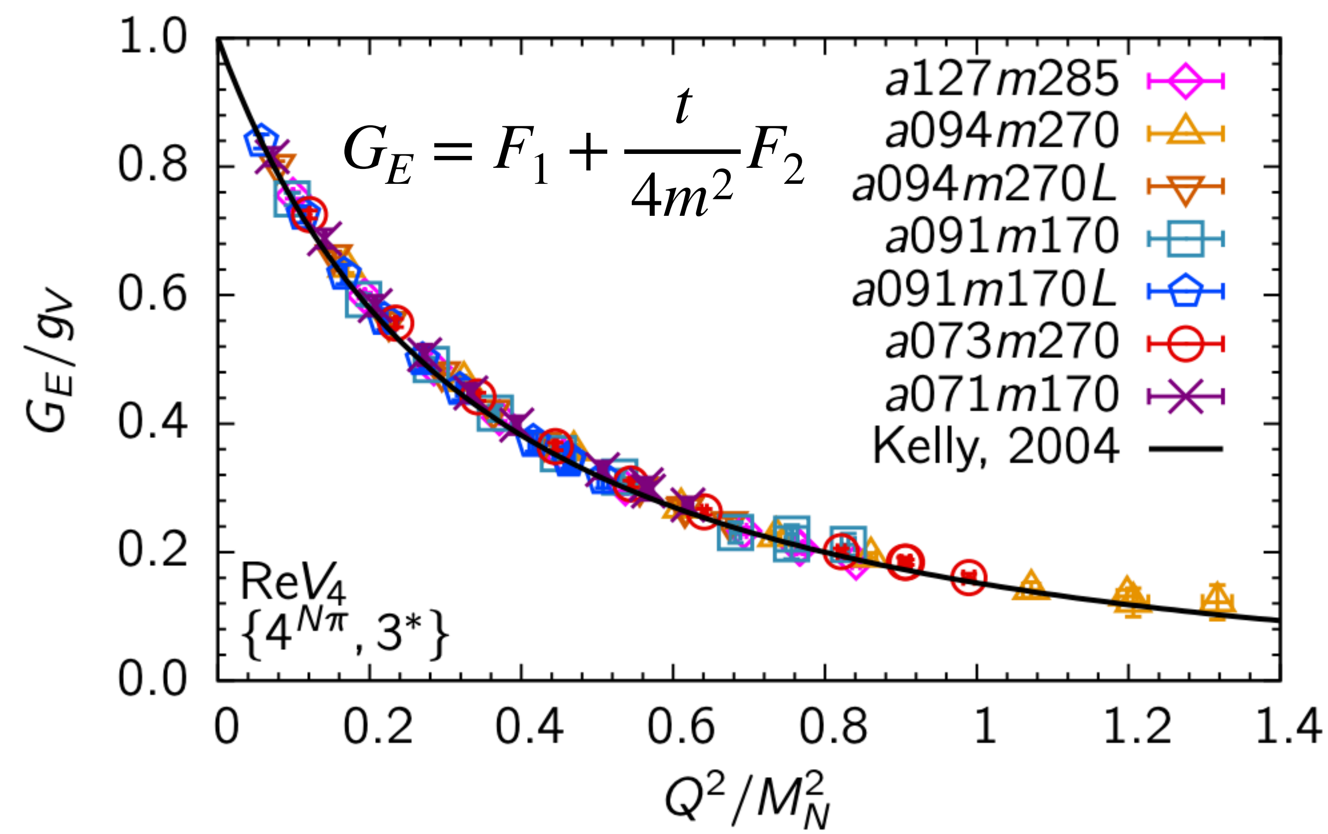
$$\sigma^{\mu q} = \sigma^{\mu\nu}q_\nu$$

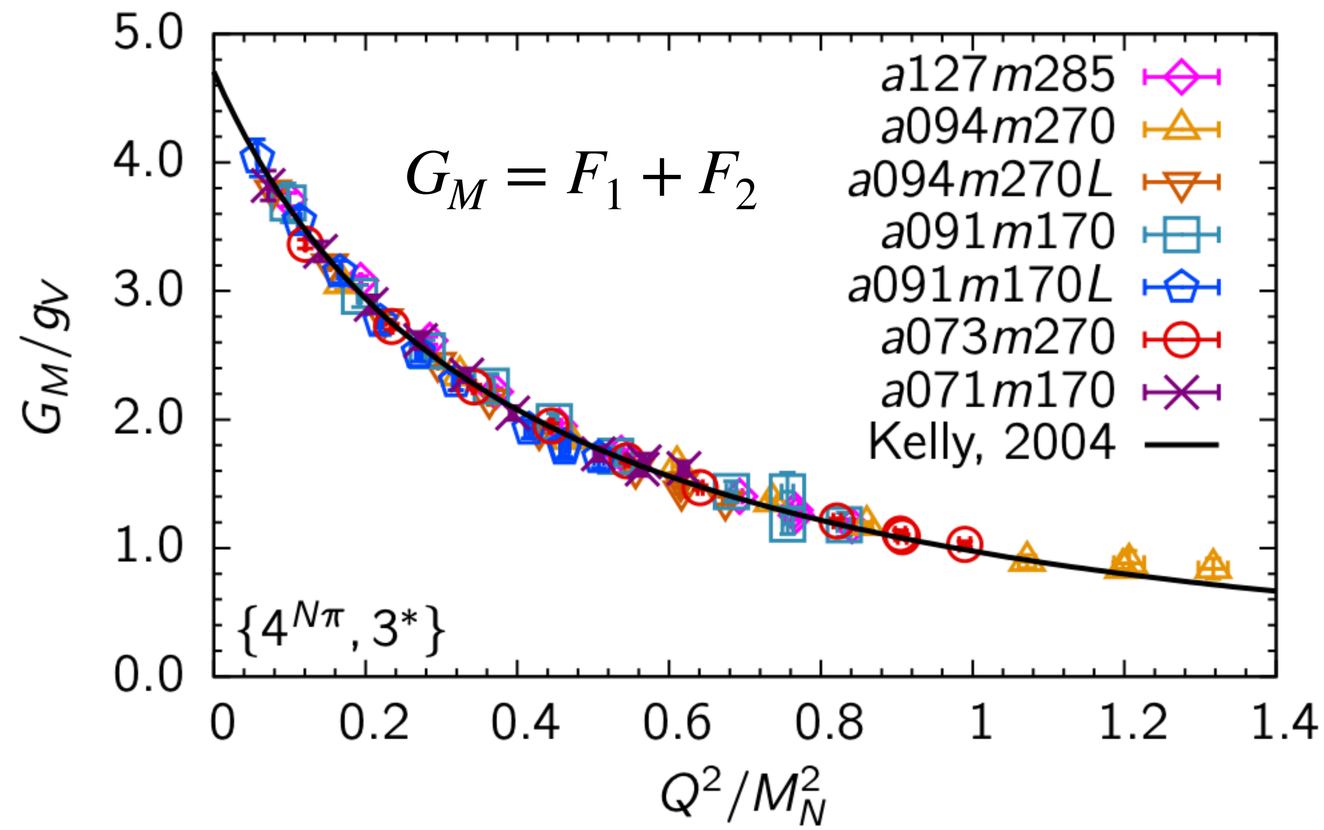
NME Collaboration PRD 105 (2021) 054505

- Electromagnetic Form Factors
- Accurate lattice results require high precision control over all systematics
- FFs are integrals of GPDs

$$F_1(t) = \int dx\, H(x, \xi = 0, t)$$

$$F_2(t) = \int dx\, E(x, \xi = 0, t)$$





Axial Form Factors

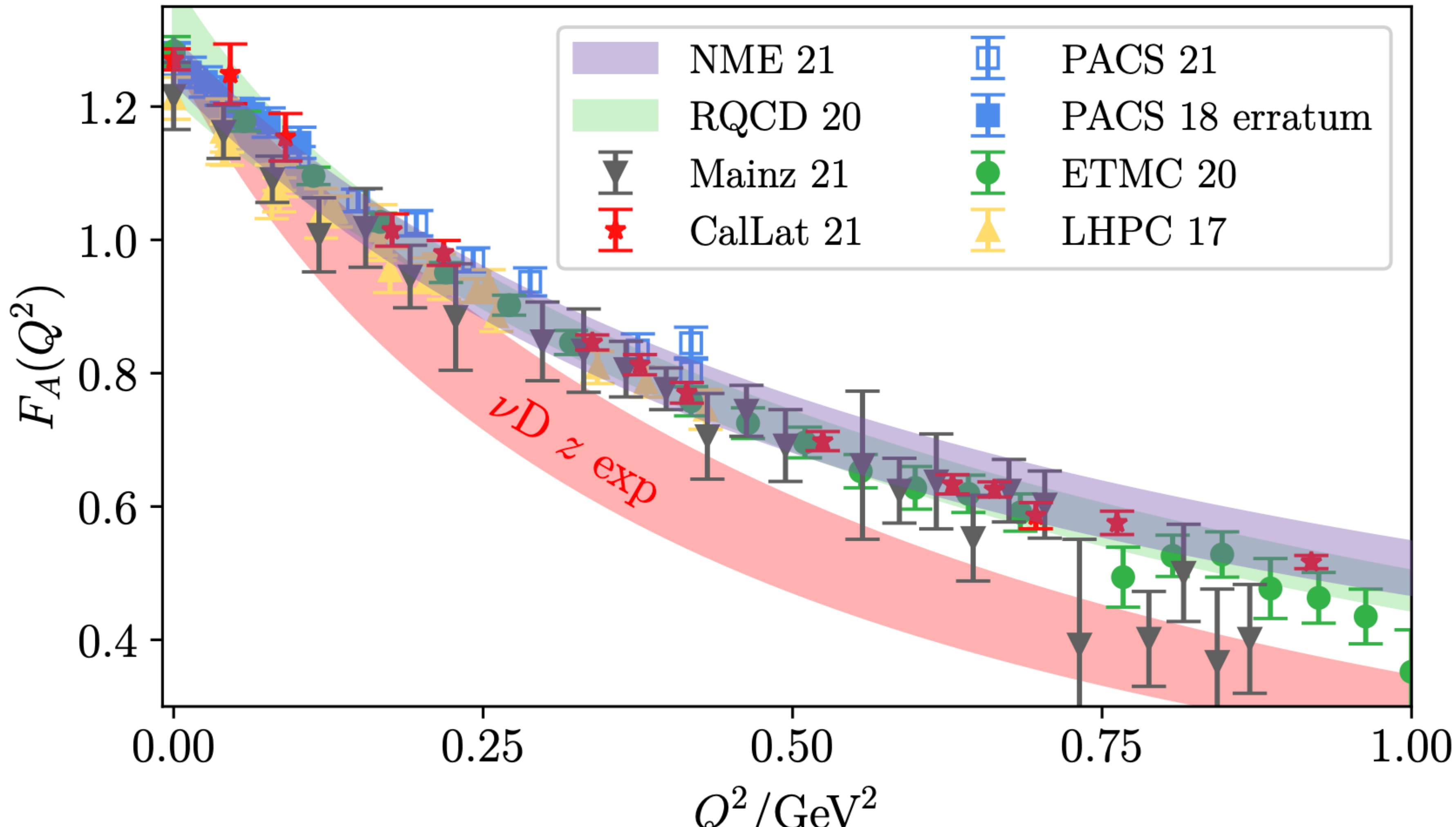
- Axial Form Factors needed for Neutrino studies

$$\langle p'|J^\mu|p\rangle = \bar{u}(p')\Big(\gamma^\mu\gamma_5 F_A(t) + \frac{q^\mu\gamma_5}{m}F_P(t)\Big)u(p)$$

A. Meyer, A. Walker-Loud, C. Wilkinson
Annu. Rev. Nucl. Part. Sci (2022) 72 205-232

- Strong agreement amongst lattice groups

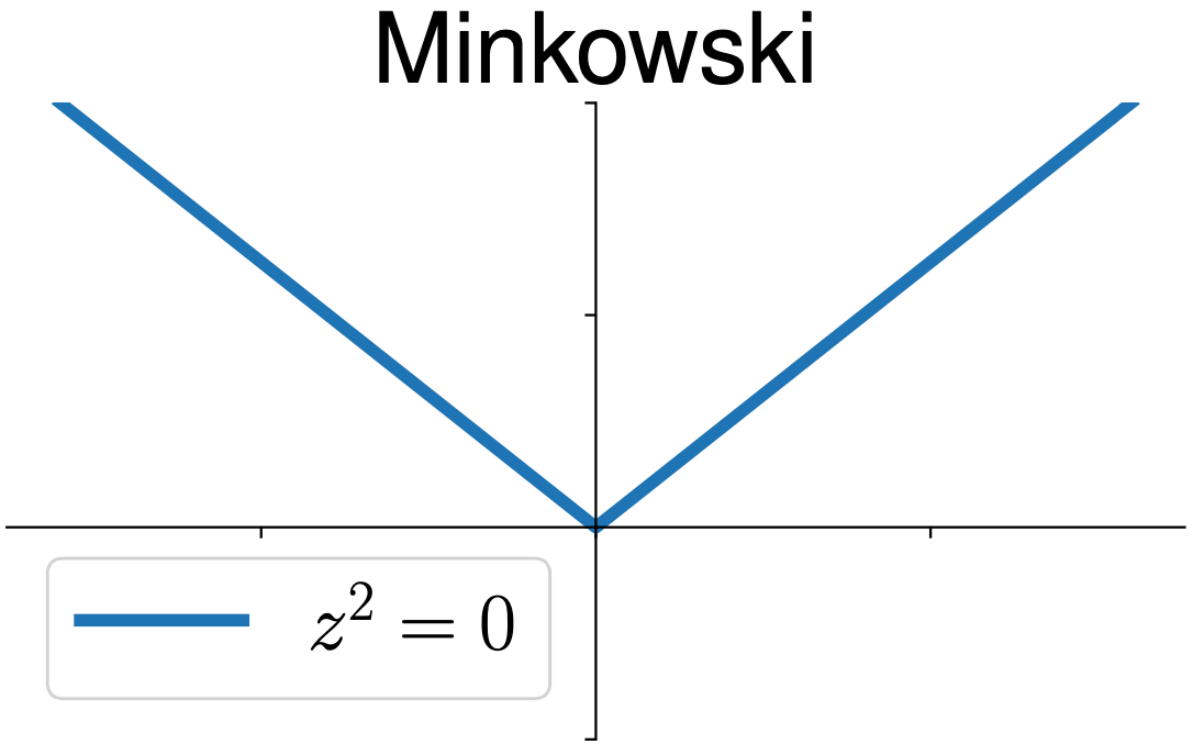
- Discrepancy could be nuclear effects in experiment



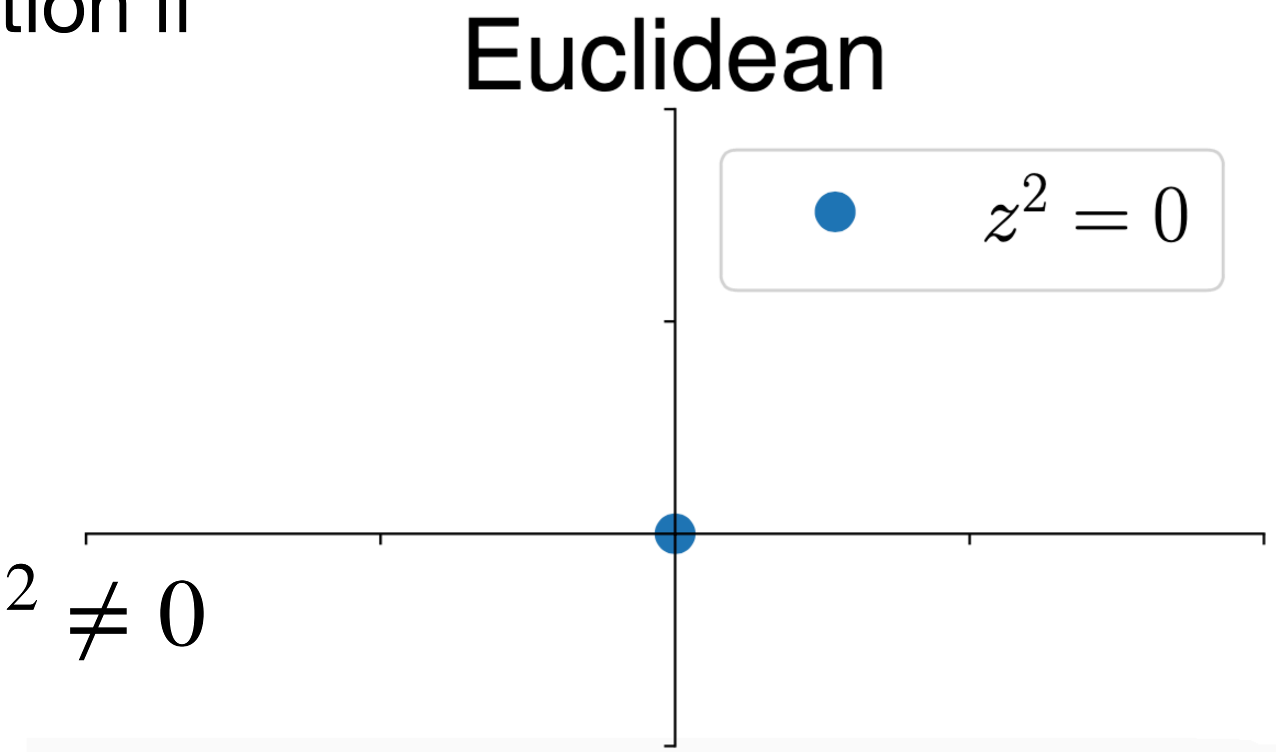
Why no PDFs from the lattice

- Parton Distributions are defined by operators with light-like separations

$$M(p,z) = \langle p \,|\, \bar{\psi}(z)\gamma^{\alpha}W(z;0)\psi(0) \,|\, p \rangle$$



- Fourier transformations of matrix elements give PDF
  Cannot integrate light cone separation if no light cone!



- Spoiler: X. Ji *Phys Rev Lett* 110 (2013) 262002
  Embrace space-like separations $z^2 \neq 0$

Mellin Moments of PDF

- OPE of Hadronic Tensor showed leading $1/Q^2$ is from operators

$$O_n^{\{\mu_1,\mu_2,\dots\mu_n\}} = \bar{q}\gamma^{\{\mu_1}D^{\mu_2}\dots D^{\mu_n\}}q$$

{ Traceless and Symmetric indices }
You'll See why later in the Lattice Cross Section example

- PDF is function whose Mellin moments are those matrix elements

$$\langle p \,|\, O_n \,|\, p \rangle = a_n = \int_{-1}^{1} dx\, x^{n-1} f(x)$$

- Local charges are just $n = 1$

- Lorentz invariant definition of PDF without need of light cone on the lattice

Symmetries of the lattice

Continuum rotation vs Lattice rotation

Continuous symmetry $O(4)$

Infinite number of Irreducible Representations (irreps) labeled by integers/half integers called spin

Spin is conserved since different irreps don't mix

Discrete and Finite symmetry $H(4)$

Hypercube symmetry group has 192 Elements with 13 irreps

Each irrep has contributions from many, but not all, spins

Mixing of spin states

"No free lunch" theorem

S. Capitani, G. Rossi (1995) arXiv:9401014
G. Beccarini, et al (1995) arXiv:9506021

Mass dimension of operator

Spin of operator

- Symmetric and Traceless operators have twist $\tau = J - M = 2$
- Bare Operators of same irrep mix under renormalization
- Bare Operators with lower J mix with higher J, but larger M needs factors of a to compensate mass dimension

$$[O_2^{43}]_b^{\rm latt}(a) = Z^{\rm latt}(a^2\mu^2)[\bar{q}\gamma^4 D^3 q]_{\mu^2}^{\rm cont} + O(a)$$

- Bare Operators with higher J mix with lower with powers of a^{-1}

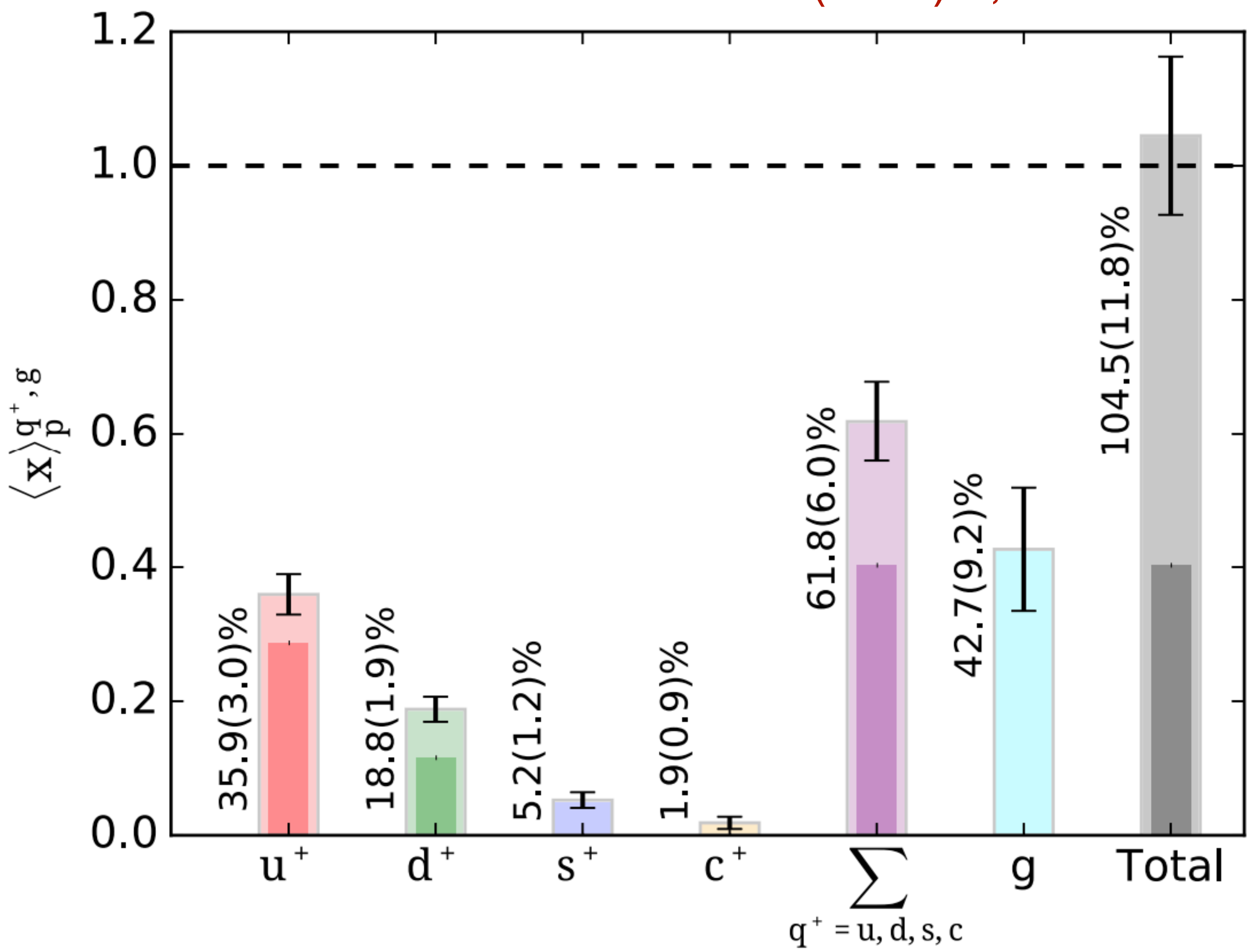
$$[O_3^{\mu\nu\rho}]_b^{\rm latt}(a) = Z_1^{\rm latt}(a^2\mu^2)[\bar{q}\gamma^\mu D^\nu D^\rho q]_{\mu^2}^{\rm cont} + \frac{1}{a^2} Z_2^{\rm latt}(a^2\mu^2) g^{\nu\rho}[\bar{q}\gamma^\mu q]_{\mu^2}^{\rm cont} + O(a)$$

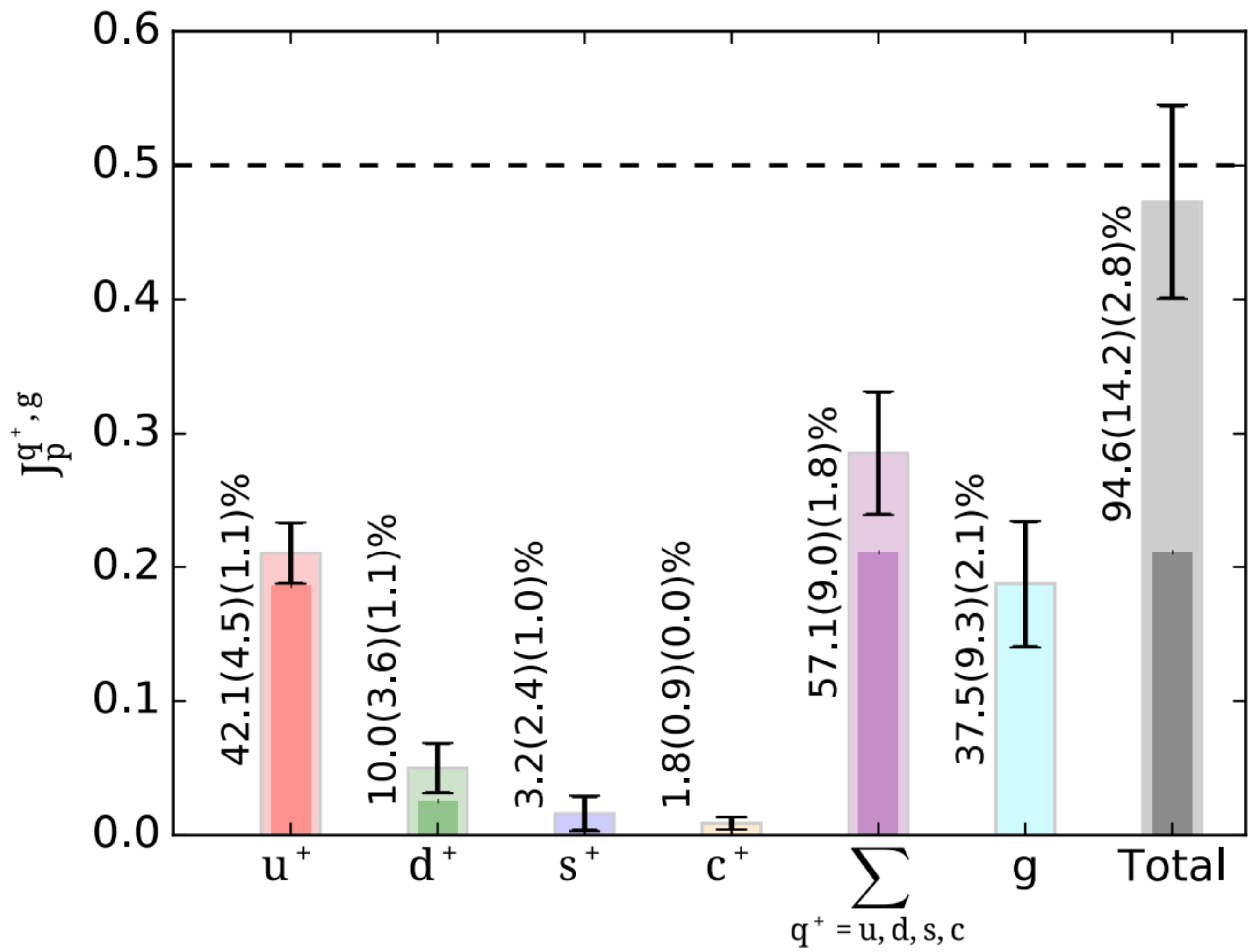
- Different choices of indices are in different irreps and mix differently

Local Moment calculations

- Quarks with $\bar{q}\gamma_{\{\mu}D_{\nu\}}q$ and $\bar{q}\gamma_5\gamma_{\{\mu}D_{\nu\}}q$
- Gluons with $F^{\mu\nu}F^{\rho\sigma}$ and $F^{\mu\nu}\tilde{F}^{\rho\sigma}$
- “Sum rules” are conservation of linear and angular momentum

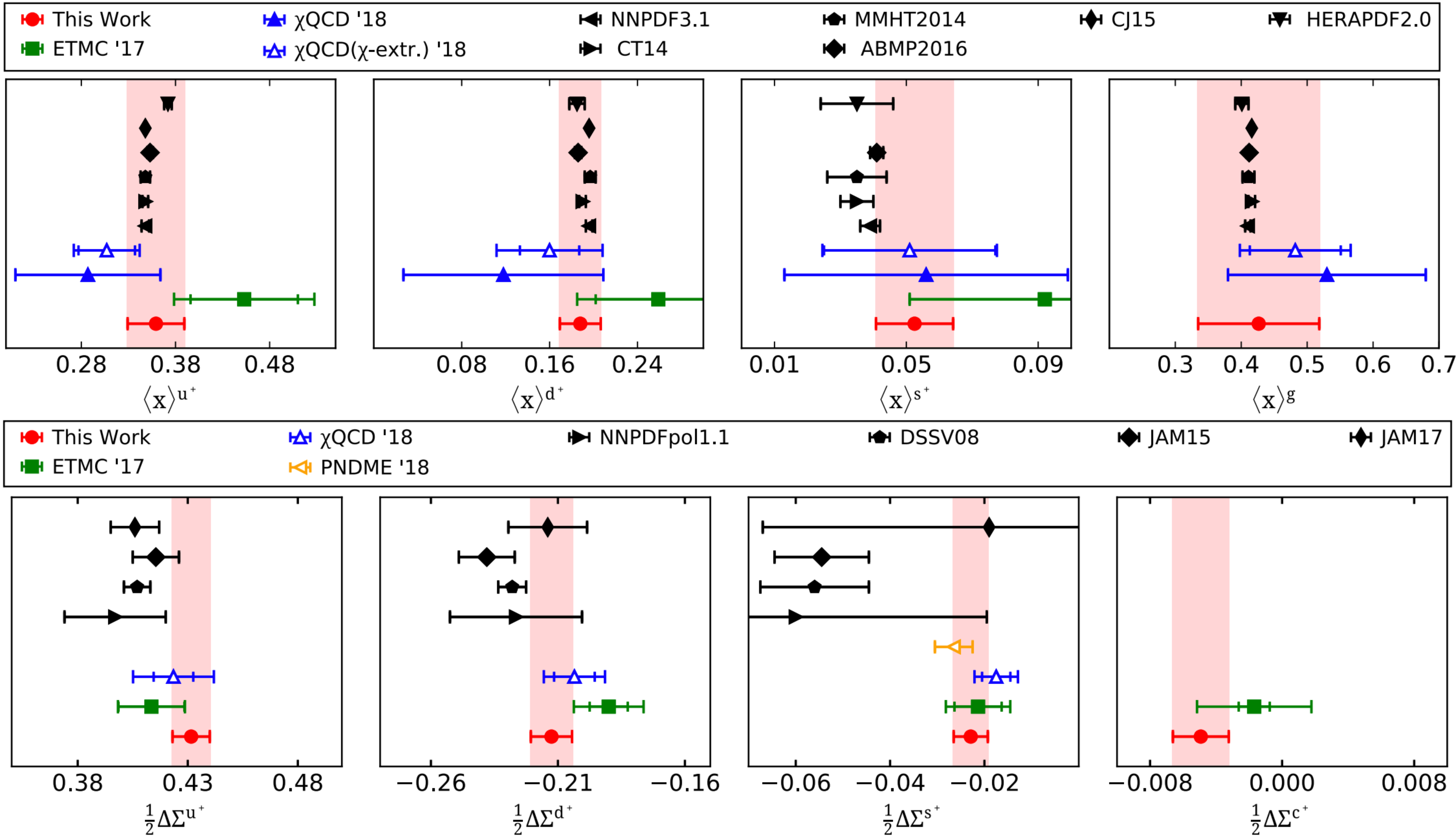
ETM Collaboration PRD 1010 (2020) 9,094513





Local Operator calculations

ETM Collaboration PRD 1010 (2020) 9,094513



Summary of local calculations

- Local Calculations are well understood numerically and theoretically

- High precision and control of systematic errors

- Direct relation to observables matched to MS-bar scheme

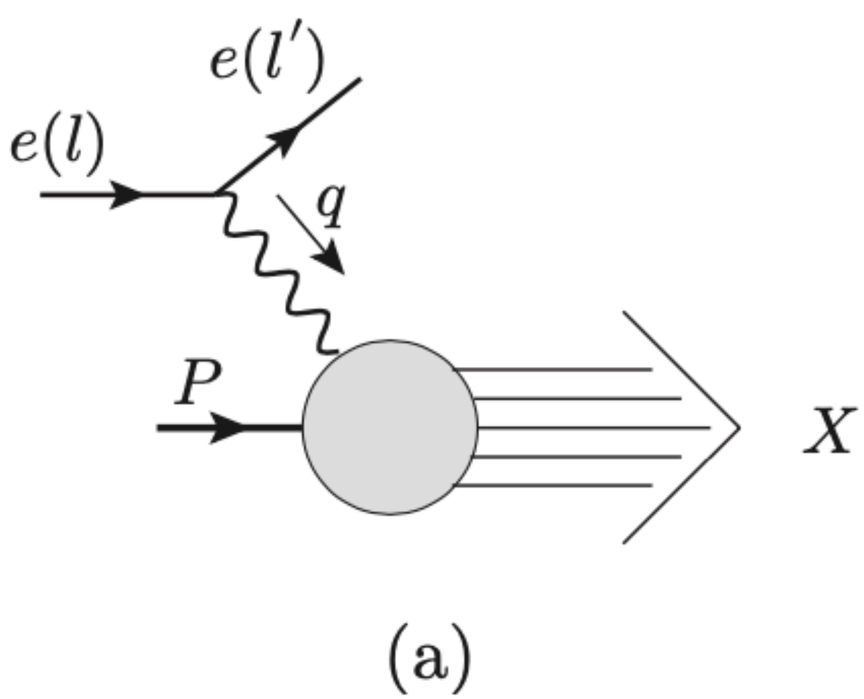
Hadronic Tensor

K.-F. Liu et al *Phys. Rev. Lett.* 72 1790 (1994)
Phys. Rev. D 62 (2000) 074501

- Minkowski Hadronic Tensor is QCD part of DIS cross section
- $E'\frac{d\sigma_{DIS}}{d^3l'} = \frac{2\alpha^2}{sQ^4}L_{\mu\nu}W^{\mu\nu}$

$$L^{\mu\nu}(l,l') = \frac{1}{2}\mathrm{Tr}\left[\gamma_\nu \not{l}\gamma_\mu \not{l}'\right]$$

$$W^{\mu\nu}(q,p) = \langle p\,|\int d^4x\, e^{iq\cdot x}J^\mu(x)J^\nu(0)\,|p\rangle$$



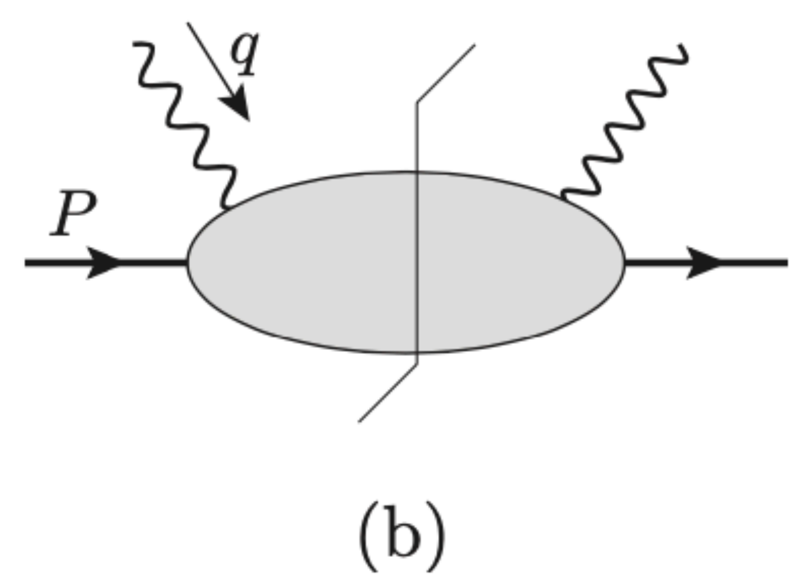


Fig. 2.4. (a) DIS amplitude to lowest order in electromagnetism. (b) Hadronic part squared and summed over final states. For the meaning of the vertical "final-state cut", see the discussion below (2.19).

Fig from "Foundations of Perturbative QCD" J. Collins

Hadronic Tensor

K.-F. Liu et al *Phys. Rev. Lett.* 72 1790 (1994)
Phys. Rev. D 62 (2000) 074501

- Minkowski Hadronic Tensor is QCD part of DIS cross section

$$E'\frac{d\sigma_{DIS}}{d^3l'} = \frac{2\alpha^2}{sQ^4} L_{\mu\nu} W^{\mu\nu}$$

$$L^{\mu\nu}(l,l') = \frac{1}{2}\mathrm{Tr}\left[\gamma_\nu l\!\!/ \gamma_\mu l\!\!/'\right]$$

$$W^{\mu\nu}(q,p) = \langle p\,|\int d^4x\, e^{iq\cdot x} J^\mu(x) J^\nu(0)\,|\,p\rangle$$

- In Euclidean space, fix the times!

$$\tilde{W}^{\mu\nu}(\vec{q},\tau,p) = \langle p\,|\int d^3x\, e^{i\vec{q}\cdot\vec{x}} J^\mu(x,\tau) J^\nu(0)\,|\,p\rangle$$

- Inverse Laplace Transform to get Minkowski HT

$$W^{\mu\nu}(\vec{q},\nu,p) = -\,i\int_{c-i\infty}^{c+i\infty} d\tau\, e^{\nu\tau}\, \tilde{W}(\vec{q},\tau,p)$$

- Requires 4 point functions!

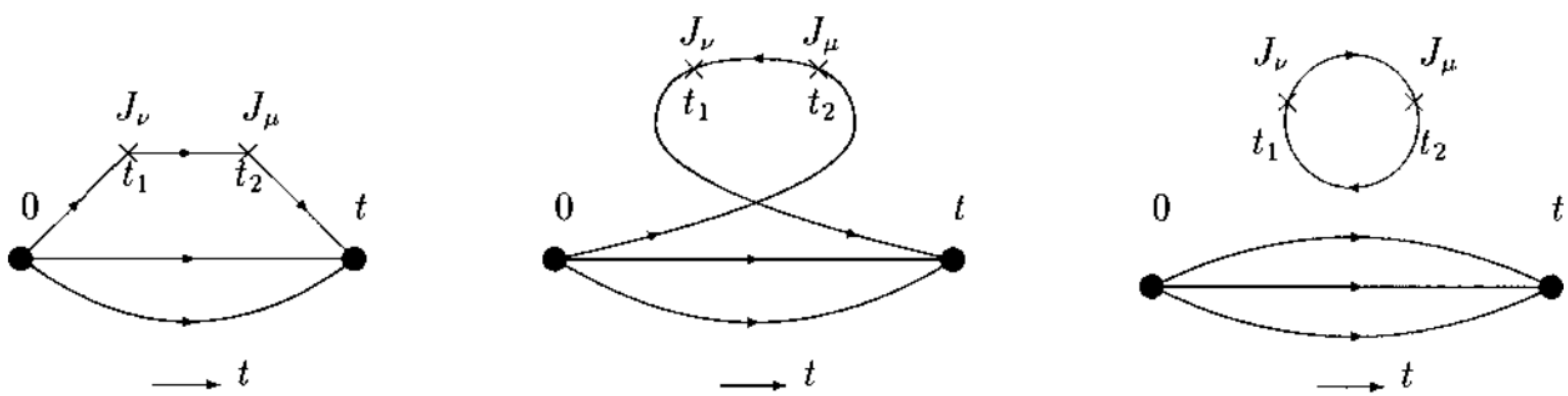
- Large $Q^2 = \nu^2 - \vec{q}^{\,2}$ limit gives PDF information, Smaller Q^2 to get resonances

Hadronic Tensor Diagrams

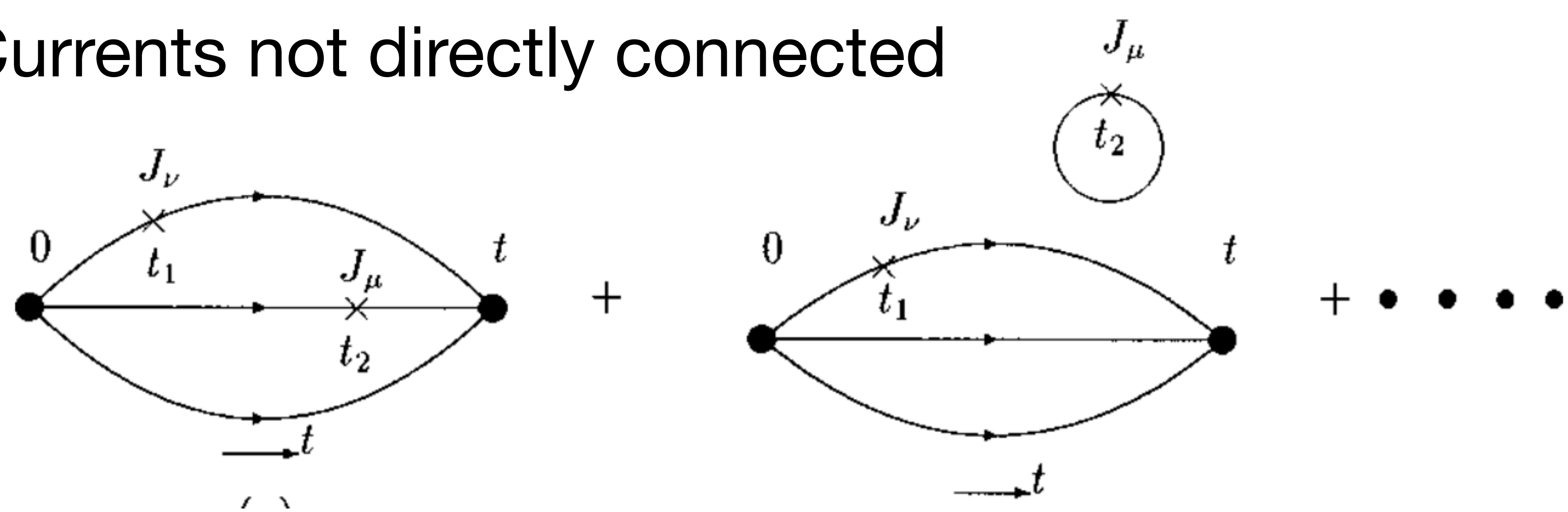
K.-F. Liu et al *Phys. Rev. Lett.* 72 1790 (1994)
Phys. Rev. D 62 (2000) 074501

$$\tilde{W}^{\mu\nu}(\vec{q},\tau) = \langle p \,|\int d^3x\, e^{i\vec{q}\cdot\vec{x}} J^\mu(x,\tau) J^\nu(0)\,|\, p\rangle$$

- Hand Bag: currents directly connected by quark line which cares hard momentum transfer in/out of currents



- Cat Ears: Currents not directly connected



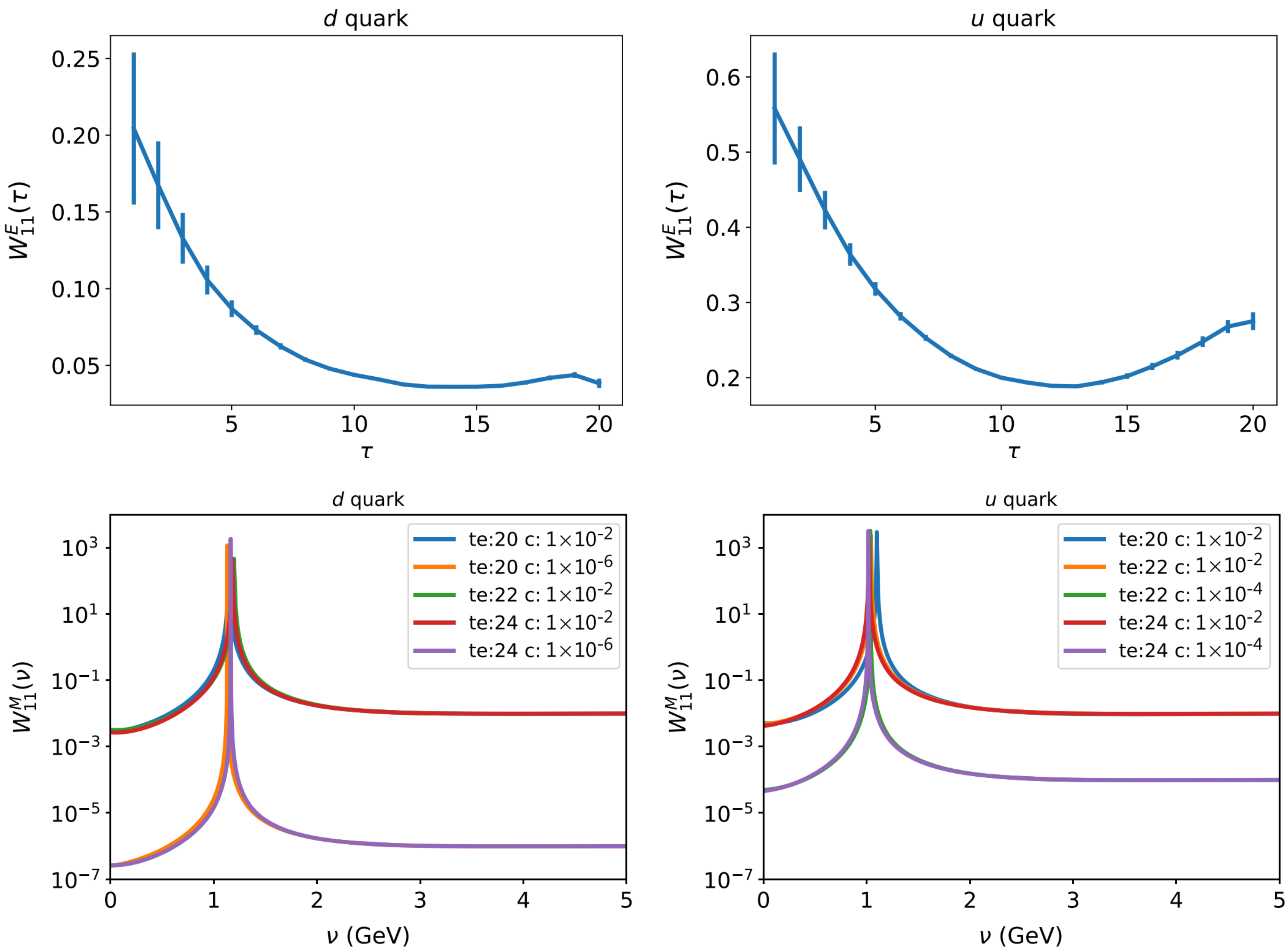
Hadronic Tensor

χQCD Collaboration PRD 101 (2020) 11, 114503

$$W^{\mu\nu}(q^2,\nu) = -i\int_{c-i\infty}^{c+i\infty} d\tau\, e^{\nu\tau}\,\tilde{W}(\vec{q},\tau)$$

Euclidean Time

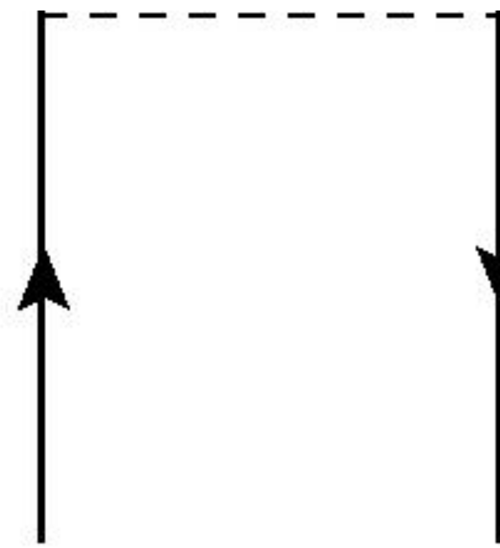
Minkowski Energy Transfer



Many non-local approaches

- **Wilson line operators**

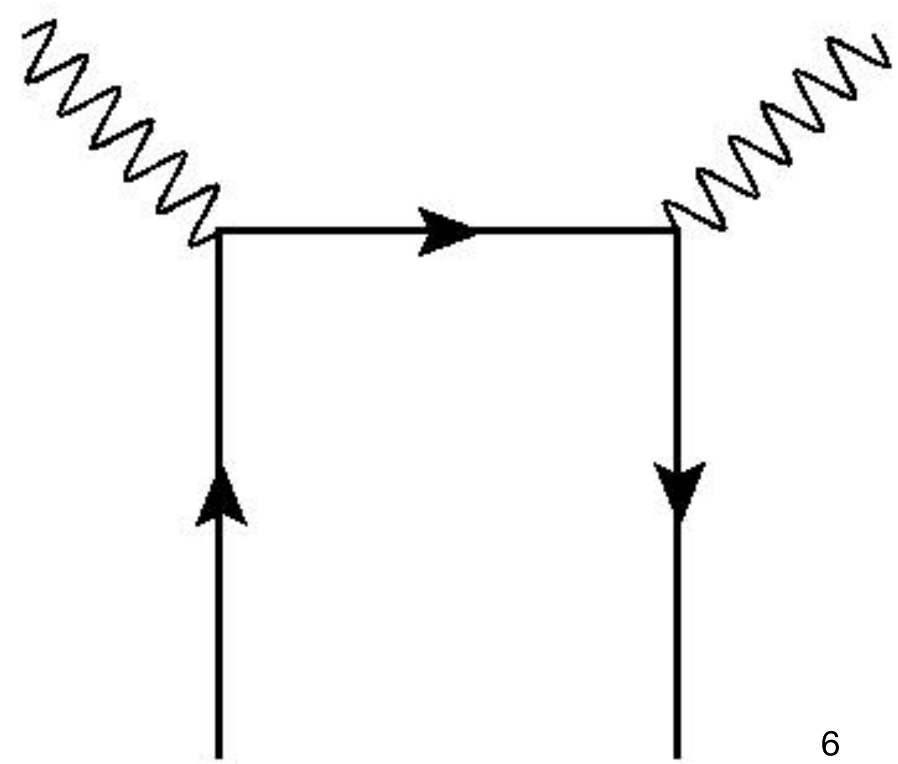
$$O_{WL}(x;z) = \bar{\psi}(x+z)\Gamma W(x+z;x)\psi(x)$$



  - LaMET X. Ji *Phys. Rev. Lett.* 110 (2013) 262002
  - Pseudo-PDF A. Radyushkin *Phys. Rev. D* 96 (2017) 3, 034025

- **Two current correlators**

$$O_{CC}(x,y) = J_{\Gamma}(x)J_{\Gamma'}(y)$$

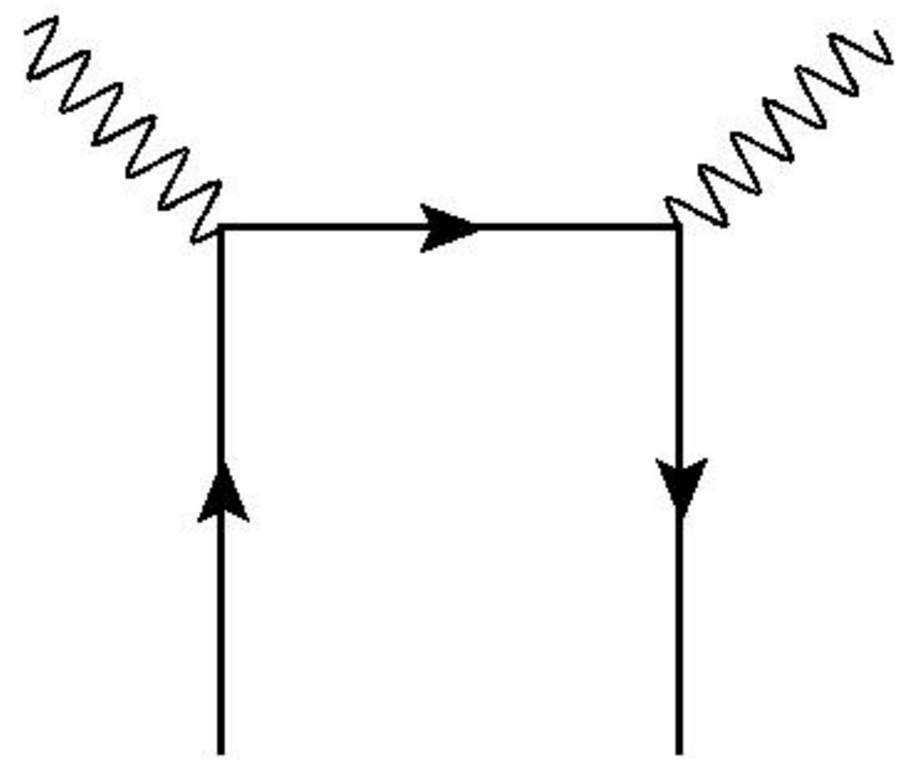


  - Hadronic Tensor
    K.-F. Liu et al *Phys. Rev. Lett.* 72 1790 (1994)
    Phys. Rev. D 62 (2000) 074501
  - HOPE
    W. Detmold and C.-J. D. Lin, *Phys. Rev. D* 73 (2006) 014501
  - Short distance OPE
    V. Braun and D. Muller *Eur. Phys. J. C* 55 (2008) 349
  - OPE-without-OPE
    A. Chambers et al, *Phys. Rev. Lett.* 118 (2017) 242001
  - Good Lattice Cross Sections
    Y.-Q. Ma and J.-W. Qiu *Phys. Rev. Lett.* 120 (2018) 2, 022003

Two Current choices

Which scale and which OPE

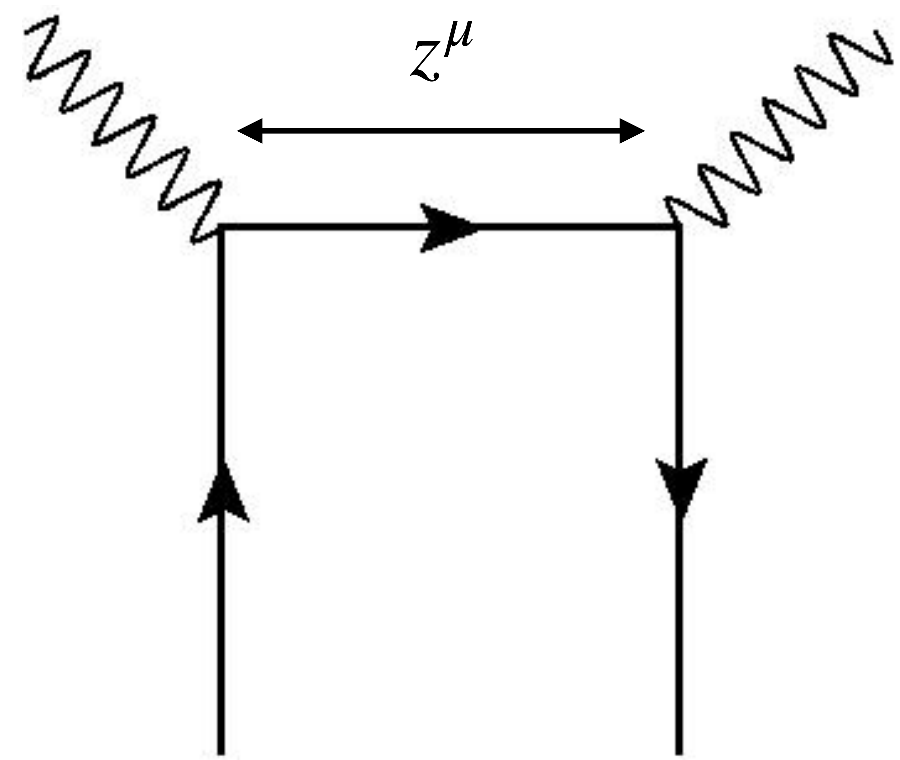
- Short Distance OPE / Good Lattice Cross Sections
  - Expand in small Lorentz invariant separation between currents
- Hadronic Tensor / OPE without OPE
  - Expand in the momentum transferred in/out of currents
- Heavy-Quark Operator Product expansion (HOPE)
  - Expand in the mass of a heavy quark between currents



Two Current choices

Which scale and which OPE

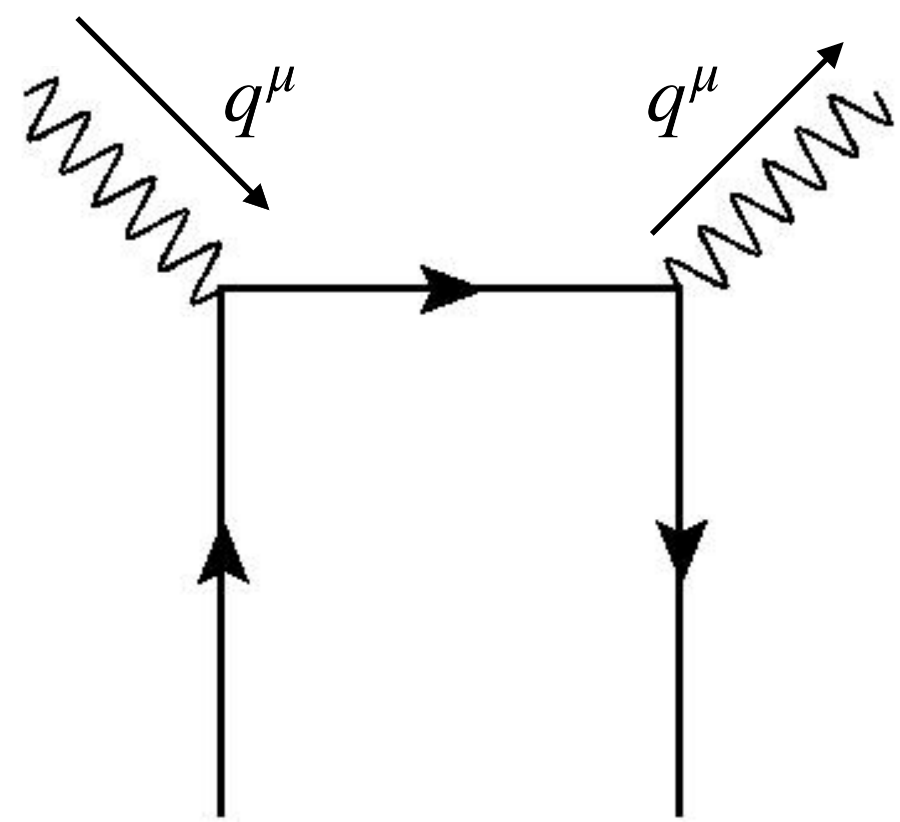
- **Short Distance OPE / Good Lattice Cross Sections**
    - Expand in small Lorentz invariant separation between currents z^2
- Hadronic Tensor / OPE without OPE
    - Expand in the momentum transferred in/out of currents
- Heavy-Quark Operator Product expansion (HOPE)
    - Expand in the mass of a heavy quark between currents



Two Current choices

Which scale and which OPE

- Short Distance OPE / Good Lattice Cross Sections
    - Expand in small Lorentz invariant separation between currents
- **Hadronic Tensor / OPE without OPE**
    - Expand in the momentum transferred in/out of currents $Q^2 = -q^2$
- Heavy-Quark Operator Product expansion (HOPE)
    - Expand in the mass of a heavy quark between currents



Two Current choices

Which scale and which OPE

- Short Distance OPE / Good Lattice Cross Sections
    - Expand in small Lorentz invariant separation between currents
- Hadronic Tensor / OPE without OPE
    - Expand in the momentum transferred in/out of currents
- **Heavy-Quark Operator Product expansion (HOPE)**
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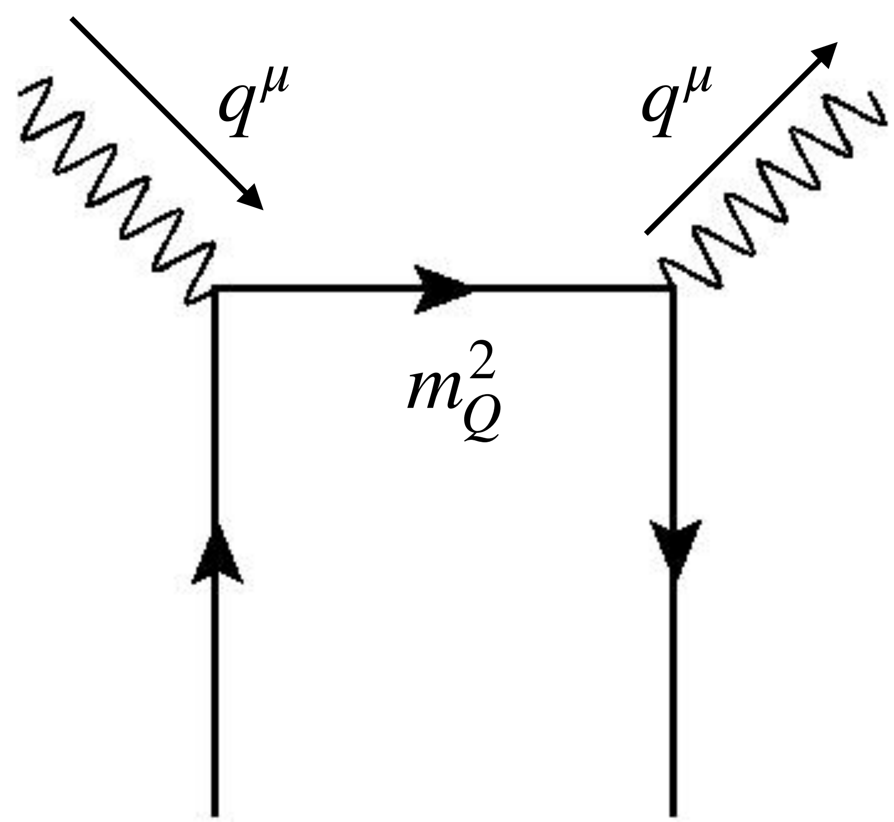

$$\tilde{Q}^2 = -q^2 + m_Q^2$$

All of this holds for pseudo-PDF

Expansion in Separation

V. Braun and D. Muller *Eur. Phys. J. C* 55 (2008) 349
Y.-Q. Ma and J.-W. Qiu *Phys. Rev. Lett.* 120 (2018) 2, 022003

- To lose indices consider scalar current j
- The matrix element can be expanded if z is sufficiently small

$$M(p,z) = \langle p \,|\, j(z) j(0) \,|\, p \rangle \qquad M(p,z) = \langle p \,|\, \bar{\psi}(z)\psi(z)\bar{\psi}(0)\psi(0) \,|\, p \rangle$$

- OPE looks like Taylor expansion in z

$$M(p,z) = \sum_n \frac{C_n(\mu^2 z^2)}{n!} z_{\mu_1} \ldots z_{\mu_n} \langle p \,|\, \bar{\psi}(0) D^{\mu_1} \ldots D^{\mu_n} \psi(0) \,|\, p \rangle_\mu^2$$

Local matrix elements proportional to $p^{\mu_1} \ldots p^{\mu_n}$ and other ``trace terms" with $g^{\mu_i \mu_j}$ factors

- Rearrange to see leading twist dominance when z^2 is small

$$M(p,z) = \sum_{n=0}^{\infty} \sum_{l=0}^{\lfloor n/2 \rfloor} \frac{C_n(\mu^2 z^2)(i\nu)^{n-2l}(\frac{z^2 m^2}{4})^l}{n!} A_{n,l}(\mu^2)$$

$l = 0$ comes from traceless symmetric operator

All of this holds for pseudo-PDF

Expansion in Separation

V. Braun and D. Muller *Eur. Phys. J. C* 55 (2008) 349

Y.-Q. Ma and J.-W. Qiu *Phys. Rev. Lett.* 120 (2018) 2, 022003

- Matching to "PDF" in different spaces

- Mellin Space $$M(p,z) = M(\nu, z^2) = \sum_{n=0}^{\infty} \sum_{l=0}^{\lfloor n/2 \rfloor} \frac{C_n(\mu^2 z^2)(i\nu)^{n-2l}(\frac{z^2 m^2}{4})^l}{n!} A_{n,l}(\mu^2)$$

$$A_{n,0}(\mu^2) = \int_{-1}^{1} dx x^{n-1} q(x,\mu^2)$$

$$C_n(\mu^2 z^2) = \int_{-1}^{1} du\, u^{n-1} C(u;\mu^2 z^2)$$

- Ioffe time Space $$M(p,z) = M(\nu, z^2) = \int_{-1}^{1} du\, C(u;\mu^2 z^2) I(u\nu,\mu^2) + O(z^2)$$

$$I(\nu,\mu^2) = \int_{-1}^{1} dx e^{i\nu x} q(x,\mu^2)$$

$$C_n(\mu^2 z^2) = 1 + O(\alpha_s) \rightarrow C(u;\mu^2 z^2) = \delta(1-u) + O(\alpha_S)$$

- Momentum Fraction Space

$$M(p,z) = M(\nu, z^2) = \int_{-1}^{1} dx\, K(x\nu;\mu^2 z^2) q(x,\mu^2) + O(z^2)$$

$$C_n(\mu^2 z^2) = 1 + O(\alpha_s) \rightarrow K(x\nu,\mu^2 z^2) = \exp[ix\nu] + O(\alpha_S) = \int_{-1}^{1} du \exp[ixu\nu] C(u,\mu^2 z^2)$$

Pause for two current summary

- Two Current objects can be factorized to parton structure
- Renormalization and Perturbatively clean
- Choices of which scales to expand in
- Hadronic Tensor could give information outside DIS regime

Wilson Line Matrix Elements

- In quasi-PDF/LaMET and pseudo-PDF/Short distance, separation and momentum swap roles

$$\nu = p \cdot z$$

  - Matrix element $$M(p,z) = \langle p \,|\, \bar{\psi}(z)\gamma^\alpha W(z;0)\psi(0) \,|\, p\rangle$$
$$= 2p^\alpha \mathscr{M}(\nu, z^2) + 2z^\alpha \mathscr{N}(\nu, z^2)$$

Wilson Line Matrix Elements

- In quasi-PDF/LaMET and pseudo-PDF/Short distance, separation and momentum swap roles

$$\nu = p \cdot z$$

  - Matrix element $M(p,z) = \langle p \,|\, \bar{\psi}(z)\gamma^\alpha W(z;0)\psi(0) \,|\, p\rangle$

$$= 2p^\alpha \mathcal{M}(\nu, z^2) + 2z^\alpha \mathcal{N}(\nu, z^2)$$

  - Quasi-PDF: $\mathcal{M}(z, p_z) = \int_{-\infty}^{\infty} \frac{p_z dy}{2\pi} e^{iyp_z z}\, \tilde{q}(y, p_z^2)$ $\qquad z^2 \neq 0$

  - Large Momentum Effective Theory: X. Ji *Phys. Rev. Lett.* 110 (2013) 262002

$$\tilde{q}(y, p_z^2) = \int \frac{dx}{|x|} K\left(\frac{y}{x}, \frac{\mu^2}{(xp_z)^2}\right) q(x, \mu^2) + O\left(\frac{\Lambda_{\text{QCD}}^2}{(xp_z)^2}, \frac{\Lambda_{\text{QCD}}^2}{((1-x)p_z)^2}\right)$$

Wilson Line Matrix Elements

- In quasi-PDF/LaMET and pseudo-PDF/Short distance, separation and momentum swap roles

$$\nu = p \cdot z$$

  - Matrix element $M(p,z) = \langle p \,|\, \bar{\psi}(z)\gamma^{\alpha} W(z;0)\psi(0) \,|\, p \rangle$
    $= 2p^{\alpha}\mathcal{M}(\nu, z^2) + 2z^{\alpha}\mathcal{N}(\nu, z^2)$

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  - Pseudo-ITD:

$$\mathcal{M}(\nu, z^2) = \int dx\, C(x\nu, \mu^2 z^2) q(x, \mu^2) + O(\Lambda_{\mathrm{QCD}}^2 z^2)$$

A. Radyushkin *Phys. Rev. D* 96 (2017) 3, 034025

Wilson Line Matrix Elements

- In quasi-PDF/LaMET and pseudo-PDF/Short distance, separation and momentum swap roles

$$\nu = p \cdot z$$

  - Matrix element $M(p,z) = \langle p | \bar{\psi}(z)\gamma^\alpha W(z;0)\psi(0) | p \rangle$

$$= 2p^\alpha \mathcal{M}(\nu, z^2) + 2z^\alpha \mathcal{N}(\nu, z^2)$$

  - Quasi-PDF: $\mathcal{M}(z, p_z) = \int_{-\infty}^{\infty} \frac{p_z dy}{2\pi} e^{iyp_z z}\, \tilde{q}(y, p_z^2)$ $\qquad z^2 \neq 0$

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$$\tilde{q}(y, p_z^2) = \int \frac{dx}{|x|} K\left(\frac{y}{x}, \frac{\mu^2}{(xp_z)^2}\right) q(x, \mu^2) + O\left(\frac{\Lambda_{\rm QCD}^2}{(xp_z)^2}, \frac{\Lambda_{\rm QCD}^2}{((1-x)p_z)^2}\right)$$

  - Pseudo-ITD:

Integral inverse problem like global analysis

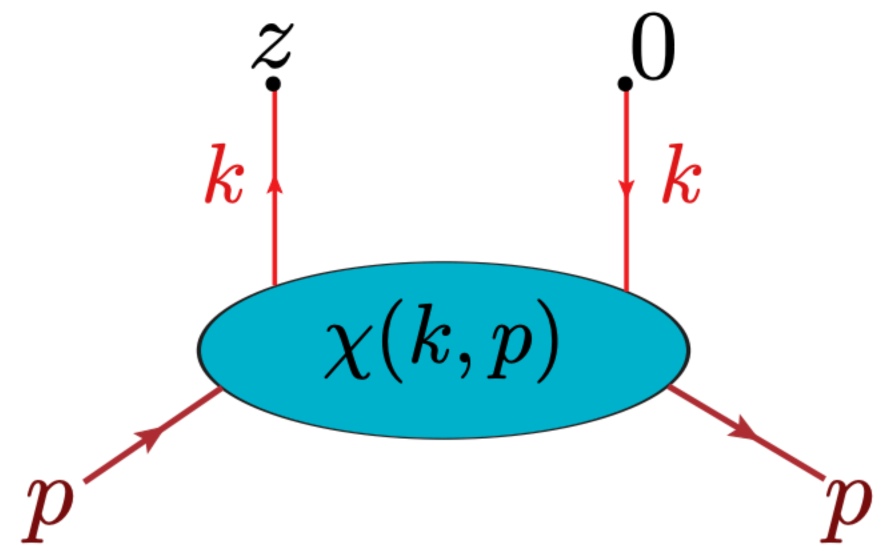
$$\mathcal{M}(\nu, z^2) = \int dx\, C(x\nu, \mu^2 z^2) q(x, \mu^2) + O(\Lambda_{\rm QCD}^2 z^2)$$

A. Radyushkin *Phys. Rev. D* 96 (2017) 3, 034025

Other Faces of WL Matrix Element

- Some are Lorentz invariant interpretations
- These interpretations nor the functions' bounds require small z^2 , only relation to light cone PDF with $z^2 = 0$ and some other regulation

Review: A. Radyushkin (2019) 1912.04244



$$i\chi_{d_i}(k,p) = i^l \frac{P(\text{c.c.})}{(4\pi i)^{2L}} \int_0^\infty \prod_{j=1}^{l} d\alpha_j [D(\alpha)]^{-2}$$
$$\times \exp\left\{ ik^2 \frac{A(\alpha)}{D(\alpha)} + i\frac{(p-k)^2 B_s(\alpha) + (p+k)^2 B_u(\alpha)}{D(\alpha)} \right\}$$
$$\times \exp\left\{ ip^2 \frac{C(\alpha)}{D(\alpha)} - i\sum_j \alpha_j (m_j^2 - i\epsilon) \right\} ,$$

$$i\chi(k,p) = \int_0^\infty d\sigma \int_{-1}^{1} dx e^{i\sigma\left[k^2 - 2x(k\cdot p) + i\epsilon\right]} V(x,\sigma)$$

$$\sigma_{d_i} = \frac{A_{d_i}(\alpha) + B_{s_{d_i}}(\alpha) + B_{u_{d_i}}(\alpha)}{D_{d_i}(\alpha)}$$

α_j are positive numbers and A, B_u, B_s, C, D are sums of products of α_j

Fourier transform to position space

$$x_{d_i} = \frac{B_{s_{d_i}}(\alpha) - B_{u_{d_i}}(\alpha)}{A_{d_i}(\alpha) + B_{s_{d_i}}(\alpha) + B_{u_{d_i}}(\alpha)}$$

$$\mathcal{M}(\nu, z^2) = \int_{-1}^{1} dx e^{i\nu x} \int_0^\infty e^{-i\sigma(z^2-\epsilon)} V(x,\sigma)$$

Other Faces of WL Matrix Element

- Some are Lorentz invariant interpretations
- These interpretations nor the functions' bounds require small z^2, only relation to light cone PDF with $z^2 = 0$ and some other regulation

Review: A. Radyushkin (2019) 1912.04244

Virtuality Distribution Function
Lorentz invariant picture
σ^{-1} pole gives log z^2
Limits from nature of Feynman diagrams

$$\mathcal{M}(\nu, z^2) = \int_{-1}^{1} dx e^{i\nu x} \int_0^\infty e^{-i\sigma(z^2-\epsilon)} V(x,\sigma)$$

pseudo-PDF
Lorentz invariant picture
log z^2 divergence from poles of TMD/VDF

$$\mathcal{M}(\nu, z^2) = \int_{-1}^{1} dx e^{i\nu x} P(x, z^2)$$

$$\tilde{q}(y, p_z^2) = \int dz \int_{-1}^{1} dx e^{ip_z z(x-y)} P(x, z^2)$$

Musch, Hagler, Negele, Schafer PRD 83 (2011) 094507

Straight Link / Primordial TMD
Frame dependent picture with nice interpretation
$1/k_T^2$ pole gives log z^2

$$z = (0, z^-, z_T) \qquad p = (p^+, \frac{m^2}{p^+}, 0_T)$$

$$\mathcal{M}(\nu, z^2) = \int_{-1}^{1} dx e^{i\nu x} \int d^2k_T e^{ik_T \cdot z_T} F(x, k_T^2)$$

Light cone PDF from regulated integral of TMD
Relate to the Lorentz invariant VDF
$1/k_T^2$ or σ^{-1} poles generate $\log \mu^2$ divergence

$$f(x, \mu^2) = \int^{\mu^2} d^2k_T F(x, k_T^2) = \int_0^\infty d\sigma \left[1 - e^{-\frac{i}{\sigma}(\mu^2 - i\epsilon)}\right] V(x, \sigma)$$

The Role of Separation and Momentum

- In quasi-PDF, pseudo-PDF, and Structure Functions, variables have common roles

Scale:

$p_z^2 \,/\, z^2 / Q^2$

- Scale for factorization to PDF
- Scale in power expansion
- Keep away from Λ_{QCD}^2
- Technically only requires single value

Dynamical variable:

$z \,/\, p_z$, or $\nu = p \cdot z$, x_B

- Variable describes non-perturbative dynamics
- Can take large or small value
- Want as many as are available
- Wider range improves the inverse problem

Pause for Wilson Line summary

- Mimics PDF's original definition but embrace space-like
- Primary advantage is 3-point function not 4-point function
- Two parameters p, z and choose one large or other small

Inverse Problems for pseudo-PDFs

- Limited range of z and p cannot approach $\nu \to \infty$ to integrate inverse

$$q(x) = \int_0^\infty d\nu\, C^{-1}(x\nu)\mathfrak{M}(\nu)$$

JK, K. Orginos, A. Rothkopf, S. Zafeiropoulos JHEP 04 (2019) 057

Inverse Problems for pseudo-PDFs

- Limited range of z and p cannot approach $\nu \to \infty$ to integrate inverse
- Forward integral to an ill-posed matrix equation

$$q(x) = \int_0^\infty d\nu \, C^{-1}(x\nu)\mathfrak{M}(\nu)$$

$$\mathfrak{M}(\nu) = \int_0^1 dx \, C(x\nu)\, q(x) \to [\mathbf{C}][\mathbf{q}]$$

JK, K. Orginos, A. Rothkopf, S. Zafeiropoulos JHEP 04 (2019) 057

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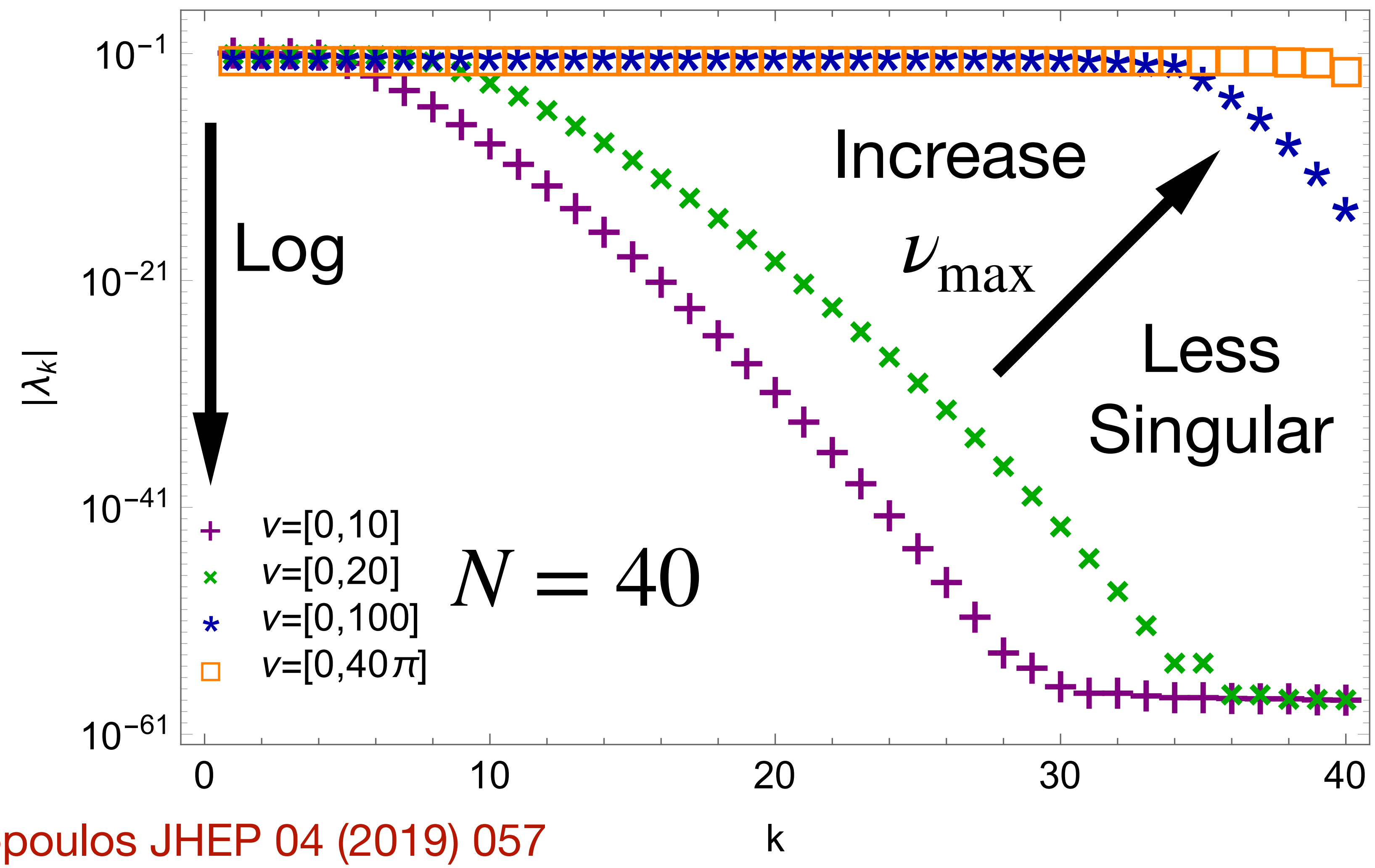

JK, K. Orginos, A. Rothkopf, S. Zafeiropoulos JHEP 04 (2019) 057

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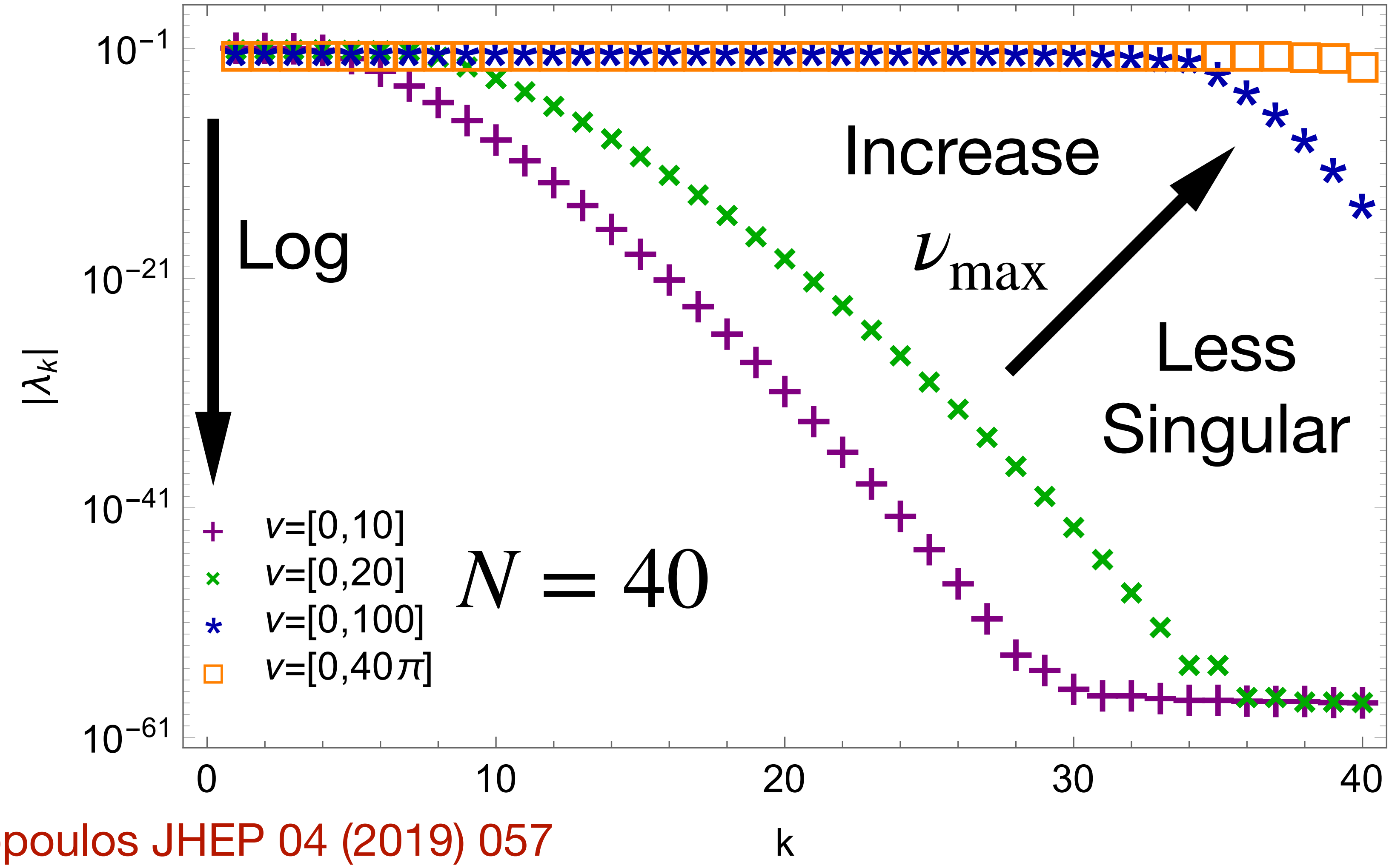
$$\mathfrak{M}(\nu) = \int_0^1 dx\, C(x\nu)\, q(x) \to [\mathbf{C}][\mathbf{q}]$$

- Must be regulated by additional information
  - Restricted functional form
  - Constraints on the PDF or parameters
  - Assumptions of smoothness, continuity, ….



JK, K. Orginos, A. Rothkopf, S. Zafeiropoulos JHEP 04 (2019) 057

Inverse Problems for Parton Physics

- **Structure Functions (from pheno)**

$$F_2(x,Q^2) = \sum_i \int_x^1 d\xi \, C(\xi, \frac{\mu^2}{Q^2}) \, q(\frac{x}{\xi}, \mu^2)$$

- **LaMET (on the lattice)**

$$M(p_z, z) = \int_{-\infty}^{\infty} p_z dy \, e^{iyp_z z} \, \tilde{q}(y, p_z^2)$$

- **pseudo-Distributions / Good Lattice Cross Sections**

$$\mathfrak{M}(\nu, z^2) = \int_{-1}^{1} dx \, C(x\nu, \mu^2 z^2) \, q(x, \mu^2)$$

- **Local Matrix elements / HOPE / OPE-without-OPE**

$$a_n(\mu^2) = \int_{-1}^{1} dx \, x^{n-1} \, q(x, \mu^2)$$

- **Hadronic Tensor**

$$\tilde{W}_{\mu\nu}(\tau) = \int d\nu \, e^{-\nu\tau} \, W_{\mu\nu}(\nu)$$

Approaches to inverse problem

- **Parametric**
  - Fit a phenomenologically motivated function
    - Method used by global fits
    - Potentially significant, but controllable model bias
  - Fit to a neural network

S. Forte, L. Garrido, J. Latorre, A. Piccione (2002) 0204232
K. Cichy, L. Del Debbio, T. Giani (2019) 1907.06037
L. Del Debbio, T. Giani, JK, K. Orginos, A. Radyushkin, S. Zafeiropoulos (2020) 2010.03996

- **Non-Parametric**
  - Backus Gilbert

For NN/BG/MEM/BR: JK, K. Orginos, A. Rothkopf, S. Zafeiropoulos (2019) 1901.05408

J. Liang, K-F. Liu, Y-B. Yang (2017) 1710.11145

  - Maximum Entropy Method / Bayesian Reconstruction

Y. Burnier and A. Rothkopf (2013) 1307.6106, J. Liang et al (2019) 1906.05312

  - Bayes-Gauss-Fourier transfrom

C. Alexandrou, G. Iannelli, K. Jansen, F. Manigrasso (2020) 2007.13800

  - Gaussian Process Regression

A. Candido, L. Del Debbio, T. Giani, G. Petrillo (2024) 2404.07573

Bayesian Solutions

- Inverse Problem Definition: Want to understand a larger possibly infinite amount of information, such as functions, from a finite amount of data

- Integral Inverse problems are interpolations and/or extrapolations

$$M(\nu) = \int dx B(\nu, x) q(x)$$

- We regulate problem by having some prior information and some data on what that function
  - Bayes's theorem $P[A \mid B, C] = \frac{P[B \mid A] \, P[A \mid C]}{P[B \mid C]}$
  - For Regression we want $\langle q \rangle = \int Dq \, q \, P[q \mid M, I]$
    - A is the function q we want to infer
    - B is the data M we want to inform our inference
    - C is the prior information I we wish to use to constrain the result

Components of the Posterior

- The inverse we wish to understand $M(\nu) = \int dx B(\nu, x) q(x)$

Data Likelihood: assumed by Central Limit Theorem

$$P[M|q] \propto \exp[-\frac{1}{2}\sum_{\nu\nu'} (M_\nu - M(\nu)) C^{-1}_{\nu\nu'} (M_{\nu'} - M(\nu'))] = \exp[-\frac{1}{2}\chi^2]$$

Posterior

$$P[q|M,I] = \frac{P[M|q]\,P[q|I]}{P[M|I]}$$

"Not using priors and minimizing χ^2" is actually setting the uniform prior $P[q|I] = 1$

Prior Distribution: Choice here decides how problem is regulated

Evidence: Normalizing factor independent of q

$$P[M|I] = \int Dq\, P[M|q] P[q|I]$$

Parameterized fits

$$P[q\,|\,M, I] = \frac{P[M\,|\,q]\,P[q\,|\,I]}{P[M\,|\,I]}$$

- Use physics or math to justify a tractable form

$$Q(x; N, \alpha, \beta) = \frac{N x^{\alpha}(1-x)^{\beta}}{B(\alpha+1, \beta+1)} \qquad Q(x; \alpha, \beta, \theta) = x^{\alpha}(1-x)^{\beta} NN(x; \theta)$$

- Prior information us a δ-function in function space

$$P[q\,|\,I] = \int dN d\alpha d\beta\, \delta\big(q - Q(\,.\,; N, \alpha, \beta)\big)\, P[N, \alpha, \beta\,|\,I]$$

- Can include priors on the parameters

$$\langle q(x) \rangle = \int dq q(x) P[q\,|\,M, I] = \int dN d\alpha d\beta\, Q(x; N, \alpha, \beta) P[N, \alpha, \beta\,|\,M, I]$$

- Maximize the posterior to get most likely parameters

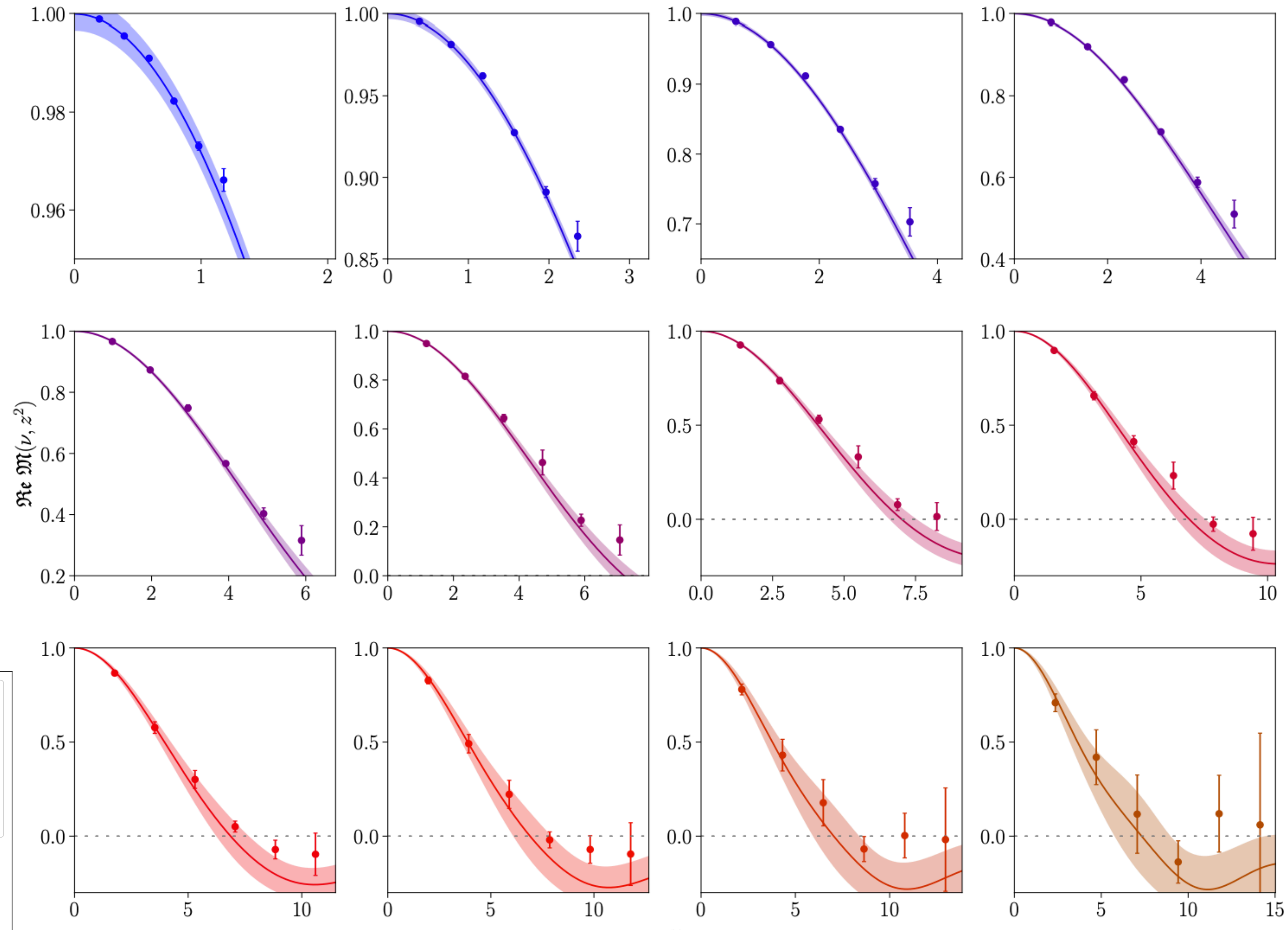
Obtaining a PDF

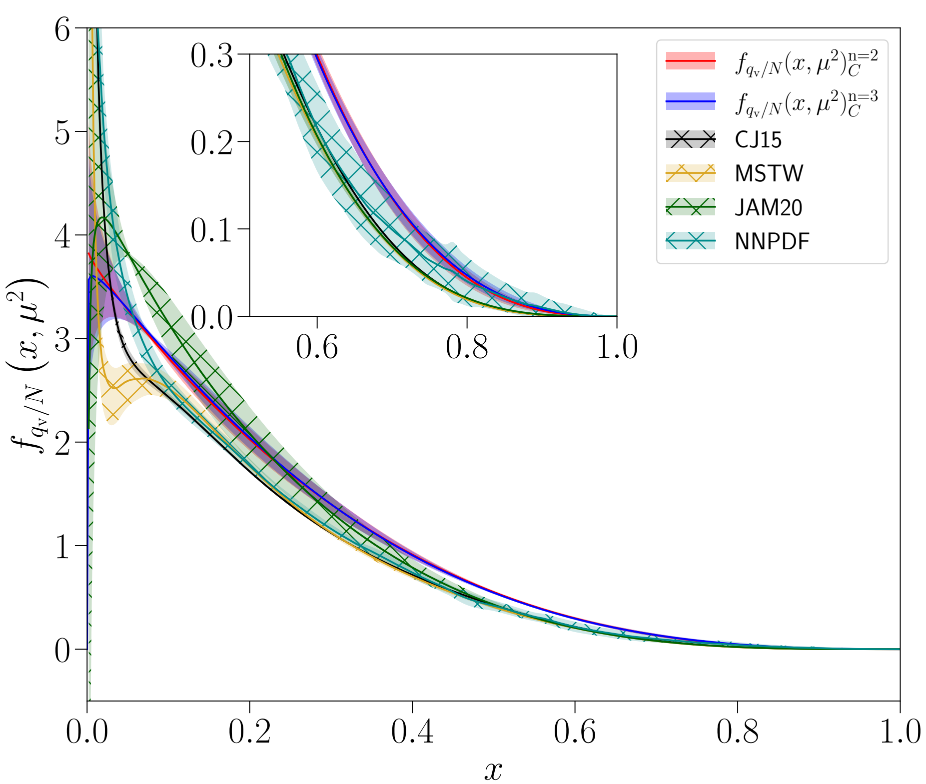
C. Egerer et al (HadStruc) 2107.05199

1. Calculate matrix elements with many p and z

2. Model (quasi-)PDF and its corrections

3. If doing LAMET, match quasi-PDF

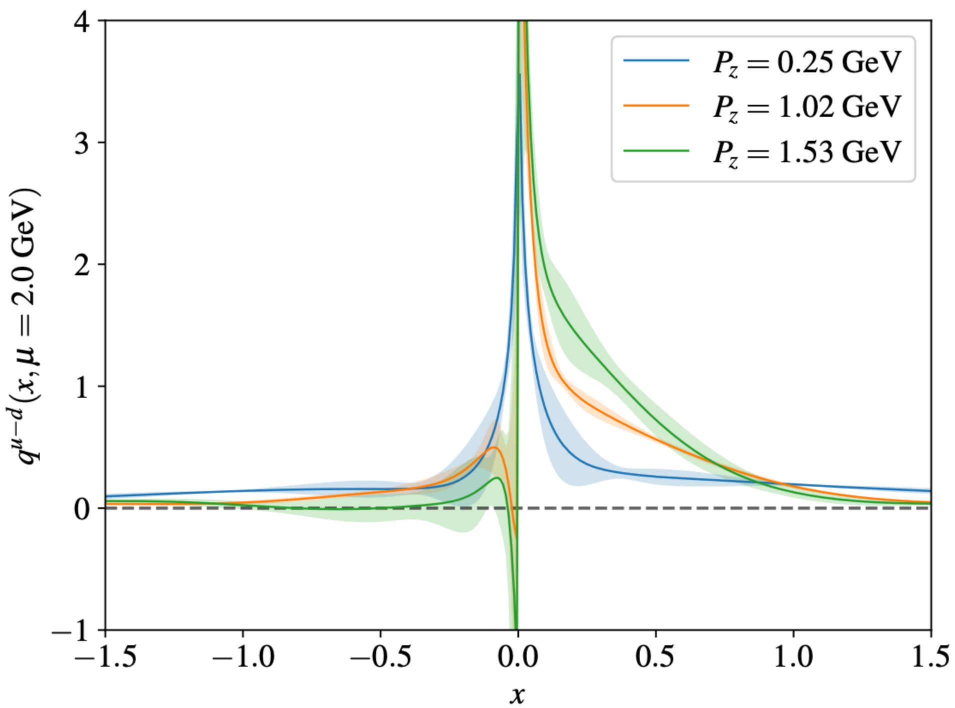




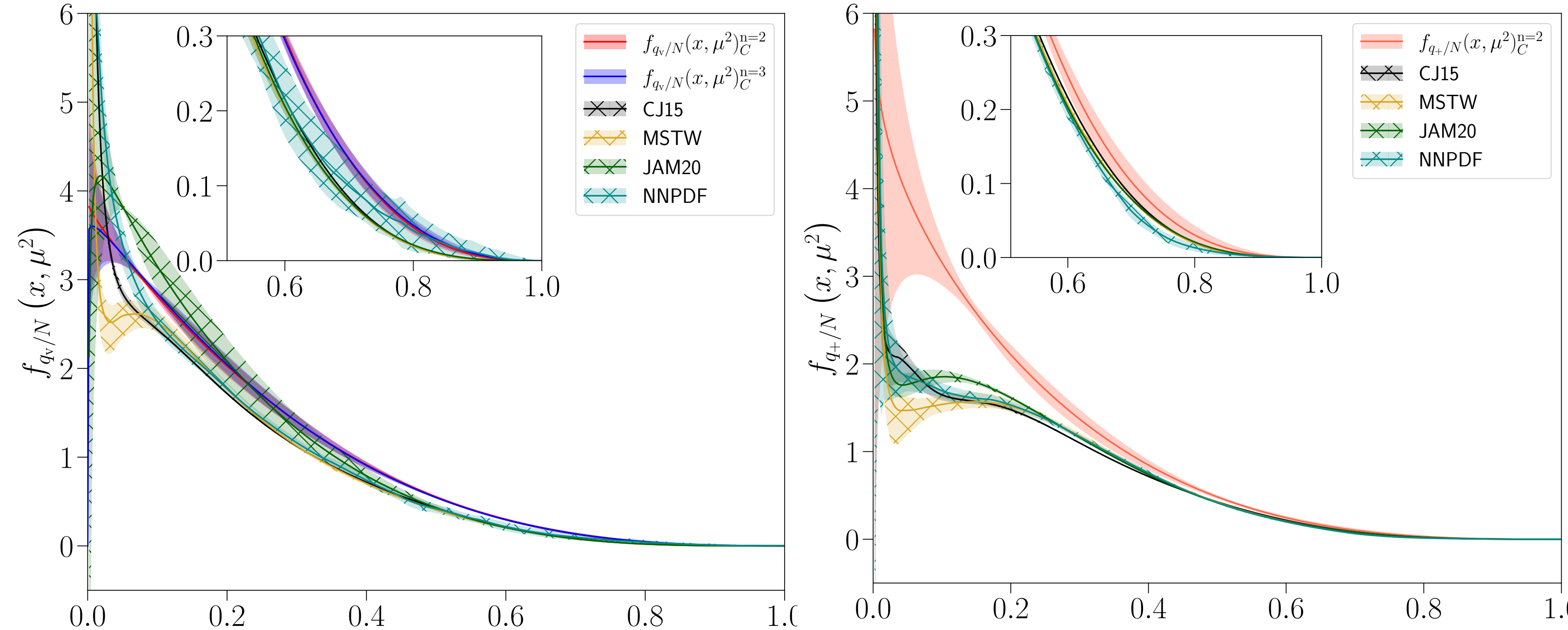
$$\mathrm{Re}\,\mathfrak{M}(\nu, z^2) = \int_0^1 dx C(\nu x, \mu^2 z^2) q(x, z^2) + O(z^2) + O(a/z) + \ldots$$

Nucleon Unpolarized Quark PDF

X. Gao et al (ANL/BNL) 2212.12569



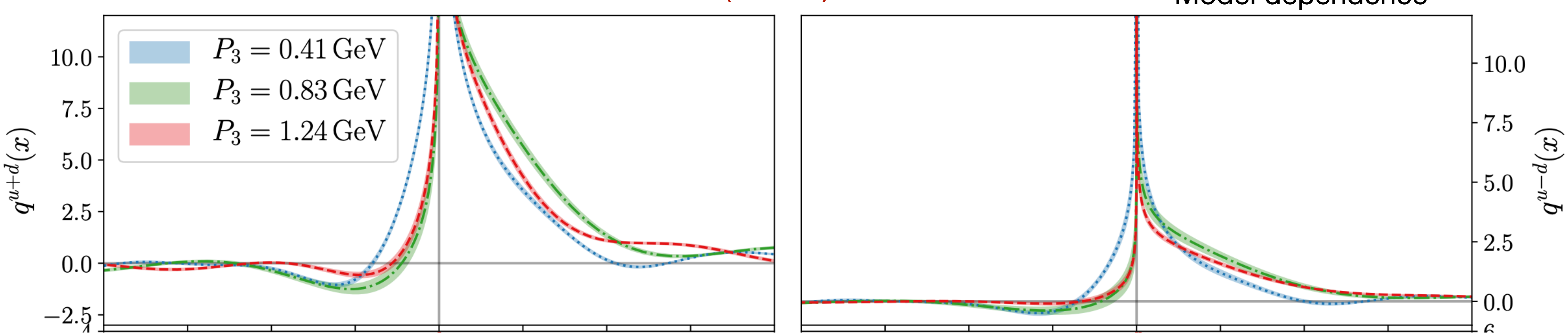
C. Egerer et al (HadStruc) 2107.05199



- Lattice spacing
- Pion mass
- Excited States
- Finite Volume
- Higher order matching
- Power Corrections
- Model dependence

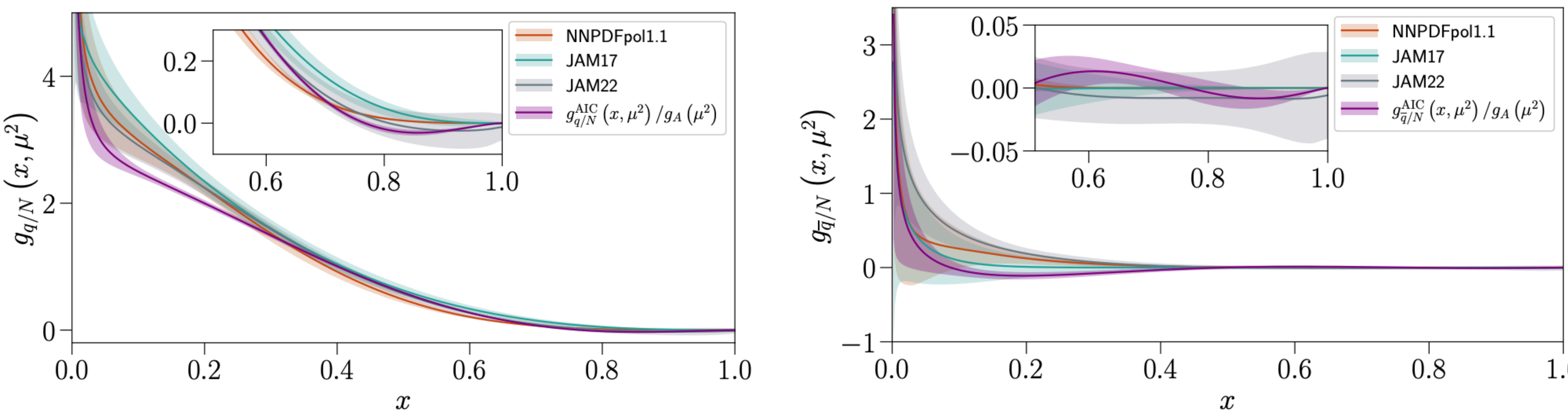
- Approaching a decade since first calculations
- Systematics have been continually improved

C. Alexandrou et al (ETMC) 2106.16065



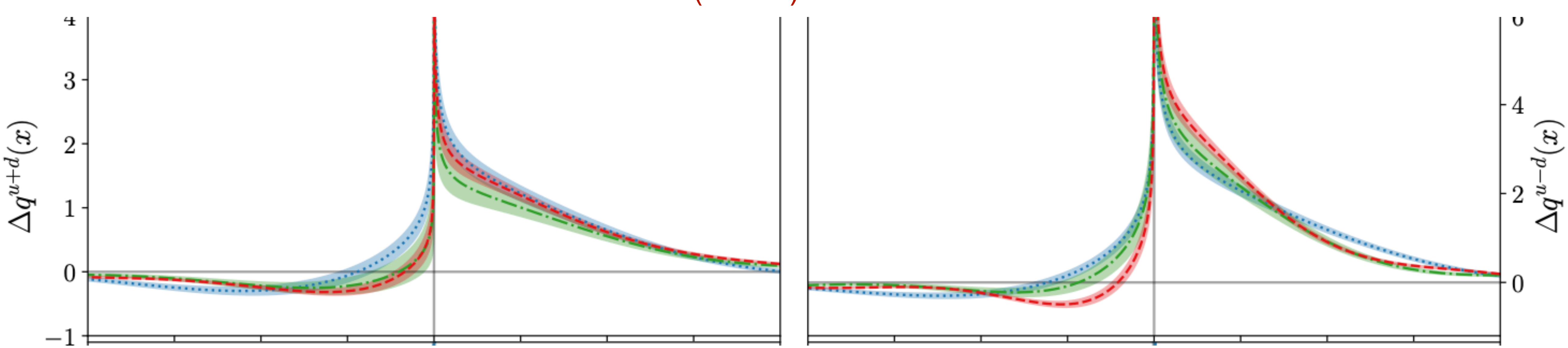
Nucleon Helicity Quark PDF

C. Egerer et al (HadStruc) 2211.04424

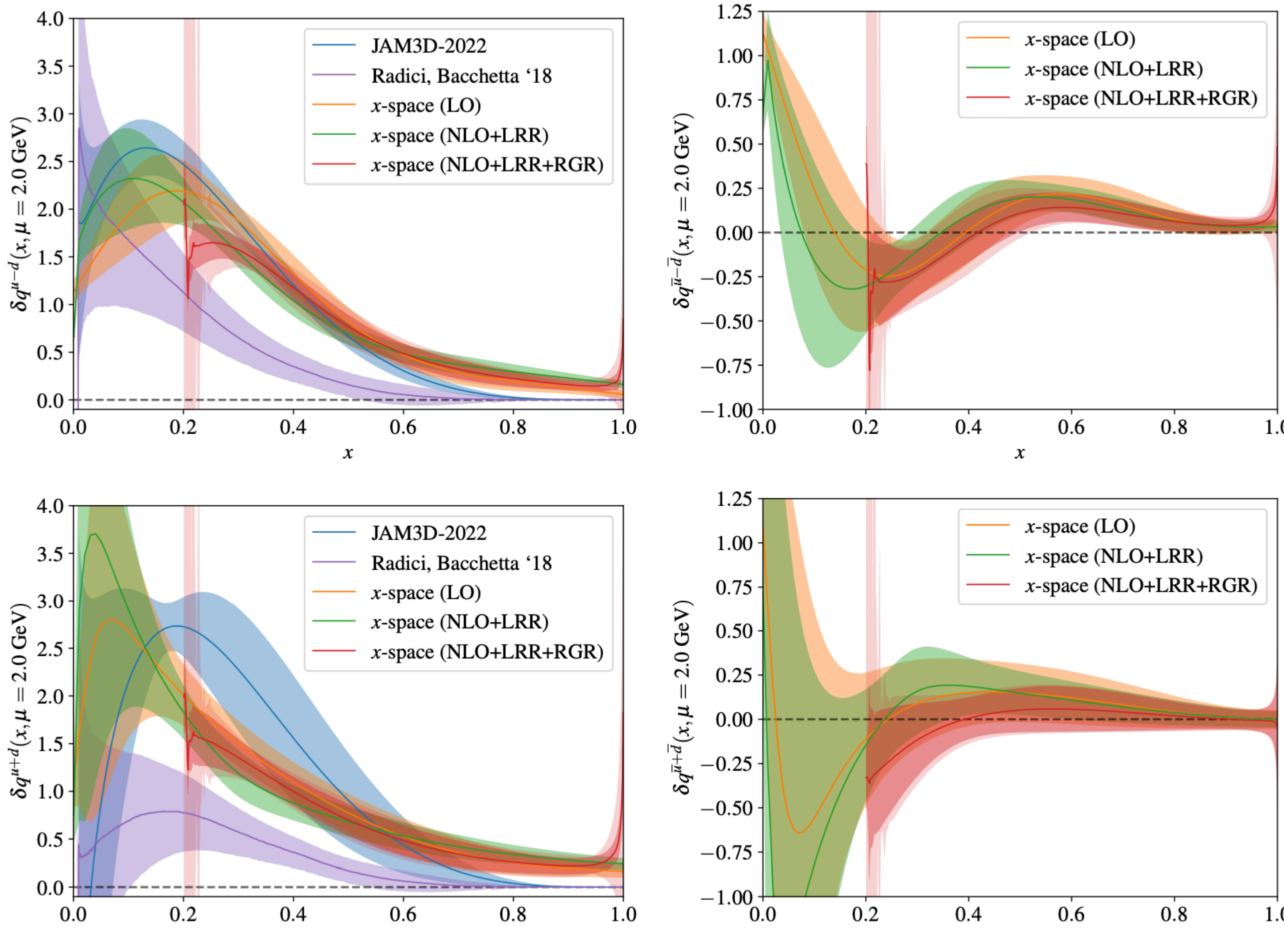


- Approaching a decade since first calculations
- Systematics have been continually improved

C. Alexandrou et al (ETMC) 2106.16065

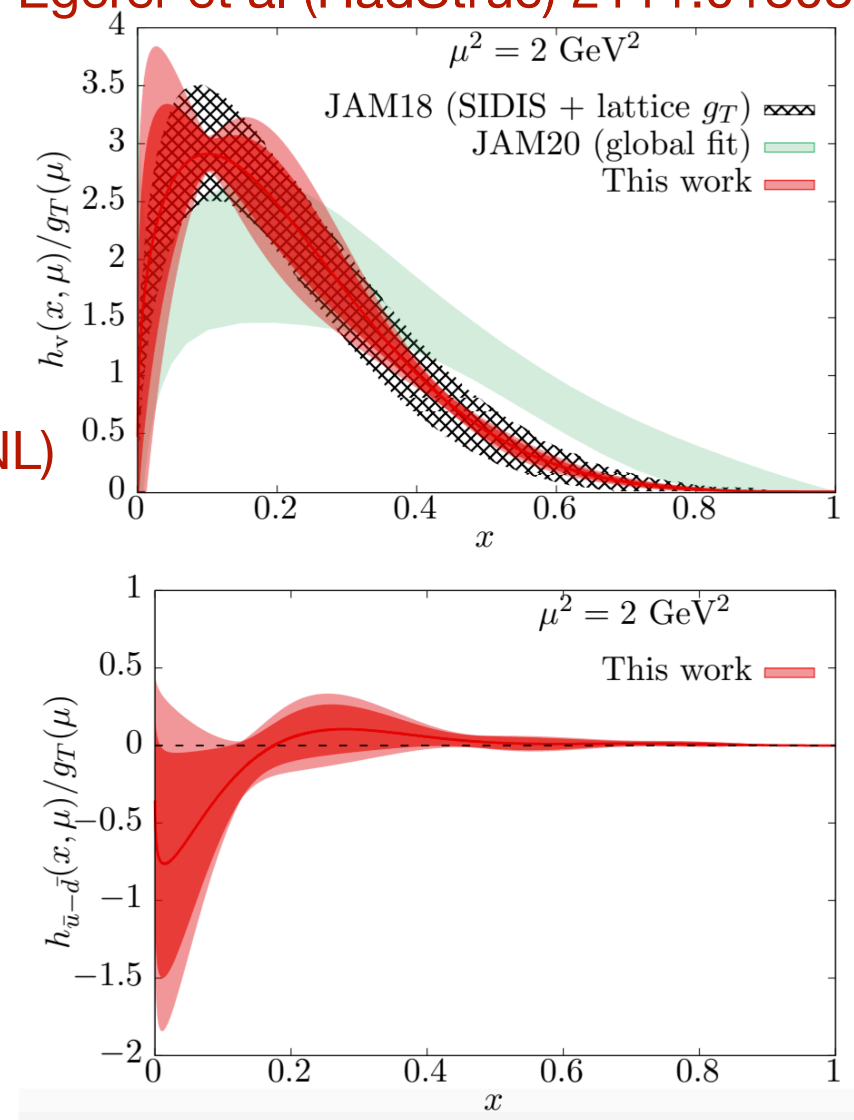


Nucleon Transversity Quark PDF



X. Gao et al (ANL/BNL) 2310.19047

C. Egerer et al (HadStruc) 2111.01808



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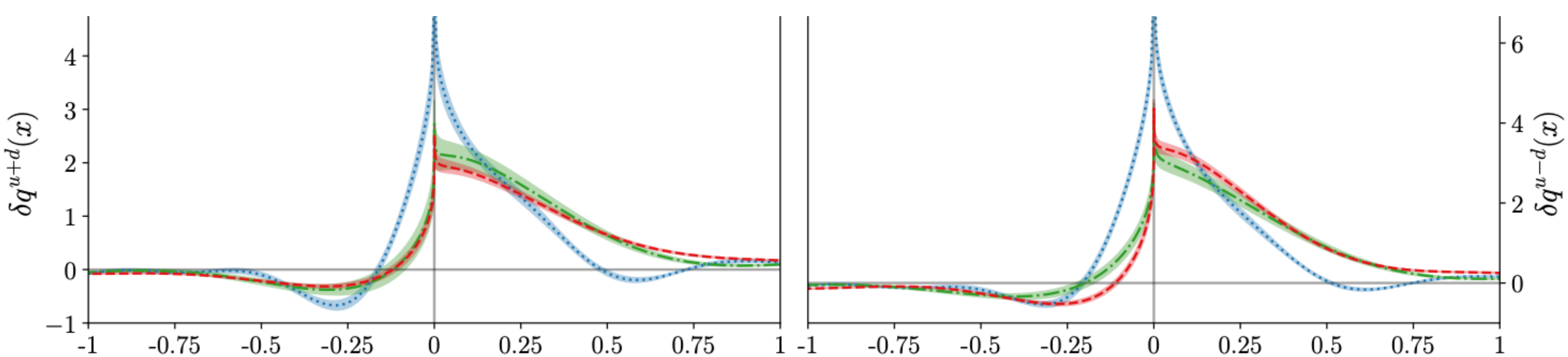


If PDFs are universal….

*If the **same** PDFs are factorizable from lattice and experiment, and if power corrections can be controlled for both*

Why not analyze both simultaneously?

- Factorization of hadronic cross sections

$$d\sigma_h = d\sigma_q \otimes f_{h/q} + P.C.$$

- Factorization of Lattice observables

$$M_h = M_q \otimes f_{h/q} + P.C.$$

Consider Lattice as a theoretical prior to the experimental Global Fit

Complementarity in Lattice and Experiment

LATTICE

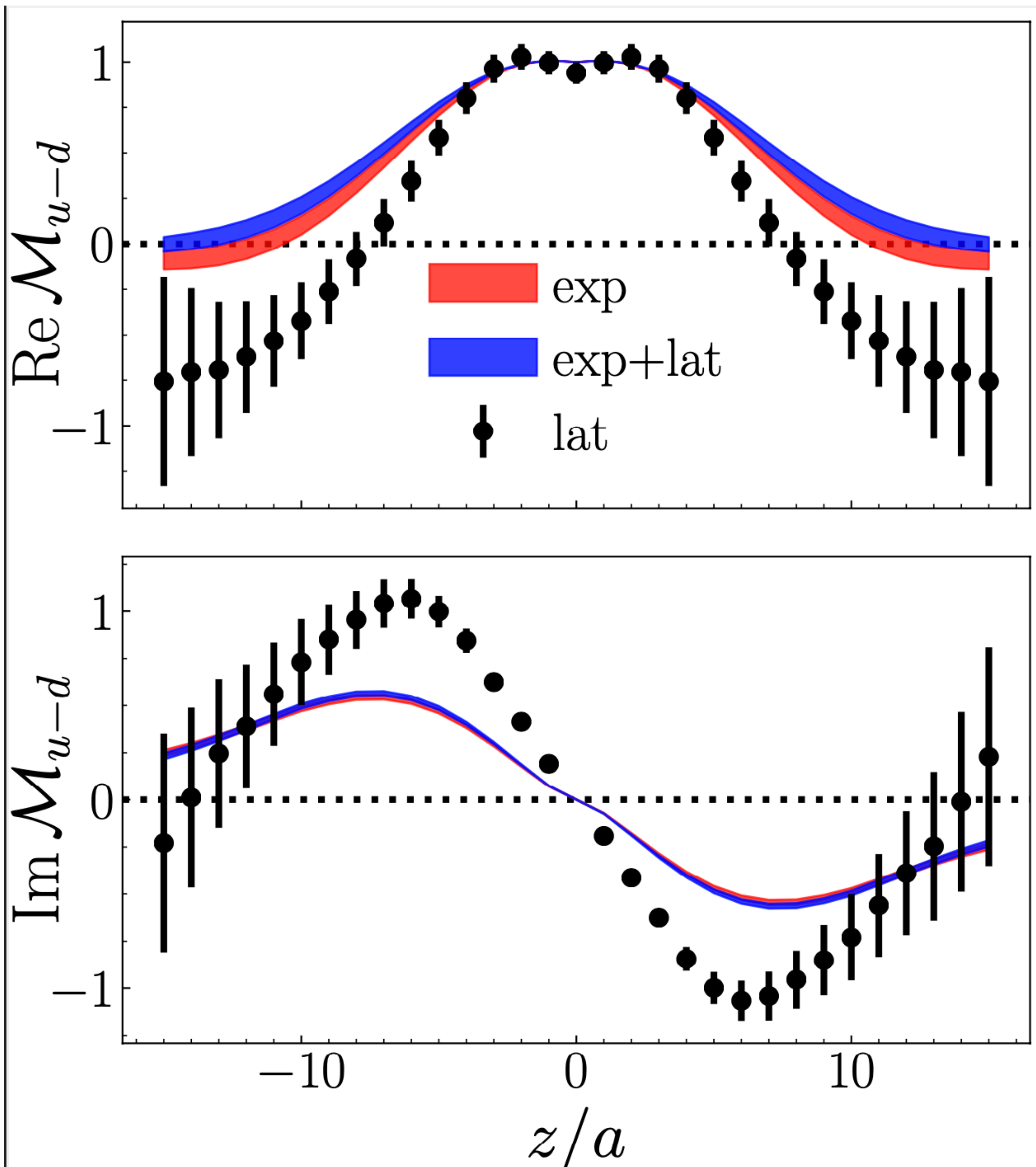
- Lattice limited to low ν, sensitive to $x \gtrsim 0.2$, but high sensitivity to large x
- Lattice matching relation is integral over all x
- Low p_z data can reach high signal-to-noise compared to experiment
- Lattice can evaluate independently each spin, flavor, and even hadron

EXPERIMENT

- Cross Sections limited to specific max but can reach very low x_B
- Cross Section matching is integral from x_B to 1
  - Creates sensitively to hard kernel in large x region
- Wealth of decades of experimental data outweigh modern lattice

First combined lattice PDF and experiment global analysis (unpol)

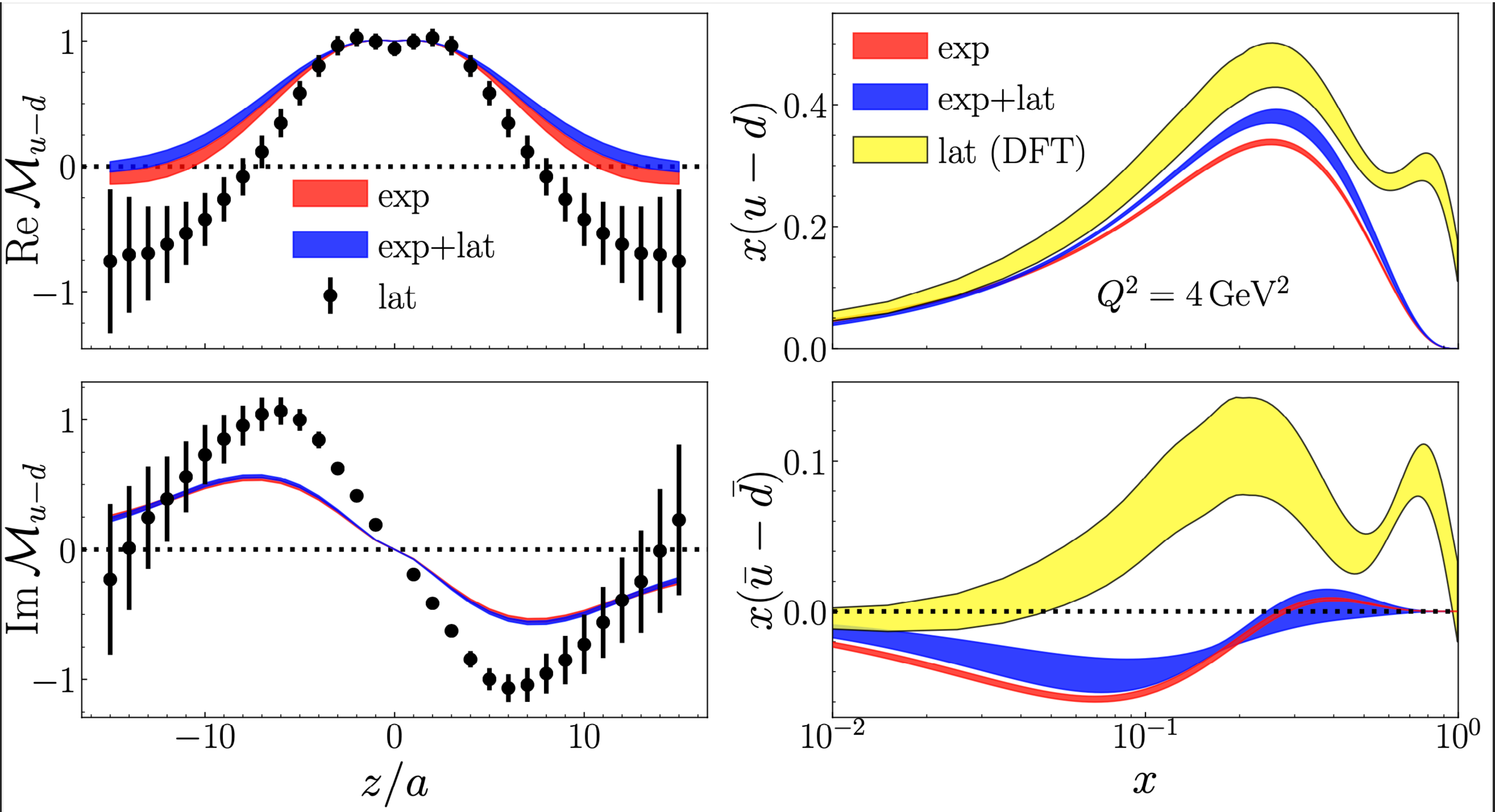
J. Bringewatt et al Phys Rev D 103, 016003 (2021)



- First study by ETMC and JAM collaborations
- Lattice data provide independent measurements of PDFs
- Can study discrepancies to understand systematic errors

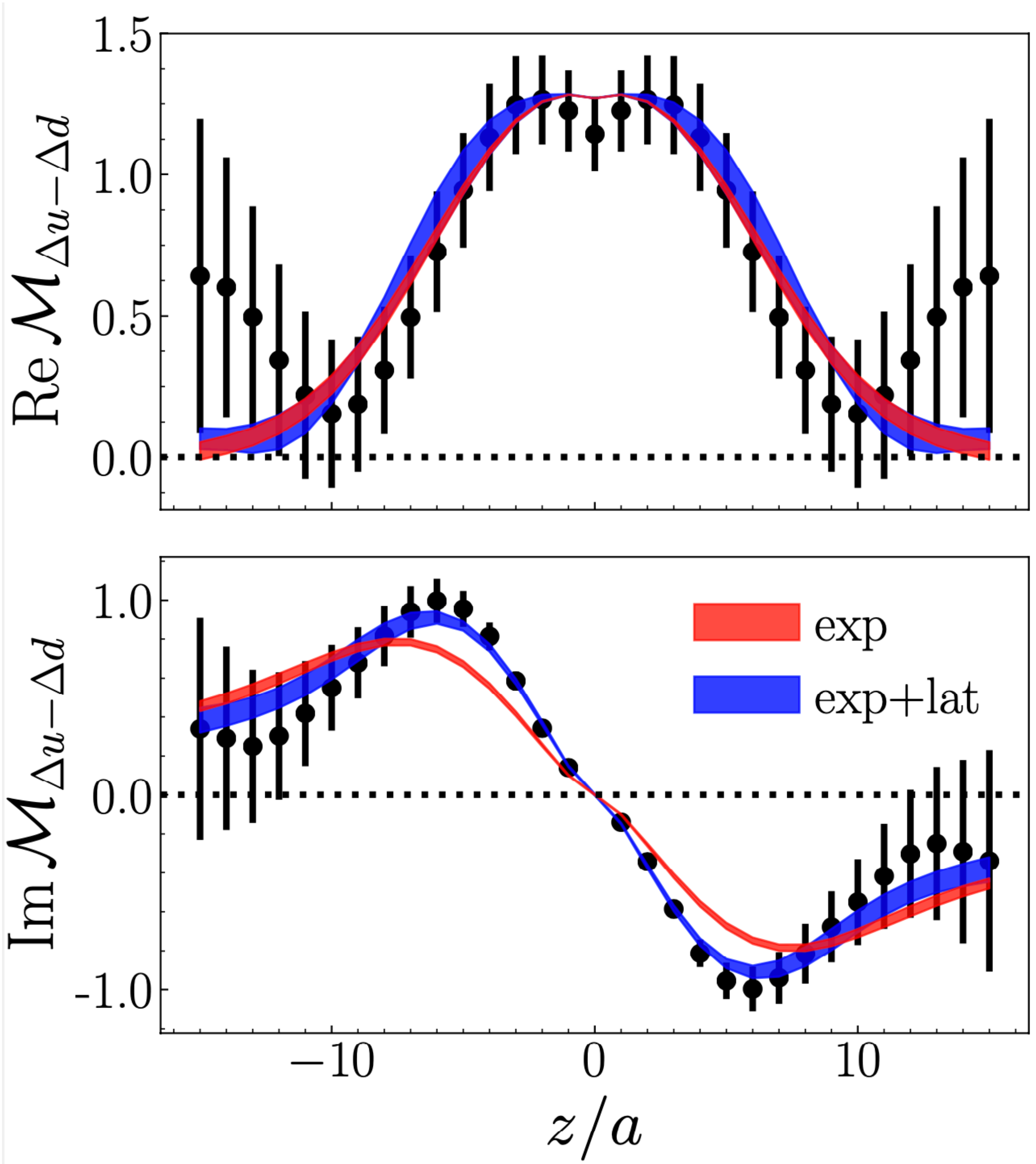
First combined lattice and experiment global analysis (unpol)

J. Bringewatt et al Phys Rev D 103, 016003 (2021)



First combined lattice and experiment global analysis (heli)

J. Bringewatt et al Phys Rev D 103, 016003 (2021)

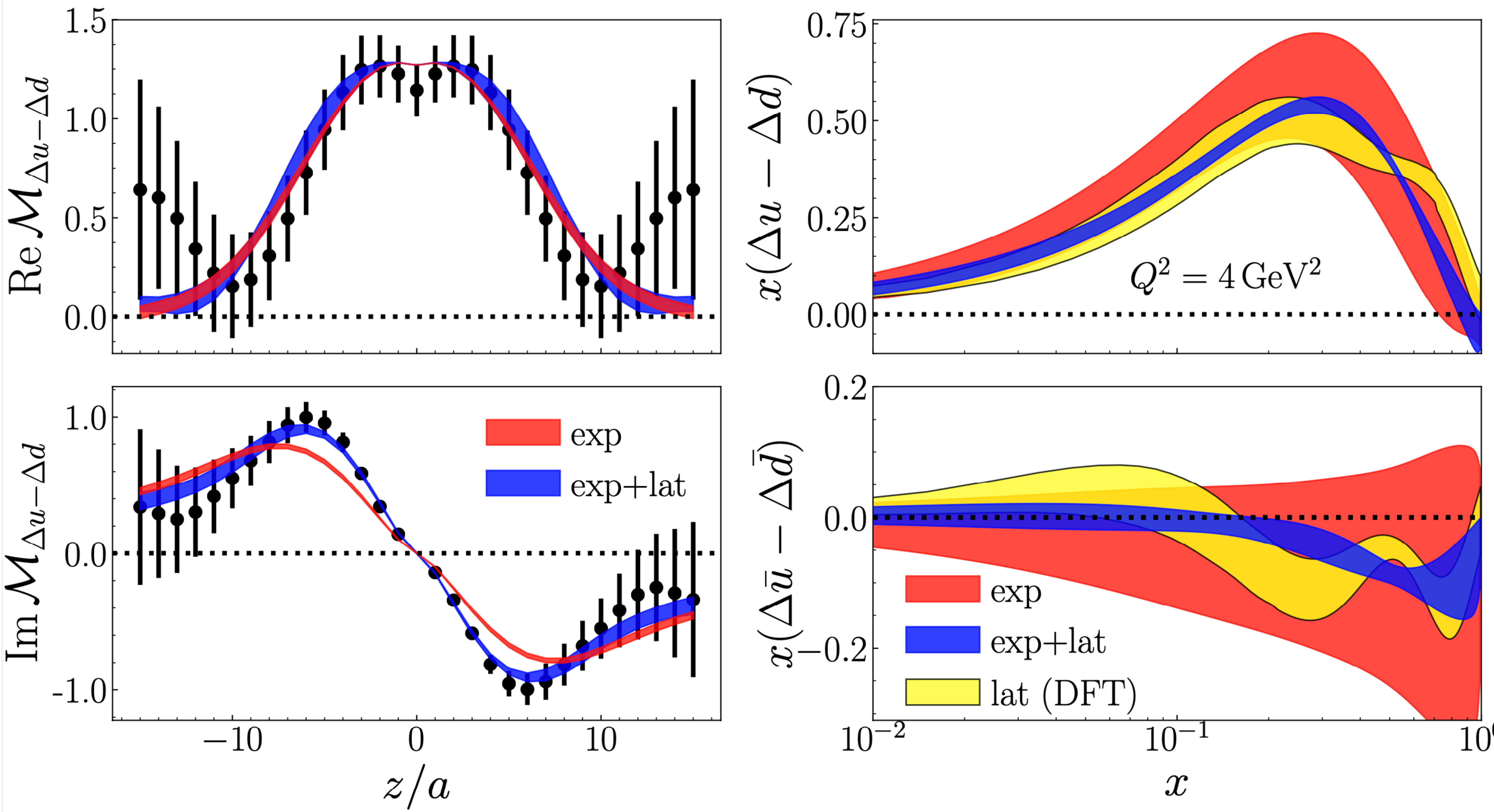


- Lattice matrix elements can give direct independent information on different spins without major modifications

- Some datapoints can be more precise than experiment and give constraining power

First combined lattice and experiment global analysis (heli)

J. Bringewatt et al Phys Rev D 103, 016003 (2021)



Strange quark distributions

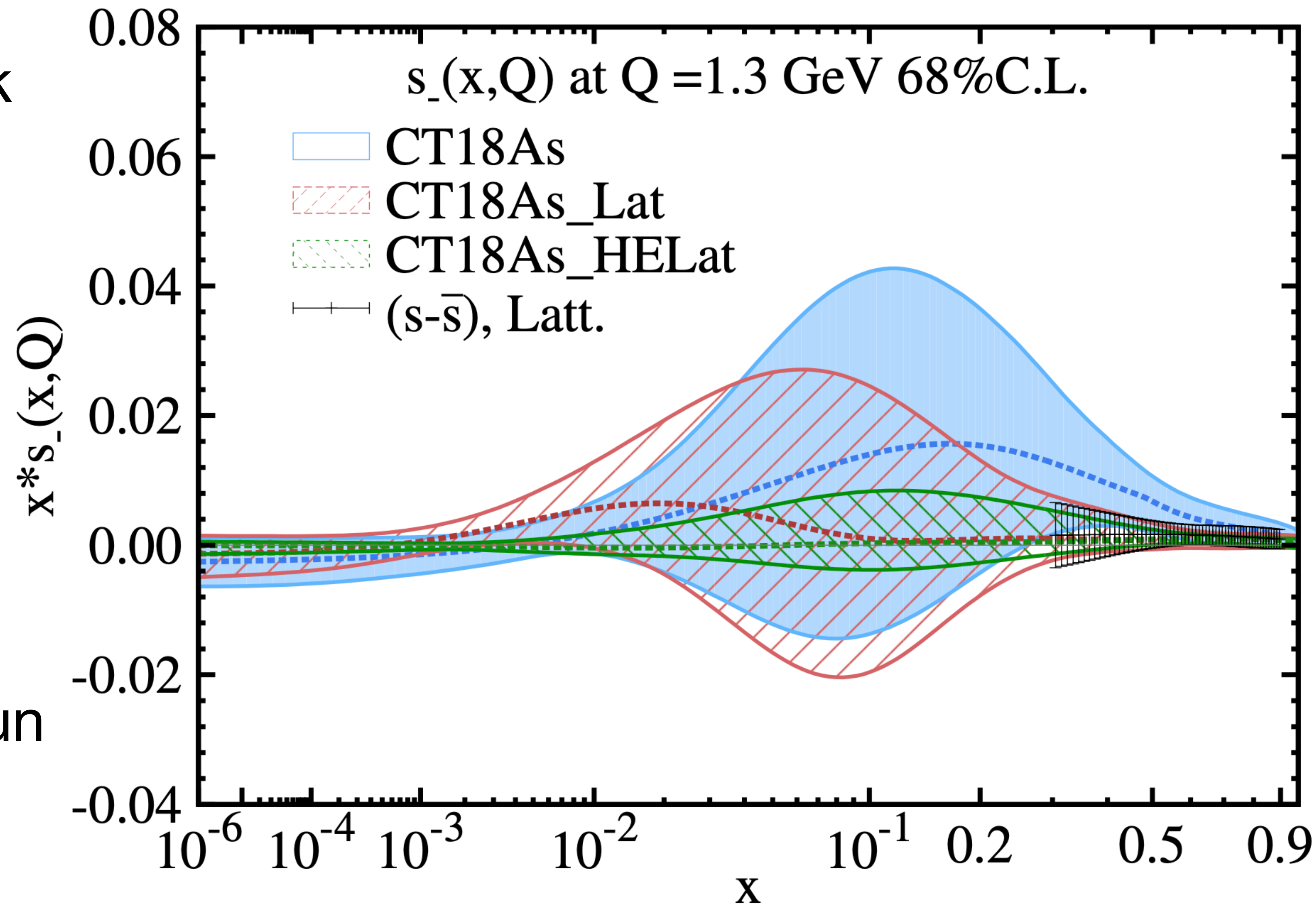
- Lattice can directly access individual quark flavors almost independently

- Flavor decomposed matrix elements have noisy "disconnected" contributions

- Studies of strange and charm PDFs have begun and give promising precision

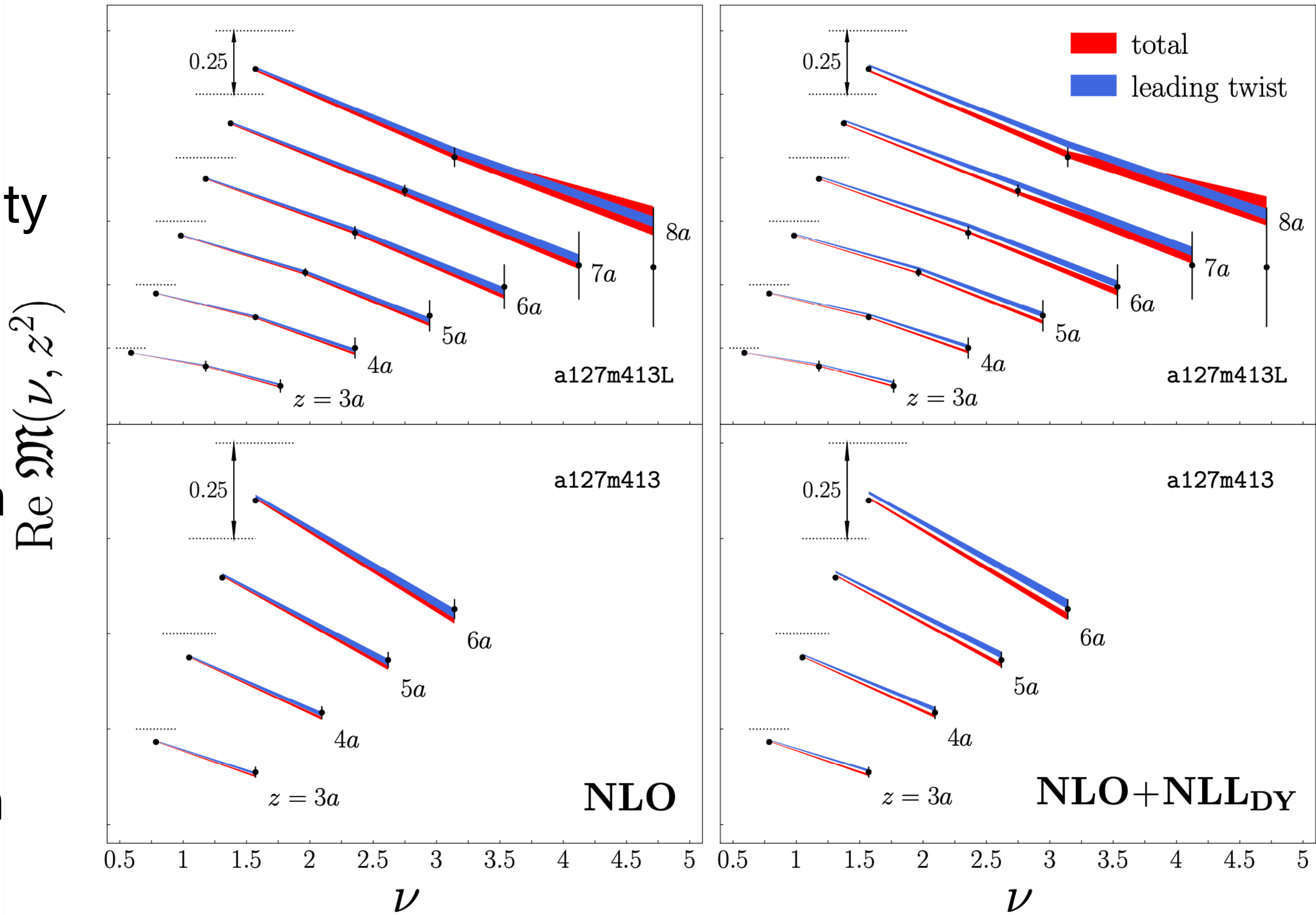
$$s\_(x) = s(x) - \bar{s}(x)$$

Hou et al arXiv 2204.07944



Complementarity in pion PDF

- Lattice can directly access different hadrons
- Lattice lacks sensitivity to threshold logs and can be used to test theoretical kernels
- Improves precision in large x where experimental data does not exist
- Low momentum pion data are extremely precise



P. Barry et al, *Phys. Rev. D* 105 (2022) 11, 114051

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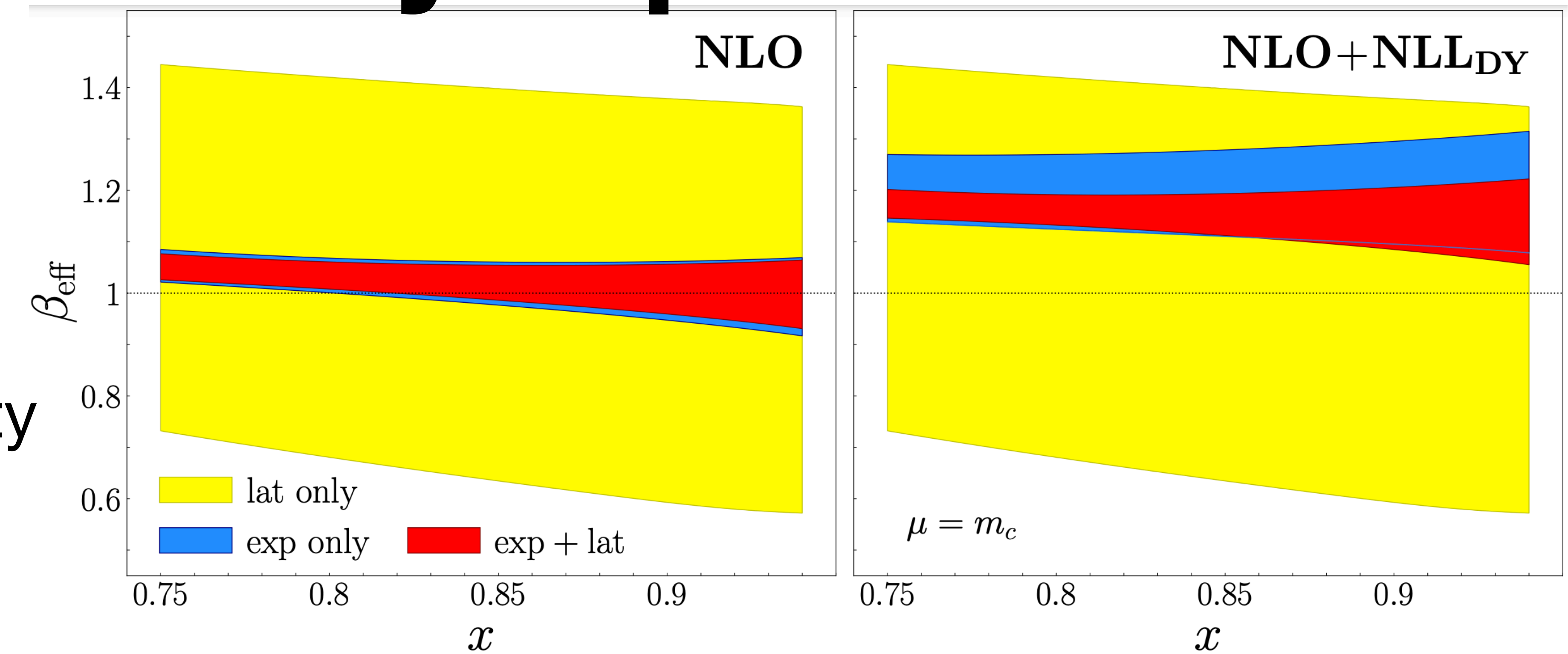

P. Barry et al, *Phys. Rev. D* 105 (2022) 11, 114051