THE INVERSE PROBLEM AND SHADOW GENERALIZED PARTON DISTRIBUTIONS (GPDS)

1ST INTERNATIONAL SCHOOL OF HADRON FEMTOGRAPHY 9-18-24

PRESENTER:
ERIC MOFFAT
ARGONNE NATIONAL LAB

The Inverse Problem

- Methods of accessing non-perturbative functions:
  - Lattice QCD (See lectures by Joe Karpie, Robert Edwards, and Eloy Romero Alcalde)
  - Global analysis (See also lectures by Marija Čuić, Yuxun Guo, and Hervé Dutrieux):
    - Extraction of non-perturbative functions from experimental data
    - Observables are related to non-perturbative functions via convolution (integrals) with perturbative factors
    - Inverse problem:
      - Trying to determine the integrand from the value of an integral

The Inverse Problem

- Particularly challenging in the case of GPDs:
  - GPDs are functions of three variables (x, ξ, and t)
  - x-dependence is lost in the integration:
    - Deeply virtual Compton scattering (DVCS):
      - Compton Form Factors:

$$\mathcal{H}(\xi, t, Q^2) = \int_{-1}^{1} dx \sum_a C^a(x, \xi, Q^2, \mu^2)\, H^a(x, \xi, t; \mu^2) \quad \mathcal{E}(\xi, t, Q^2) = \int_{-1}^{1} dx \sum_a C^a(x, \xi, Q^2, \mu^2)\, E^a(x, \xi, t; \mu^2)$$

    - Same is true for Deeply virtual meson production (DVMP)
  - While a fit could obtain a GPD:
    - Does the x-dependence represent the true GPD?
    - There is an infinite number of functions that can give the same CFF
    - Shadow GPDs (SGPDS) (Bertone, et. al. Phys.Rev.D 103 (2021) 11, 114019):

Properties of GPDs

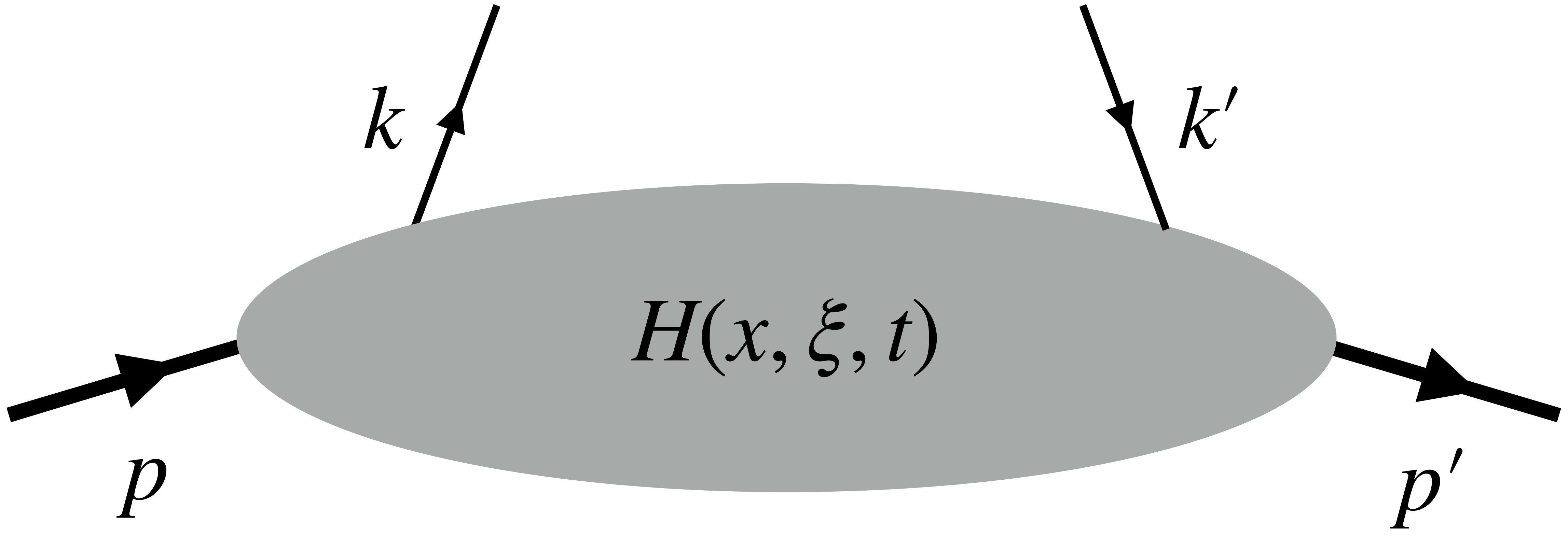
* Definition:

$$P\cdot n\int\frac{d\lambda}{2\pi}\,e^{ixP\cdot n\lambda}\left\langle p'\left|\bar\psi^q(-\tfrac{1}{2}\lambda n)\,\not n\,\psi^q(\tfrac{1}{2}\lambda n)\right|p\right\rangle=\bar u(p')\left[\;H^q(x,\xi,t;\mu^2)\,\not n+\;E^q(x,\xi,t;\mu^2)\,\frac{i\sigma^{n\Delta}}{2M}\right]u(p),$$

$$n_\mu n_\nu\int\frac{d\lambda}{2\pi}\,e^{ixP\cdot n\lambda}\left\langle p'\left|G^{\mu\alpha}(-\tfrac{1}{2}\lambda n)\,G_\alpha{}^{\nu}(\tfrac{1}{2}\lambda n)\right|p\right\rangle=\bar u(p')\left[x\,H^g(x,\xi,t;\mu^2)\,\not n+x\,E^g(x,\xi,t;\mu^2)\,\frac{i\sigma^{n\Delta}}{2M}\right]u(p),$$

* Functions of x, ξ, and t:

$$x=\frac{k^+ + k'^+}{p^+ + p'^+}\quad \xi=\frac{p'^+ - p^+}{p^+ + p'^+}\quad t=(p'-p)^2$$



Properties of GPDs

- Forward Limit ($\xi, t \to 0$):
  - H GPD:

$$H^q(x,0,0) = q(x)\,\Theta(x) - \bar{q}(-x)\,\Theta(-x),$$

$$2\,H^g(x,0,0) = g(x)\,\Theta(x) - g(-x)\,\Theta(-x),$$

  - Forward limit of E does not map to known functions

Properties of GPDs

- Polynomiality:

$$\int_{-1}^{1} \mathrm{d}x\, x^s H^a(x,\xi,t;\mu^2) = \sum_{i=0\,(\mathrm{even})}^{s} (2\xi)^i\, A^a_{s+1,i}(t,\mu^2) + \mathrm{mod}(s,2)\,(2\xi)^{s+1}\, C^a_{s+1}(t,\mu^2),$$

$$\int_{-1}^{1} \mathrm{d}x\, x^s E^a(x,\xi,t;\mu^2) = \sum_{i=0\,(\mathrm{even})}^{s} (2\xi)^i\, B^a_{s+1,i}(t,\mu^2) - \mathrm{mod}(s,2)\,(2\xi)^{s+1}\, C^a_{s+1}(t,\mu^2),$$

Properties of GPDs

- First moments (s=0) give electromagnetic form factors:

$$\int_{-1}^{1} dx\, H^q(x,\xi,t;\mu^2) = A_{10}^a(t;\mu^2) = F_1^q(t;\mu^2)$$

$$\int_{-1}^{1} dx\, E^q(x,\xi,t;\mu^2) = B_{10}^a(t;\mu^2) = F_2^q(t;\mu^2)$$

Properties of GPDs

- Second moments give gravitational form factors:

$$\int_{-1}^{1} dx\, xH^a(x,\xi,t;\mu^2) = A_{20}^a(t;\mu^2) + 4\xi^2 C_2^a(t;\mu^2) \quad \int_{-1}^{1} dx\, xE^a(x,\xi,t;\mu^2) = B_{20}^a(t;\mu^2) - 4\xi^2 C_2^a(t;\mu^2)$$

- Ji sum rule:

$$2J^a(\mu^2) = A_{20}^a(0,\mu^2) + B_{20}^a(0,\mu^2) = \int_{-1}^{1} dx\, x\left[H^a(x,\xi,0;\mu^2) + E^a(x,\xi,0;\mu^2)\right]$$

- C_2^a is related to internal stresses

Properties of GPDs

- Evolution:
  - GPDs change with the energy scale in accordance with evolution equations of the general form:

$$\frac{\mathrm{d}H^a(x,\xi,t)}{\mathrm{d}\ln Q^2} = \int \mathrm{d}x P^a(x,\xi) H^a(x,\xi,t;Q_0^2)$$

Shadow GPDs

- The difference between one of the multiple solutions to the inverse problem and the true GPD:

$$F_S^a(x,\xi;\mu^2) = F_F^a(x,\xi;\mu^2) - F_T^a(x,\xi;\mu^2)$$

- Can rule out any F_F^a that do not satisfy the properties of GPDs, therefore SGPDs:
  - Must satisfy polynomiality
  - Zero contribution to CFF:

$$\sum_a C^a(x,\xi,Q^2,\mu^2) \otimes F_S^a(x,\xi;\mu^2) = 0$$

  - Forward Limit: $H_S^a(x,0,0) = 0$

Examples of Shadow GPDs

- Start from a double distribution (DD):

$$F_{DD}(\alpha,\beta) = \sum_{m\ \text{even},n\ \text{odd}}^{m+n\leq N} c_{mn}\alpha^m\beta^n$$

- SGPD is a Radon transform of the DD:

$$H_S(x,\xi) = \int_{-1}^{1} d\beta \int_{-1+|\beta|}^{1-|\beta|} d\alpha\delta(x-\beta-\alpha\xi)F_{DD}(\alpha,\beta)$$

- This guarantees the SGPDs satisfy polynomiality

$$H^{q(+)}(x,\xi) = \sum_{u=1,v=0}^{N+1}\left[\frac{1}{(1+\xi)^u}+\frac{1}{(1-\xi)^u}\right]q_{uv}x^v \qquad q_{uv} = \sum_{m,n} R_{uv}^{mn} c_{mn} \qquad R_{uv}^{mn} = \sum_{j=0}^{n}\frac{(-1)^{u+v+j}}{m+j+1}\binom{n}{j}\binom{j}{m-u+j+1}\times\binom{m+j+1}{v-n+j}.$$

Bertone, et. al. Phys.Rev.D 103 (2021) 11, 114019

Examples of Shadow GPDs

- SGPD conditions give a set of equations that can be solved for the unknowns (c_{mn}):
  - CFF zero contribution:
    - At leading order (LO) in α_S:

$$H^{q(+)}(\xi,\xi) = \sum_{w=1}^{N+1} \frac{1}{(1+\xi)^w} \sum_{u,v} C_w^{uv} q_{uv} \qquad C_w^{uv} = (-1)^{u+v+w} \binom{v}{u-w}$$

    - At next-to-leading order (NLO):

$$\mathrm{Im}C^q_{\mathrm{coll}} \otimes H^q = \frac{e_q^2 C_F}{2} \left(H^{q(+)}(\xi,\xi) \left[\frac{3}{2} + \log\left(\frac{1-\xi}{2\xi}\right) \right] + \int_\xi^1 \mathrm{d}x \frac{H^{q(+)}(x,\xi) - H^{q(+)}(\xi,\xi)}{x-\xi} \right)$$

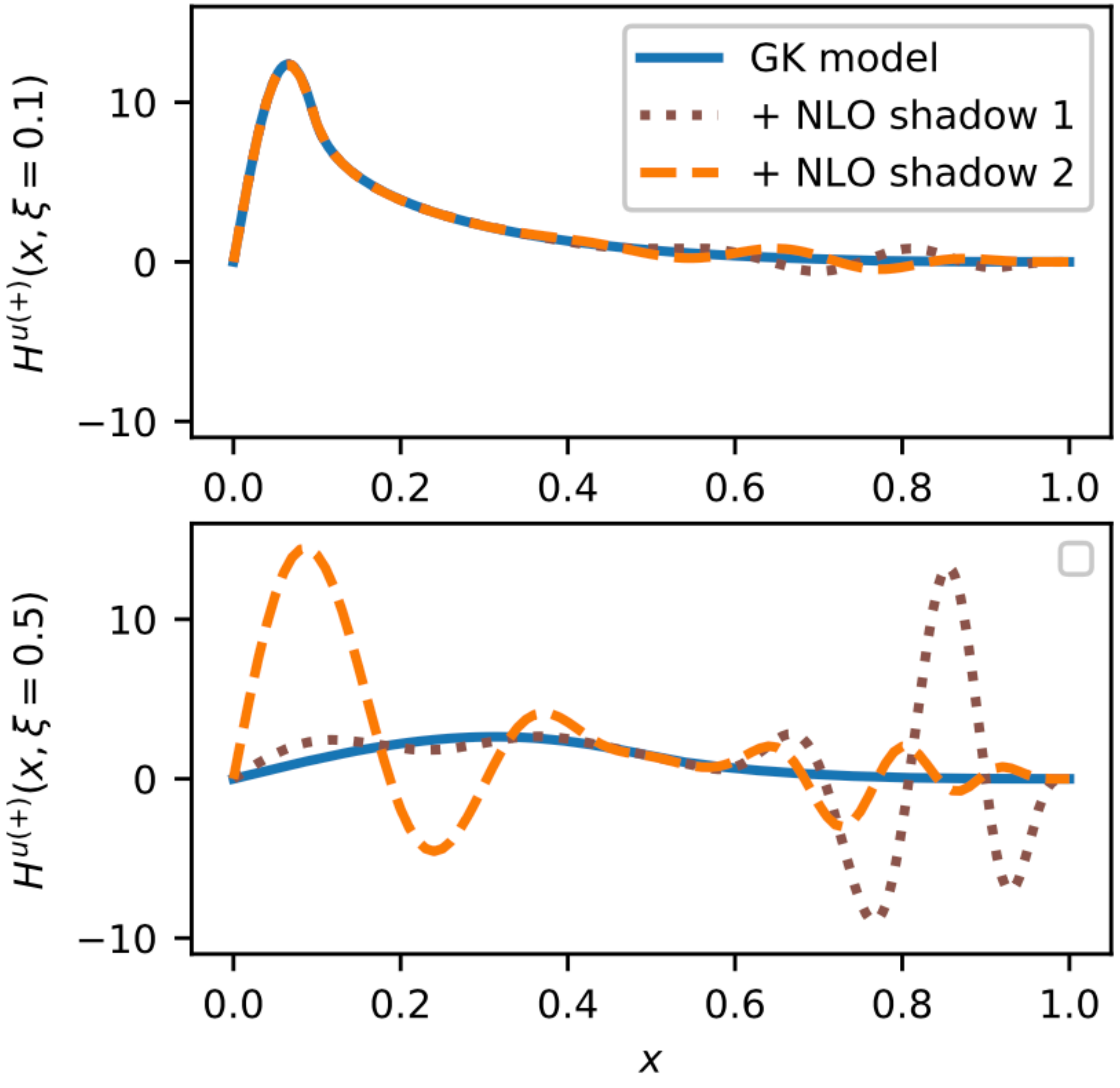
$$\int_\xi^1 \mathrm{d}x \frac{H^{q(+)}(x,\xi) - H^{q(+)}(\xi,\xi)}{x-\xi} = \sum_{w=1}^{N+1} \frac{\sum_{u,v} D_w^{uv} q_{uv}}{(1+\xi)^w} \qquad D_w^{uv} = (-1)^{u+v+w} \sum_{k=1}^{v} \frac{(-1)^k}{k} \binom{v-k}{u-w} - \frac{1}{k} \binom{v}{u-w}$$

Bertone, et. al. Phys.Rev.D 103 (2021) 11, 114019

Examples of Shadow GPDs

- SGPD conditions give a set of equations that can be solved for the unknowns (c_{mn}):
  - Forward Limit constraint:

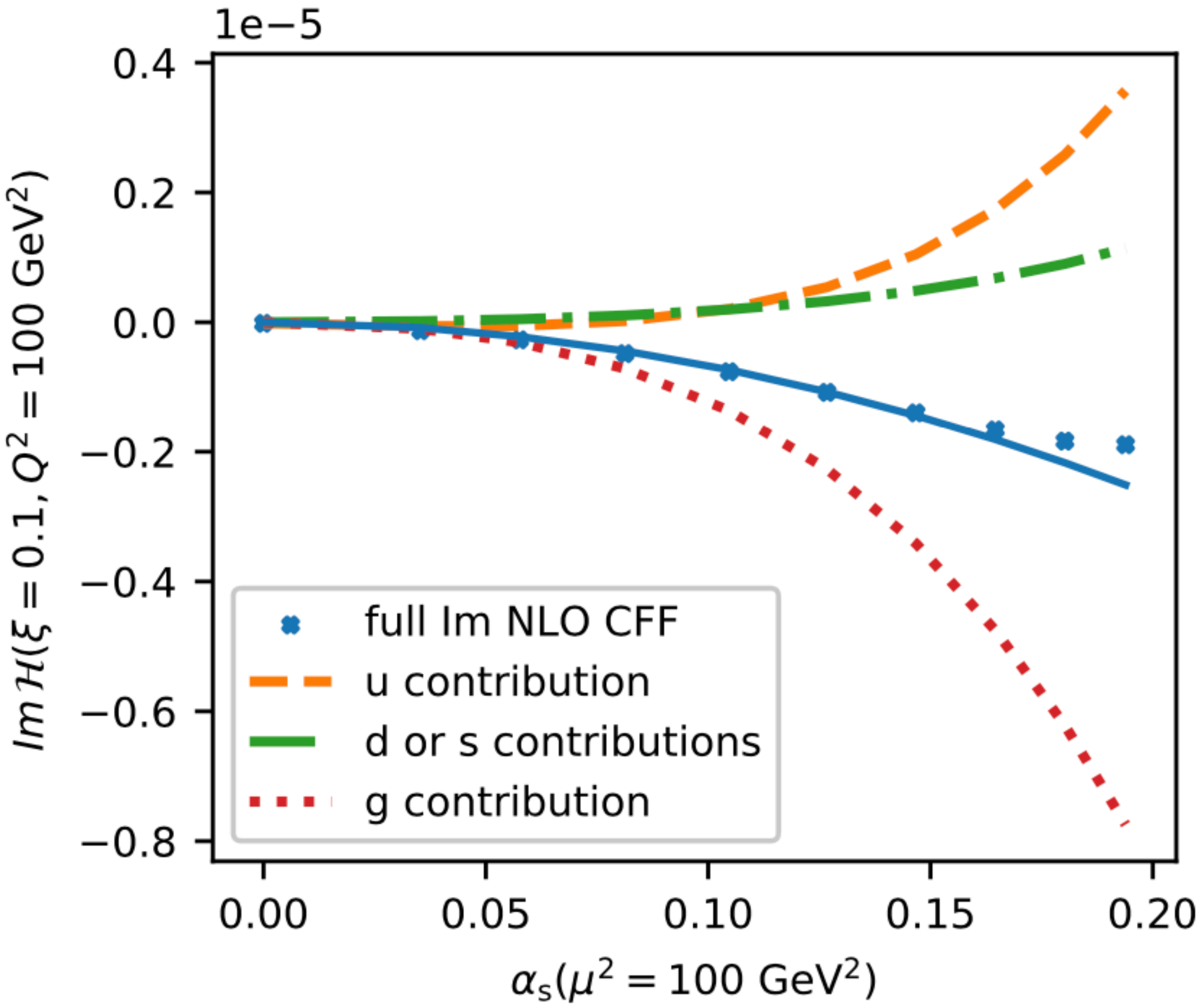
$$H^{q(+)}(x,0) = \sum_{w=0}^{N+1} x^w \sum_{u,v} Q_w^{uv} q_{uv}$$



Bertone, et. al. Phys.Rev.D 103 (2021) 11, 114019

Evolution and SGPDs

- Example SGPDs explored in Bertone, et. al. Phys.Rev.D 103 (2021) 11, 114019 give very small but non-zero CFF after evolution to a different energy scale.
- SGPDs can be multiplied by any factor and the result would still be a SGPD at the input scale
- Non-zero CFF after evolution would be multiplied by this factor
- Data spanning a range of energy scales would give a limit to the possible scaling factors



Bertone, et. al. Phys.Rev.D 103 (2021) 11, 114019

Evolution and SGPDs

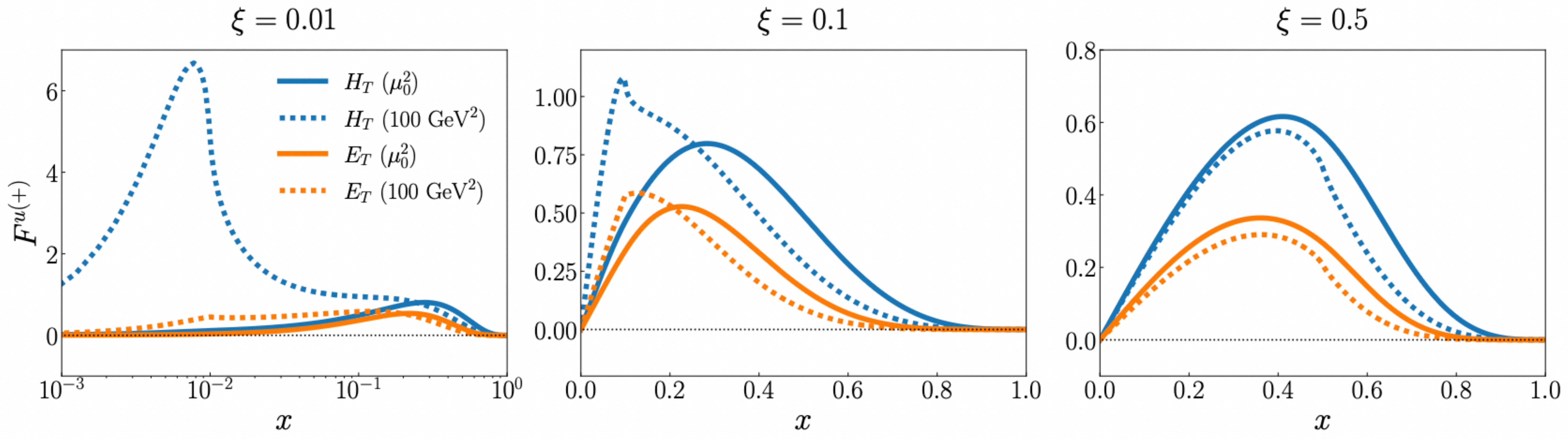
- EM, et. al. Phys.Rev.D 108 (2023) 3, 036027
  - Explored impact of evolution on SGPDs:
    - Generated simulated CFF data spanning a range of energy scales and skewness using a model
    - Calculated how this data constrains a Monte Carlo sampling of SGPDs

“True” GPDs

- Use VGG model as a proxy for the “true” GPD:
  - Vanderhaeghen, et. al., Phys. Rev. Lett. 80, 5064 (1998)
  - Vanderhaeghen, et. al., Phys. Rev. D 60, 094017 (1999)
  - Goeke, et. al., Prog. Part. Nucl. Phys. 47, 401 (2001)
  - Guidal, et. al., Phys. Rev. D 72, 054013 (2005)
- Use Parton Distribution Functions (PDFs) from JAM20-SIDIS (EM, et. al., Phys. Rev. D 104, 016015 (2021))

EM, et. al. Phys.Rev.D 108 (2023) 3, 036027

“True” GPDs



EM, et. al. Phys.Rev.D 108 (2023) 3, 036027

Calculating Shadow GPDs

- For SGPDs derived this way we can impose the forward limit in two ways:
  - Type A:
    - Consistent with Bertone, et. al. Phys.Rev.D 103 (2021) 11, 114019:

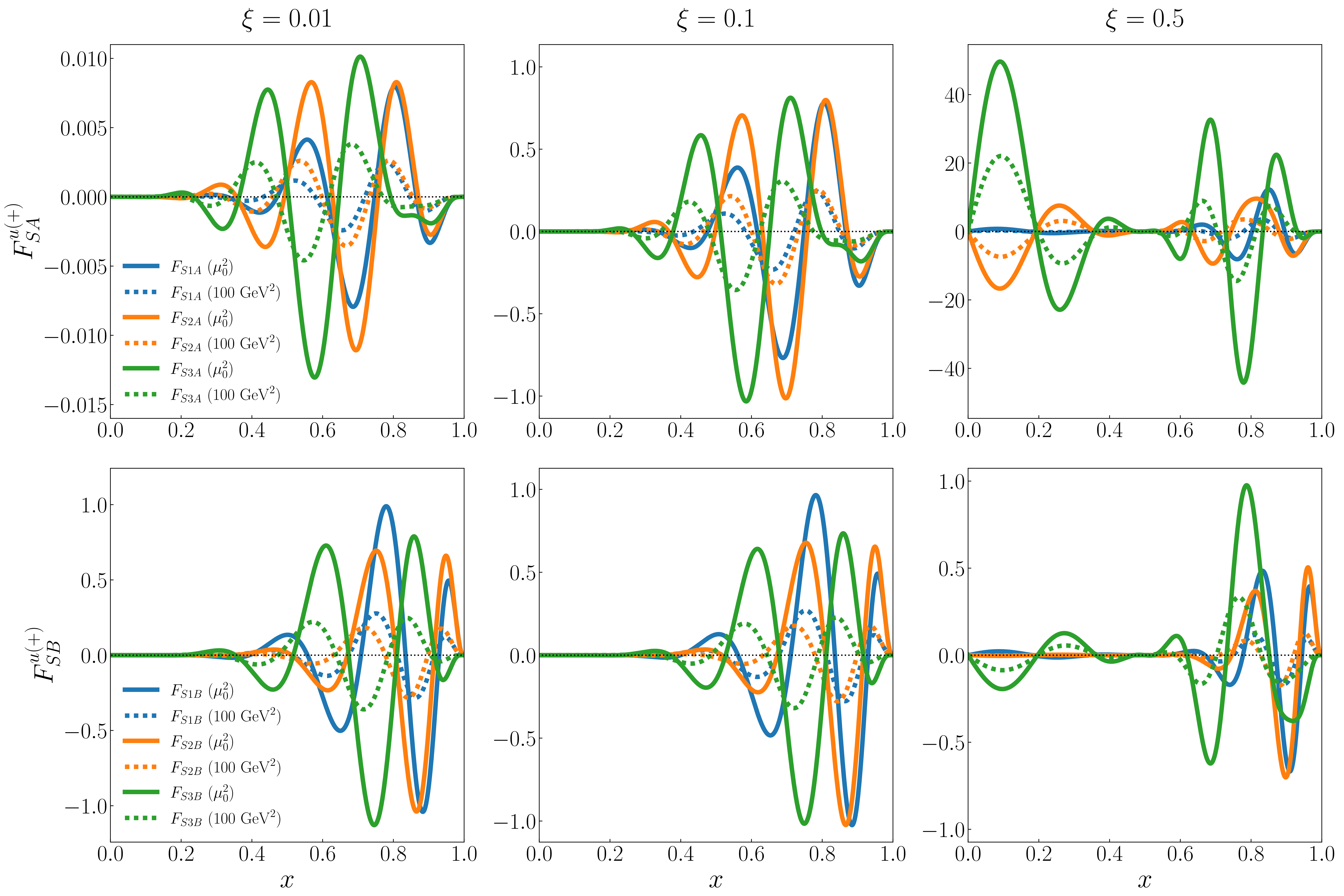
$$H_S^{u(+)}(x,0;\mu_0)) = 0$$

  - Type B:
    - Could also multiply F_{DD} by a function of t that is zero when $t = 0$

$$H_S^{u(+)}(x,0;\mu_0)) \neq 0$$

EM, et. al. Phys.Rev.D 108 (2023) 3, 036027

Example Shadow GPDs



EM, et. al. Phys.Rev.D 108 (2023) 3, 036027

Exploring SGPDs and Evolution

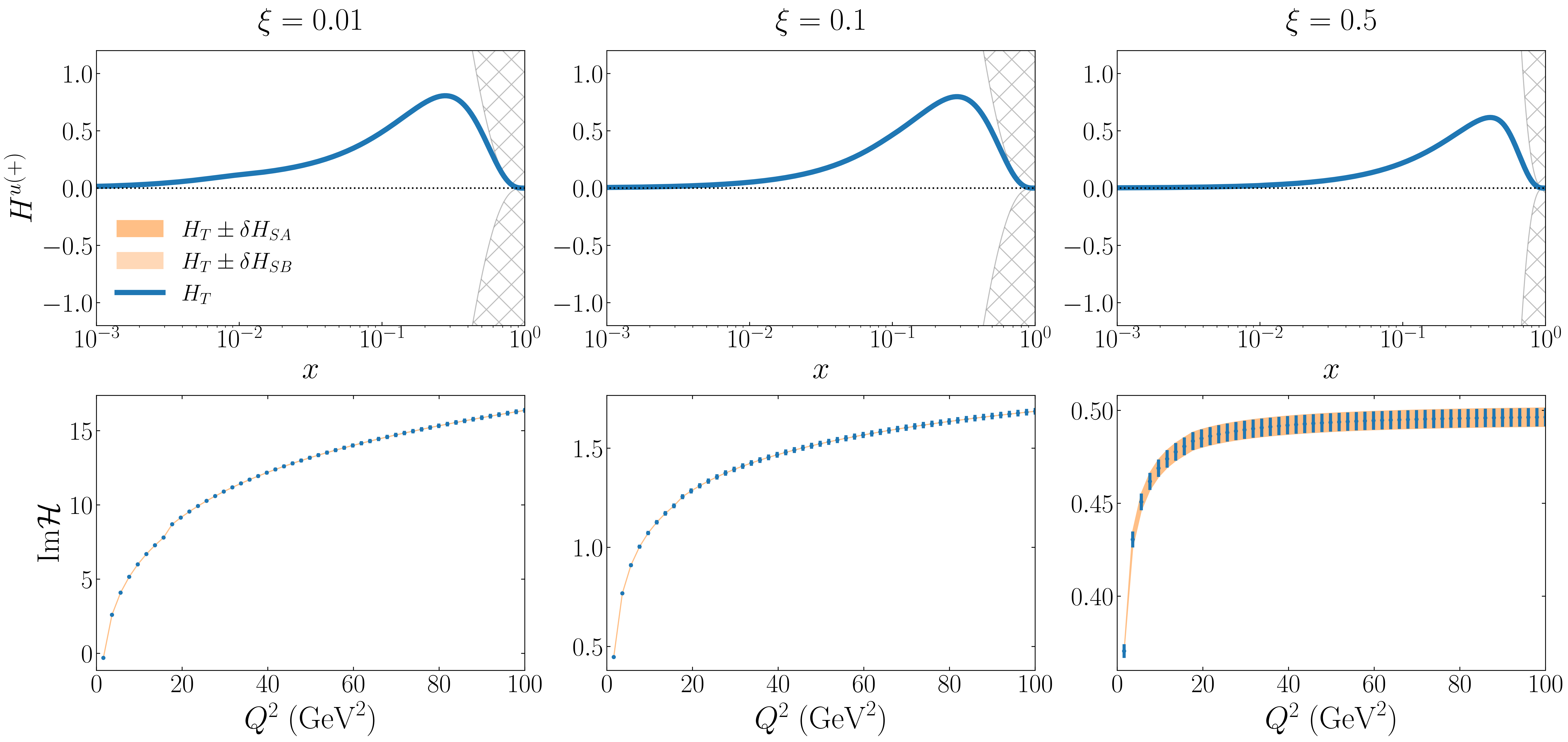
- Use Monte Carlo sampling to generate replicas that are linear combinations of three SGPDs:

$$H^{u(+)}(x,\xi;\mu^2,\lambda) = H_T^{u(+)}(x,\xi;\mu^2) + \lambda_1 H_{S1}^{u(+)}(x,\xi;\mu^2) + \lambda_2 H_{S2}^{u(+)}(x,\xi;\mu^2) + \lambda_3 H_{S3}^{u(+)}(x,\xi;\mu^2)$$

  - Randomly select the scaling factors until we get many replicas that all give CFFs that are within 1% of the simulated data from the model.
  - Plot the region δH_S: Outer boundary of all replicas

EM, et. al. Phys.Rev.D 108 (2023) 3, 036027

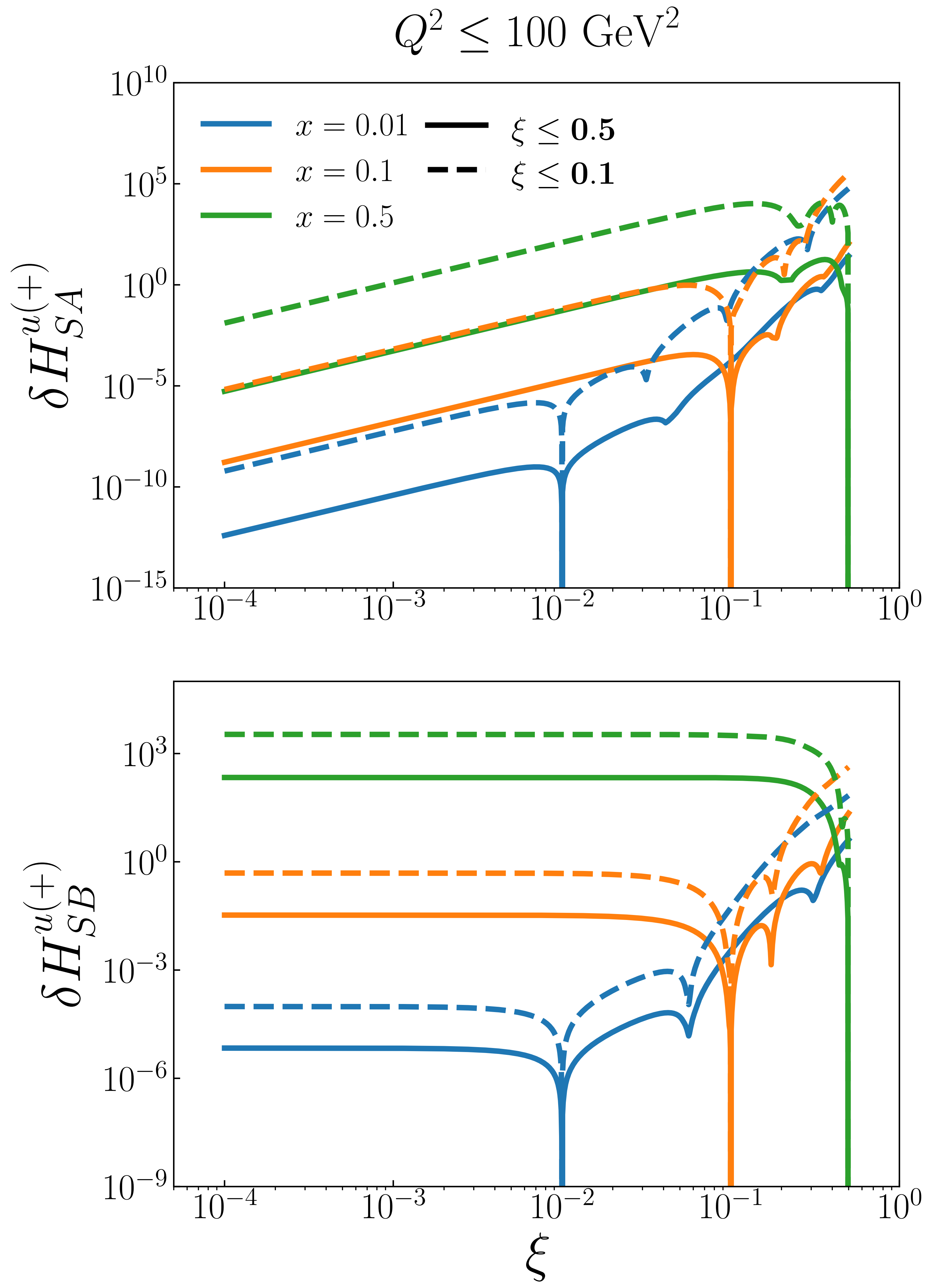
Exploring SGPDs and Evolution



EM, et. al. Phys.Rev.D 108 (2023) 3, 036027

Exploring SGPDs and Evolution

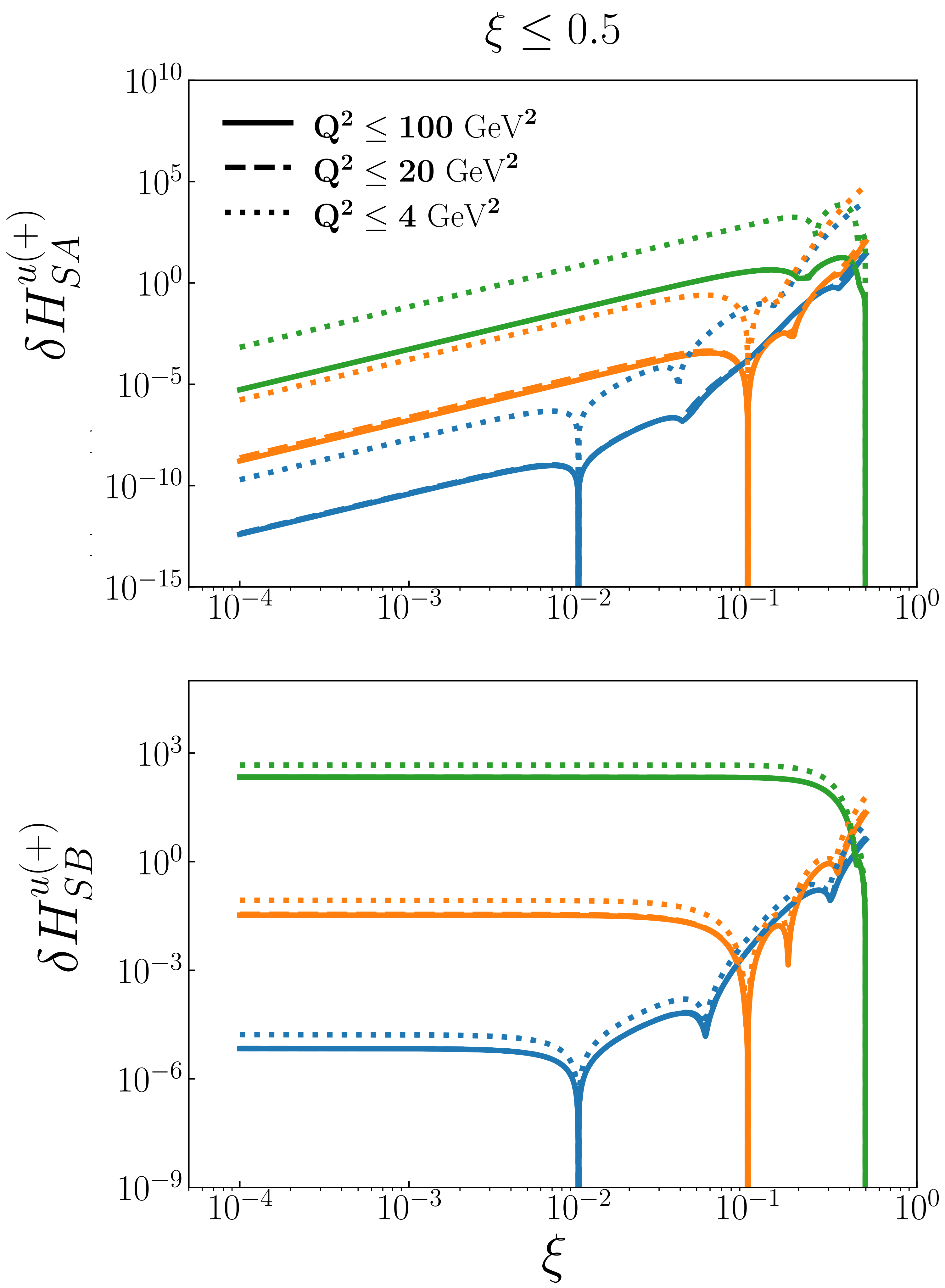
- Inclusion of higher ξ data leads to better constraint of SGPDs at smaller ξ
  - True over the full range of x when $H_S^{u(+)}(x,0;\mu_0)) = 0$
  - Only true for low x when $H_S^{u(+)}(x,0;\mu_0)) \neq 0$



EM, et. al. Phys.Rev.D 108 (2023) 3, 036027

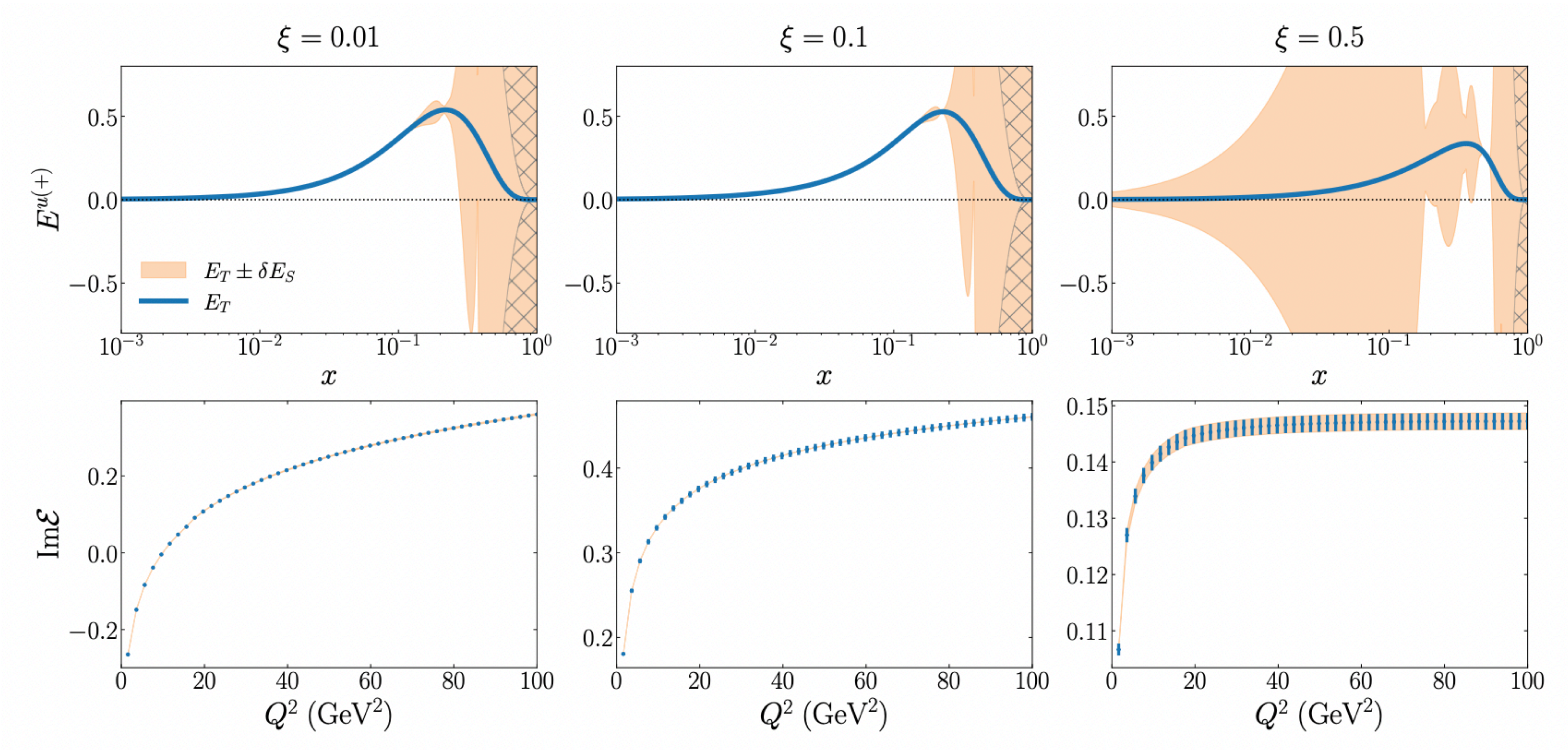
Exploring SGPDs and Evolution

* Some range of Q^2 is necessary for evolution to constrain the SGPDs but a large range is not as necessary as having large ξ data.



EM, et. al. Phys.Rev.D 108 (2023) 3, 036027

Exploring SGPDs and Evolution

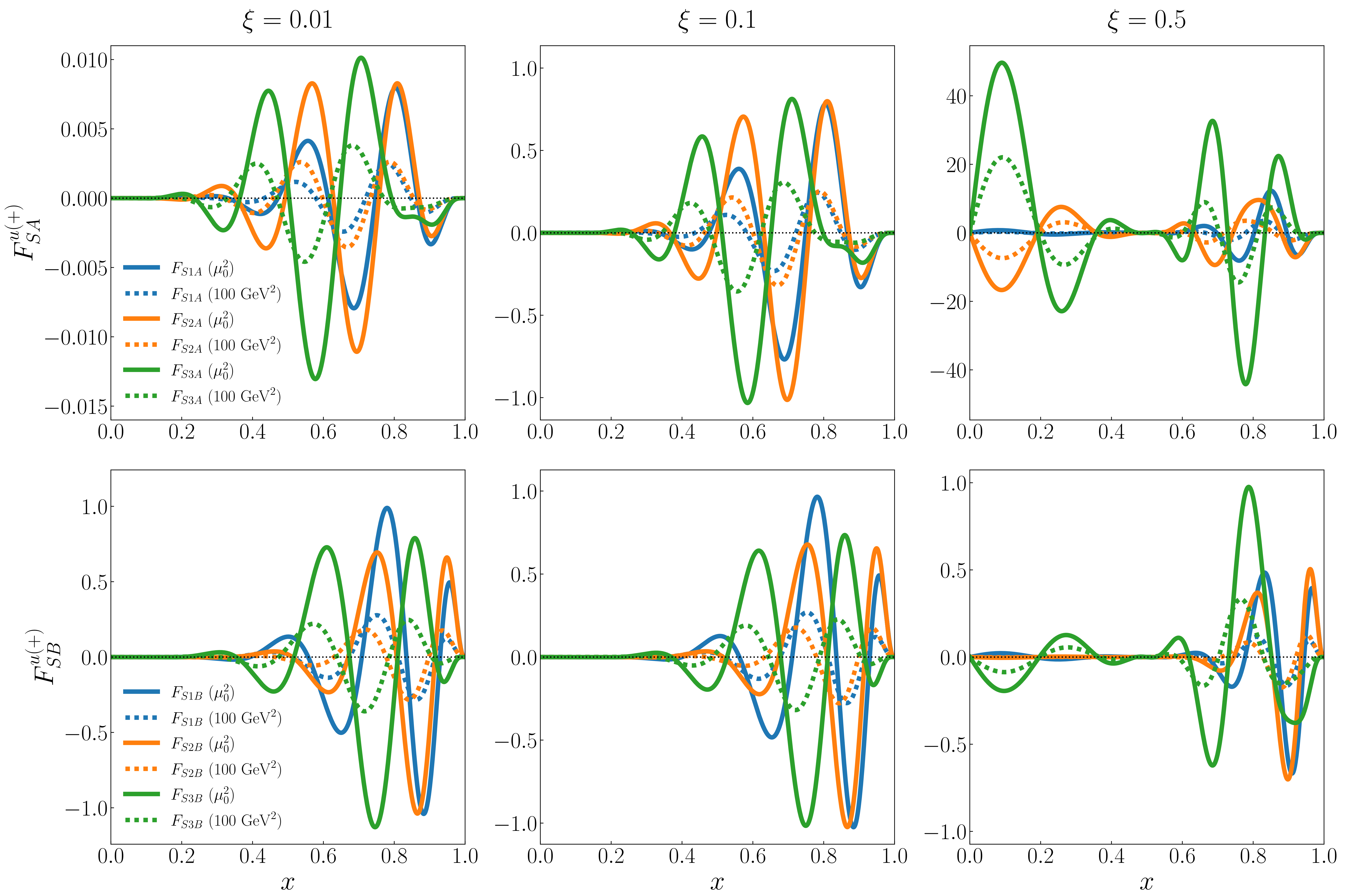


EM, et. al. Phys.Rev.D 108 (2023) 3, 036027

Exploring SGPDs and Evolution

- The trend of larger ξ data leading to better constrained SGPDs at smaller ξ is a direct result of the ξ dependence of the SGPDs
  - Independent of the model used as a proxy for the “true” GPD
  - Independent of the chosen uncertainty
  - May be intrinsic to the polynomial model used to construct the example SGPDs
    - Need a much more general sampling of SGPDs to determine if these trends are generally true (using neural networks (NNs) for example)

Example Shadow GPDs



EM, et. al. Phys.Rev.D 108 (2023) 3, 036027

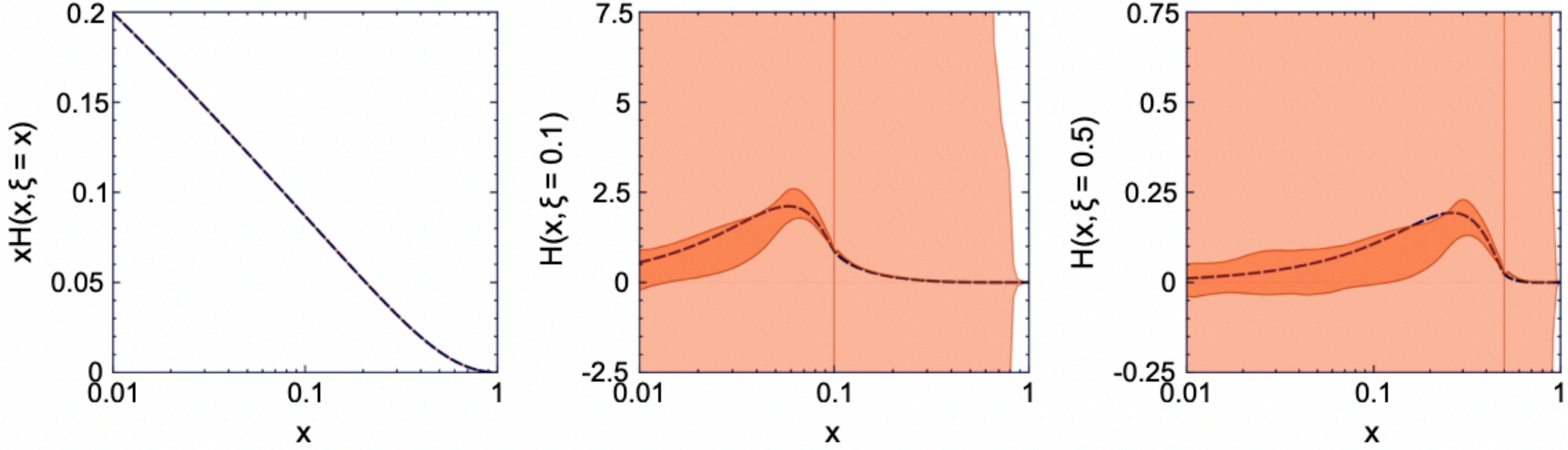
Exploring SGPDs with NNs

- Dutrieux, et. al. *Eur.Phys.J.C* 82 (2022) 3, 252:
  - Utilized NNs to model H GPD:

$$(1-x^2)F_C(\beta,\alpha)+(x^2-\xi^2)F_S(\beta,\alpha)+\xi F_D(\beta,\alpha)$$

    - F_C - "Classic" GPD double distribution
    - F_S - SGPD part of the double distribution
    - F_D - Used for a D-term
  - Trained the NNs utilizing ImCFF pseudodata generated using the Goloskokov-Kroll (GK) model:
    - Goloskokov, Kroll, Eur. Phys. J. C 42, 281 (2005)
    - Goloskokov, Kroll, Eur. Phys. J. C 53, 367 (2008)
    - Goloskokov, Kroll, Eur. Phys. J. C 65, 137 (2010)
  - Note: All pseudodata is at the same energy scale (No evolution)

Exploring SGPDs with NNs



Dutrieux, et. al. *Eur.Phys.J.C* 82 (2022) 3, 252

Positivity constraints and SGPDs

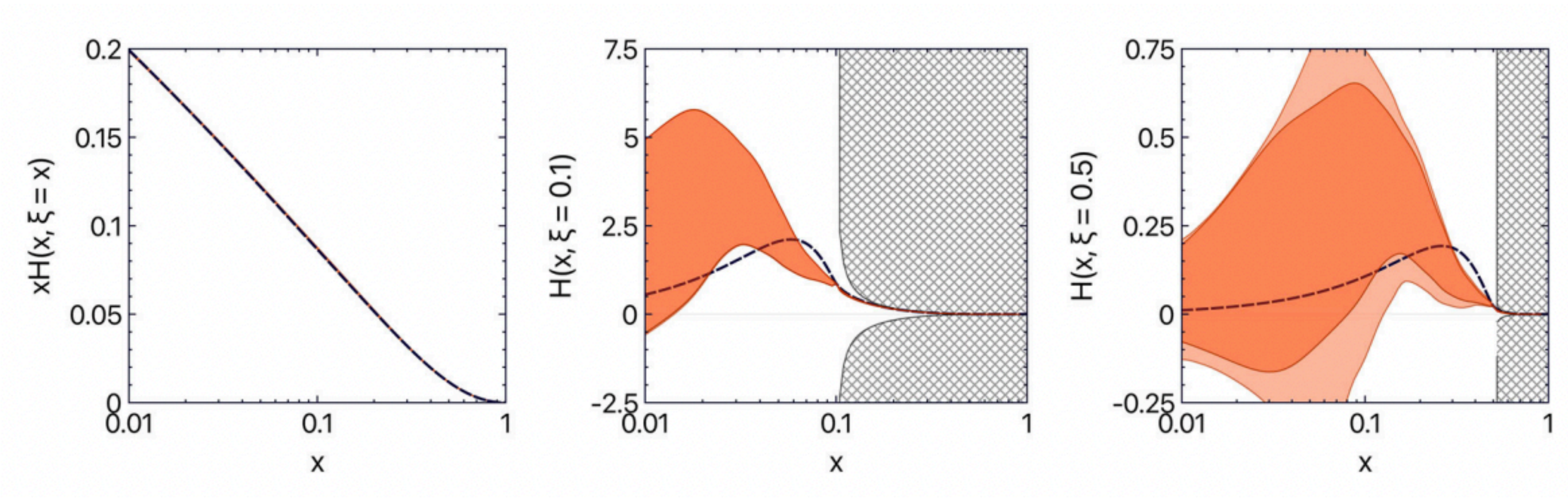
- Positivity constraints:
  - Probability distribution interpretation of PDFs can yield limits on the GPDs

$$|H(x,\xi,t)| \leq \sqrt{q\left(\frac{x+\xi}{1+\xi}\right) q\left(\frac{x-\xi}{1-\xi}\right) \frac{1}{1-\xi^2}}$$

    - Radyushkin, Phys. Rev. D 59, 014030 (1999)
    - Pire, Soffer, Teryaev, Eur. Phys. J. C 8, 103 (1999)

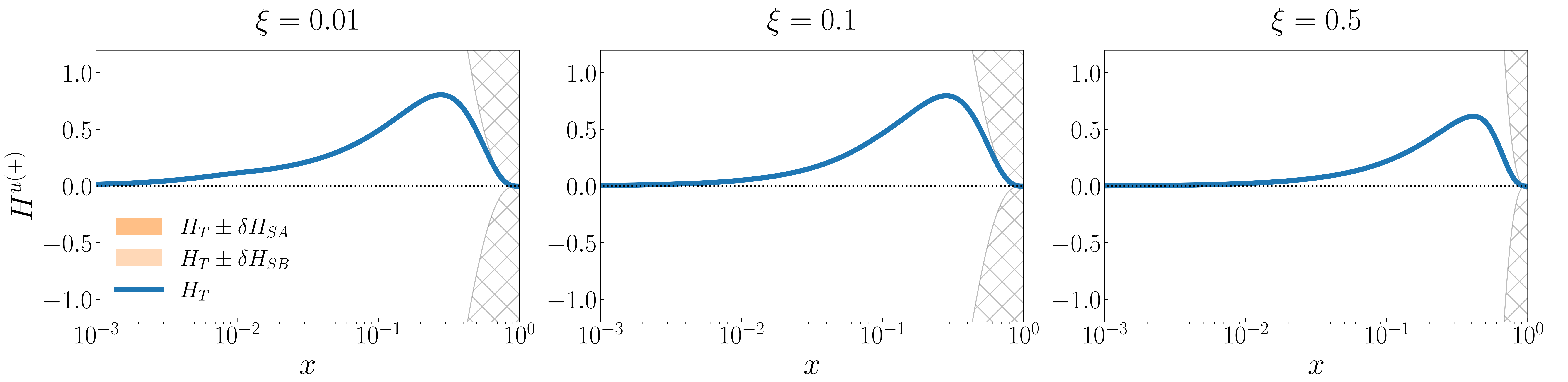
Dutrieux, et. al. *Eur.Phys.J.C* 82 (2022) 3, 252

Positivity constraints and SGPDs



Dutrieux, et. al. *Eur.Phys.J.C* 82 (2022) 3, 252

Positivity constraints and SGPDs

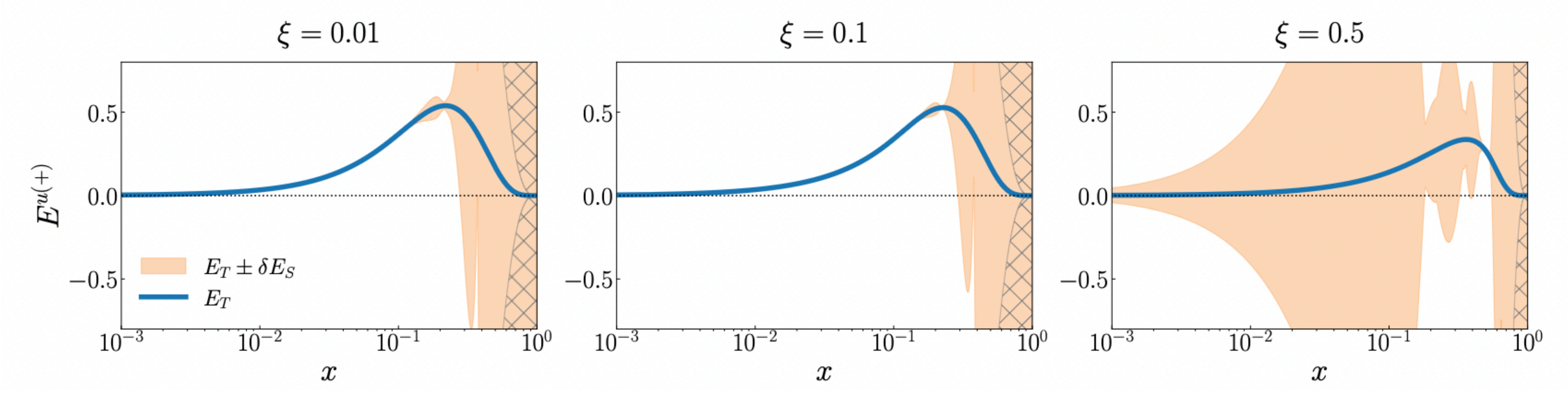


EM, et. al. Phys.Rev.D 108 (2023) 3, 036027

$$\left|H^q(x,\xi,t;\mu^2)\right| \leqslant \sqrt{\left(1-\frac{t_{\min}\,\xi^2}{t_{\min}-t}\right)\frac{q(x_{\text{in}};\mu^2)\,q(x_{\text{out}};\mu^2)}{1-\xi^2}},$$

Pobylitsa, Phys. Rev. D 65, 114015 (2002)

Positivity constraints and SGPDs



EM, et. al. Phys.Rev.D 108 (2023) 3, 036027

$$\left|E^q(x,\xi,t;\mu^2)\right| \leqslant \frac{2M}{\sqrt{t_{\text{min}}-t}}\sqrt{q(x_{\text{in}};\mu^2)\,q(x_{\text{out}};\mu^2)}.$$

Pobylitsa, Phys. Rev. D 65, 114015 (2002)

Positivity constraints and SGPDs

- Positivity constraints can help to better constrain SGPDs
- Care must be taken since these inequalities can be violated by regularization and renormalization effects in QCD (Collins, Rogers, Sato, Phys. Rev. D 105, 076010 (2022))

SGPDs and Spin

- Ji sum rule:

$$2J^a(\mu^2) = A^a_{20}(0,\mu^2) + B^a_{20}(0,\mu^2) = \int_{-1}^{1} dx\, x\left[H^a(x,\xi,0;\mu^2) + E^a(x,\xi,0;\mu^2)\right]$$

- For H:

$$A^q_{20}(0) = \int_{-1}^{1} dx x H^q(x,0,0;\mu^2) = \int_{0}^{1} dx x(q(x;\mu^2) + \bar{q}(x;\mu^2))$$

$$A^g_{20}(0) = \int_{-1}^{1} dx x H^g(x,0,0;\mu^2) = 2\int_{0}^{1} dx x g(x;\mu^2)$$

  - Since this contribution can be determined from the PDFs, H SGPDs would not contribute.

- For E:

  - E SGPDs can contribute to the spin because the forward limit is not known

EM, et. al. Phys.Rev.D 108 (2023) 3, 036027

SGPDs and Spin

- Calculating the spin contributions:
  - H_T: $J^{u+} = 0.389$
  - E_T: $J^{u+} = 0.219$
  - δE_S: $J^{u+} = 0.009$
- The contribution of E SGPDs to the spin is ~4%.
- Knowledge of the forward limit of the E GPD from lattice would reduce the possible E SGPDs to those for which the forward limit gives zero.

EM, et. al. Phys.Rev.D 108 (2023) 3, 036027

SGPDs and Internal Stresses

- Internal stresses in the hadron are connected to C_2^a in the second moment of the GPD.
- This contribution comes from the D-term portion of the GPD
- The SGPDs explored here have no D-term and so would not affect internal stress calculations
- Dutrieux, et. al., Eur. Phys. J. C 81, 300 (2021):
  - Found different D-terms that fit data equally well (shadow D-terms)
  - Can result in significantly different internal stresses

SGPDs and Tomography

- Transverse spatial distribution can be obtained from a transverse Fourier transform of the H GPD at $\xi = 0$.
  - Requires accurate knowledge of t-dependence at $\xi = 0$.
  - Not accessible experimentally.
  - Must extrapolate from t-dependence at non-zero ξ.
- Impact of Type A SGPDs would be minimal since they get smaller as $\xi \to 0$
- Impact of Type B SGPDs would be minimal at small x but could be substantial at large x
- Quantitative analysis of the impact SGPDs could have on tomography requires a thorough exploration of possible t-dependent SGPDs. (Work utilizing NNs in progress by EM and collaborators)

Summary

- Integration relating GPDs to currently available observable data (DVCS and DVMP) leads to an infinite number of GPD-like functions that would fit the data equally well
- A SGPD is the difference between one of these GPD-like functions and the true GPD
  - The existence of such functions must be accounted for in the estimated uncertainties of any GPDs extracted from DVCS and DVMP data alone
  - Those explored so far seem to gain limited constraint due to evolution. Further exploration is needed with a more general sampling of possible SGPDs to verify generality of this finding (Exploration with NNs in progress by EM and collaborators)
  - Positivity constraints can help to limit SGPDs