GPD global analysis with GUMP

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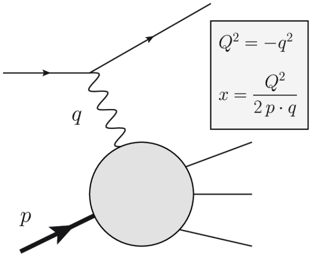
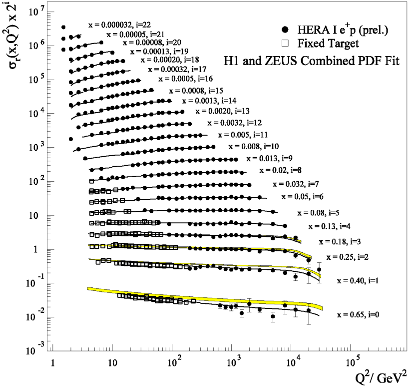
Contents

Introduction of parton distributions

Bjorken scaling

Inclusive deep inelastic scattering (DIS): high-energy electron (virtual photon) penetrate the nucleon and measure the total inclusive cross-section.





Observation: for medium x, the cross-sections do not depend on Q^2.

Feynman parton model and parton distributions

The photon virtuality Q can be considered as the resolution of the probe.

$$\text{Higher } Q^2 \Leftrightarrow \text{Finer spatial resolution}$$

Conjecture: there are some point-like structures in the nucleon — no extra feature probed when increasing the resolution.

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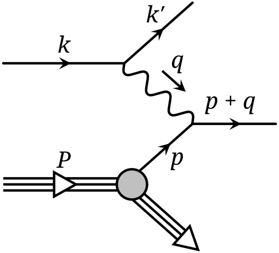

Feynman Parton model:

If the knock-out process happens so fast that the interaction among the constituent particles themselves can be neglected, an almost free particle that carries a fraction x of the total momentum will be probed: $p = xP$.

The final particle is almost on-shell, so $(p+q)^2 = (xP+q)^2 \approx 0$
$\to x \approx -q^2/(2P\cdot q)$

Parton distribution functions (PDFs)

Following the parton model, one can immediately write down the first ‘factorized’ formula for inclusive DIS:

$$\sigma \approx \int \mathrm{d}x' f(x', Q)\sigma(x', Q)\delta(x - x') \ ,$$

1. $\sigma(x', Q)$: simple elastic partonic cross-section
2. $\delta(x - x')$: hard scattering coefficient
3. $f(x', Q)$: PDF, probability of finding a parton with momentum fraction x'

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Bjorken scaling: PDF $f(x', Q)$ almost do not depend on Q.

Well, why would/should the $f(x', Q)$ depend on Q at all?

Answer: Quantum corrections.

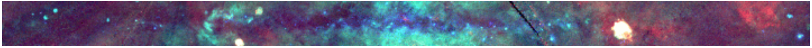
Resolution dependence of PDFs

In a quantum theory, probes of different resolution scales perceive the quantum fluctuations differently. E.g. Multiwavelength Milky Way

Optical

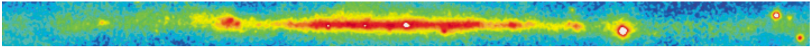
1x, 1x Grid, 4x, 4x Grid, 8x, 8x Grid

X-Ray



1x, 1x Grid, 4x, 4x Grid, 8x, 8x Grid

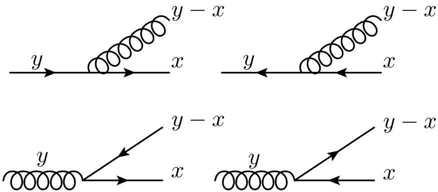
Gamma Ray



In the case of Quantum Chromodynamics (QCD), quarks and gluons fluctuate into each others. Therefore, we see different numbers of quarks and gluons at different resolution scales.

QCD evolution of PDFs: DGLAP equation

When the interaction is weak, the quantum fluctuations of quarks and gluons can be perturbatively calculated, which diagrammatically look like[1]:



[1] Altarelli and Parisi, "Asymptotic Freedom in Parton Language".

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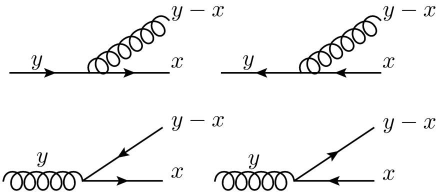


Therefore, we have the so-called DGLAP equation:

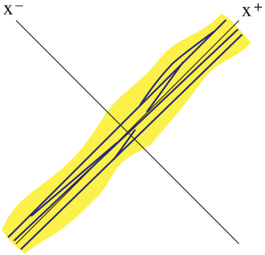
$$\frac{\mathrm{d}}{\mathrm{d}\log Q^2} f^i(x, Q) = \frac{\alpha_S(Q)}{2\pi} \int_x^1 \frac{\mathrm{d}y}{y} \sum_{j=q,\bar{q},g} P^{ij}(x, y) f^j(y, Q) + \mathcal{O}(\alpha_S^2) ,$$

and we have $P^{ij}(x, y) = P^{ij}(x/y)$!

[1] Altarelli and Parisi, "Asymptotic Freedom in Parton Language".

Partons in the infinite momentum frame

To understand this, let's switch to another frame, where the proton is moving at high speed in one direction z. (Infinite Momentum Frame)



World line of partons in fast-moving nucleon[a].

[a] Soper, "Parton distribution functions".

Light cone coordinates are defined as

$$x^{\pm} = (x^0 \pm x^3)/\sqrt{2} \ ,$$

so x^+ is light-like in the $+z$ direction and x^- is light-like in the $-z$ direction. And the inner product reads:

$$x \cdot P = x^- P^+ + x^+ P^- x_\perp \cdot P_\perp \ ,$$

Each constituent particle moves almost at the speed of light, so its interactions can be ignored (asymptotic freedom).

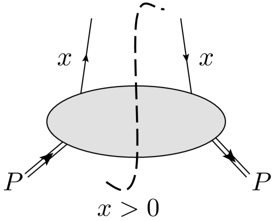
Partons are effective objects moving at the speed of light.

Operator definition of PDFs

Let's also quick review the operator definition of PDFs:

$$f_q(x) = \frac{1}{2x(2\pi)^3}\int \mathrm{d}^2k_\perp \sum_s \langle P \,|\, b_s^\dagger(xP^+,k_\perp) b_s(xP^+,k_\perp) \,|\, P\rangle$$
$$= \frac{1}{2}\int \frac{\mathrm{d}y^-}{2\pi} e^{ixP^+y^-} \left\langle P \left| \bar\psi\left(-\frac{y^-}{2}\right)\gamma^+\psi\left(\frac{y^-}{2}\right)\right| P\right\rangle ,$$

with implicit gauge links between the two fields. Diagrammatically, it looks



when $x<0$, it represents antiquark moving with momentum fraction $-x$.

Generalized Parton Distributions (GPDs)

GPDs are the generalization of PDFs with non-zero momentum transfer.

$$F_q = \frac{1}{2}\int \frac{\mathrm{d}y^-}{2\pi} e^{ixP^+y^-} \left\langle P' \left| \bar{\psi}\left(-\frac{y^-}{2}\right)\gamma^+\psi\left(\frac{y^-}{2}\right)\right| P \right\rangle \ ,$$

This matrix element can be parameterized with two scalar functions:

$$F_q = \frac{1}{2\bar{P}^+}\bar{u}(P')\left[H_q(x,\xi,t)\gamma^+ + E_q(x,\xi,t)\frac{i\sigma^{+\alpha}\Delta_\alpha}{2M_N}\right]u(P) \ ,$$

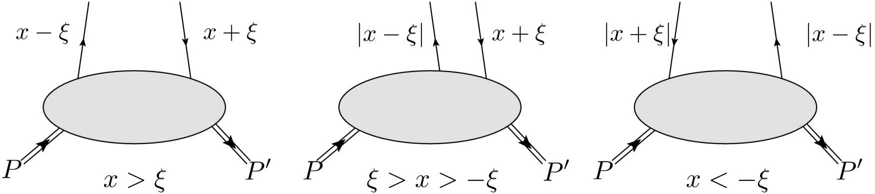
where we define: $\bar{P} \equiv (P+P')/2$, $\Delta \equiv P' - P$, $t \equiv \Delta^2$ and $\xi \equiv -n\cdot\Delta/(2n\cdot P)$. We also use the light-cone vector n and its conjugate $\bar{n}$ such that $n^2 = \bar{n}^2 = 0$ and $n\cdot\bar{n} = 1$. For an arbitrary vector V, $V^+ \equiv n\cdot V$ and $V^- \equiv \bar{n}\cdot V$

$$V^\mu = n^\mu(V\cdot\bar{n}) + \bar{n}^\mu(V\cdot n) + V_\perp \ ,$$

The light-cone structure is unchanged under $n \to e^\lambda n^\mu$, $\bar{n} \to e^{-\lambda}\bar{n}^\mu$.

Partonic picture of GPDs

The partonic picture of GPDs resembles PDFs except for one region:



when $\xi > x > -\xi$, the two fields annihilate a quark-antiquark pair.
GPDs resemble distribution amplitudes (DAs) in the nucleons in this DA-like region, whereas in the PDF-like $x > \xi$ and $x < -\xi$ regions, GPDs resemble the quark and antiquark PDFs.

This partonic picture also affects the evolution of GPDs.

Parton evolution in moment space

Why do me need moments, and what are them?

Complicate equations can be solved simply with Fourier transform.

For example: a scalar Yukawa theory has

$$\mathcal{L}_{\text{Yukawa}} = \bar{\psi}(i\partial\!\!\!/ - m_f)\psi + \frac{1}{2}(\partial_t^2 - \partial_r^2 + m^2)\phi^2 - g\bar{\psi}\psi\phi$$

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The Yukawa potential $V(r)$ is given by

$$(\partial_{\boldsymbol{r}}^2 - m^2)\,V(\boldsymbol{r}) = 4\pi g^2 \delta^{(3)}(\boldsymbol{r}) \Leftrightarrow V(\boldsymbol{r}) = -4\pi g^2 \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3}\, e^{i\boldsymbol{k}\cdot\boldsymbol{r}} \frac{1}{\boldsymbol{k}^2 + m^2}$$

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More often we do calculation in momentum space because the substitution

$$-i\partial_{\boldsymbol{r}} \Leftrightarrow \boldsymbol{k}$$

will turn a differential equation into an ordinary equation.

Exer. Prove that the two expressions of $V(r)$ are equivalent.

Fourier transform and convolution theorem

Another interesting property of the Fourier transform is the convolution theorem. If we define the Fourier convolution of two functions as:

$$(\tilde{f} * \tilde{g})(k) \equiv \int \mathrm{d}k' \ \tilde{f}(k')\tilde{g}(k-k') \ ,$$

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The proof can be done by simply inserting the definition, we can write

$$(\tilde{f} * \tilde{g})(k) = \int \mathrm{d}k'\mathrm{d}k'' \, \tilde{f}(k')\tilde{g}(k'')\delta(k'+k''-k) \ ,$$

therefore we have

$$\mathcal{F}\left[f(x)g(x)\right] = (\tilde{f} * \tilde{g})(k) = \int \mathrm{d}k'\mathrm{d}k'' \, \tilde{f}(k')\tilde{g}(k'')\delta(k'+k''-k) \ .$$

From Fourier to Mellin transform*

Now we consider a slightly different convolution:

$$(f \otimes g)(x) \equiv \int_0^\infty \frac{\mathrm{d}y}{y} g\left(\frac{x}{y}\right) f(y) \ ,$$

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$$(f\otimes g)(x)\equiv\int_0^\infty \frac{\mathrm{d}y}{y}g\left(\frac{x}{y}\right)f(y)\ ,$$

we can define $k_x\equiv\log x$, $k_y\equiv\log y$ and we also define two new functions:

$$\bar{f}(k_x)\equiv f(e^{k_x})=f(x)\quad\text{and}\quad \bar{g}(k_y)\equiv g(e^{k_y})=g(y)\ ,$$

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and the convolution becomes a Fourier convolutions:

$$\overline{f\otimes g}(k_x)=\int_{-\infty}^{+\infty}\mathrm{d}k_y\bar{g}(k_x-k_y)\bar{f}(k_y)\ .$$

What does the Fourier transform means?

$$\int_{-\infty}^{+\infty}\mathrm{d}k_x\bar{f}(k_x)e^{ik_x(-in)}=\int_0^\infty\frac{\mathrm{d}x}{x}f(x)x^n\ ,$$

where I redefine the Fourier conjugate of k_x to be $-in$.

Mellin transform and Mellin convolution

Now that we can formally introduce the Mellin transform:

$$f_n \equiv \mathcal{M}\left[f(x)\right] = \int_0^\infty \mathrm{d}x\ x^{n-1} f(x)\ ,$$

and we have the Mellin convolution theorem:

$$\mathcal{M}\left[(f \otimes g)(x)\right] = f_n \times g_n$$

The Mellin transform can be considered as the Laplace transform, or the Fourier transform, with change of variable:

$$f_n = \int_0^\infty \mathrm{d}k_x\ e^{k_x n} f(x)\ ,$$

where $k_x \equiv \log x$. (If you further redefine $n \to -in$, you get Fourier transform.)

Evolution of PDFs in Mellin space

Recall the DGLAP equation that reads:

$$\frac{\mathrm{d}}{\mathrm{d}\log Q^2} f^i(x,Q) = \frac{\alpha_S(Q)}{2\pi}\int_x^1 \frac{\mathrm{d}y}{y}\sum_{j=q,\bar{q},g} P^{ij}(x,y) f^j(y,Q) + \mathcal{O}(\alpha_S^2)\ ,$$

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Therefore, the DGLAP evolution in the Mellin space becomes multiplicative:

$$\frac{\mathrm{d}}{\mathrm{d}\log Q^2} f^i_n(Q) = \frac{\alpha_S(Q)}{2\pi}\sum_{j=q,\bar{q},g} P^{ij}_n\, f^j_n(Q) + \mathcal{O}(\alpha_S^2)\ ,$$

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which is much easier to solve.

However, everything came with a price.

Inverse Mellin transform

Like the case of Fourier transform, the price is the inverse transform. Recall that the Mellin transform can be considered as a Laplace transform:

$$f_n = \int_0^\infty \mathrm{d}k_x \; e^{k_x n} f(x) = \int_0^\infty \mathrm{d}k_x \; e^{k_x n} \bar{f}(k_x) \; ,$$

where $k_x \equiv \log x$. The inverse Laplace transform reads,

$$\bar{f}(k_x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \mathrm{d}n \; e^{-k_x n} f_n \; ,$$

$$\Rightarrow f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \mathrm{d}n \; x^{-n} f_n$$

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1. Ignoring the c, inverse Laplace transform is just inverse Fourier transform with a rotation of the variable $k_x \to i k_x$.
2. Inverse Mellin transform is inverse Laplace transform with redefinition.
3. n must be a complex number that has imaginary part!

Evolution of GPDs

Now we move on to the GPD evolution, which reads,[2]

$$\frac{\mathrm{d}}{\mathrm{d}\ln Q^2}F\left(x,\xi,t,Q^2\right)=\frac{\alpha_s(Q)}{2\pi}\int_{-1}^{1}\frac{\mathrm{d}x'}{|\xi|}\left[V\left(\frac{x}{\xi},\frac{x'}{\xi}\right)\right]_+F\left(x',\xi,t,Q^2\right)\ .$$

The evolution of GPDs has two limits in the PDF- and DA-like regions.

$$\lim_{\xi\to 0}\frac{1}{|\xi|}\left[V\left(\frac{x}{\xi},\frac{1}{\xi}\right)\right]_+=P(x)\ ,$$

with $P(x)$ the DGLAP splitting kernel. And

$$V(2x-1,2y-1)_{0<x,y<1}=V_{\mathrm{ERBL}}(x,y)\ .$$

where $V_{\mathrm{ERBL}}(x,y)$ the ERBL kernel for the DA evolution.

[2] Belitsky et al., "On the leading logarithmic evolution of the off forward distributions".

Eigenfunctions of GPD evolutions

The so-called Gegenbauer polynomials diagonalize the LO evolution

$$\int_{-1}^{1}\frac{\mathrm{d}x'}{|\xi|}\left[V^{(0)}\left(\frac{x}{\xi},\frac{x'}{\xi}\right)\right]_{+}C_j^{\frac{3}{2}}\left(\frac{x}{\xi}\right)=\gamma_j C_j^{\frac{3}{2}}\left(\frac{x'}{\xi}\right)\ ,$$

Thus, expanding $F(x,\xi,t)$ Gegenbauer polynomials will diagonalize it.

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Thus, expanding $F(x,\xi,t)$ Gegenbauer polynomials will diagonalize it.
But before getting to it, let's review some quantum mechanics:

$$\hat{H}\left|\psi_n\right\rangle = E_n\left|\psi_n\right\rangle$$

so that any solution at energy E can be written as.

$$\left|F\right\rangle=\sum_i f_i\left|\psi_i\right\rangle \text{ such that } E=\sum_i f_i E_i$$

and the wave function can be obtained by:

$$F(x)\equiv\left\langle x|F\right\rangle=\sum_i f_i\left\langle x|\psi_i\right\rangle=\sum_i f_i\psi_i(x) \text{ and } f_i=\left\langle\psi_i|F\right\rangle\ .$$

Conformal expansion of GPDs

Then we can write the **formal** decomposition of GPDs as[3]

$$F(x,\xi,t)=\sum_{j=0}^{\infty}(-1)^{j}p_{j}(x,\xi)\mathcal{F}_{j}(\xi,t)$$

Now that we know the following transform diagonalize evolution:

$$\mathcal{F}_{j}(\xi,t)\equiv\int_{-1}^{1}\mathrm{d}x c_{j}(x,\xi)F(x,\xi,t) \text{ with } c_{j}(x,\xi)\equiv\xi^{j}\frac{\Gamma\left(\frac{3}{2}\right)\Gamma(j+1)}{2^{j}\Gamma\left(\frac{3}{2}+j\right)}C_{j}^{\frac{3}{2}}\left(\frac{x}{\xi}\right)$$

[3] Mueller and Schafer, "Complex conformal spin partial wave expansion of generalized parton distributions and distribution amplitudes".

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the prefactor is defined such that

$$\lim_{\xi\to 0}\mathcal{F}_j(\xi,t)=\int_{-1}^{1}\mathrm{d}x\, x^j F(x,\xi,t)\text{ or }\quad\lim_{\xi\to 0}c_j(x,\xi)=x^j$$

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the $p_j(x,\xi)$ can be constructed as

$$p_j(x,\xi)\equiv\xi^{-j-1}\frac{2^j\Gamma\left(\frac{5}{2}+j\right)}{\Gamma\left(\frac{3}{2}\right)\Gamma(j+3)}\left[1-\left(\frac{x}{\xi}\right)^2\right]C_j^{\frac{3}{2}}\left(\frac{x}{\xi}\right)$$

<u>such that $\int \mathrm{d}x\; c_j(x,\xi)(-1)^k p_k(x,\xi)$</u> $=\delta^{jk}$ analogous to $\langle\psi_j|\psi_k\rangle=\delta^{jk}$

[3] Mueller and Schafer, "Complex conformal spin partial wave expansion of generalized parton distributions and distribution amplitudes".

GPD evolution with conformal moments

Note that Gegenbauer polynomials have a **weight** function $w^{3/2}(x) = 1 - x^2$:

$$\int_{-1}^{1} \mathrm{d}x\, w(x) C_n^{\frac{3}{2}}(x) C_m^{\frac{3}{2}}(x) \propto \delta_{nm}$$

With the equations in the previous slides

$$\frac{\mathrm{d}}{\mathrm{d}\ln Q^2} \mathcal{F}_j\left(\xi, t, Q^2\right) = \frac{\alpha_s(Q)}{2\pi} \gamma^j F_j\left(\xi, t, Q^2\right) + \mathcal{O}(\alpha_s^2)\ ,$$

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$$\int_{-1}^{1} \mathrm{d}x \; w(x) C_n^{\frac{3}{2}}(x) C_m^{\frac{3}{2}}(x) \propto \delta_{nm}$$

With the equations in the previous slides

$$\frac{\mathrm{d}}{\mathrm{d}\ln Q^2} \mathcal{F}_j \left(\xi, t, Q^2\right) = \frac{\alpha_s(Q)}{2\pi} \gamma^j F_j \left(\xi, t, Q^2\right) + \mathcal{O}(\alpha_s^2) \; ,$$

the evolution are diagonalized!

1. Even if it's **not** diagonalized beyond LO, you still get evolution equations without integral in x, except that you will need an evolution matrix E^{jk}.
2. Any transform of the variable x/ξ (not x) should do this trick, but Gegenbauer polynomials diagonalized the LO.
3. Again, anything come with a price — inverse transform issue.

Inverse transform (resummation) of GPDs moments

To obtain the GPD, one needs to resum the expression:

$$F(x,\xi,t)=\sum_{j=0}^{\infty}(-1)^j p_j(x,\xi)\mathcal{F}_j(\xi,t)$$

Unfortunately, you cannot truncate the sum, which is divergent!

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Unfortunately, you cannot truncate the sum, which is divergent!

We need an analytic tick that adds ALL moments together analytically.

$$1 + 2 + 3 + 4 + \cdots \overset{?}{=} -\frac{1}{12}$$

This is defined with the Riemann $\zeta(s)$ function

$$\zeta(s) = \sum_{s=1}^{\infty} \frac{1}{n^s} \ ,$$

when $\mathrm{Re}(s) > 1$, convergent and well-defined summation.

Mellin-Barnes integral

For any s the Riemann $\zeta(s)$ can be defined according to the integral form:

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \mathrm{d}x \frac{x^{s-1}}{e^x - 1} = \frac{1}{\Gamma(s)} \int_0^\infty \mathrm{d}x \frac{x^{s-1} e^{-x}}{1 - e^{-x}}$$
$$= \frac{1}{\Gamma(s)} \int_0^\infty \mathrm{d}x \sum_{n=0}^{\infty} x^{s-1} e^{-x} e^{-nx} = \sum_{n=0}^{\infty} \frac{1}{(n+1)^s}$$

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The following Mellin-Barnes integral forms an analytical continuation of the formal summation of GPDs:

$$F(x,\xi,t) = \frac{1}{2i} \int_{c-i\infty}^{c+i\infty} dj \frac{p_j(x,\xi)}{\sin(\pi[j+1])} \mathcal{F}_j(\xi,t) \ ,$$

where $-1 < c < 0$ assuming that $F_j(\xi,t)$ has no pole when $\mathrm{Re} j > -1$. This allows resummation of GPDs if the analytical expression of $F_j(\xi,t)$ is known.

Phenomenology with moment space evolution

PDFs fit with moment space method

Doing phenomenology with a moment space approach is not complicated.

$$\text{Parameters} \Rightarrow \text{Moments} \Rightarrow \text{Evolution} \Rightarrow \text{Observables}$$

PDFs fit with moment space method

Doing phenomenology with a moment space approach is not complicated.

Parameters $\Rightarrow$ Moments $\Rightarrow$ Evolution $\Rightarrow$ Observables

1. The differential equation can be solved analytically for a given order.

$$\frac{\mathrm{d}}{\mathrm{d}\log Q^2} f_n^i(Q) = \frac{\alpha_S(Q)}{2\pi} \sum_{j=q,\bar{q},g} P_n^{ij}\, f_n^j(Q) + \mathcal{O}(\alpha_S^2)$$

2. The moments are multiplicative renormalizable.
3. Non-parametric forms are harder to implement — we need moments in complex plane. Numeric functions need to be extrapolated.
4. Observables should be either calculated or transformed to moment space. Not necessary to go back to x-space to calculate the observables.

$$\int \mathrm{d}x C(x) f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \mathrm{d}s'\, C_{1-s'} f_{s'} \ ,$$

A simple example of PDFs

Let's first choose an ansatz for PDFs $f(x,\mu_0)$. We know that PDF must vanish when $x\to 1$ and it has some scaling $x^{-\alpha}$ according to small-x physics.

$$f(x,\mu_0)=Nx^{-\alpha}(1-x)^{\beta}P(x).$$

where $P(x)$ is an extra piece to give $f(x,\mu_0)$ more flexibility for medium-x[4].

$$\Rightarrow f_n(\mu_0)=NB(n-\alpha,1+\beta)\quad \text{where}\quad B(z_1,z_2)\equiv\frac{\Gamma(z_1)\Gamma(z_2)}{\Gamma(z_1+z_2)}\ ,$$

where we set $P(x)=1$ and B is the Euler beta function.

[4] Hou et al., "New CTEQ global analysis of quantum chromodynamics with high-precision data from the LHC".

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where we set $P(x)=1$ and B is the Euler beta function.
Now we can solve the moment space evolution:

$$f_n(Q)=\left[\frac{\alpha_S(\mu_0)}{\alpha_S(Q)}\right]^{P_n/\beta_0}f_n(\mu_0)$$

The β_0 is the leading order β function of QCD.

[4] Hou et al., "New CTEQ global analysis of quantum chromodynamics with high-precision data from the LHC".

A simple example of PDFs (Continued)

Of course, we can include the flavor of the parton as well, so we have

$$f_n^{ij}(Q) = \left[\frac{\alpha_S(\mu_0)}{\alpha_S(Q)}\right]^{P_n^{ij}/\beta_0} f_n^j(\mu_0)$$

where $i, j = q, \bar{q}, g$. Commonly, we consider the evolution basis[5]:

$$\text{Singlet:} \quad \Sigma = \sum_{i=1}^{n_\mathrm{f}} (q_i + \bar{q}_i), \tag{3.8}$$

$$\text{Non-singlet:} \quad q_{ij}^{\pm} = (q_i \pm \bar{q}_i) - (q_j \pm \bar{q}_j), \tag{3.9}$$

$$\text{Valence (non-singlet):} \quad q^{\mathrm{V}} = \sum_{i=1}^{n_\mathrm{f}} (q_i - \bar{q}_i), \tag{3.10}$$

$$\text{Gluon:} \quad g = g. \tag{3.11}$$

Flavor mixing only happens between singlet (Σ) and gluon (g).

[5] Herrmann, *Evolution of parton distribution functions*.

Constraints on GPD

Now we move on to the GPD analysis, the first thing we need is the ansatz. What do we know about GPDs $F(x,\xi,t)$ in general?

1. It vanishes when $x=1$
2. It reduces to PDF when $\xi=t=0$
3. It has non-analyticity at $x=\xi$.
4. Polynomiality condition: $\int \mathrm{d}x\, x^{n-1}F(x,\xi,t)$ must be polynomials of ξ
5. It has positivity constrains
6. $\cdots$

Unfortunately, most of them do not directly constrain GPDs.

The overall behaviors of GPDs are largely undetermined.

Especially true noting that they are 3D functions.

Phenomenological parameterization of GPDs

Now we start to build a phenomenological modeling of GPDs in terms of their moment $\mathcal{F}_j(\xi,t)$. We start with the polynomiality condition[6]:

$$\int_{-1}^{1} \mathrm{d}x x^{n-1} H_q(x,\xi,t) = \sum_{i=0}^{i\leq(n-1)/2} (2\xi)^{2i} A^q_{n,2i} + \mathrm{Mod}(n+1,2)(2\xi)^n C^q_n \ ,$$

$$\int_{-1}^{1} \mathrm{d}x x^{n-1} E_q(x,\xi,t) = \sum_{i=0}^{i\leq(n-1)/2} (2\xi)^{2i} B^q_{n,2i} - \mathrm{Mod}(n+1,2)(2\xi)^n C^q_n \ .$$

where $\mathrm{Mod}(n+1,2)=1$ for even n and $=0$ for odd n.
Therefore, we can write the moments as

$$\mathcal{F}_j(\xi,t) = \sum_k \mathcal{F}_{j,k}(t)\xi^{2k} \ ,$$

Polynomiality condition obtained and the ξ-dependence modeled simply.

[6] Ji, "Off forward parton distributions".

The x- and t-dependence of GPDs[7]

Let's start by considering the $\xi \to 0$ pieces of GPDs. Since GPDs reduce to PDFs when $t = 0$ as well, let just write:

$$F_{j,0}(t=0) = NB(j+1-\alpha, 1+\beta).$$

This corresponds to the ansatz of PDFs.

Now we add extra t-dependence with some overall t-dependent factor:

$$F_{j,0}(t) = NB(j+1-\alpha, 1+\beta) \times f(j,t)$$

The t-dependent part consists of two pieces

$$f(j,t) = \frac{j+1-k-\alpha}{j+1-k-\alpha(t)} r(t) \ ,$$

The first part is the Regge term that produce $x^{-\alpha(t)}$ with $\alpha(t) = \alpha + \alpha' t$ observed in experiments, and the other part $r(t)$ is a general t-dependent term.

[7] Kumerički and Mueller, "Deeply virtual Compton scattering at small x_B and the access to the GPD H".

The ξ-dependent terms of GPDs

Furthermore, the ξ-dependence is constrained by the polynomiality condition:

$$\mathcal{F}_j(\xi,t)=\mathcal{F}_{j,0}(t)+\xi^2\mathcal{F}_{j,2}(t)+\xi^4\mathcal{F}_{j,4}(t)+\mathcal{O}(\xi^6)\ ,$$

which means that moments are analytical and expandable in ξ (NOT the case for the GPD themselves). We consider truncating the series for small ξ.

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The off-forward moments $F_{j,2}(t), F_{j,4}(t)$ can be completely independently modeled. One simple choice is to let me be proportional to the forward ones.

$$F_{j,2}(t) = R_{\xi^2}F_{j-2,0}(t) \quad \text{and} \quad F_{j,4}(t) = R_{\xi^4}F_{j-4,0}(t)$$

There could be smarter choices (and there should be).

GUMP program for GPD extraction

GUMP program

The GUMP (GPDs through Universal Moment Parameterization) program makes use of the above constructions for GPD analysis.

We aim to put together the constraints from:

1. Global analysis of PDFs / directly fitting to DIS experiments
2. Global analysis of charge form factors/ directly fitting to measurements
3. Exclusive processes like DVCS/DVMP
4. Other possible exclusive productions
5. Lattice calculated GPD moments
6. Lattice calculated x-dependence GPD
7. $\cdots$

to obtain the best constraints on GPDs.

GUMP for t-dependent PDFs (tPDFs) (2207.05768)

One simple example is the tPDFs that correspond to the GPDs at $\xi = 0$.
The $f(x, t)$ can be constrained by the PDF $f(x)$, the corresponding form factors, and lattice calculation of tPDFs as well:

GUMP for t-dependent PDFs (tPDFs) (2207.05768)

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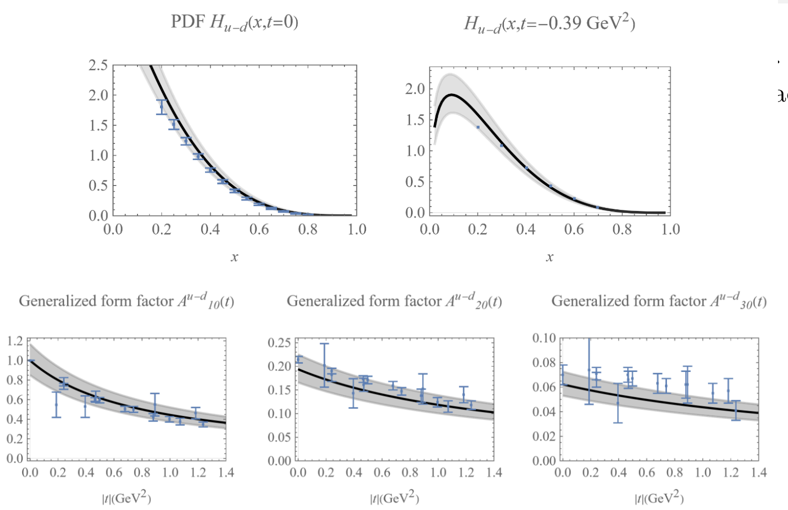


GUMP for tPDFs (continued)

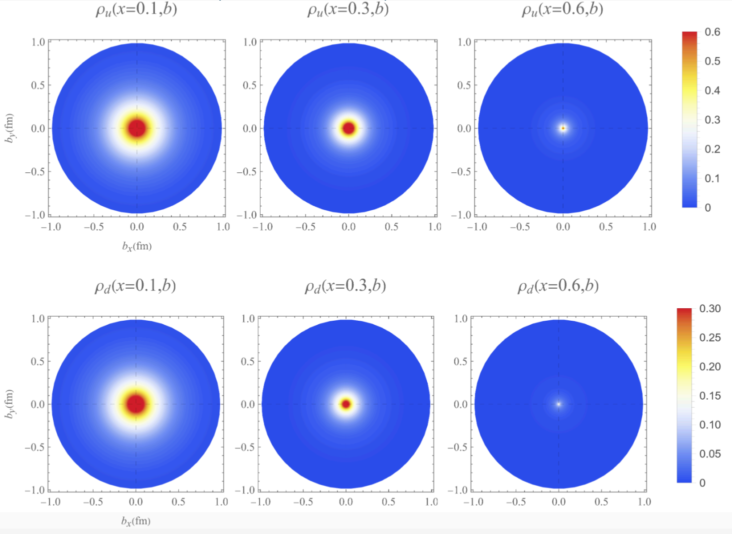
We can then obtain a 3-D image of the nucleon with the obtained tPDFs:

$$\rho_q(x,\boldsymbol{b}) = \int \frac{\mathrm{d}^2\boldsymbol{\Delta}}{(2\pi)^2} e^{-i\boldsymbol{\Delta}\cdot\boldsymbol{b}} H_q(x,-\boldsymbol{\Delta}^2) = \mathscr{H}_q(x,\boldsymbol{b}) \ ,$$

And we have the following image based on the extraction

GUMP for tPDFs (continued)

We can ... Fs:



And we

GUMP for tPDFs (continued)

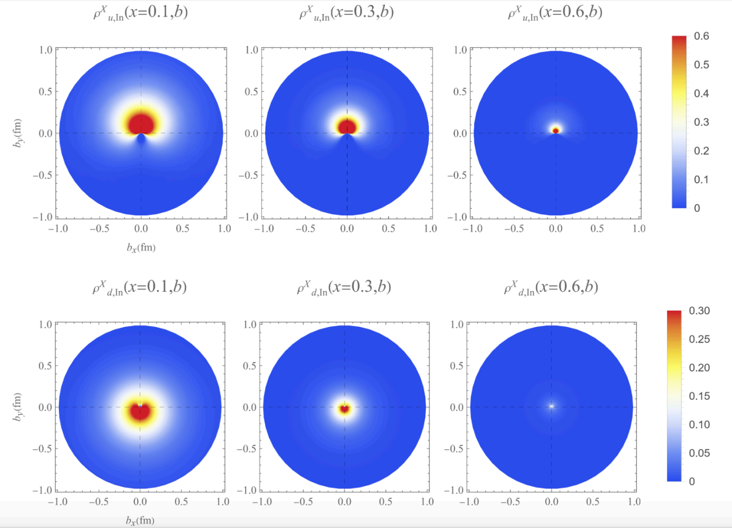
The same can be done for polarized nucleon:

$$\rho_q^X(x,\boldsymbol{b}) = \int \frac{\mathrm{d}^2\boldsymbol{\Delta}}{(2\pi)^2} e^{-i\boldsymbol{\Delta}\cdot\boldsymbol{b}} \left(H_q(x,-\boldsymbol{\Delta}^2) + \frac{i\Delta_y}{2M} E_q(x,-\boldsymbol{\Delta}^2) \right) ,$$
$$= \mathscr{H}_q(x,\boldsymbol{b}) - \frac{1}{2M}\frac{\partial}{\partial b^y}\mathscr{E}_q(x,\boldsymbol{b}) ,$$

of which the corresponding image looks

GUMP for tPDFs (continued)

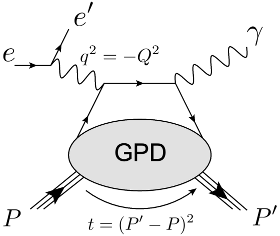
The san



of which

GUMP for DVCS (2302.07279)

Extending to the off-forward case, we consider the DVCS measurements for the quark GPDs, which measures the so-called Compton form factors.



$$\mathcal{H}_{CFF}(\xi,t) = -Q_q^2\int_{-1}^{1}\mathrm{d}x\left(\frac{1}{x-\xi+i\epsilon}\right)H_q^{(+)}(x,\xi,t)$$

This process provides us sensitivities of quark GPDs.

GUMP for DVCS (2302.07279)

Extendi ... cs for

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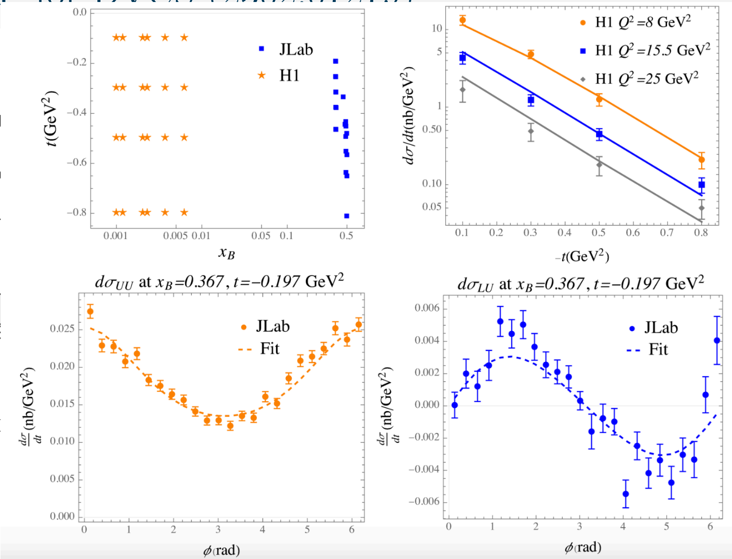
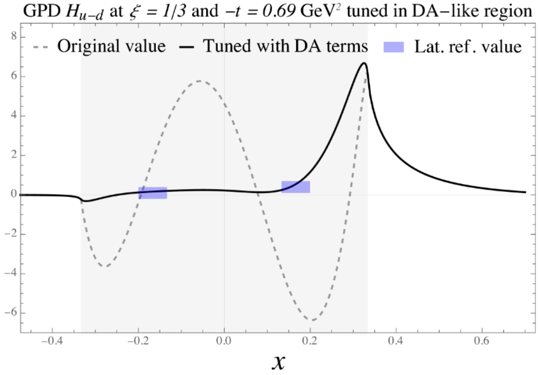


$x, \xi, t)$

This pro

Ambiguity when inverting CFF to GPD

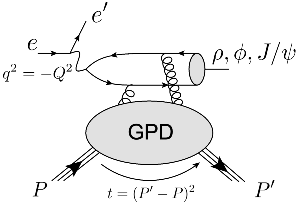
It is well-known that the shape of GPDs is not uniquely determined by the sole input of CFFs. It can be further constrained by lattice input:



Lattice simulations of GPDs provide complementary information!

GUMP for DV$J/\psi$P (2409.****)

The heavy meson production provides the sensitivity to the gluon GPDs.



$$\mathcal{H}_g(\xi,t)=\int_0^1 \mathrm{d}x\mathrm{d}z\left(\frac{1}{x-\xi+i\epsilon}\right)H_g(x,\xi,t)\frac{\Phi(z)}{1-z}\,,$$

GUMP for DV$J/\psi$P (2409.****)

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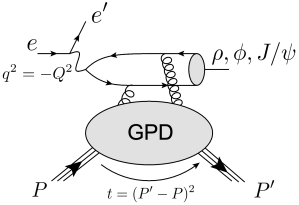
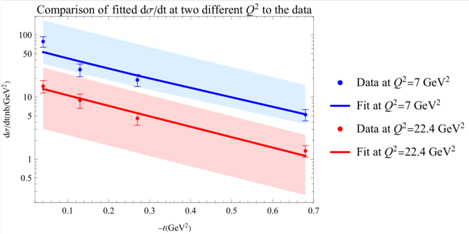
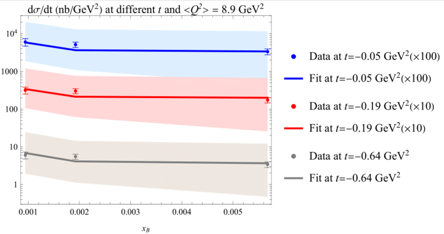


$$\mathcal{H}_g(\xi, t) = \int_0^1 \mathrm{d}x\mathrm{d}z \left(\frac{1}{x - \xi + i\epsilon}\right) H_g(x, \xi, t) \frac{\Phi(z)}{1 - z} ,$$





Summary

Here we summarize the main results discussed in this talk.

1. Phenomenology in the extraction of parton distributions (PDFs, GPDs)
2. Moment space approach facilitates the parton evolution equations.
3. Inverse transform are needed for moment space treatments.
4. Particularly, the so-called conformal moment expansion helps parameterize the GPDs and allows for simple evolution.
5. Phenomenological application of such methods.
6. Still, one requires complementary inputs (from lattice or different processes) to improve the determination of GPDs.

Of course, many future developments in both theory and phenomenology!

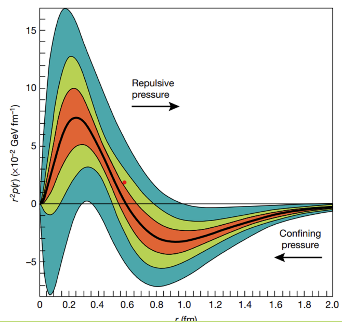
Some extra discussions (if time allows)

Dispersion relation and D-terms

The amplitudes are analytical functions which satisfied the so-called dispersion relation:

$$\mathcal{H}_{gC}(\xi,t)=\frac{1}{\pi}\int_0^{\xi_{\rm th}}\mathrm{d}\xi'\frac{2\xi'\mathrm{Im}\mathcal{H}_{gC}(\xi,t)}{(\xi-\xi'-i0)\,(\xi+\xi'+i0)}+\mathcal{C}_g(t)\ ,$$

Then the so-called $C_g(t)$ form factors can be determined when both the real and imaginary parts are determined.



An extraction of the quark $C_q(t)$ form factors based on dispersion analysis.[a]

[a] Burkert, Elouadrhiri, and Girod, "The pressure distribution inside the proton".

Threshold J/ψ productions

Interestingly for large ξ, the imaginary part of the amplitudes will be suppressed, and the real parts are dominated by gravitational form factors:[8]

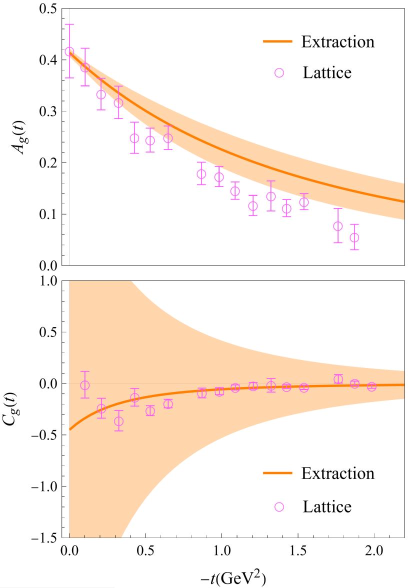
$$\mathrm{Re}\mathcal{H}_{g\mathrm{C}}(\xi,t)=\mathcal{C}_g(t)+\xi^{-2}\mathcal{A}_g^{(2)}(t)+\xi^{-4}\mathcal{A}_g^{(4)}(t)+\cdots\ ,$$

$$\mathrm{Re}\mathcal{E}_{g\mathrm{C}}(\xi,t)=-\mathcal{C}_g(t)+\xi^{-2}\mathcal{B}_g^{(2)}(t)+\xi^{-4}\mathcal{B}_g^{(4)}(t)+\cdots\ ,$$

such that they can be extracted from threshold J/ψ productions with large ξ.

[8] Guo, Ji, and Yuan, "Proton's gluon GPDs at large skewness and gravitational form factors from near threshold heavy quarkonium photoproduction".

Threshold J/ψ



Interestingly for large … tudes will be suppressed, and the r… ional form factors:[8]

$$\mathrm{Re}\mathcal{H}_{gC}(\xi, \quad \ldots (t) + \cdots \ ,$$

$$\mathrm{Re}\mathcal{E}_{gC}(\xi, \quad \ldots (t) + \cdots \ ,$$

such that they can be … luctions with large ξ.

[8] Guo, Ji, and Yuan, "Proton's gluon GPDs at large skewness and gravitational form factors from near threshold heavy quarkonium photoproduction".

Some mathematical subtleties

The conformal wave expansion method is nice when just looking at it, but somewhat strange when looking into it. The summation:

$$F(x,\xi,t)=\sum_{j=0}^{\infty}(-1)^j p_j(x,\xi)\mathcal{F}_j(\xi,t)$$

is divergent. Moreover, $p_j(x,\xi)$ vanishes when $x>\xi$ for all integer j.
The analytical continuation not only performs a resummation of the divergent summation. This process also extends GPDs to the PDF-like region $x>\xi$.
Thus, we have two statements based on this

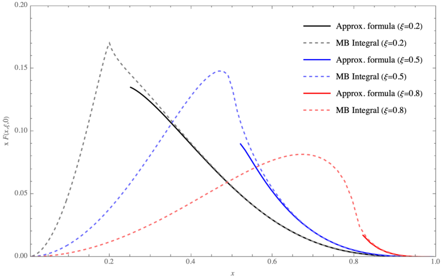
1. The PDF-like region is non-zero only when the summation diverges.
2. Modifying finite moments do not affect GPDs in the PDF-like region
3. GPDs in the PDF-like regions depend on the divergent (asymptotic) behaviors of summation/moments when $j\to\infty$.

Some mathematical subtleties (Continued)

In fact, GPDs in the PDF-like region ($x > \xi$) can be related to[9]:

$$F(x,\xi,0) \approx -2\sqrt{2}C(z)\,\mathrm{Res}\left[x^{-j-1}\left[\frac{2}{1+\sqrt{1-z}}\right]^{j}\mathcal{F}_j(\xi,t);\infty\right]\ ,$$
$$= -C_p(z)\,\mathrm{Res}\left[x_p(z)^{-(j+1)}\mathcal{F}_j(\xi,t);\infty\right]\ ,$$

with which one obtains an estimate of GPDs in the PDF region:



[9] Zhang and Ji, "On convergence properties of GPD expansion through Mellin/conformal moments and orthogonal polynomials"