GPD analysis

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Part II: GPD multichannel analysis

Conformal moments of GPDs

Motivation

GPDs depend on scale, evolution equations (at LO):

$$\mu \frac{d}{d\mu} F^q\left(x, \xi, \Delta^2, \mu^2\right) = -\frac{\alpha_s(\mu)}{2\pi} \gamma^{(0)} \otimes F^q\left(x, \xi, \Delta^2, \mu^2\right),$$

Convolution, hard to implement numerically,
integral kernel complicated and mixes GPD components

Can we linearize the evolution equations?

Gegenbauer polynomials

quark GPDs: $F_n^q\left(\xi,\Delta^2\right)=\int_{-1}^{1}\mathrm{d}x c_n^{3/2}(x,\xi)F^q\left(x,\xi,\Delta^2\right)\rightarrow c_n^{3/2}(x,\xi)=\xi^n\frac{\Gamma(3/2)\Gamma(n+1)}{2^n\Gamma(n+3/2)}C_n^{3/2}\left(\frac{x}{\xi}\right)$

gluon GPDs: $F_n^{\mathrm{G}}\left(\xi,\Delta^2\right)=\int_{-1}^{1}\mathrm{d}x c_{n-1}^{5/2}(x,\xi)F^{\mathrm{G}}\left(x,\xi,\Delta^2\right)\rightarrow c_{n-1}^{5/2}(x,\xi)=\xi^{n-1}\frac{\Gamma(3/2)\Gamma(n+1)}{2^n\Gamma(n+3/2)}\frac{3}{n}\underbrace{C_{n-1}^{5/2}\left(\frac{x}{\xi}\right)}$

Gegenbauer polynomials

↓

Evolution equations (LO): $\mu\frac{d}{d\mu}F_j^q\left(\xi,t,\mu^2\right)=-\frac{\alpha_s(\mu)}{2\pi}\gamma_j^{(0)}F_j^q\left(\xi,t,\mu^2\right)$

At NLO the anomalous dimension is a matrix, so there is mixing between components, but it is matrix multiplication which is simpler than integrating over a complicated evolution kernel.

Inverting into x-space

Conformal moments: $p_n(x,\xi) = \xi^{-n-1}\theta(1-|x/\xi|)\dfrac{2^n\Gamma\left(\frac{5}{2}+n\right)}{\Gamma\left(\frac{3}{2}\right)\Gamma(3+n)}\left(1-(x/\xi)^2\right)C_n^{\frac{3}{2}}(-x/\xi)$

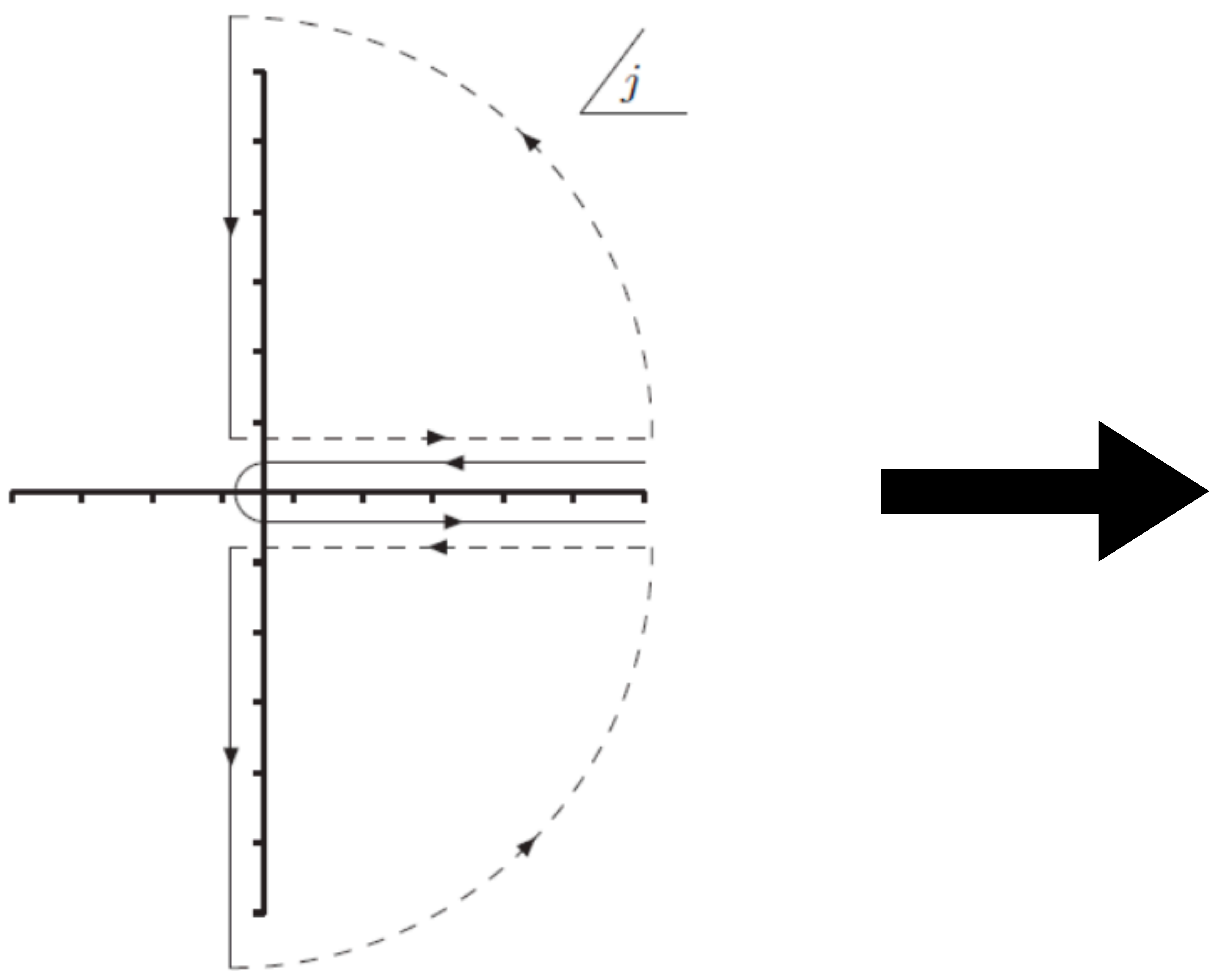
+ orthogonality relation: $\int_{-1}^{1} dx p_n(x,\xi)c_m(x,\xi) = (-1)^n\delta_{nm}$

= x-space GPD: $F(x,\xi,t) = \sum_{n=0}^{\infty}(-1)^n p_n(x,\xi)F_n(\xi,t)$

➡ Series does not converge due to $1/\xi^{n+1}$, we need to perform an analytic continuation: integer $n \to$ complex number j

Sommerfeld-Watson representation

GPDs: $F(x,\xi,t) = \frac{1}{2i}\oint dj \frac{1}{\sin(\pi j)} p_j(x,\xi) F_j(\xi,t)$



Integral contour

Carlson's theorem guarantees the uniqueness of the analytic continuation if the functions p and F are bounded by an exponential function.

If we avoid poles of the conformal moments and the GPDs, we only have poles from the sine function, which are at $j = n$. If we ignore the contribution from the arches, we get the Mellin-Barnes integral:

$$F(x,\xi,t) = \frac{1}{2i}\int_{c-i\infty}^{c+i\infty} dj \frac{1}{\sin(\pi j)} p_j(x,\xi) F_j(\eta,t)$$

Constant c has to be chosen so that all poles of p and F are to the left of it on the real line.

Conformal moments

Schläfli integral for representing conformal moments: $$p_j(x,\xi) = -\frac{\Gamma(5/2+j)}{\Gamma(1/2)\Gamma(2+j)}\frac{1}{2i\pi}\oint_{-1}^{1} du \frac{\left(u^2-1\right)^{j+1}}{(x+u\xi)^{j+1}}$$

For general x and ξ: $$p_j(x,\xi) = \theta(\xi-|x|)\xi^{-j-1}\mathscr{P}_j\left(\frac{x}{\xi}\right) + \theta(x-\xi)\xi^{-j-1}\mathscr{Q}_j\left(\frac{x}{\xi}\right)$$

$$\mathscr{P}_j^{\lambda}(y) = \frac{2^{j+\lambda-1/2}\Gamma(j+\lambda+1)}{\Gamma(1/2)\Gamma(j+1)\Gamma(\lambda+1/2)}(1+y)^{\lambda-1/2}{}_2F_1\left(\begin{matrix}-j-\lambda+1/2 & j+\lambda+1/2 \\ & \lambda+1/2\end{matrix}\middle|\frac{1+y}{2}\right)$$

$$Q_j^{\lambda}(y) = -\frac{\sin(\pi j)}{\pi}y^{-j-1}{}_2F_1\left(\begin{matrix}(j+1)/2 & (j+2)/2 \\ & j+\lambda+1\end{matrix}\middle|\frac{1}{y^2}\right)$$

↓

Hypergeometric function

Zero skewness example: $$p_j^{\lambda}(x,\xi=0) = (-1)^{\lambda+1/2}x^{-j-1}\frac{\sin\left[\pi\left(j+\lambda-1/2\right)\right]}{\pi} \Rightarrow F_j$$ Mellin moments of PDFs

Mellin-Barnes representation of scattering amplitudes

CFFs:

$$\mathscr{F}^{S}\left(\xi,\Delta^2,Q^2\right) = \int_{-1}^{1}\frac{\mathrm{d}x}{2\xi}{}^{A}T\left(x,\xi\middle|\alpha_s\left(\mu_R\right),\frac{Q^2}{\mu_F^2}\right)F^{A}\left(x,\xi,\Delta^2,\mu_F^2\right)$$

Conformal space

$$\mathscr{F}^{S}\left(\xi,\Delta^2,Q^2\right) = \frac{1}{2i}\int_{c-i\infty}^{c+i\infty}\mathrm{d}j\xi^{-j-1}\left[i\pm\begin{Bmatrix}\tan\\ \cot\end{Bmatrix}\left(\frac{\pi j}{2}\right)\right]\overline{\boldsymbol{T}}_{j}^{\mathrm{I}}\left(Q^2,\mu_0^2\right)\boldsymbol{F}_{j}\left(\xi,\Delta^2,\mu_0^2\right)$$

TFFs:

$$\mathscr{F}^{A}\left(\xi,\Delta^2,Q^2\right) = \frac{fC_F}{QN_c}\int_{-1}^{1}\frac{\mathrm{d}x}{2\xi}\int_{0}^{1}\mathrm{d}v\varphi(v){}^{A}T\left(x,v,\xi\middle|\alpha_s\left(\mu_R\right),\frac{Q^2}{\mu_F^2},\frac{Q^2}{\mu_\varphi^2},\frac{Q^2}{\mu_R^2}\right)F^{A}\left(x,\xi,\Delta^2,\mu_F^2,\mu_R^2\right)$$

Conformal space

$$\mathscr{F}^{A}\left(\xi,\Delta^2,Q^2\right) = \frac{fC_F}{QN_c}\frac{1}{2i}\int_{c}\mathrm{d}j\xi^{-j-1}\left[i\pm\begin{Bmatrix}\tan\\ \cot\end{Bmatrix}\left(\frac{\pi j}{2}\right)\right]\sum_{k,\ \mathrm{even}}\varphi_k\left(\mu_0^2\right)\overline{T}_{jk}^{\mathrm{I}}\left(Q^2,\mu_0^2\right)F_j^{A}\left(\xi,\Delta^2,\mu_0^2\right)$$

Kumerički-Müller GPD model

t-channel SO(3) partial wave expansion:

Doubly expanded GPDs

Angular momentum J complex?

$$F_j^a(\xi,t) = \sum_{\substack{J=J_{\min} \\ \text{even}}}^{j+1} F_{j,J}^a(t)\xi^{j+1-J}\hat{d}_{\alpha,\beta}^J(\xi), \quad J = j+1, j-1, j-3, \ldots, \quad a \in \{q, \mathrm{G}\}$$

Wigner's reduced rotation matrices

This expansion comes from the crossed channel $\gamma^*(q) + \gamma\left(-q'\right) \to h\left(p'\right) + \bar{h}(-p)$, where the expansion is done in terms of orbital angular momentum (Legendre polynomials), but we need to introduce the substitution $\cos\theta_t \to -\dfrac{1}{\xi} + \mathcal{O}\left(1/Q^2\right)$

Only **two** Wigner matrices contribute:

$$\hat{d}_{0,0}^J(\xi) = \frac{\Gamma(1/2)\Gamma(J+1)}{2^J\Gamma(J+1/2)}\xi^J C_J^{1/2}\left(\frac{1}{\xi}\right) = \frac{\Gamma(1/2)\Gamma(J+1)}{2^J\Gamma(J+1/2)}\xi^J {}_2F_1\left(\left.\begin{matrix}-J & J+1 \\ & 1\end{matrix}\,\right|\frac{\xi-1}{2\xi}\right)$$

$$\hat{d}_{0,1}^J(\xi) = \frac{\Gamma(1/2)\Gamma(J)}{2^J\Gamma(J+1/2)}\xi^{J-1} C_{J-1}^{3/2}\left(\frac{1}{\xi}\right) = \frac{\Gamma(3/2)\Gamma(J+1)}{2^J\Gamma(J+1/2)}\xi^{J-1} {}_2F_1\left(\left.\begin{matrix}-J+1 & J+2 \\ & 2\end{matrix}\,\right|\frac{\xi-1}{2\xi}\right)$$

For $\xi \leq 0.3$, which is valid for most kinematics, we can take $\hat{d}_{\alpha,\beta}^J(\xi) \approx 1$ since it makes no numerical difference

What are $F^a_{j,J}(t)$?

In the forward limit we expect to recover PDFs: $F^a_{j,j+1}(0) = f^a_j = \int_0^1 \mathrm{d}x x^j f^a(x)$

We include the standard PDF Ansatz into our model: $f^a(x) = \frac{N_a}{B\left(2-\alpha_0^a, \beta^a+1\right)} x^{-\alpha_0^a}(1-x)^{\beta^a} \quad \Rightarrow \quad f^a_j = N_a \frac{B\left(1-\alpha_0^a+j, \beta^a+1\right)}{B\left(2-\alpha_0^a, \beta^a+1\right)}$

For low x we only model sea quarks and gluons, we fix the parameters that regulate high-x behavior, $\beta^{\text{sea}} = 8$, $\beta^{\text{G}} = 6$.

Normalization constants are defined as average longitudinal momentum of parton a, so momentum conservation gives: $N_{\text{sea}} + N_{\text{val}} + N_{\text{G}} = 1$.

We use Regge theory to model the low-x factorized t and x dependence. We first complete the Regge trajectory $\alpha_0^a \to \alpha^a(t) = \alpha_0^a + \alpha'^a t$.

Regge theory predicts the behavior of the cross section, and models it by particle exchange that follow Regge trajectories in the complex J space. When a threshold for particle energy is crossed, the cross section contains a pole, which we model with a monopole function:

$$\frac{1}{1-\frac{t}{(m^a_j)^2}}, \quad (m^a_j)^2 = \frac{1+j-\alpha_0^a}{\alpha'^a}.$$

We model the form factor type t dependence with a dipole $\beta(t) = \left(1 - \frac{t}{m_a^2}\right)^{-2}$.

Final GPD form

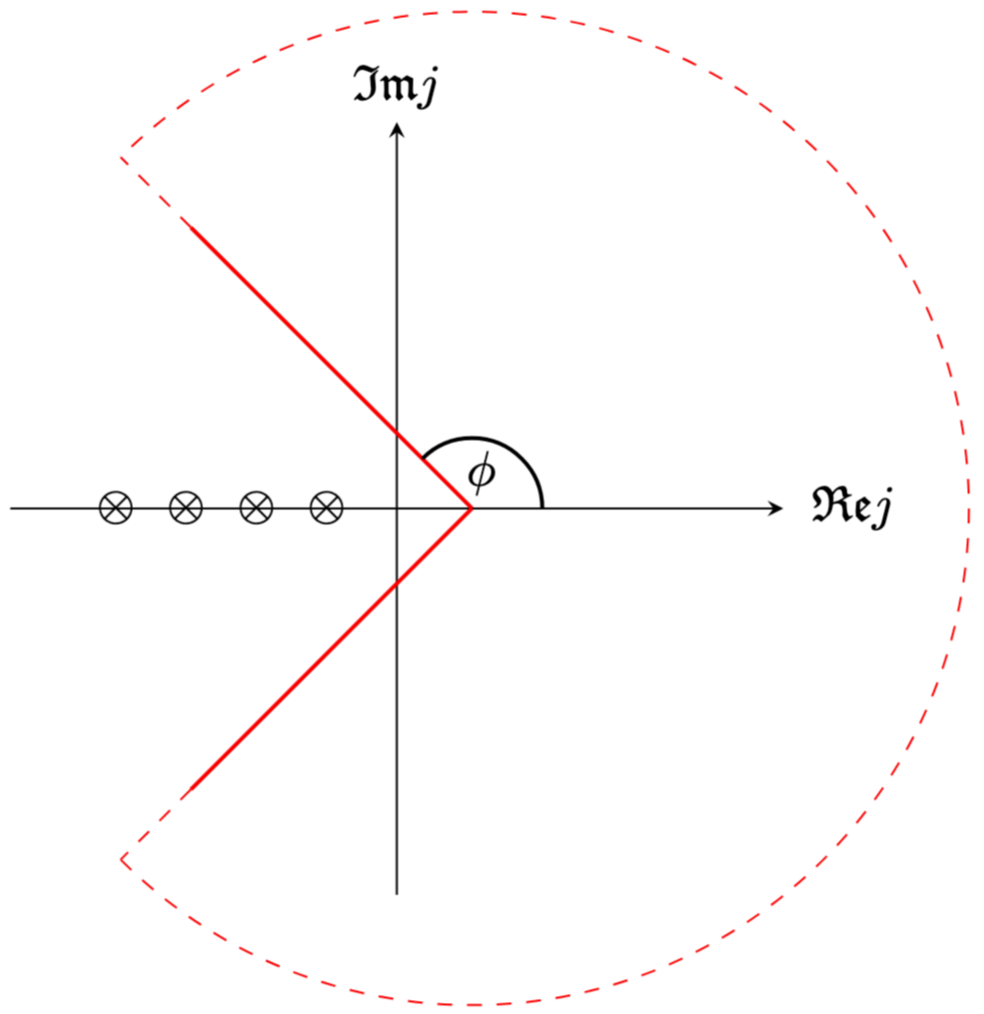
$$F^a_{j,j+1}(t) \equiv f^a_j(t) = f^a_j \frac{1+j-\alpha^a_0}{1+j-\alpha^a_0-\alpha'^a t}\left(1-\frac{t}{m_a^2}\right)^{-2}$$ → We multiply all of our "ingredients" for the final GPD

$$\Rightarrow F^a_j(\xi,t) = \left(1+s^a_2\xi^2+s^a_4\xi^4\right) f^a_j(t)$$ → We truncate the series after the third wave, and set all of the subleading waves to be proportional to the leading one. s^a_2 and s^a_4 parameters obtained from fits.

CFFs: $$\mathscr{F} = \frac{1}{2i}\int_{c-i\infty}^{c+i\infty} \mathrm{d}j\,\xi^{-j-1}\left[i+\tan\left(\frac{\pi j}{2}\right)\right]\left[\overline{\boldsymbol{T}}^{\mathrm{I}}_j + \boldsymbol{s_2}\overline{\boldsymbol{T}}^{\mathrm{I}}_{j+2} + \boldsymbol{s_4}\overline{\boldsymbol{T}}^{\mathrm{I}}_{j+4}\right]\boldsymbol{F}_j \rightarrow \boldsymbol{F}_j = \begin{pmatrix} F^{\mathrm{sea}} \\ F^{\mathrm{G}} \end{pmatrix}$$

↓ shift by $j-2$ ↓ shift by $j-4$

To evaluate the integral, we introduce the variable $j = c + ye^{i\phi}$ and deform the integration contour. This provides numerical stability if $\phi > \pi/2$. c is chosen to avoid GPD and hard scattering amplitude poles.



$$\mathscr{F}^A\left(\xi,\Delta^2,\mathcal{Q}^2\right) = \mathfrak{Im}\, e^{i\phi}\int_0^\infty \mathrm{d}y\,\xi^{-j-1}\tan\left(\frac{\pi j}{2}\right)\boldsymbol{C}_j\left(Q^2/\mu^2,\alpha_s(\mu)\right)\boldsymbol{F}^A_j\left(\xi,\Delta^2,\mu^2\right)$$
$$+ i\,\mathfrak{Im}\, e^{i\phi}\int_0^\infty \mathrm{d}y\,\xi^{-j-1}\boldsymbol{C}_j\left(Q^2/\mu^2,\alpha_s(\mu)\right)\boldsymbol{F}^A_j\left(\xi,\Delta^2,\mu^2\right)\Big|_{j=c+ye^{i\phi}}$$

Cross section and data

$$\text{DVCS: } \frac{d\sigma^{\gamma^* N \to \gamma N}}{d\Delta^2} \approx \frac{\pi \alpha_{em}^2}{(W^2+Q^2)^2} \left[|\mathcal{H}|^2 + |\widetilde{\mathcal{H}}|^2 - \frac{\Delta^2}{4M^2} |\mathcal{E}|^2 \right]$$

$$\text{DVMP: } \frac{d\sigma^{\gamma^* N \to VN}}{d\Delta^2} \approx \frac{4\pi^2 \alpha_{em} x_B^2}{Q^4} \left[|\mathcal{H}|^2 - \frac{\Delta^2}{4M^2} |\mathcal{E}|^2 \right]$$

For $|\Delta^2| < 1\,\text{GeV}^2$ CFF $\mathcal{E}$ suppressed by $-\frac{\langle \Delta^2 \rangle}{4M^2} \approx 5 \times 10^{-2}$. For $\widetilde{\mathcal{H}}$ Regge intercept $\alpha(0) \approx 1/2$, for $\mathcal{H}$ $\alpha(0) \approx 1$, $\widetilde{\mathcal{H}}$ also suppressed. We ignore valence contributions, only singlet sea quark and gluon GPDs. We take the asymptotic distribution amplitude, dominant term in conformal space $\varphi_0 \approx 1$. We use dispersion relations without a subtraction constant.

- DIS data: H1 F_2
- DVCS: H1 and ZEUS data, $Q^2 \geq 5.0\,\text{GeV}^2$
- DVMP: H1 and ZEUS ρ^0 production, $Q^2 \geq 10.0\,\text{GeV}^2$
- no t dependence

Dataset	N_{pts}	LO-			NLO-		
		DVCS	DVMP	DVCS-DVMP	DVCS	DVMP	DVCS-DVMP
DIS	85	0.6	0.6	0.6	0.8	0.8	0.8
DVCS	27	0.4	$\gg 1$	0.6	0.6	$\gg 1$	0.8
DVMP	45	$\gg 1$	3.1	3.3	$\gg 1$	1.5	1.8
Total	157	$\gg 1$	$\gg 1$	1.4	3.7	$\gg 1$	1.1

Fitting procedure

MINUIT routine for finding the best fit by minimizing χ^2. Fitting is usually performed in two steps. We first fit to DIS data, and release parameters pertaining to the PDF portion of the GPD model. Then we fit to DVCS and DVMP points and release parameters pertaining to the t and ξ dependence of the GPD.

```
f = g.MinuitFitter(DISpoints, th)
f.release_parameters('ns', 'alos', 'alog')
f.fit()
f.fix_parameters('ALL')
f = g.MinuitFitter(DVCSpoints+DVMPpoints, th)
f.release_parameters('ms2', 'alps', 'secs', 'this', 'mg2', 'alpg', 'secg', 'thig')
f.limit_parameters(pars_range)
f.fit()
```

We needed to implement constraints on the parameters so they take on physically plausible values. It is easy to obtain a good fit with a flexible model, but does it have physical predictions?

Initial version of fits with no constraints:

$$\chi^2/N_{d.o.f}\Big|_{\text{NLO all}} = 1.279$$

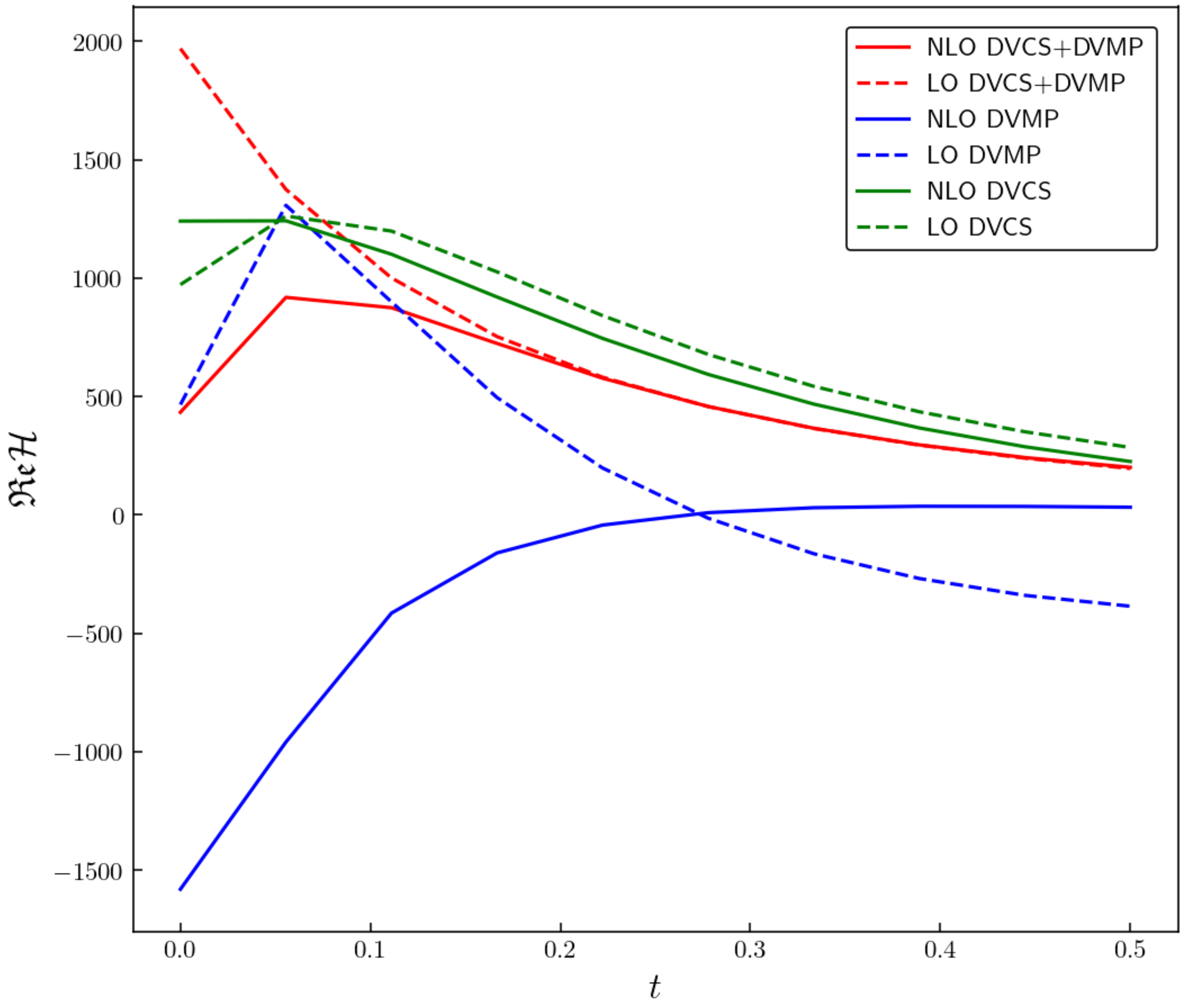
$$\chi^2/N_{d.o.f}\Big|_{\text{LO all}} = 1.012$$

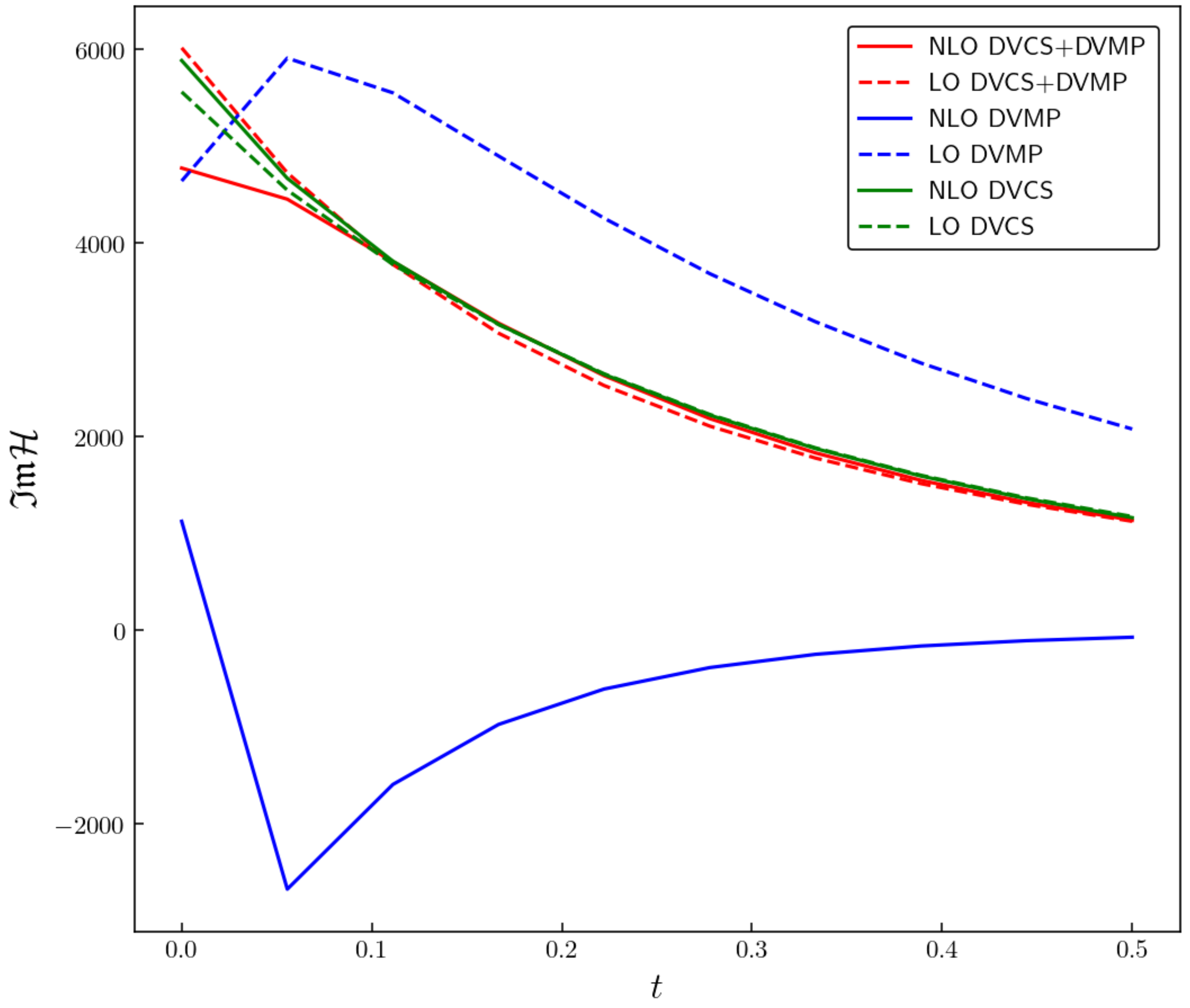
$$\chi^2/N_{d.o.f}\Big|_{\text{NLO DVCS}} = 0.786$$

$$\chi^2/N_{d.o.f}\Big|_{\text{LO DVCS}} = 0.659$$

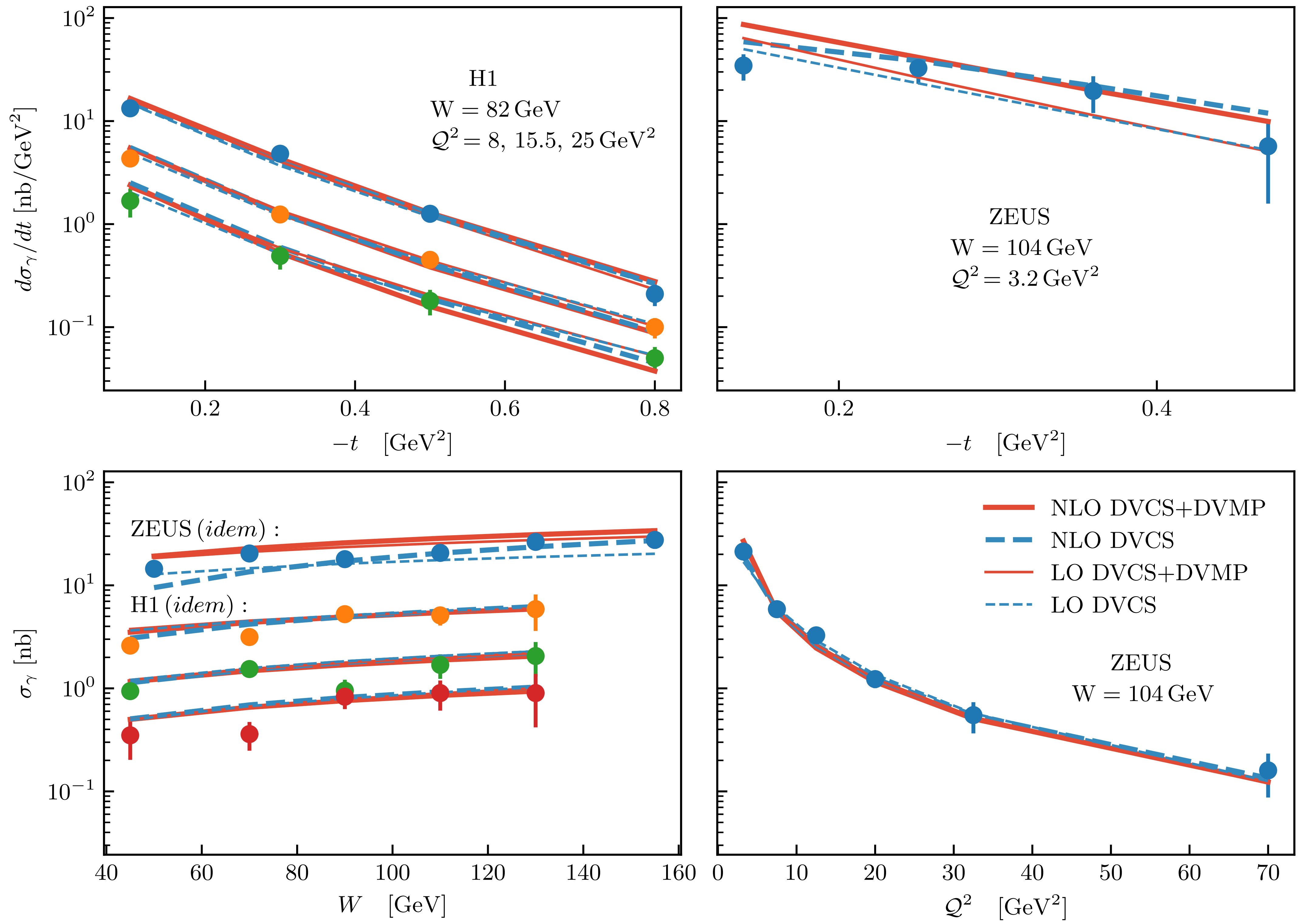
$$\chi^2/N_{d.o.f}\Big|_{\text{NLO DVMP}} = 1.135$$

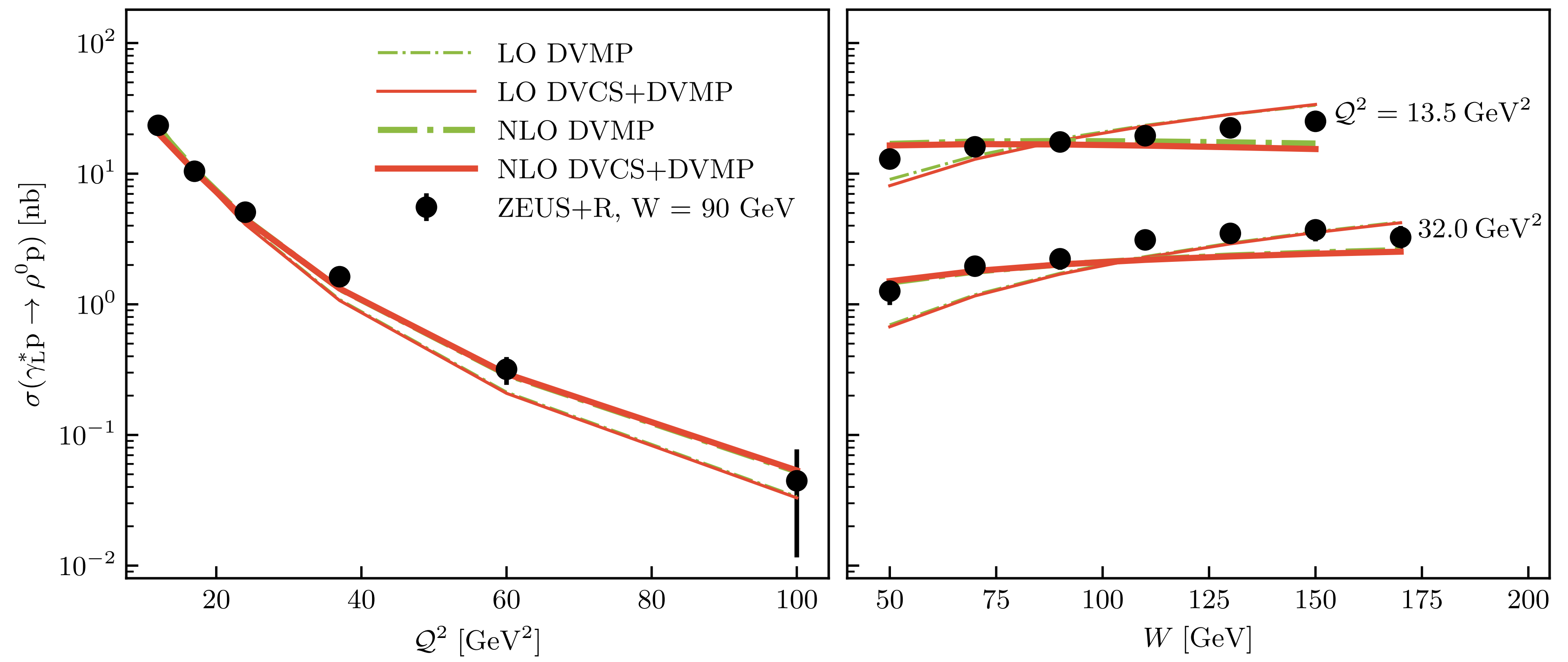
$$\chi^2/N_{d.o.f}\Big|_{\text{LO DVMP}} = 0.879$$

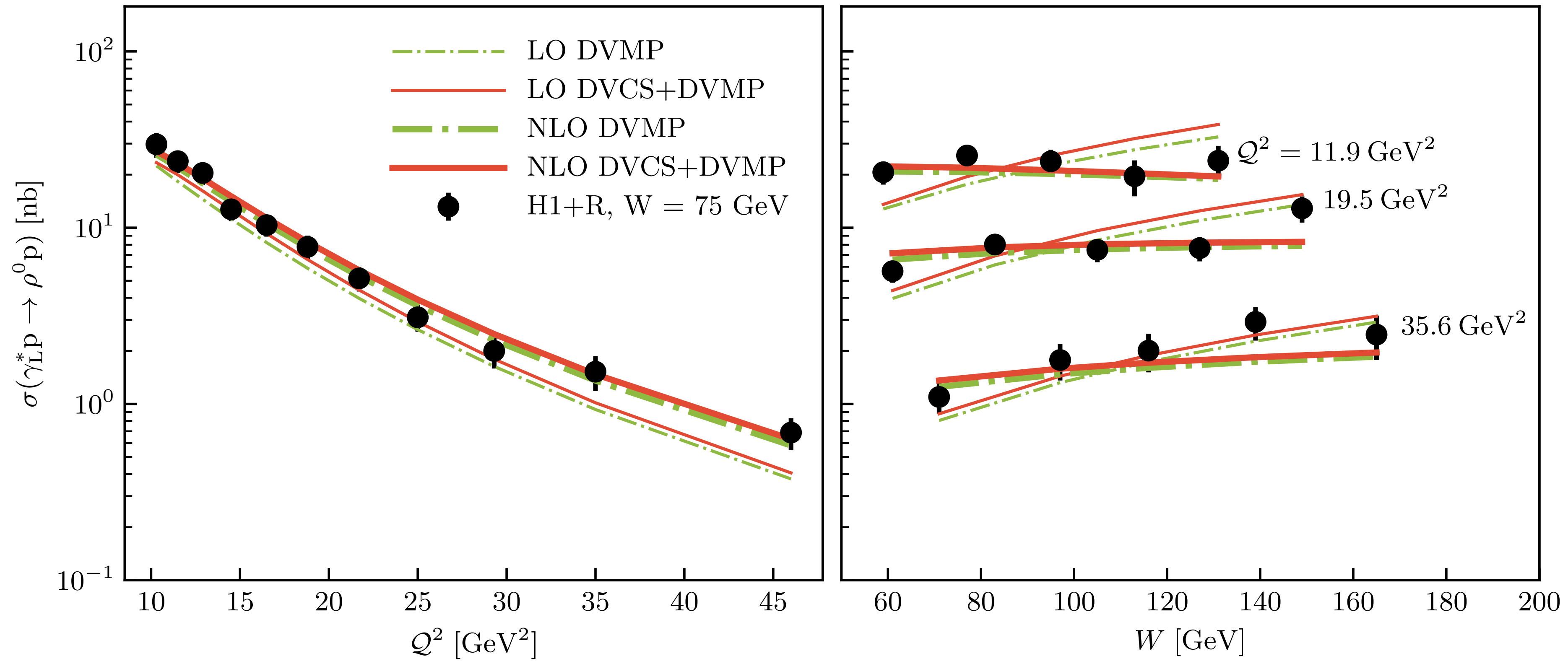


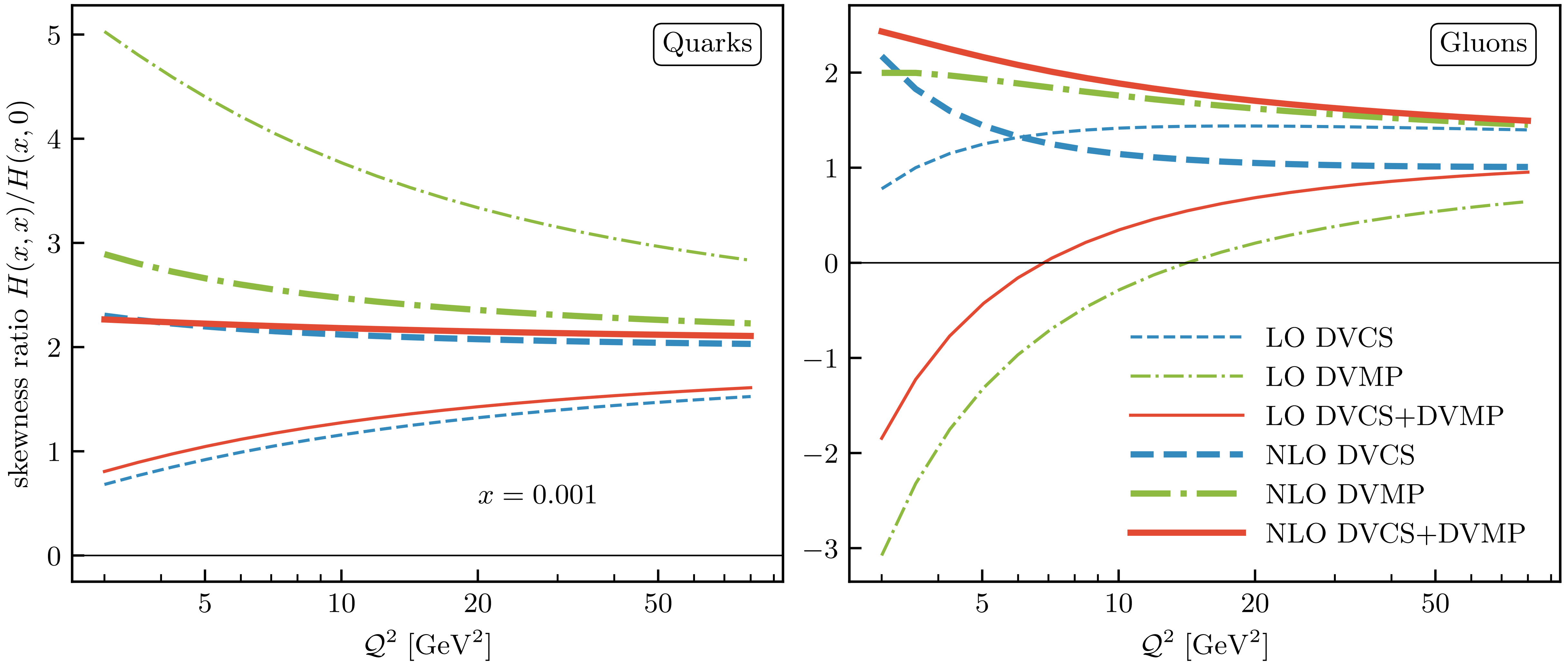


Results


H1
W = 82 GeV
$Q^2 = 8,\ 15.5,\ 25\ \mathrm{GeV}^2$
ZEUS
W = 104 GeV
$Q^2 = 3.2\ \mathrm{GeV}^2$
$d\sigma_\gamma/dt\ [\mathrm{nb}/\mathrm{GeV}^2]$
$-t\ [\mathrm{GeV}^2]$
ZEUS (idem) :
H1 (idem) :
σ_γ [nb]
W [GeV]
NLO DVCS+DVMP
NLO DVCS
LO DVCS+DVMP
LO DVCS
ZEUS
W = 104 GeV
$Q^2\ [\mathrm{GeV}^2]$


LO DVMP
LO DVCS+DVMP
NLO DVMP
NLO DVCS+DVMP
ZEUS+R, W = 90 GeV
$\sigma(\gamma^*_L p \to \rho^0 p)$ [nb]
$Q^2\ [\mathrm{GeV}^2]$
W [GeV]
$Q^2 = 13.5\ \mathrm{GeV}^2$
$32.0\ \mathrm{GeV}^2$


LO DVMP
LO DVCS+DVMP
NLO DVMP
NLO DVCS+DVMP
H1+R, W = 75 GeV
$\sigma(\gamma^*_L p \to \rho^0 p)$ [nb]
$Q^2\ [\mathrm{GeV}^2]$
W [GeV]
$Q^2 = 11.9\ \mathrm{GeV}^2$
$19.5\ \mathrm{GeV}^2$
$35.6\ \mathrm{GeV}^2$

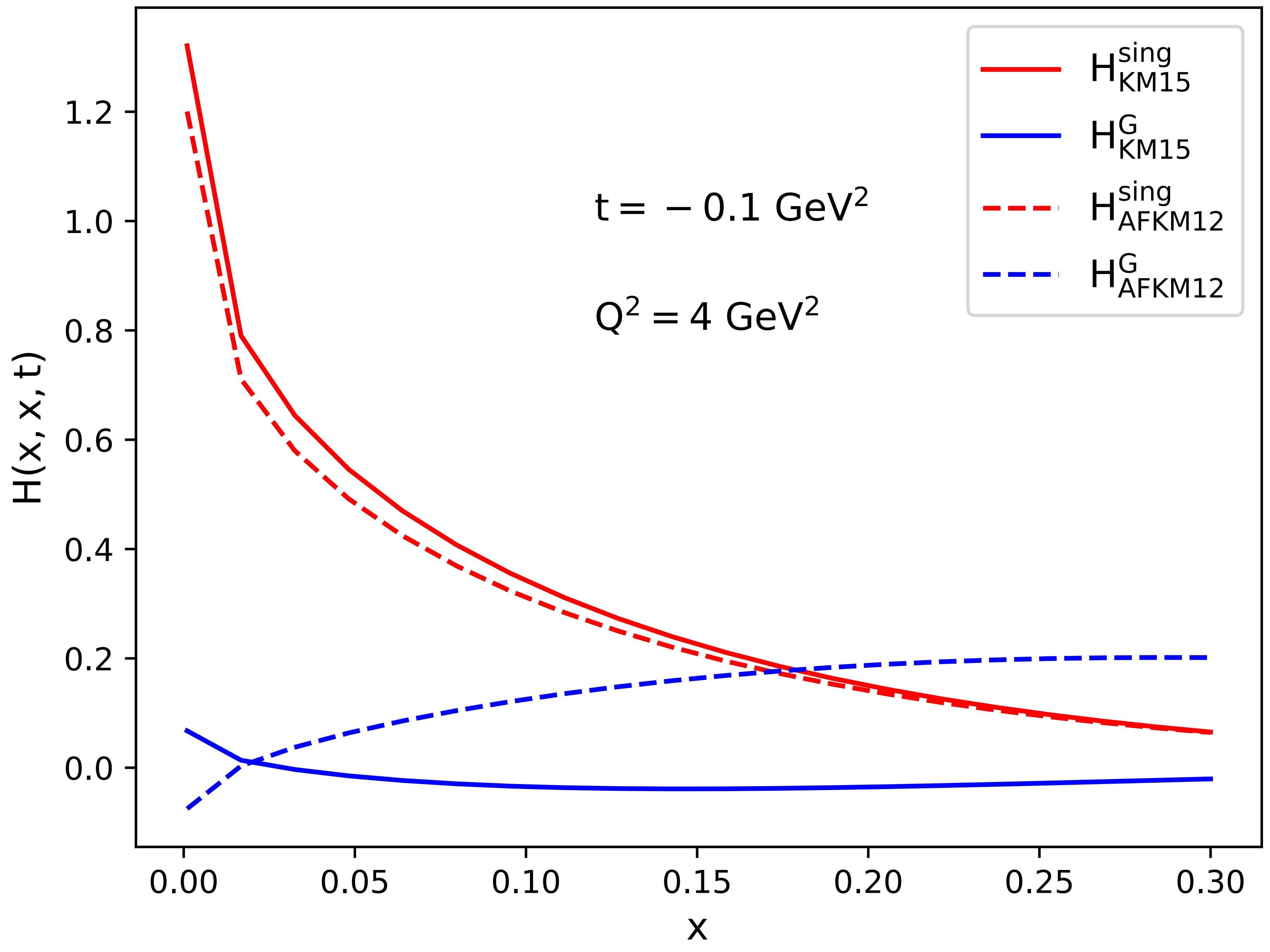

Quarks
Gluons
skewness ratio $H(x,x)/H(x,0)$
$x = 0.001$
$Q^2\ [\mathrm{GeV}^2]$
LO DVCS
LO DVMP
LO DVCS+DVMP
NLO DVCS
NLO DVMP
NLO DVCS+DVMP

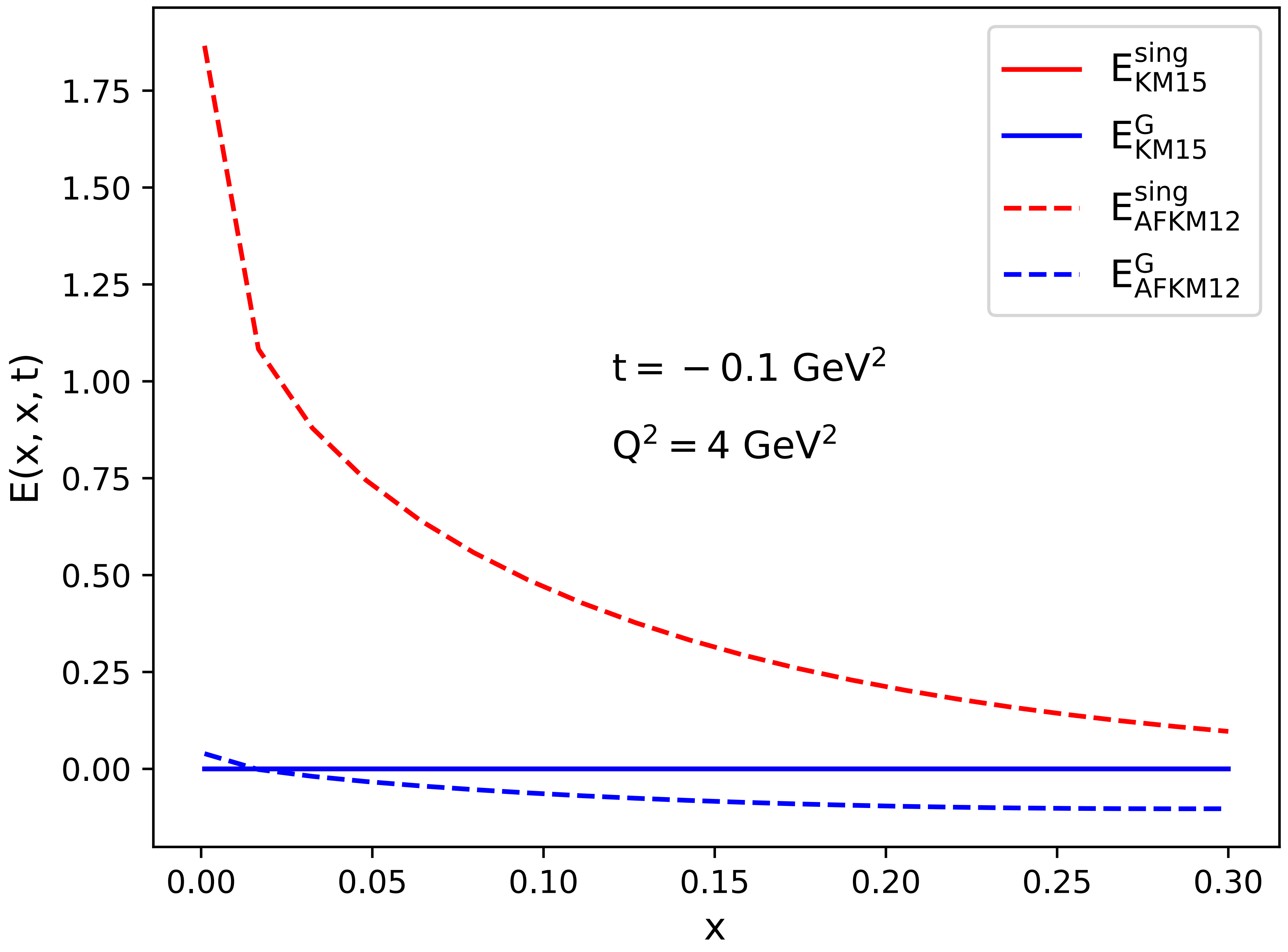
Hybrid models

KM-type models have a hybrid implementation of GPDs: valence sector is modeled in the x-space at the crossover line, there is no evolution; sea and gluon sector is implemented in the conformal space, LO evolution. The valence sector is defined as:

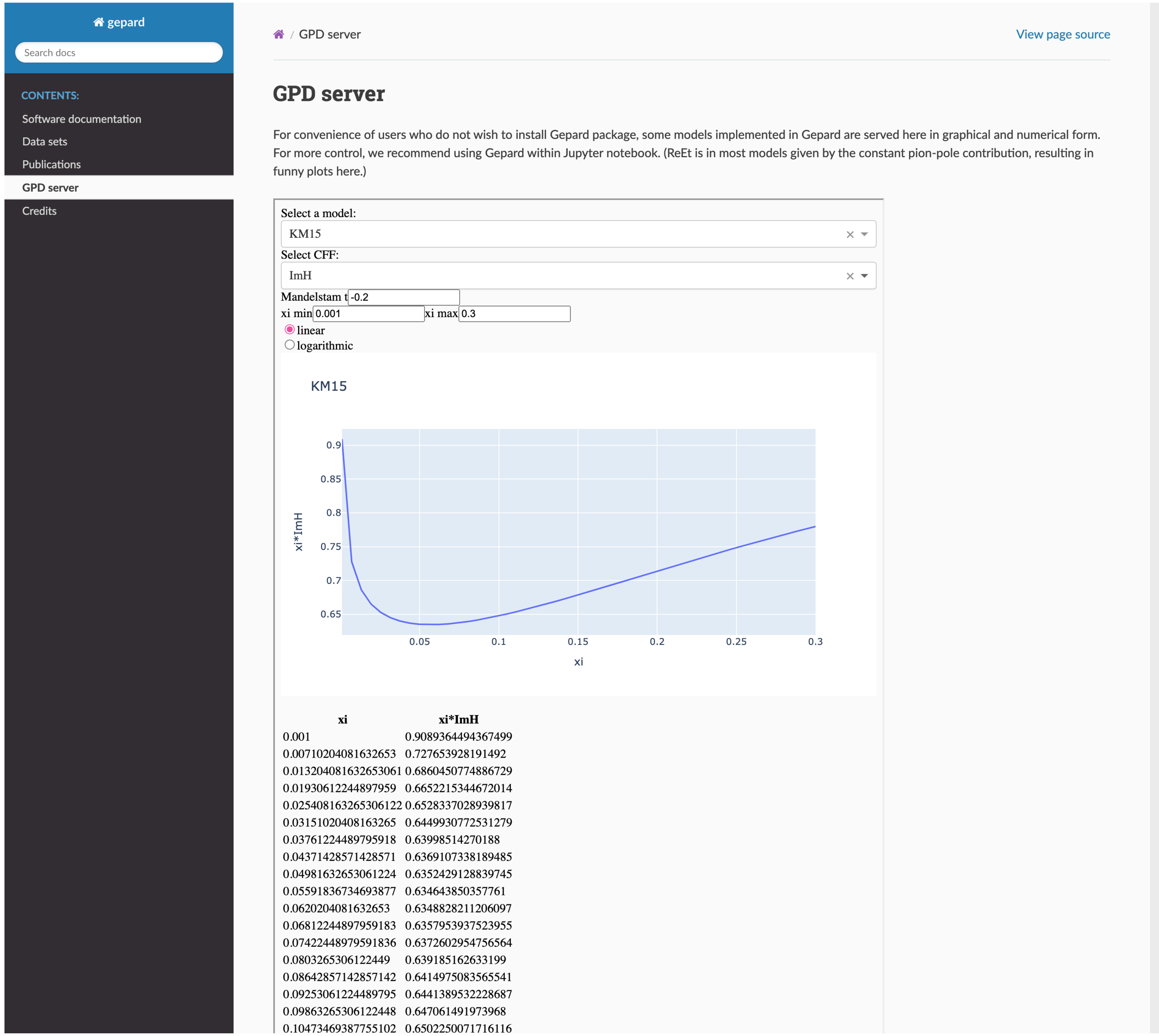
$$\text{CFF: } \mathfrak{Im}\mathscr{H}(\xi,t) = \pi\left[\frac{4}{9}H^{u_{\text{val}}}(\xi,\xi,t) + \frac{1}{9}H^{d_{\text{val}}}(\xi,\xi,t) + \frac{2}{9}H^{\text{sea}}(\xi,\xi,t)\right] \quad + \quad \text{subtraction constant: } \Delta(t) = \frac{C}{\left(1-\frac{t}{M_C^2}\right)^2}$$

$$\text{GPDs: } H_q^{\text{val}}(x,x,t) = \frac{n_q r_q}{1+x}\left(\frac{2x}{1+x}\right)^{-\alpha_v(t)}\left(\frac{1-x}{1+x}\right)^{b_q}\frac{1}{1-\frac{1-x}{1+x}\frac{t}{M_q^2}}, \quad q = u, d$$





Gepard



xi	xi*ImH
0.001	0.9089364494367499
0.00710204081632653	0.727653928191492
0.013204081632653061	0.6860450774886729
0.01930612244897959	0.6652215344672014
0.025408163265306122	0.6528337028939817
0.03151020408163265	0.6449930772531279
0.03761224489795918	0.63998514270188
0.04371428571428571	0.6369107338189485
0.04981632653061224	0.6352429128839745
0.05591836734693877	0.634643850357761
0.0620204081632653	0.6348828211206097
0.06812244897959183	0.6357953937523955
0.07422448979591836	0.6372602954756564
0.0803265306122449	0.639185162633199
0.08642857142857142	0.6414975083565541
0.09253061224489795	0.6441389532228687
0.09863265306122448	0.647061491973968
0.10473469387755102	0.6502250071716116

Software for calculating observables for exclusive processes: https://gepard.phy.hr/index.html developed by Krešimir Kumerički. It can be downloaded as a Python package, but also hosts a GPD server which directly outputs values of CFFs for a variety of models. This software also includes neural network models implemented in PyTorch. They are available on the Torch branch on GitHub. The DVCS+DVMP models are available on the Devel branch.