$\boxed{\text{Ex 1}}$

$$\langle \vec{r}\,'|\rho(\vec{x})|\vec{r}\rangle = \int \frac{d^3p'}{(2\pi)^3}\frac{d^3p}{(2\pi)^3}\, e^{i\vec{p}\,'\cdot\vec{r}\,'} e^{-i\vec{p}\cdot\vec{r}} \langle \vec{p}\,'|\rho(\vec{x})|\vec{p}\rangle$$

- $\langle \vec{p}\,'|\rho(\vec{x})|\vec{p}\rangle = \langle \vec{p}\,'| e^{-i\hat{\vec{P}}\cdot\vec{x}}\rho(\vec{0})e^{i\hat{\vec{P}}\cdot\vec{x}}|\vec{p}\rangle = e^{-i\vec{\Delta}\cdot\vec{x}}\langle \vec{p}\,'|\rho(\vec{0})|\vec{p}\rangle$
  (translation symmetry)

- $\vec{p}\,'\cdot\vec{r}\,' - \vec{p}\cdot\vec{r} = (\vec{P}+\frac{\vec{\Delta}}{2})\cdot\vec{r}\,' - (\vec{P}-\frac{\vec{\Delta}}{2})\cdot\vec{r}$
  $= \vec{P}\cdot(\vec{r}\,'-\vec{r}) + \vec{\Delta}\cdot\frac{\vec{r}\,'+\vec{r}}{2}$

$$\begin{vmatrix} \frac{\partial P_z}{\partial p_z} & \frac{\partial P_z}{\partial p_z'} \\ \frac{\partial \Delta_z}{\partial p_z} & \frac{\partial \Delta_z}{\partial p_z'} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ -1 & 1 \end{vmatrix} = 1$$

- $d^3p'\,d^3p = d^3P\,d^3\Delta$

$\Rightarrow$ $$\langle \vec{r}\,'|\rho(\vec{x})|\vec{r}\rangle = \int \frac{d^3P}{(2\pi)^3}\frac{d^3\Delta}{(2\pi)^3}\, e^{i\vec{P}\cdot(\vec{r}\,'-\vec{r})} e^{-i\vec{\Delta}\cdot(\vec{x}-\frac{\vec{r}\,'+\vec{r}}{2})} \langle \vec{P}+\tfrac{\vec{\Delta}}{2}|\rho(\vec{0})|\vec{P}-\tfrac{\vec{\Delta}}{2}\rangle$$

$\boxed{\text{Ex 2}}$ Light-front spinors

$a^{\pm} = \frac{1}{\sqrt{2}}(a^0 \pm a^3)$ $\qquad a_{R,L} = a^1 \pm i a^2$

$$u(p,+) = \mathcal{N}\begin{pmatrix} \sqrt{2}p^+ + M_N \\ p_R \\ \sqrt{2}p^+ - M_N \\ p_R \end{pmatrix}, \qquad u(p,-) = \mathcal{N}\begin{pmatrix} -p_L \\ \sqrt{2}p^+ + M_N \\ p_L \\ -\sqrt{2}p^+ + M_N \end{pmatrix}, \qquad \mathcal{N}^2 = \frac{1}{2\sqrt{2}p^+}$$

$\Rightarrow$ $\bar{u}(p,+) = \mathcal{N}\left(\sqrt{2}p^+ + M_N,\ p_L,\ -\sqrt{2}p^+ + M_N,\ -p_L\right)$

$$v(p,\pm) = -u(p,\mp)\big|_{M_N \to -M_N}$$

$\bar{u}(p,-) = \mathcal{N}\left(-p_R,\ \sqrt{2}p^+ + M_N,\ -p_R,\ \sqrt{2}p^+ - M_N\right)$

- $$\gamma^+ u(p,+) = \frac{\mathcal{N}}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}\begin{pmatrix} \sqrt{2}p^+ + M_N \\ p_R \\ \sqrt{2}p^+ - M_N \\ p_R \end{pmatrix} = \mathcal{N}\, 2p^+ \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}$$

- $$\gamma^+ u(p,-) = \frac{\mathcal{N}}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}\begin{pmatrix} -p_L \\ \sqrt{2}p^+ + M_N \\ p_L \\ -\sqrt{2}p^+ + M_N \end{pmatrix} = \mathcal{N}\, 2p^+ \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

$\Rightarrow$ $\bar{u}(p,\lambda_q')\gamma^+ u(p,\lambda_q) = 2p^+\,\delta_{\lambda_q'\lambda_q}$

Ex 3 Poincaré algebra

$[\downarrow, \rightarrow]$	J^j	K^j	P^j	P^0
J^i	$i\varepsilon^{ijk}J^k$	$i\varepsilon^{ijk}K^k$	$i\varepsilon^{ijk}P^k$	0
K^i		$-i\varepsilon^{ijk}J^k$	$-i\delta^{ij}P^0$	$-iP^i$
P^i			0	0
P^0				0

$K^i = M^{0i}$

$J^i = \frac{1}{2}\varepsilon^{ijk}M^{jk}$

$e^{-i\vec{\theta}\cdot\vec{J}}\, e^{-i\vec{\eta}\cdot\vec{K}}$

$$B_\perp^i = M^{+i} = \frac{1}{\sqrt{2}}(M^{0i} + M^{3i}) = \frac{1}{\sqrt{2}}\left(K_\perp^i + \varepsilon^{3ij}J_\perp^j\right) \qquad (i = 1,2)$$

$$\mathcal{J}_\perp^i = \varepsilon^{3ij}M^{-j} = \frac{1}{\sqrt{2}}\varepsilon^{3ij}(M^{0j} - M^{3j}) = \frac{1}{\sqrt{2}}\left(J_\perp^i + \varepsilon^{3ij}K_\perp^j\right)$$

$$\hookrightarrow [J^3, \mathcal{J}_\perp^i] = i\varepsilon^{3ij}\mathcal{J}_\perp^j$$

$$[J^3, B_\perp^i] = i\varepsilon^{3ij}B_\perp^j$$

$$[J^3, P^\pm] = 0$$

$$\begin{aligned}
[B_\perp^i, P^+] &= \frac{1}{2}\left[K_\perp^i + \varepsilon^{3ij}J_\perp^j, P^0 + P^3\right] \\
&= \frac{1}{2}\left([K_\perp^i, P^0] + \underbrace{[K_\perp^i, P^3]}_{=0}\right) + \frac{1}{2}\varepsilon^{3ij}\left(\underbrace{[J_\perp^j, P^0]}_{=0} + [J_\perp^j, P^3]\right) \\
&= \frac{1}{2}(-i)P_\perp^i + \frac{1}{2}\varepsilon^{3ij}\, i\varepsilon^{j3k}P_\perp^k \\
&= 0
\end{aligned}$$

$$\hookrightarrow [B_\perp^i, P^-] = -iP_\perp^i$$

$$\begin{aligned}
[B_\perp^i, P_\perp^j] &= \frac{1}{\sqrt{2}}\left[K_\perp^i + \varepsilon^{3ik}J_\perp^k, P_\perp^j\right] \\
&= \frac{1}{\sqrt{2}}\left([K_\perp^i, P_\perp^j] + \varepsilon^{3ik}[J_\perp^k, P_\perp^j]\right) \\
&= \frac{1}{\sqrt{2}}\left(-i\delta_\perp^{ij}P^0 + \varepsilon^{3ik}\, i\varepsilon^{kj3}P^3\right) \\
&= -i\delta_\perp^{ij}P^+
\end{aligned}$$

$$\begin{aligned}
[B_\perp^i, B_\perp^j] &= \frac{1}{2}\left([K_\perp^i, K_\perp^j] + \varepsilon^{3jk}[K_\perp^i, J_\perp^k] + \varepsilon^{3ik}[J_\perp^k, K_\perp^j] + \varepsilon^{3ik}\varepsilon^{3jl}[J_\perp^k, J_\perp^l]\right) \\
&= \frac{1}{2}\left(\cancel{-i\varepsilon^{ij3}J^3} + \cancel{\varepsilon^{3jk}(-i)\varepsilon^{ki3}K^3} + \cancel{\varepsilon^{3ik}\, i\varepsilon^{kj3}K^3} + \cancel{\varepsilon^{3ik}\varepsilon^{3jl}\, i\varepsilon^{kl3}J^3}\right) = 0
\end{aligned}$$

$\boxed{\text{Ex } 4}$

$$\langle p'^+, \vec{r}'_\perp | \int dx^- J^+(x) | p^+, \vec{r}_\perp \rangle \overset{x^+=0}{=} \int dx^- \frac{d^2P_\perp}{(2\pi)^2} \frac{d^2\Delta_\perp}{(2\pi)^2} e^{i\vec{P}_\perp\cdot(\vec{r}'_\perp - \vec{r}_\perp)}$$

$$\times e^{i\Delta^+ x^-} e^{-i\vec{\Delta}_\perp\cdot\left(\vec{x}_\perp - \frac{\vec{r}'_\perp + \vec{r}_\perp}{2}\right)} \langle P^+ + \tfrac{\Delta^+}{2}, \vec{P}_\perp + \tfrac{\vec{\Delta}_\perp}{2} | J^+(0) | P^+ - \tfrac{\Delta^+}{2}, \vec{P}_\perp - \tfrac{\vec{\Delta}_\perp}{2} \rangle$$

- $\int dx^- e^{i\Delta^+ x^-} = 2\pi\delta(p'^+ - p^+) = 2P^+ 2\pi\delta(p'^+ - p^+) \frac{1}{2P^+}$ $\qquad P^+ = \frac{p'^+ + p^+}{2} \quad \Delta^+ = p'^+ - p^+$
- $\langle P^+, \vec{P}_\perp + \frac{\vec{\Delta}_\perp}{2} | J^+(0) | P^+, \vec{P}_\perp - \frac{\vec{\Delta}_\perp}{2} \rangle = \langle P^+, \frac{\vec{\Delta}_\perp}{2} | J^+(0) | P^+, -\frac{\vec{\Delta}_\perp}{2} \rangle$ $\qquad$ (2D Galilean symmetry)
- $\int \frac{d^2P_\perp}{(2\pi)^2} e^{i\vec{P}_\perp\cdot(\vec{r}'_\perp - \vec{r}_\perp)} = \delta^{(2)}(\vec{r}'_\perp - \vec{r}_\perp)$

$\hookrightarrow \langle p'^+, \vec{r}'_\perp | \int dx^- J^+(x) | p^+, \vec{r}_\perp \rangle \overset{x^+=0}{=} 2P^+ 2\pi\delta(p'^+ - p^+) \delta^{(2)}(\vec{r}'_\perp - \vec{r}_\perp) \int \frac{d^2\Delta_\perp}{(2\pi)^2} e^{-i\vec{\Delta}_\perp\cdot(\vec{x}_\perp - \vec{r}_\perp)} \frac{\langle P^+, \frac{\vec{\Delta}_\perp}{2} | J^+(0) | P^+, -\frac{\vec{\Delta}_\perp}{2} \rangle}{2P^+}$

$\boxed{\text{Ex } 5}$

$$\rho_{LF}(\vec{b}_\perp) = \underbrace{\int \frac{d^2\Delta_\perp}{(2\pi)^2} e^{-i\vec{\Delta}_\perp\cdot\vec{b}_\perp} F_1(\vec{\Delta}_\perp^2)}_{\rho_1(\vec{b}_\perp^2)} - \frac{\vec{S}\times\vec{\nabla}_\perp}{M_N} \underbrace{\int \frac{d^2\Delta_\perp}{(2\pi)^2} e^{-i\vec{\Delta}_\perp\cdot\vec{b}_\perp} F_2(\vec{\Delta}_\perp^2)}_{\rho_2(\vec{b}_\perp^2)}$$

- $\int d^2b_\perp\, b^i_\perp\, \rho_1(\vec{b}_\perp^2) = 0$ $\qquad$ (axial symmetry)
- $\int d^2b_\perp\, b^i_\perp\, \nabla^k_\perp \rho_2(\vec{b}_\perp^2) = -\delta^{ik}_\perp \int d^2b_\perp\, \rho_2(\vec{b}_\perp^2) = -\delta^{ik}_\perp F_2(0)$

$$\hookrightarrow \langle d^i_\perp \rangle = \varepsilon^{3ji} \frac{S^j}{M_N} F_2(0)$$

$$= -\varepsilon^{3ij} \frac{\hat{S}^j}{2M_N} \kappa_N \qquad \vec{S} = \tfrac{1}{2}\hat{S}$$

$\hookrightarrow$ unit vector

Ex 6 $\quad (x^0 = 0)$

$$\langle \psi | O(x) | \psi \rangle = \int \frac{d^3p'}{(2\pi)^3} \frac{d^3p}{(2\pi)^3} \frac{\langle \psi | p' \rangle}{\sqrt{2p'^0}} \frac{\langle p | \psi \rangle}{\sqrt{2p^0}} \frac{\langle p' | O(x) | p \rangle}{\sqrt{2p'^0 2p^0}}$$

$$= \int \frac{d^3P}{(2\pi)^3} \frac{d^3\Delta}{(2\pi)^3} \tilde{\psi}^*(P+\tfrac{\Delta}{2}) \tilde{\psi}(P-\tfrac{\Delta}{2}) \frac{\langle P+\frac{\Delta}{2} | O(x) | P-\frac{\Delta}{2} \rangle}{\sqrt{4(P^0)^2-(\Delta^0)^2}}$$

$$= \int \frac{d^3P}{(2\pi)^3} \frac{d^3\Delta}{(2\pi)^3} d^3R \, e^{i\vec{\Delta}\cdot\vec{R}} \rho_\psi(\vec{R},\vec{P}) \frac{\langle P+\frac{\Delta}{2} | O(x) | P-\frac{\Delta}{2} \rangle}{\sqrt{4(P^0)^2-(\Delta^0)^2}}$$

$$= \int \frac{d^3P}{(2\pi)^3} d^3R \, \rho_\psi(\vec{R},\vec{P}) \underbrace{\int \frac{d^3\Delta}{(2\pi)^3} e^{-i\vec{\Delta}\cdot(\vec{x}-\vec{R})} \frac{\langle P+\frac{\Delta}{2} | O(0) | P-\frac{\Delta}{2} \rangle}{\sqrt{4(P^0)^2-(\Delta^0)^2}}}_{= \langle O \rangle_{\vec{R},\vec{P}}(\vec{x})}$$

Ex 7 $\quad$ Boost along z-direction $\quad p^\mu \to p'^\mu = (\Lambda_z)^\mu{}_\nu \, p^\nu$

$$D^{(1/2)}(p,\Lambda_z) = \cos\frac{\theta}{2} \, \mathbb{1} + i \sin\frac{\theta}{2} (\vec{e}_z \times \hat{p}) \cdot \vec{\sigma}$$

$$= \begin{pmatrix} c & \hat{p}_L s \\ -\hat{p}_R s & c \end{pmatrix}$$

$c = \cos\frac{\theta}{2} \qquad \hat{p} = \vec{p}/|\vec{p}|$

$s = \sin\frac{\theta}{2} \qquad p_{R,L} = p^1 \pm i p^2$

$$\vec{P}'_{\perp B} = -\vec{P}_{\perp B} = \frac{\vec{\Delta}_\perp}{2}$$

$\Rightarrow$

- $$D^{\dagger(1/2)}(p'_B,\Lambda_z) \, \mathbb{1} \, D^{(1/2)}(p_B,\Lambda_z) = \begin{pmatrix} c & -\hat{\Delta}_L s \\ \hat{\Delta}_R s & c \end{pmatrix} \begin{pmatrix} c & -\hat{\Delta}_L s \\ \hat{\Delta}_R s & c \end{pmatrix}$$

$$= \begin{pmatrix} c^2-s^2 & -\hat{\Delta}_L 2sc \\ \hat{\Delta}_R 2sc & c^2-s^2 \end{pmatrix} = \cos\theta \, \mathbb{1} + i\sin\theta (\vec{\sigma}\times\hat{\Delta}_\perp)_z$$

- $$D^{\dagger(1/2)}(p'_B,\Lambda_z) \, i(\vec{\sigma}\times\hat{\Delta}_\perp)_z \, D^{(1/2)}(p_B,\Lambda_z) = \begin{pmatrix} c & -\hat{\Delta}_L s \\ \hat{\Delta}_R s & c \end{pmatrix} \begin{pmatrix} 0 & -\hat{\Delta}_L \\ \hat{\Delta}_R & 0 \end{pmatrix} \begin{pmatrix} c & -\hat{\Delta}_L s \\ \hat{\Delta}_R s & c \end{pmatrix}$$

$$= \begin{pmatrix} c & -\hat{\Delta}_L s \\ \hat{\Delta}_R s & c \end{pmatrix} \begin{pmatrix} -s & -\hat{\Delta}_L c \\ \hat{\Delta}_R c & -s \end{pmatrix} = \begin{pmatrix} -2sc & -\hat{\Delta}_L(c^2-s^2) \\ \hat{\Delta}_R(c^2-s^2) & -2sc \end{pmatrix}$$

$$= -\sin\theta \, \mathbb{1} + i\cos\theta (\vec{\sigma}\times\hat{\Delta}_\perp)_z$$

- $D^{\dagger(1/2)}(p'_B,\Lambda_2)\,(\vec{\sigma}_\perp\cdot\hat{\Delta}_\perp)\,D^{(1/2)}(p_B,\Lambda_2) = \begin{pmatrix} c & -\hat{\Delta}_L s \\ \hat{\Delta}_R s & c \end{pmatrix}\begin{pmatrix} 0 & \hat{\Delta}_L \\ \hat{\Delta}_R & 0 \end{pmatrix}\begin{pmatrix} c & -\hat{\Delta}_L s \\ \hat{\Delta}_R s & c \end{pmatrix}$

$$= \begin{pmatrix} c & -\hat{\Delta}_L s \\ \hat{\Delta}_R s & c \end{pmatrix}\begin{pmatrix} s & \hat{\Delta}_L c \\ \hat{\Delta}_R c & -s \end{pmatrix} = \begin{pmatrix} 0 & \hat{\Delta}_L (c^2+s^2) \\ \hat{\Delta}_R (c^2+s^2) & 0 \end{pmatrix}$$

$$= (\vec{\sigma}_\perp\cdot\vec{\Delta}_\perp)$$

- $D^{\dagger(1/2)}(p'_B,\Lambda_2)\,\sigma_z\,D^{(1/2)}(p_B,\Lambda_2) = \begin{pmatrix} c & -\hat{\Delta}_L s \\ \hat{\Delta}_R s & c \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} c & -\hat{\Delta}_L s \\ \hat{\Delta}_R s & c \end{pmatrix}$

$$= \begin{pmatrix} c & -\hat{\Delta}_L s \\ \hat{\Delta}_R s & c \end{pmatrix}\begin{pmatrix} c & -\hat{\Delta}_L s \\ -\hat{\Delta}_R s & -c \end{pmatrix} = \begin{pmatrix} c^2+s^2 & 0 \\ 0 & -(c^2+s^2) \end{pmatrix}$$

$$= \sigma_z$$

$\boxed{Ex\ 8}$ $\quad A^+=0 \Rightarrow \mathcal{W}=\mathbb{1}$

$$\int dx\, x^n \int \frac{dz^-}{2\pi} e^{ixP^+z^-}\,\bar{\psi}\left(-\tfrac{z^-}{2}\right)\Gamma\,\psi\left(\tfrac{z^-}{2}\right)$$

$$= \frac{1}{(P^+)^n}\int dx\,\frac{dz^-}{2\pi}\left[\left(-i\frac{d}{dz^-}\right)^n e^{ixP^+z^-}\right]\bar{\psi}\left(-\tfrac{z^-}{2}\right)\Gamma\,\psi\left(\tfrac{z^-}{2}\right)$$

$$= \frac{1}{(P^+)^n}\int dx\,\frac{dz^-}{2\pi}\, e^{ixP^+z^-}\left(i\frac{d}{dz^-}\right)^n \bar{\psi}\left(-\tfrac{z^-}{2}\right)\Gamma\,\psi\left(\tfrac{z^-}{2}\right)$$

$$= \frac{1}{(P^+)^n}\int \frac{dz^-}{2\pi}\, 2\pi\,\delta(P^+z^-)\;\bar{\psi}\left(-\tfrac{z^-}{2}\right)\Gamma\,\tfrac{i}{2}\overleftrightarrow{\partial}^+\psi\left(\tfrac{z^-}{2}\right)$$

$\partial^+ = \partial_- = \dfrac{\partial}{\partial x^-}$

$\overleftrightarrow{\partial}^+ = \overrightarrow{\partial}^+ - \overleftarrow{\partial}^+$

$$= \frac{1}{(P^+)^{n+1}}\;\bar{\psi}(0)\Gamma\,\tfrac{i}{2}\overleftrightarrow{\partial}^+\psi(0)$$

More generally

$$\frac{d}{dz^-}\mathcal{W}\left(-\tfrac{z^-}{2},\tfrac{z^-}{2}\right) = -\frac{i}{2}g\left[A^+\left(-\tfrac{z^-}{2}\right)\mathcal{W}\left(-\tfrac{z^-}{2},\tfrac{z^-}{2}\right) + \mathcal{W}\left(-\tfrac{z^-}{2},\tfrac{z^-}{2}\right)A^+\left(\tfrac{z^-}{2}\right)\right]$$

$$\overrightarrow{\partial}^+ \mapsto \overrightarrow{D}^+ = \overrightarrow{\partial}^+ - ig\,A^+\left(\tfrac{z^-}{2}\right)$$

$$\overleftarrow{\partial}^+ \mapsto \overleftarrow{D}^+ = \overleftarrow{\partial}^+ + ig\,A^+\left(-\tfrac{z^-}{2}\right)$$

Ex 9

$$\frac{1}{2}\int \frac{dz^-}{2\pi}\, e^{ixP^+z^-} \langle p',s'|\bar{\psi}(-\tfrac{z}{2})\gamma^+ w\, \psi(\tfrac{z}{2})|p,s\rangle$$

$$= \frac{1}{2P^+}\, \bar{u}(p',s')\left[\gamma^+ H_q(x,\xi,t) + \frac{i\sigma^{+\perp}\Delta_\perp}{2M_N} E_q(x,\xi,t)\right] u(p,s)$$

↳ $$\int dx\ \frac{1}{2}\int \frac{dz^-}{2\pi}\, e^{ixP^+z^-} \langle p',s'|\bar{\psi}(-\tfrac{z}{2})\gamma^+ w\, \psi(\tfrac{z}{2})|p,s\rangle$$

$$= \frac{1}{2P^+}\langle p',s'|\bar{\psi}(0)\gamma^+\psi(0)|p,s\rangle$$

$$= \frac{1}{2P^+}\, \bar{u}(p',s')\left[\gamma^+ F_1^q(t) + \frac{i\sigma^{+\perp}\Delta_\perp}{2M_N} F_2^q(t)\right] u(p,s)$$

↳ $\int dx\, H_q(x,\xi,t) = F_1^q(t)$ & $\int dx\, E_q(x,\xi,t) = F_2^q(t)$

Ex 10

$$\langle p'|\partial_\mu T^{\mu\nu}(x)|p\rangle = \partial_\mu \langle p'|T^{\mu\nu}(x)|p\rangle$$

$$= \partial_\mu \langle p'|e^{i\hat{P}\cdot x}\, T^{\mu\nu}(0)\, e^{-i\hat{P}\cdot x}|p\rangle$$

$$= \partial_\mu e^{i\Delta\cdot x}\langle p'|T^{\mu\nu}(0)|p\rangle$$

$$= i\Delta_\mu\, e^{i\Delta\cdot x}\langle p'|T^{\mu\nu}(0)|p\rangle$$

$$= i\Delta_\mu \langle p'|T^{\mu\nu}(x)|p\rangle$$

Ex 11

$$\langle p|P_a^\mu|p\rangle = \langle p|\int d^3x\, T_a^{0\mu}(x)|p\rangle = \int d^3x\ \langle p|T_a^{0\mu}(0)|p\rangle$$

$$\int d^3x = (2\pi)^3\delta^{(3)}(\vec{0}) = \frac{1}{2p^0}\, 2p^0(2\pi)^3\delta^{(3)}(\vec{p}-\vec{p}) = \frac{1}{2p^0}\langle p|p\rangle$$

Ex 12

$$T^{\mu\nu} = \bar{T}^{\mu\nu} + \hat{T}^{\mu\nu}$$

$$\bar{T}^{\mu\nu} = T^{\mu\nu} - \frac{1}{4} g^{\mu\nu} T^{\alpha}{}_{\alpha} \quad \Rightarrow \quad \bar{T}^{00} = T^{00} - \frac{1}{4} T^{\alpha}{}_{\alpha} = \frac{3}{4} T^{00} + \frac{1}{4} \sum_i T^{ii}$$

$$\hat{T}^{\mu\nu} = \frac{1}{4} g^{\mu\nu} T^{\alpha}{}_{\alpha} \quad \Rightarrow \quad \hat{T}^{00} = \frac{1}{4} T^{\alpha}{}_{\alpha} = \frac{1}{4} T^{00} + \frac{1}{4} \sum_i T^{ii}$$

$$\left\langle \int d^3x \, T^{ii} \right\rangle = 0 \quad \Rightarrow \quad \langle \bar{T}^{00} \rangle = 3 \langle \hat{T}^{00} \rangle$$

Ex 13 ($x^0 = 0$)

$$\langle P + \tfrac{\Delta}{2} | \int d^3x \, x^i O(x) | P - \tfrac{\Delta}{2} \rangle = \int d^3x \, x^i e^{-i \vec{\Delta} \cdot \vec{x}} \langle P + \tfrac{\Delta}{2} | O(0) | P - \tfrac{\Delta}{2} \rangle$$

$$= \left[i \nabla_\Delta^i (2\pi)^3 \delta^{(3)}(\vec{\Delta}) \right] \langle P + \tfrac{\Delta}{2} | O(0) | P - \tfrac{\Delta}{2} \rangle$$

$$= i \nabla_\Delta^i \left[(2\pi)^3 \delta^{(3)}(\vec{\Delta}) \langle P + \tfrac{\Delta}{2} | O(0) | P - \tfrac{\Delta}{2} \rangle \right]$$

$$+ (2\pi)^3 \delta^{(3)}(\vec{\Delta}) \left[-i \nabla_\Delta^i \langle P + \tfrac{\Delta}{2} | O(0) | P - \tfrac{\Delta}{2} \rangle \right]$$

$$= \langle P | O(0) | P \rangle (2\pi)^3 \, i \nabla_\Delta^i \delta^{(3)}(\vec{\Delta})$$

$$+ (2\pi)^3 \delta^{(3)}(\vec{\Delta}) \left[-i \nabla_\Delta^i \langle P + \tfrac{\Delta}{2} | O(0) | P - \tfrac{\Delta}{2} \rangle \right]$$

$$\Rightarrow \lim_{\Delta \to 0} \frac{\langle P + \tfrac{\Delta}{2} | \int d^3x \, x^i O(x) | P - \tfrac{\Delta}{2} \rangle}{\langle P | P \rangle} = \frac{\langle P | O(0) | P \rangle}{\langle P | P \rangle} (2\pi)^3 \, i \nabla^i \delta^{(3)}(\vec{0})$$

$$+ \frac{\cancel{(2\pi)^3 \delta^{(3)}(\vec{0})}}{\cancel{(2\pi)^3} 2p^0 \cancel{\delta^{(3)}(0)}} \left[-i \nabla_\Delta^i \langle P + \tfrac{\Delta}{2} | O(0) | P - \tfrac{\Delta}{2} \rangle \right]_{\Delta = 0}$$

Identity: $\delta'(x) f(x) = f(0) \delta'(x) - f'(0) \delta(x)$

Proof: $\int dx \, \delta'(x) f(x) g(x) = - \int dx \, \delta(x) \left[f'(x) g(x) + f(x) g'(x) \right]$

$$= - f'(0) g(0) - f(0) g'(0)$$

$$= - f'(0) \int dx \, \delta(x) g(x) + f(0) \int dx \, \delta'(x) g(x)$$

$\boxed{\text{Ex 14}}$ $\quad x^0 = 0$

$$\left\langle \int d^3x \, x^i \, O(x) \right\rangle_{\vec{R},\vec{P}} = \int d^3x \int \frac{d^3\Delta}{(2\pi)^3} e^{-i\vec{\Delta}\cdot(\vec{x}-\vec{R})} \, x^i \, \frac{\langle P+\frac{\Delta}{2}|O(0)|P-\frac{\Delta}{2}\rangle}{\sqrt{4(P^0)^2-(\Delta^0)^2}}$$

$\vec{r} = \vec{x} - \vec{R}$

$$= R^i \int d^3r \int \frac{d^3\Delta}{(2\pi)^3} e^{-i\vec{\Delta}\cdot\vec{r}} \frac{\langle P+\frac{\Delta}{2}|O(0)|P-\frac{\Delta}{2}\rangle}{\sqrt{4(P^0)^2-(\Delta^0)^2}}$$

$$+ \int d^3r \int \frac{d^3\Delta}{(2\pi)^3} \left(i\nabla_\Delta^i e^{-i\vec{\Delta}\cdot\vec{r}} \right) \frac{\langle P+\frac{\Delta}{2}|O(0)|P-\frac{\Delta}{2}\rangle}{\sqrt{4(P^0)^2-(\Delta^0)^2}}$$

NB: $\Delta^0 = \dfrac{\vec{P}\cdot\vec{\Delta}}{P^0}$

$$= R^i \frac{\langle p|O(0)|p\rangle}{2p^0} - \frac{1}{2p^0}\left[-i\nabla_\Delta^i \langle P+\tfrac{\Delta}{2}|O(0)|P-\tfrac{\Delta}{2}\rangle \right]_{\Delta=0}$$

$\boxed{\text{Ex 15}}$ $\quad \vec{P} = P_z \vec{e}_z$, longitudinal pol:

$$\langle L_z^a \rangle_{\vec{0},\vec{P}} = \left\langle \int d^3r \, \varepsilon^{3ij} r^i T_a^{0j}(r) \right\rangle_{\vec{R},\vec{P}} = \frac{1}{2p^0} \varepsilon^{3ij} \left[-i\nabla_\Delta^i \langle P+\tfrac{\Delta}{2}, s'| T_a^{0j}(0) |P-\tfrac{\Delta}{2}, s\rangle \right]_{\Delta=0}$$

- $$\langle P+\tfrac{\Delta}{2}, s'|T_a^{0j}(0)|P-\tfrac{\Delta}{2}, s\rangle = \bar{u}(p,s')\Big\{ \frac{P^0P^j}{M} A_a(t) + \frac{\Delta^0\Delta^j - g^{0j}\Delta^2}{M} C_a(t) + M g^{0j} \bar{C}_a(t)$$
$$+ \frac{P^0 \, i\sigma^{j\lambda}\Delta_\lambda}{2M}\left[J_a(t) - S_a(t)\right] + \frac{P^j \, i\sigma^{0\lambda}\Delta_\lambda}{2M}\left[J_a(t)+S_a(t)\right]\Big\} u(p,s)$$
$$= \frac{P^0}{2M} \bar{u}(p,s') \, i\sigma^{j\lambda}\Delta_\lambda \, u(p,s) \left[J_a(0) - S_a(0)\right] + O(\Delta^2)$$

$\Delta^0 = \dfrac{P_z\Delta_z}{P^0}$ $\Rightarrow$ $\left[-i\nabla_\Delta^i \langle P+\tfrac{\Delta}{2}, s'|T_a^{0j}(0)|P-\tfrac{\Delta}{2}, s\rangle \right]_{\Delta=0} = \dfrac{P^0}{2M} \bar{u}(p,s)\, \sigma^{ij} u(p,s) \left[J_a(0)-S_a(0)\right]$

- $\frac{1}{2}\varepsilon^{3ij}\sigma^{ij} = \Sigma^3 = \begin{pmatrix} \sigma_z & 0 \\ 0 & \sigma_z \end{pmatrix}$ $\qquad u(p,s) = \sqrt{p^0+M} \begin{pmatrix} \chi_s \\ \frac{\vec{p}\cdot\vec{\sigma}}{p^0+M}\chi_s \end{pmatrix}$ $\quad \chi_\uparrow = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \chi_\downarrow = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$\Rightarrow \bar{u}(p,s')\Sigma^3 u(p,s) = (p^0+M)\left(\chi_{s'}^\dagger, -\chi_{s'}^\dagger \frac{p_z\sigma_z}{p^0+M} \right) \begin{pmatrix} \sigma_z & 0 \\ 0 & \sigma_z \end{pmatrix} \begin{pmatrix} \chi_s \\ \frac{p_z\sigma_z}{p^0+M}\chi_s \end{pmatrix}$$

$$= (p^0+M)\left(\chi_{s'}^\dagger \sigma_z \chi_s\right)\left(1 - \frac{p_z^2}{(p^0+M)^2}\right) \qquad p_z^2 = (p^0)^2 - M^2$$

$$= (\sigma_z)_{s's}\left[p^0+M-(p^0-M)\right]$$

$$= 2M(\sigma_z)_{s's}$$

$$\Rightarrow \langle L_z^a \rangle_{\vec{0},\vec{P}} = \frac{1}{2p^0}\frac{P^0}{M} 2M (\sigma_z)_{s's}\left[J_a(0)-S_a(0)\right] = (\sigma_z)_{s's}\left[J_a(0)-S_a(0)\right]$$