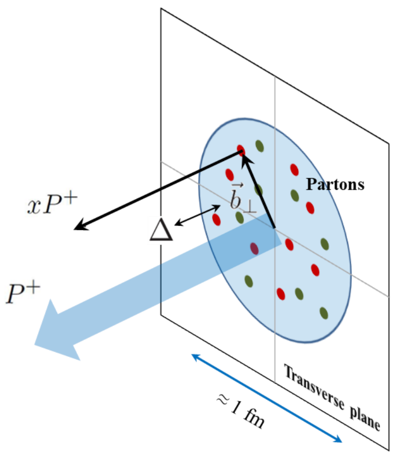


Hadron structure with GPDs

Cédric Lorcé

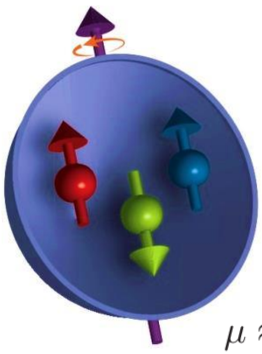
CPHT Centre de Physique Théorique
l'X École Polytechnique
IP Paris
cnrs

Nucleon structure

Non-relativistic picture

dominated by constituents

Spectroscopy



$\mu \approx \Lambda_{\text{QCD}}$

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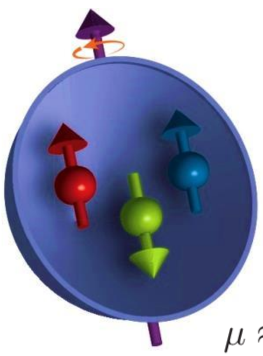

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Mass

$$M_N c^2 \sim \sum_Q M_Q c^2 + E_{\text{binding}}$$

~ 102 %   ~ - 2 %

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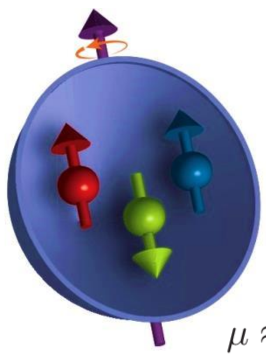

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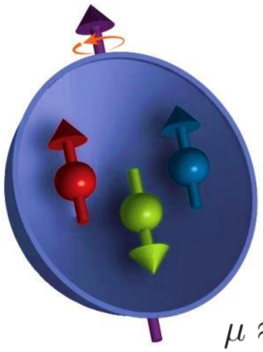

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$\mu \approx 2\,{\rm GeV}$

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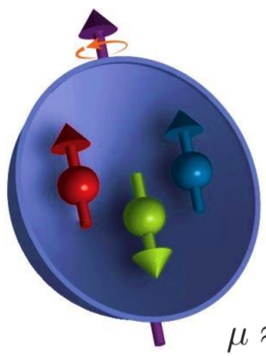

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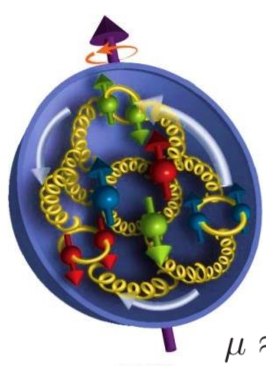


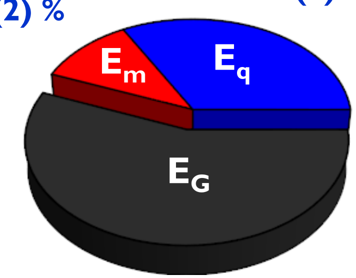
$\mu \approx 2\,{\rm GeV}$

Quark mass & QCD condensate

~ 11(2) %

Quark kinetic+potential energies

~ 33(2) %



?

Gluon kinetic+potential energies

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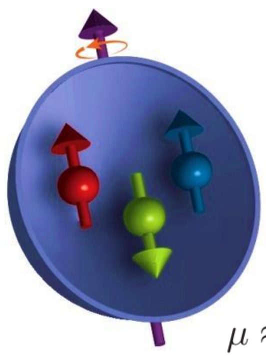

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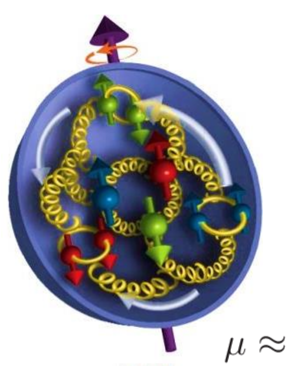

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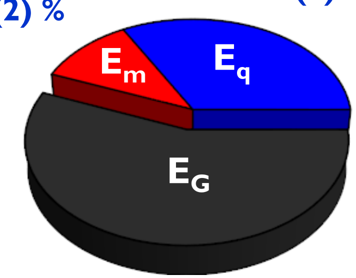


?

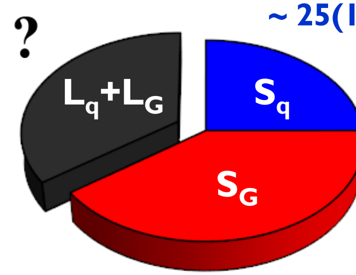
Gluon kinetic+potential energies

Orbital angular momentum

?

Quark spin

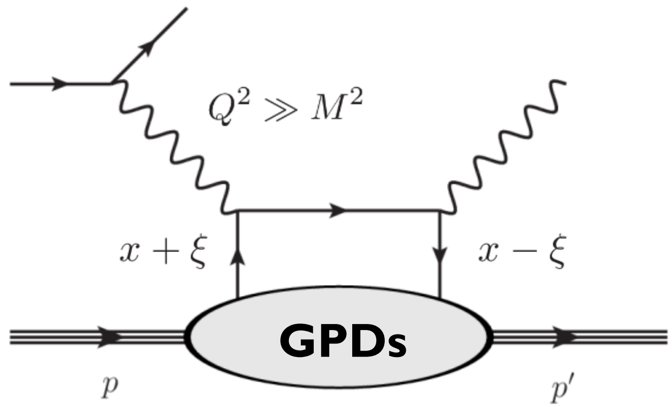
~ 25(10) %



~ 40(?) %

Gluon spin

Objective of these lectures



Deeply virtual Compton scattering (DVCS)

$$\Downarrow$$

Off-forward matrix element

$$\int \frac{\mathrm{d}z^-}{2\pi}\, e^{ixP^+z^-} \langle p', s'|\overline{\psi}(-\tfrac{z}{2})\Gamma\mathcal{W}(-\tfrac{z}{2},\tfrac{z}{2})\psi(\tfrac{z}{2})|p, s\rangle\big|_{z^+=|\vec{z}_\perp|=0}$$

Non-local operator

What is the physical meaning/content of such a correlator ?

Outline

Mo	• Spatial imaging
Tu-We	• Light-front & phase-space pictures
We	• Energy-momentum tensor
Th-Fr	• Mass & spin decompositions
Fr	• Mechanical properties

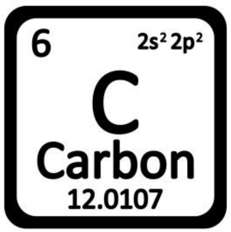
Do not hesitate to ask questions !

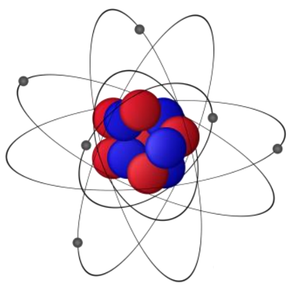
Spatial imaging

Spatial structure

3D structure is fundamental to understand physical properties

i.e. thermal, electrical, mechanical, …





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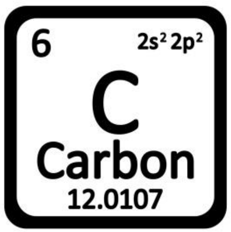

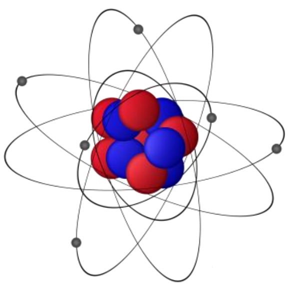

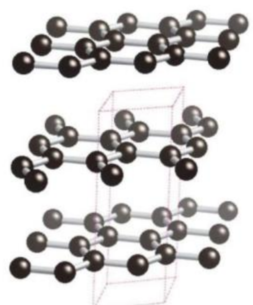

graphite

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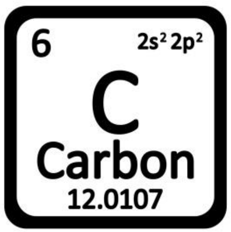

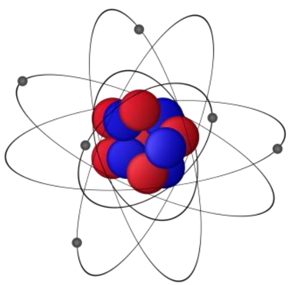

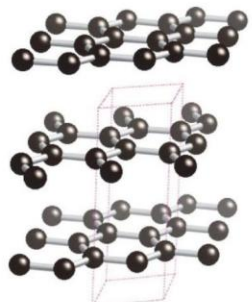

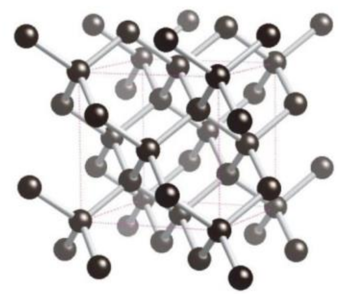

graphite

diamond

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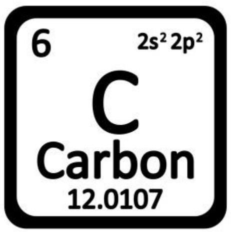
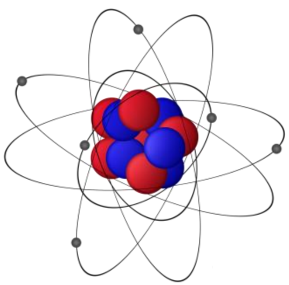
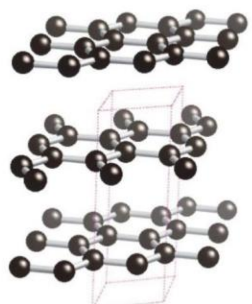
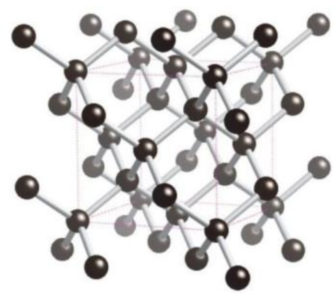


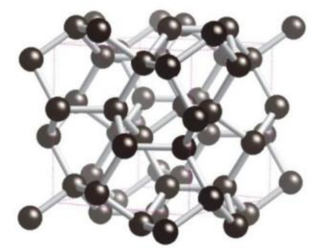




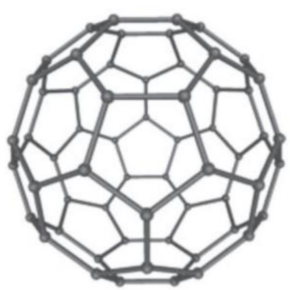
graphite



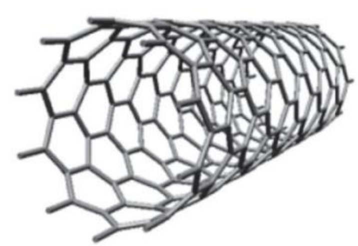
diamond



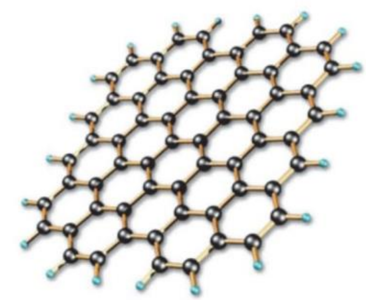
BC8



fullerene



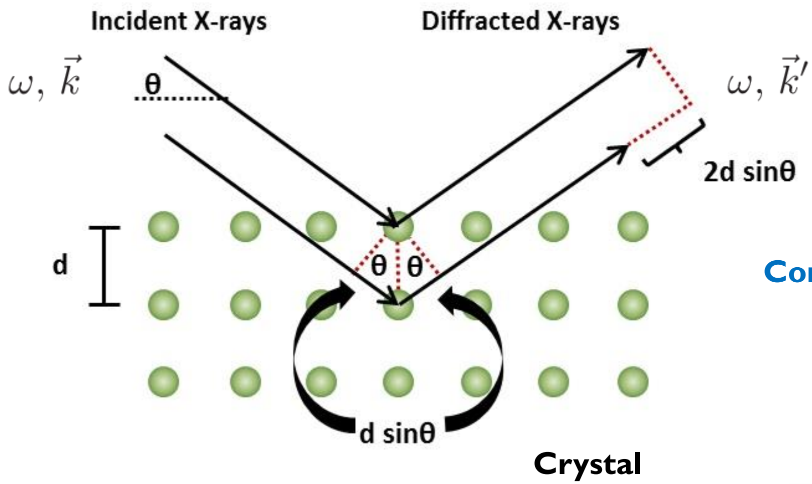
nanotube



graphene

Spatial structure through elastic scattering

<u>Example</u>: X-ray diffraction



Constructive interference
(Bragg's law)

$$2d\,\sin\theta = n\lambda$$

$$\boxed{\lambda \sim d}$$

Spatial structure through elastic scattering

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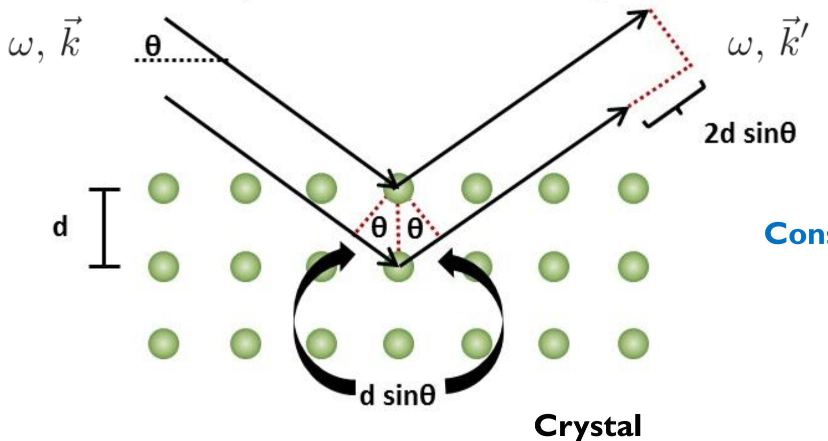

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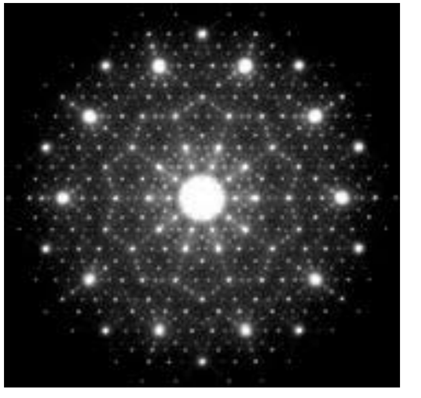
Diffraction pattern



$\propto |A_{\rm scatt}|^2$

Spatial structure through elastic scattering

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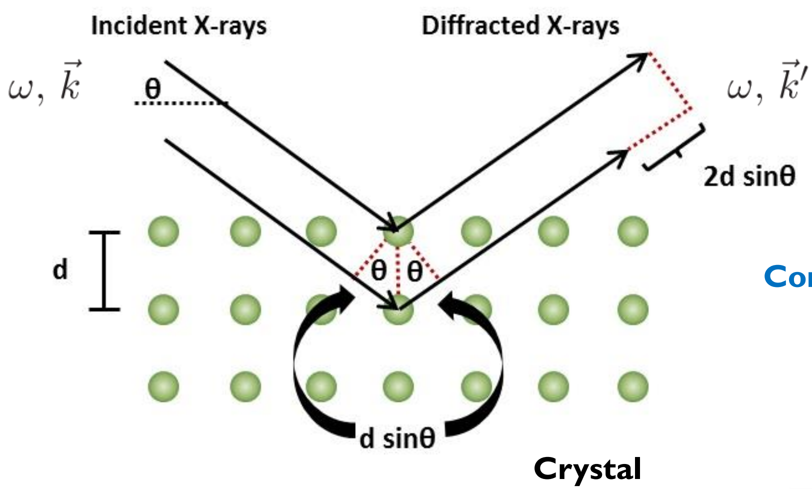

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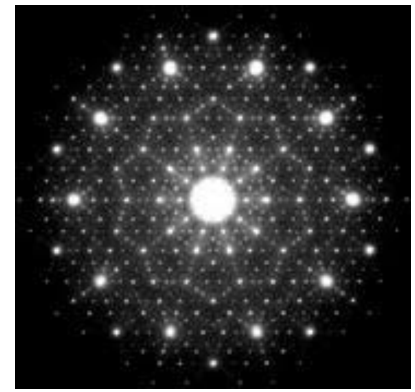

$\propto |A_{\rm scatt}|^2$

Scattered amplitude

$$A_{\rm scatt} \propto F(\vec{q}) = \int \mathrm{d}^3 r\, e^{i\vec{q}\cdot\vec{r}}\, \rho(\vec{r}) \qquad \vec{q} = \vec{k} - \vec{k}'$$

Form factor **Scatterer distribution**

Nuclear elastic scattering

Crystals & atoms $d \approx 10^{-10}\,\mathrm{m} \quad \Rightarrow \quad \hbar\omega \approx 10^4\,\mathrm{eV}$ ➡ **X-rays**

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⚠ **Large recoil for light nuclei!**

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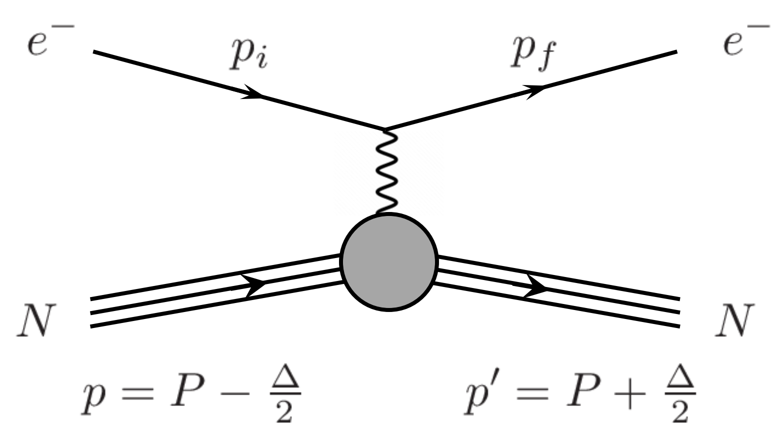
Relativistic treatment
in Born approximation

Spin-0 target

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} \Big/ \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\Big|_{\text{pointlike}} = [F(Q^2)]^2$$



$Q^2 = -\Delta^2$

[Rosenbluth, PR79 (1950) 615]
[Hofstadter, RMP28 (1956) 214]
[Yennie, Lévy, Ravenhall, RMP29 (1957) 144]

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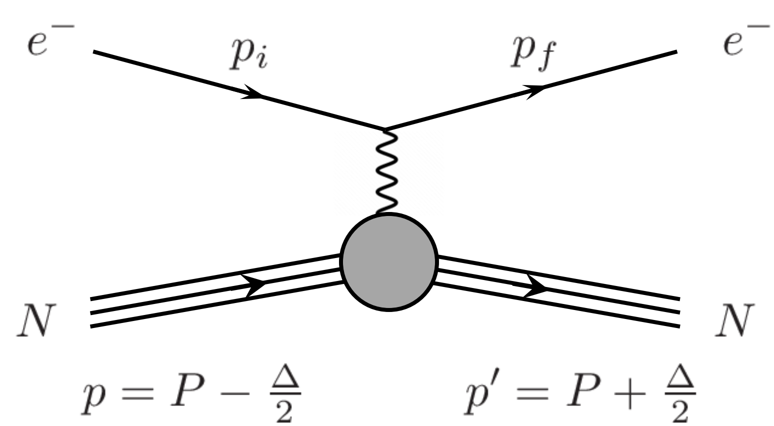

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Spin-1/2 target

$$= \left\{ \left[G_E(Q^2)\right]^2 + \frac{\tau}{\epsilon}\left[G_M(Q^2)\right]^2 \right\} \frac{1}{1+\tau}$$

Electric **form factor** — **Magnetic** **form factor**

$Q^2 = -\Delta^2$

$\tau = Q^2/4M_N^2$

$\epsilon = (1 + 2(1+\tau)\tan^2 \frac{\theta_e}{2})^{-1}$

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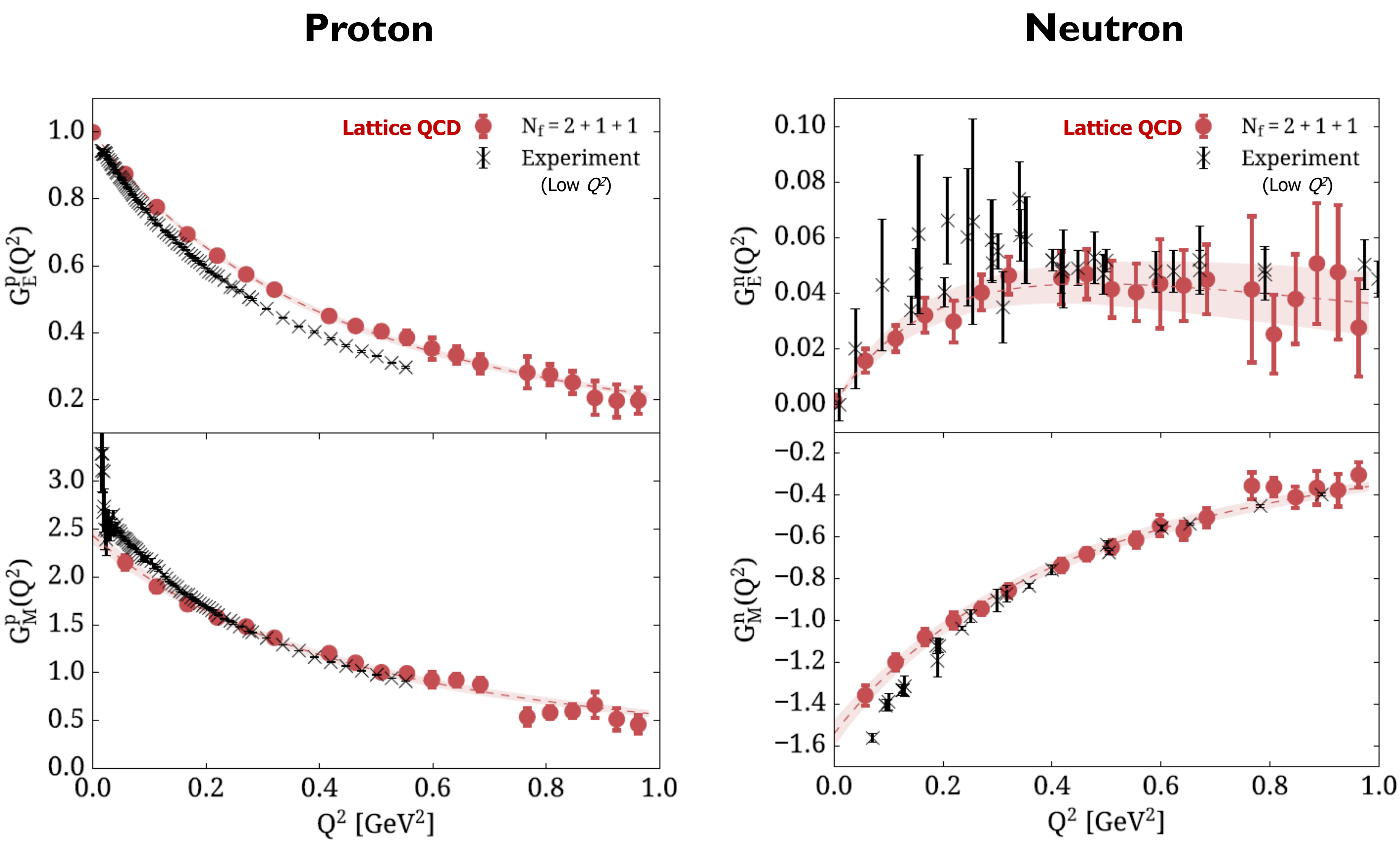
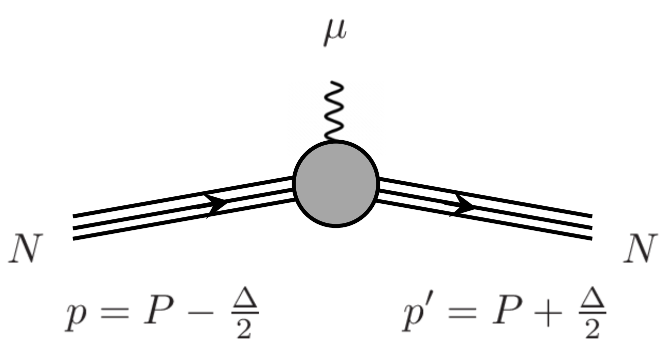
Nucleon form factors



[Alexandrou *et al.*, PRD100 (2019) 014509]

Electromagnetic current matrix elements

Normalization $\langle p'|p\rangle = (2\pi)^3 2p^0\,\delta^{(3)}(\vec{p}\,' - \vec{p})$



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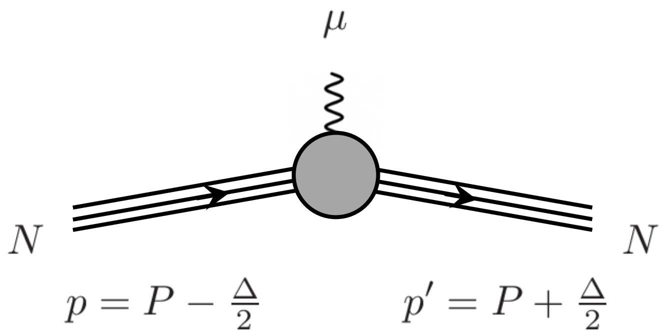

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Dirac form factor **Pauli form factor**

$$F_1(0) = q_N, \qquad F_2(0) = \kappa_N$$

Electric charge **Anomalous magnetic moment**

[Foldy, PR87 (1952) 688]
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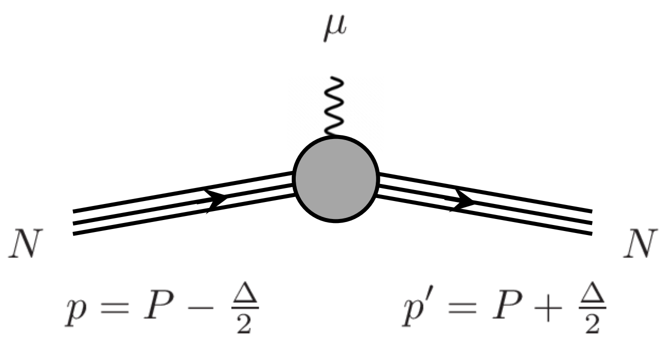

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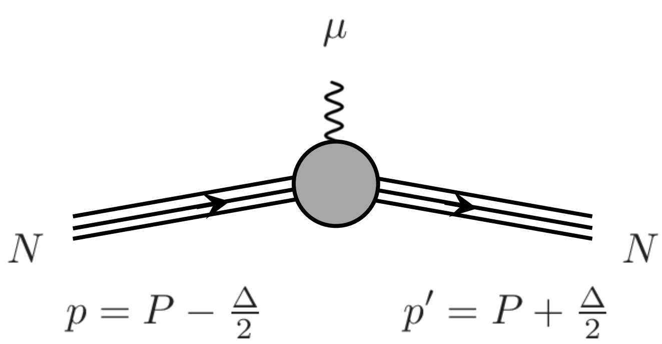

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Mass shell constraints

$$(P \pm \tfrac{\Delta}{2})^2 = M_N^2 \quad \Rightarrow \quad \boxed{\begin{aligned} P\cdot\Delta &= 0 \\ P^2 &= M_N^2(1+\tau)\end{aligned}}$$

[Foldy, PR87 (1952) 688]
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Non-relativistic interpretation

Localized states $$|\vec{r}\rangle = \int \frac{\mathrm{d}^3 p}{(2\pi)^3}\, e^{-i\vec{p}\cdot\vec{r}}\, |\vec{p}\rangle$$

Normalization $\langle \vec{p}\,'|\vec{p}\rangle = (2\pi)^3\, \delta^{(3)}(\vec{p}\,' - \vec{p})$

$$\langle \vec{r}\,'|\rho(\vec{x})|\vec{r}\rangle =$$

Charge density

Non-relativistic interpretation

Localized states $|\vec{r}\rangle = \int \frac{\mathrm{d}^3 p}{(2\pi)^3}\, e^{-i\vec{p}\cdot\vec{r}}\,|\vec{p}\rangle$

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Localized states $$|\vec{r}\rangle = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\, e^{-i\vec{p}\cdot\vec{r}}\, |\vec{p}\rangle$$

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$$\langle \vec{r}\,'|\rho(\vec{x})|\vec{r}\rangle = \int \frac{\mathrm{d}^3P}{(2\pi)^3}\,\frac{\mathrm{d}^3\Delta}{(2\pi)^3}\, e^{i\vec{P}\cdot(\vec{r}\,'-\vec{r})}\, e^{-i\vec{\Delta}\cdot(\vec{x}-\frac{\vec{r}\,'+\vec{r}}{2})}\, \langle \vec{P} + \tfrac{\vec{\Delta}}{2}|\rho(\vec{0})|\vec{P} - \tfrac{\vec{\Delta}}{2}\rangle$$

Charge density

Galilean symmetry $$\langle \vec{P} + \tfrac{\vec{\Delta}}{2}|\rho(\vec{0})|\vec{P} - \tfrac{\vec{\Delta}}{2}\rangle = \langle \tfrac{\vec{\Delta}}{2}|\rho(\vec{0})| - \tfrac{\vec{\Delta}}{2}\rangle$$

Non-relativistic boost

$\vec{p} \mapsto \vec{p} + M_N\vec{v}$

Non-relativistic interpretation

Localized states $$|\vec{r}\rangle = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\, e^{-i\vec{p}\cdot\vec{r}}\, |\vec{p}\rangle$$

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Charge density

Galilean symmetry $$\langle\vec{P}+\tfrac{\vec{\Delta}}{2}|\rho(\vec{0})|\vec{P}-\tfrac{\vec{\Delta}}{2}\rangle = \langle\tfrac{\vec{\Delta}}{2}|\rho(\vec{0})|-\tfrac{\vec{\Delta}}{2}\rangle$$

Non-relativistic boost
$\vec{p}\mapsto\vec{p}+M_N\vec{v}$

$$\langle\vec{r}\,'|\rho(\vec{x})|\vec{r}\rangle = \delta^{(3)}(\vec{r}\,'-\vec{r})\int \frac{\mathrm{d}^3\Delta}{(2\pi)^3}\, e^{-i\vec{\Delta}\cdot(\vec{x}-\vec{r})}\,\langle\tfrac{\vec{\Delta}}{2}|\rho(\vec{0})|-\tfrac{\vec{\Delta}}{2}\rangle$$

Non-relativistic interpretation

Localized states $$|\vec{r}\rangle = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\, e^{-i\vec{p}\cdot\vec{r}}\, |\vec{p}\rangle$$

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$$\underbrace{\langle\vec{r}\,'|\rho(\vec{x})|\vec{r}\rangle}_{\text{Charge density}} = \int \frac{\mathrm{d}^3P}{(2\pi)^3}\,\frac{\mathrm{d}^3\Delta}{(2\pi)^3}\, e^{i\vec{P}\cdot(\vec{r}\,'-\vec{r})}\, e^{-i\vec{\Delta}\cdot(\vec{x}-\frac{\vec{r}\,'+\vec{r}}{2})}\,\langle\vec{P}+\tfrac{\vec{\Delta}}{2}|\rho(\vec{0})|\vec{P}-\tfrac{\vec{\Delta}}{2}\rangle$$

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Non-relativistic boost
$\vec{p}\mapsto\vec{p}+M_N\vec{v}$

$$\langle\vec{r}\,'|\rho(\vec{x})|\vec{r}\rangle = \delta^{(3)}(\vec{r}\,'-\vec{r})\,\underbrace{\int\frac{\mathrm{d}^3\Delta}{(2\pi)^3}\, e^{-i\vec{\Delta}\cdot(\vec{x}-\vec{r})}\,\langle\tfrac{\vec{\Delta}}{2}|\rho(\vec{0})|-\tfrac{\vec{\Delta}}{2}\rangle}_{\equiv\,\rho(\vec{x}-\vec{r})}$$

Internal distribution

Non-relativistic interpretation

Localized states $|\vec{r}\rangle = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\, e^{-i\vec{p}\cdot\vec{r}}\, |\vec{p}\rangle$

Normalization $\langle\vec{p}\,'|\vec{p}\rangle = (2\pi)^3\,\delta^{(3)}(\vec{p}\,'-\vec{p})$

$$\langle\vec{r}\,'|\rho(\vec{x})|\vec{r}\rangle = \int \frac{\mathrm{d}^3P}{(2\pi)^3}\,\frac{\mathrm{d}^3\Delta}{(2\pi)^3}\, e^{i\vec{P}\cdot(\vec{r}\,'-\vec{r})}\, e^{-i\vec{\Delta}\cdot(\vec{x}-\frac{\vec{r}\,'+\vec{r}}{2})}\,\langle\vec{P}+\tfrac{\vec{\Delta}}{2}|\rho(\vec{0})|\vec{P}-\tfrac{\vec{\Delta}}{2}\rangle$$

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Galilean symmetry $\langle\vec{P}+\frac{\vec{\Delta}}{2}|\rho(\vec{0})|\vec{P}-\frac{\vec{\Delta}}{2}\rangle = \langle\frac{\vec{\Delta}}{2}|\rho(\vec{0})|-\frac{\vec{\Delta}}{2}\rangle$

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Internal distribution

Generic expectation value $\langle\psi|\psi\rangle = 1$

Wave packet $\psi(\vec{r}) = \langle\vec{r}|\psi\rangle$

$$\Rightarrow\quad \langle\rho\rangle_\psi(\vec{x}) = \langle\psi|\rho(\vec{x})|\psi\rangle = \int \mathrm{d}^3r\,|\psi(\vec{r})|^2\,\rho(\vec{x}-\vec{r})$$

Probabilistic interpretation

Semi-relativistic interpretation (Sachs approach)

Generic expectation value $\langle\psi|\psi\rangle = 1$

Normalization $\langle\vec{p}\,'|\vec{p}\rangle = (2\pi)^3\,\delta^{(3)}(\vec{p}\,' - \vec{p})$

Wave packet $\tilde{\psi}(\vec{p}) = \langle\vec{p}|\psi\rangle$

$$\langle\psi|O(x)|\psi\rangle = \int \frac{\mathrm{d}^3P}{(2\pi)^3}\,\frac{\mathrm{d}^3\Delta}{(2\pi)^3}\,\tilde{\psi}^*(\vec{P}+\tfrac{\vec{\Delta}}{2})\tilde{\psi}(\vec{P}-\tfrac{\vec{\Delta}}{2})\,\langle\vec{P}+\tfrac{\vec{\Delta}}{2}|O(x)|\vec{P}-\tfrac{\vec{\Delta}}{2}\rangle$$

[Sachs, PR126 (1962) 2256]
[Burkardt, PRD62 (2000) 071503]
[Belitsky, Ji, Yuan, PRD69 (2004) 074014]

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Crucial assumption $\tilde{\psi}(\vec{P}\pm\tfrac{\vec{\Delta}}{2}) \approx \tilde{\psi}(\vec{P})$

$$\langle\psi|O(x)|\psi\rangle \overset{x^0=0}{\approx} \int \frac{\mathrm{d}^3P}{(2\pi)^3}\,|\tilde{\psi}(\vec{P})|^2\,\underbrace{\int \frac{\mathrm{d}^3\Delta}{(2\pi)^3}\,e^{-i\vec{\Delta}\cdot\vec{x}}\,\langle\vec{P}+\tfrac{\vec{\Delta}}{2}|O(0)|\vec{P}-\tfrac{\vec{\Delta}}{2}\rangle}_{\text{Internal distribution}}$$

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Probabilistic interpretation

Validity domain $1/D \ll |\vec{\Delta}| \ll |\delta\vec{p}| \ll P^0$

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$$\langle\psi|O(x)|\psi\rangle \overset{x^0=0}{\approx} \int\frac{\mathrm{d}^3P}{(2\pi)^3}\,|\tilde{\psi}(\vec{P})|^2\,\underbrace{\int\frac{\mathrm{d}^3\Delta}{(2\pi)^3}\,e^{-i\vec{\Delta}\cdot\vec{x}}\,\langle\vec{P}+\tfrac{\vec{\Delta}}{2}|O(0)|\vec{P}-\tfrac{\vec{\Delta}}{2}\rangle}_{\text{Internal distribution}}$$

Probabilistic interpretation

Validity domain $1/D \ll |\vec{\Delta}| \ll |\delta\vec{p}| \ll P^0$

Breit frame $|\vec{P}| = 0 \quad \Rightarrow \quad P^0 \approx M_N$

$|\psi(\vec{P})|^2 \to (2\pi)^3\delta^{(3)}(\vec{P})$

Clash with $\tilde{\psi}(\vec{P}\pm\tfrac{\vec{\Delta}}{2}) \approx \tilde{\psi}(\vec{P})$ **!**

[Sachs, PR126 (1962) 2256]
[Burkardt, PRD62 (2000) 071503]
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Probabilistic interpretation

Validity domain $1/D \ll |\vec{\Delta}| \ll |\delta\vec{p}| \ll P^0$

Hydrogen $M_H D_H \approx 10^5$ ✓

Nucleon $M_N D_N \approx 4$ ⚠

Breit frame $|\vec{P}| = 0 \quad\Rightarrow\quad P^0\approx M_N$

$|\psi(\vec{P})|^2\to(2\pi)^3\delta^{(3)}(\vec{P})$ ⚠ **Clash with** $\tilde{\psi}(\vec{P}\pm\frac{\vec{\Delta}}{2})\approx\tilde{\psi}(\vec{P})$ **!**

[Sachs, PR126 (1962) 2256]
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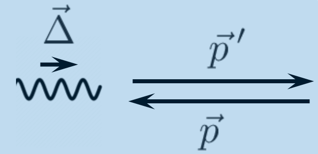
[Belitsky, Ji, Yuan, PRD69 (2004) 074014]

Semi-relativistic interpretation (Sachs approach)

Breit (aka brick-wall) **frame**



$$\vec{P} = \vec{0} \quad \Rightarrow \quad \Delta^0 = \frac{\vec{P} \cdot \vec{\Delta}}{P^0} = 0$$

[Sachs, PR126 (1962) 2256]
[Friar, Negele, In *Adv. Nucl. Phys., Vol.8* (1975) 219]

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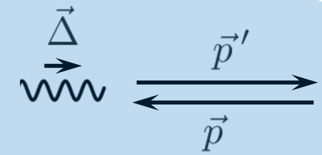

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$$\langle p', s'|J^0(0)|p, s\rangle\big|_{\mathrm{BF}} = 2M_N \delta_{s's}\, G_E(Q^2)$$

$$\langle p', s'|\vec{J}(0)|p, s\rangle\big|_{\mathrm{BF}} = i(\vec{\sigma}_{s's} \times \vec{\Delta})\, G_M(Q^2)$$

Same structure as in non-relativistic case !

$$Q^2\big|_{\mathrm{BF}} = \vec{\Delta}^2$$

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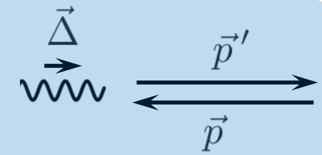

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3D charge distribution

$$\boxed{\rho_E^{\mathrm{BF}}(\vec{r}) = \int \frac{\mathrm{d}^3\Delta}{(2\pi)^3}\, e^{-i\vec{\Delta}\cdot\vec{r}}\, \frac{G_E(Q^2)}{\sqrt{1+\tau}}}$$

Relativistic recoil corrections ?

[Sachs, PR126 (1962) 2256]
[Friar, Negele, In *Adv. Nucl. Phys., Vol.8* (1975) 219]

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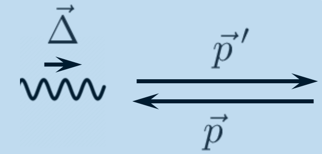

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Relativistic recoil corrections ?

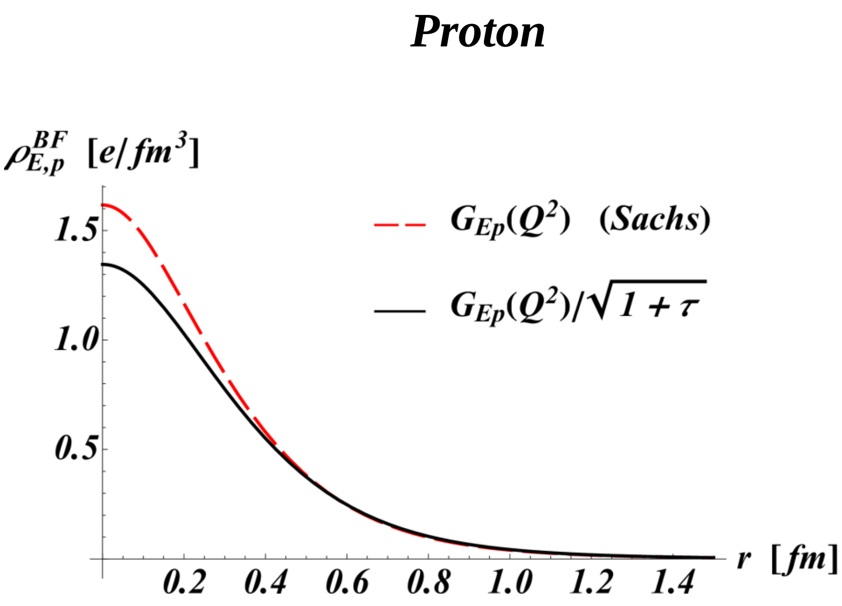
$$P^0\big|_{\mathrm{BF}} = M_N\sqrt{1+\tau}$$

responsible for the Darwin term in the non-relativistic expansion

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\Big/\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\Big|_{\mathrm{pointlike}} = \left\{\left[G_E(Q^2)\right]^2 + \frac{\tau}{\epsilon}\left[G_M(Q^2)\right]^2\right\}\frac{1}{1+\tau}$$

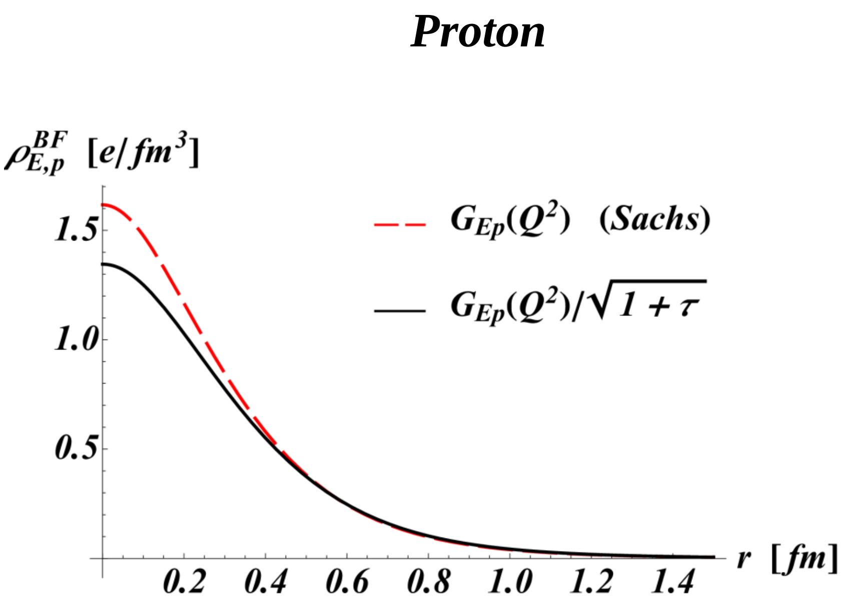
[Sachs, PR126 (1962) 2256]
[Friar, Negele, In *Adv. Nucl. Phys., Vol.8* (1975) 219]

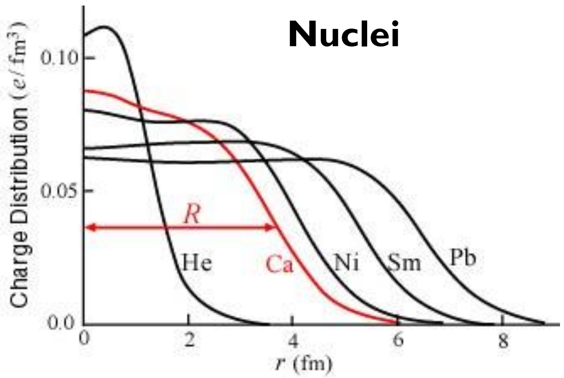
Breit frame distributions



[C.L., PRL125 (2020) 232002]

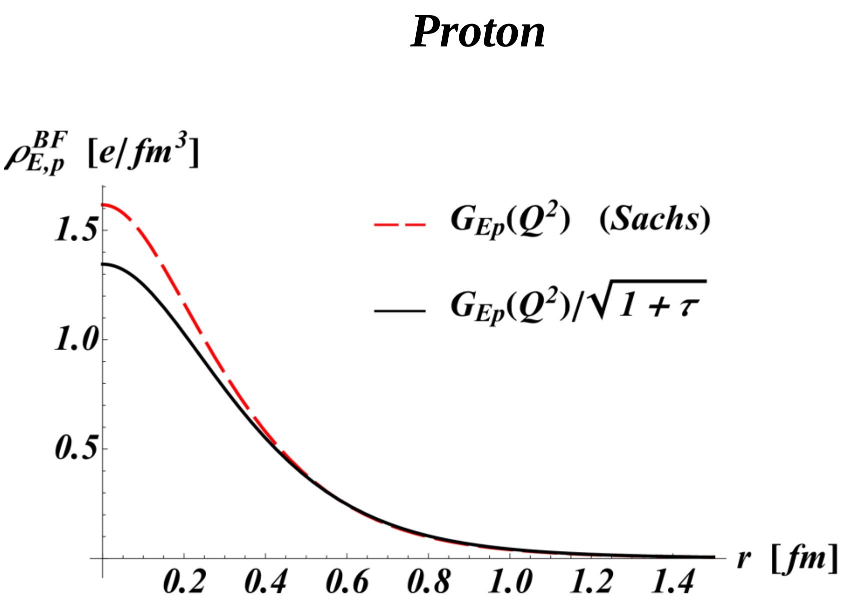
Breit frame distributions

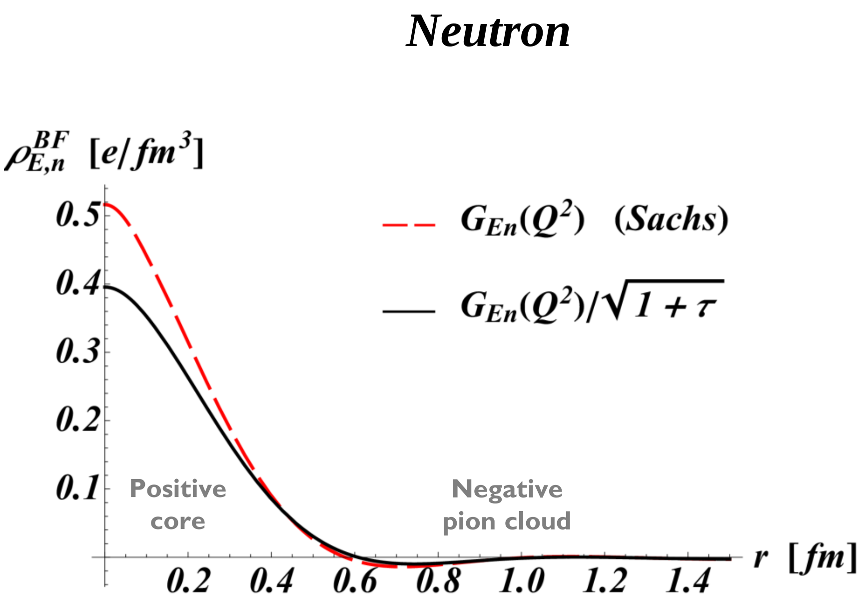


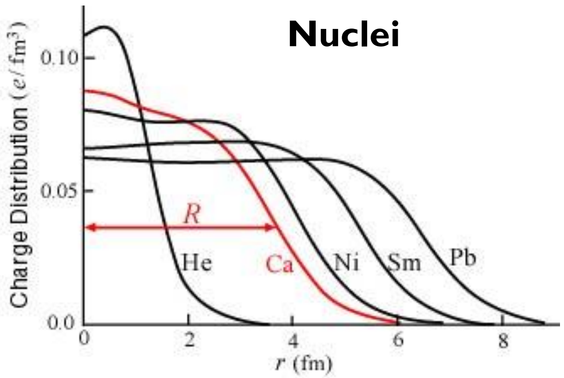


[C.L., PRL125 (2020) 232002]

Breit frame distributions



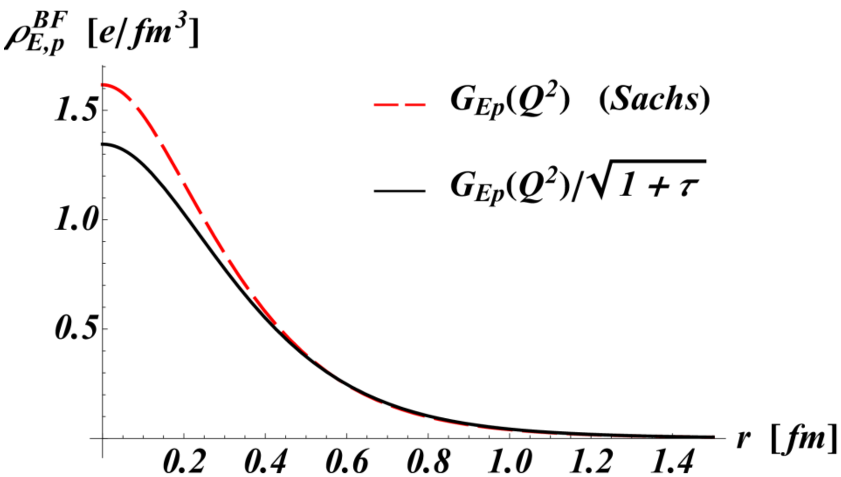




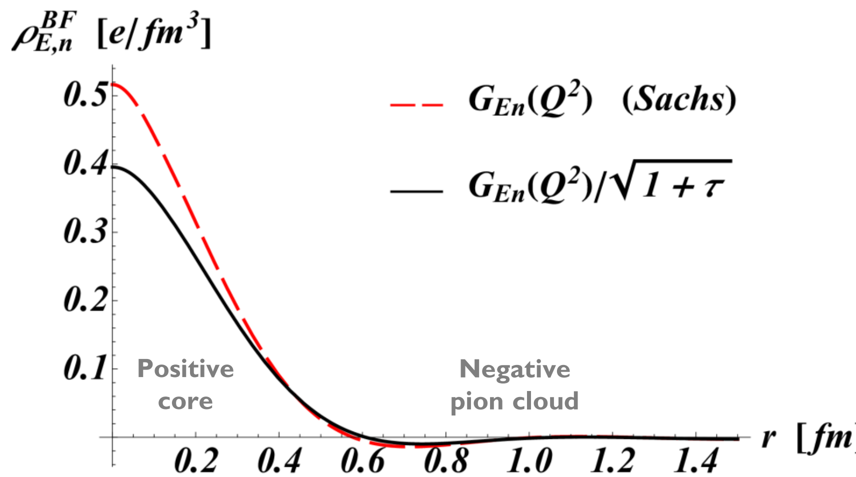
[C.L., PRL125 (2020) 232002]

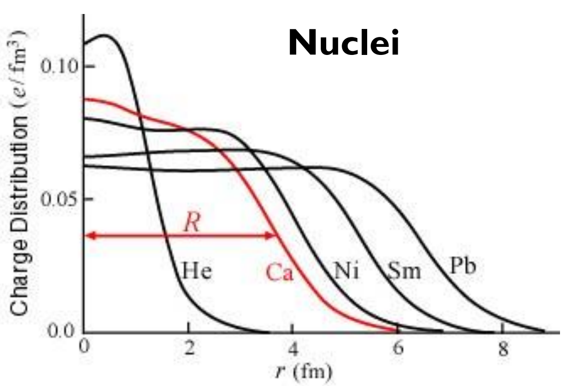
Breit frame distributions

Proton

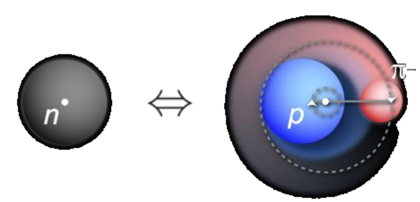


Neutron





Proton-pion fluctuation



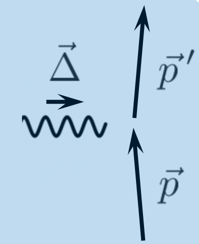
[C.L., PRL125 (2020) 232002]

Relativistic interpretation (IMF approach)

Probabilistic interpretation

Validity domain $1/D \ll |\vec{\Delta}| \ll |\delta\vec{p}| \ll P^0$

Infinite-momentum frame



$$P_z \to \infty \quad \Rightarrow \quad \Delta^0 \approx \Delta_z \ll P^0$$

[Bouchiat, Fayet, Meyer, NPB34 (1971) 157]
[Soper, PRD15 (1977) 1141]
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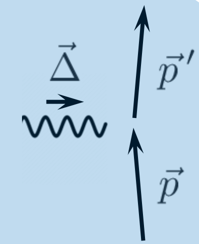

$$P_z \to \infty \quad \Rightarrow \quad \Delta^0 \approx \Delta_z \ll P^0$$

$$\langle p', \lambda' | J^0(0) | p, \lambda \rangle\big|_{\mathrm{IMF}} = 2P^0 \left[\delta_{\lambda'\lambda} \, F_1(Q^2) + \frac{i(\vec{\sigma}_{\lambda'\lambda} \times \vec{\Delta})_z}{2M_N} \, F_2(Q^2) \right] \qquad Q^2\big|_{\mathrm{IMF}} = \vec{\Delta}_\perp^2$$

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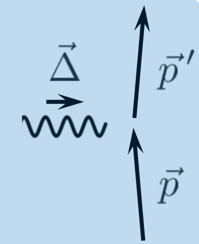

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2D charge distribution

$$\boxed{\rho_E^{\text{IMF}}(\vec{b}_\perp) = \int \frac{\mathrm{d}^2\Delta_\perp}{(2\pi)^2} \, e^{-i\vec{\Delta}_\perp \cdot \vec{b}_\perp} \, F_1(Q^2) - \frac{(\vec{S} \times \vec{\nabla})_z}{M_N} \int \frac{\mathrm{d}^2\Delta_\perp}{(2\pi)^2} \, e^{-i\vec{\Delta}_\perp \cdot \vec{b}_\perp} \, F_2(Q^2)}$$

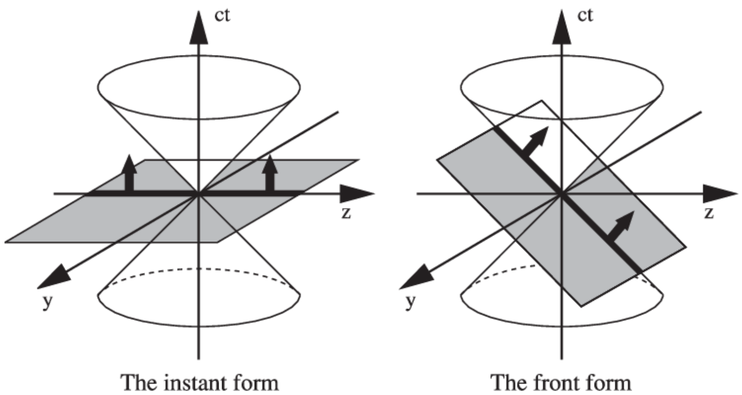
Galilean symmetry under finite boosts ➡ **No recoil correction !**

[Bouchiat, Fayet, Meyer, NPB34 (1971) 157]
[Soper, PRD15 (1977) 1141]
[Burkardt, PRD62 (2000) 071503]

Other approaches with similar results

Light-front quantization and Drell-Yan frame $\Delta^+ = 0$ (no need to consider IMF)

$$a = [a^+, a^1, a^2, a^-], \quad a^\pm = \frac{a^0 \pm a^3}{\sqrt{2}}$$

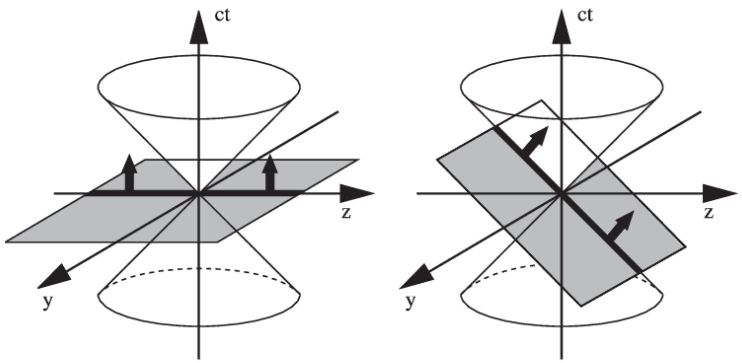


[Ralston, Jain, Buniy, AIP Conf. Proc. 549 (2000) 1, 302]
[Burkardt, IJMPA 18 (2003) 2, 173]
[Miller, PRL99 (2007) 11200]
[Carlson, Vanderhaeghen, PRL100 (2008) 032004]
[Freese, Miller, PRD105 (2022) 1, 014003]

Other approaches with similar results

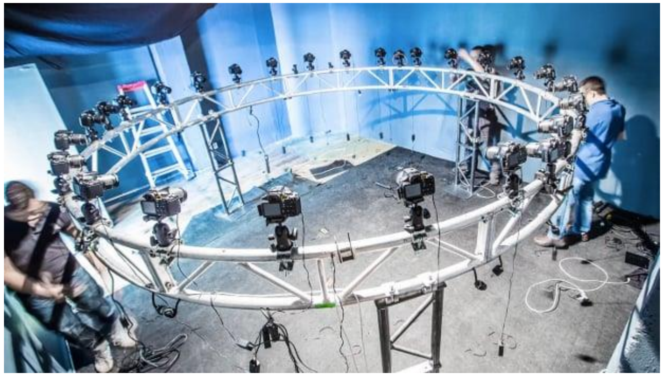
Light-front quantization and Drell-Yan frame $\Delta^+ = 0$ (no need to consider IMF)

$$a = [a^+, a^1, a^2, a^-], \quad a^\pm = \frac{a^0 \pm a^3}{\sqrt{2}}$$



[Ralston, Jain, Buniy, AIP Conf. Proc. 549 (2000) 1, 302]
[Burkardt, IJMPA 18 (2003) 2, 173]
[Miller, PRL99 (2007) 11200]
[Carlson, Vanderhaeghen, PRL100 (2008) 032004]
[Freese, Miller, PRD105 (2022) 1, 014003]

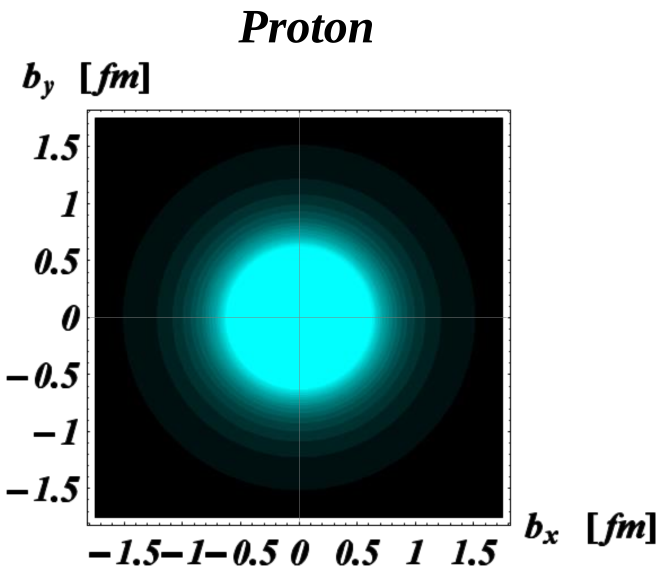
Method of dimensional counting (IMF averaged over all directions)



[Fleming, In *Phys. Reality & Math. Descrip.* (1974) 357]
[Epelbaum, Gegelia, Lange, Meissner, Polyakov, PRL129 (2022) 012001]
[Panteleeva, Epelbaum, Gegelia, Meissner, PRD106 (2022) 5, 056019]

IMF distributions

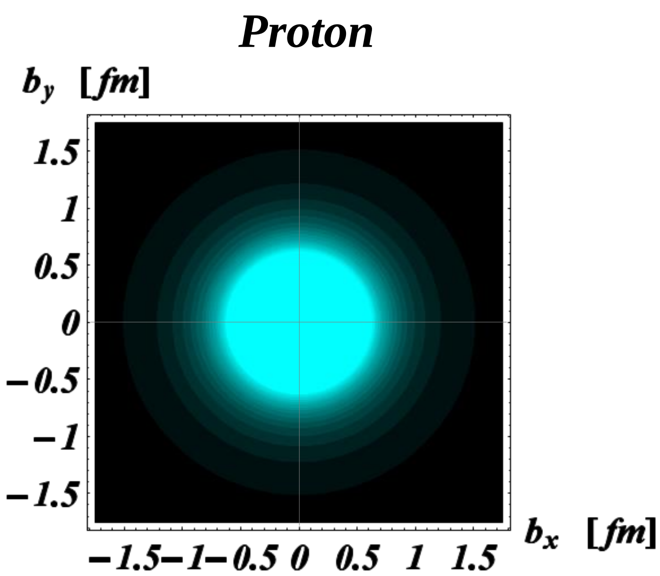
$$\vec{S} = \frac{\hbar}{2}\,\vec{e}_z$$

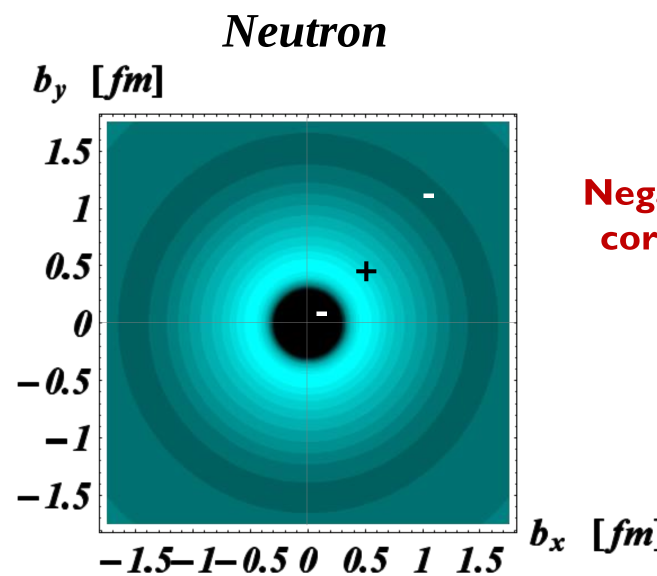


[Miller, PRL99 (2007) 11200]
[Carlson, Vanderhaeghen, PRL100 (2008) 032004]

IMF distributions

$$\vec{S} = \frac{\hbar}{2}\,\vec{e}_z$$

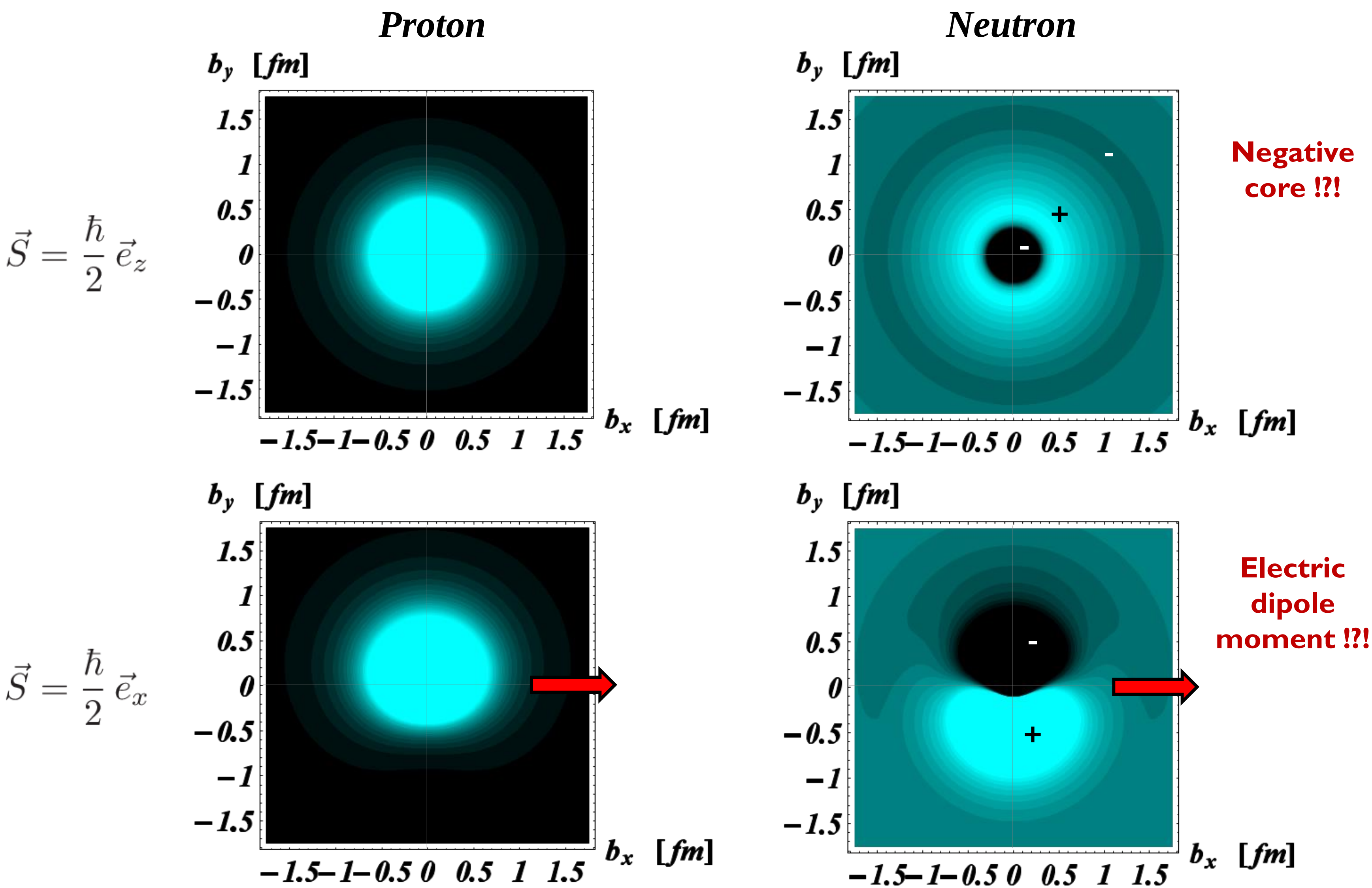




Negative core !?!

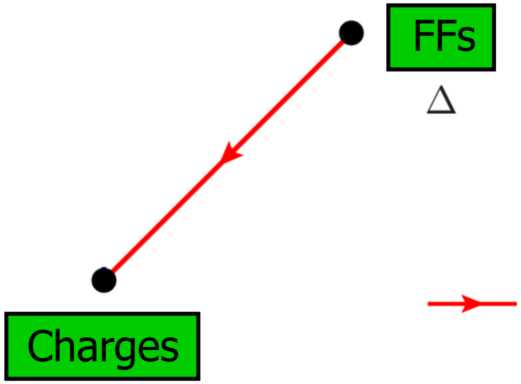
[Miller, PRL99 (2007) 11200]
[Carlson, Vanderhaeghen, PRL100 (2008) 032004]

IMF distributions



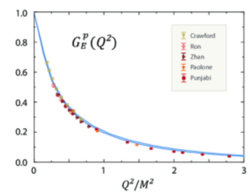
[Miller, PRL99 (2007) 11200]
[Carlson, Vanderhaeghen, PRL100 (2008) 032004]

Generalized parton distributions



$\langle p'|O|p\rangle$

Off-forward matrix element

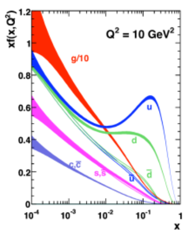


$\langle p|O|p\rangle$

$\Delta = 0$

$$O \sim \overline{\psi}(0) \cdots \psi(0)$$

Generalized parton distributions

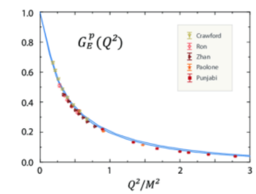


$\langle p|O(x)|p\rangle$ **PDFs** x

Non-local operator

FFs Δ $\langle p'|O|p\rangle$

Off-forward matrix element



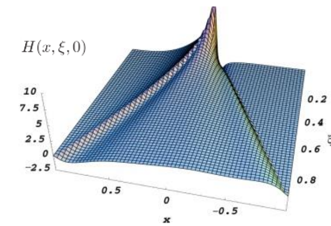
$\langle p|O|p\rangle$ **Charges**

$\Delta = 0$

$\int \mathrm{d}x$

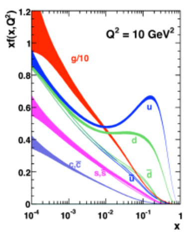
$$O(x) \sim \int \frac{\mathrm{d}z^-}{2\pi}\, e^{ixP^+z^-}\, \overline{\psi}(-\tfrac{z}{2}) \cdots \psi(\tfrac{z}{2})\big|_{z^+=|\vec{z}_\perp|=0} \xrightarrow{\int \mathrm{d}x} O \sim \overline{\psi}(0) \cdots \psi(0)$$

Generalized parton distributions



GPDs $\langle p'|O(x)|p\rangle$

x, Δ

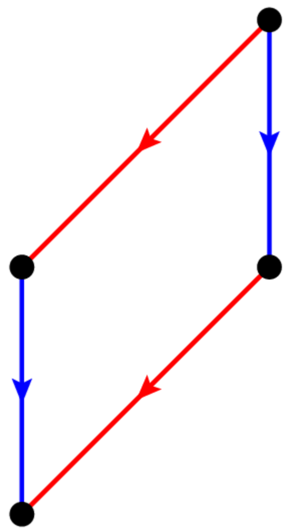


$\langle p|O(x)|p\rangle$

Non-local operator

PDFs

x

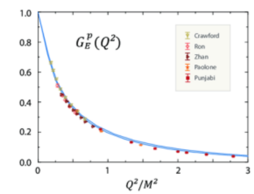


FFs

Δ

$\langle p'|O|p\rangle$

Off-forward matrix element



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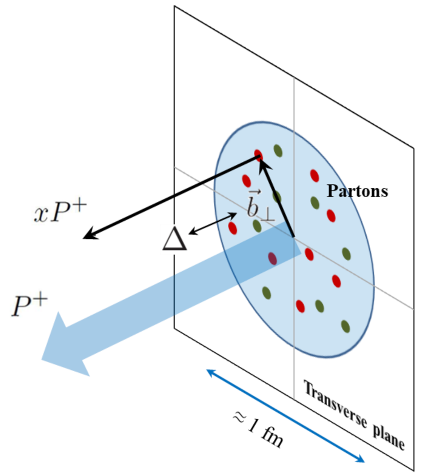
Hadron tomography

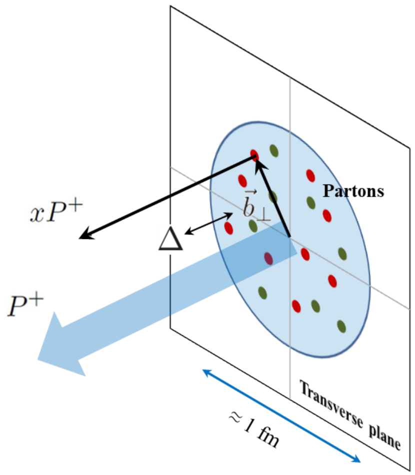
(2+1)D picture $\Delta^+ = 0$



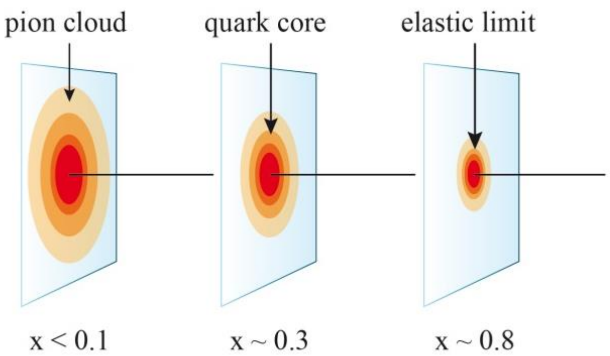
[Soper, PRD15 (1977) 1141]
[Burkardt, PRD62 (2000) 071503]
[Burkardt, IJMPA18 (2003) 2, 173]
[Diehl, Hägler, EPJC44 (2005) 87]

Hadron tomography

(2+1)D picture $\Delta^+ = 0$



tomê **= « cut »**
graphein **= « draw/write »**



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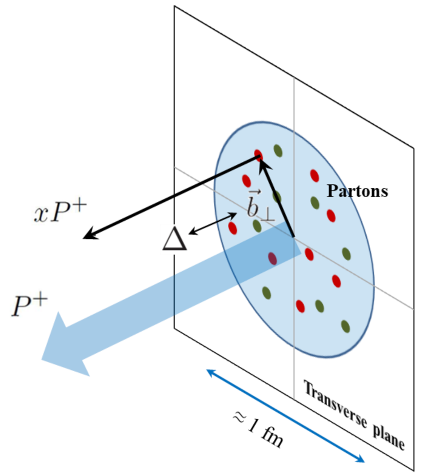

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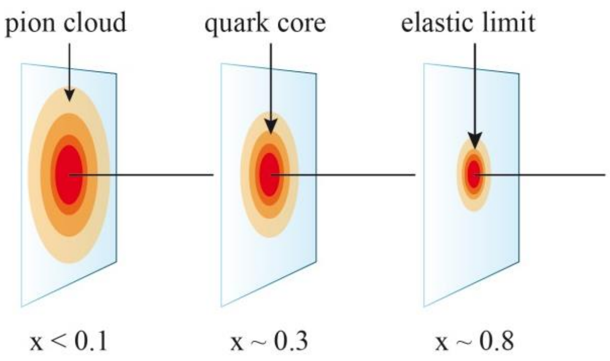
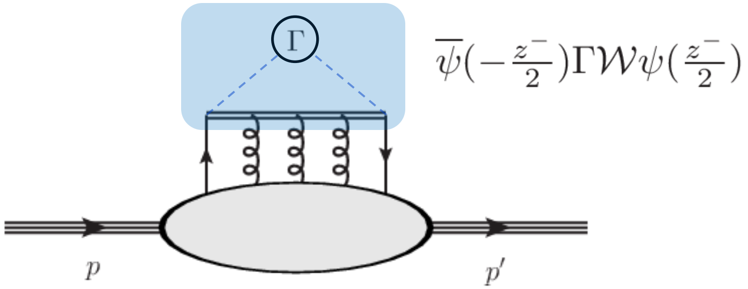
graphein **= « draw/write »**



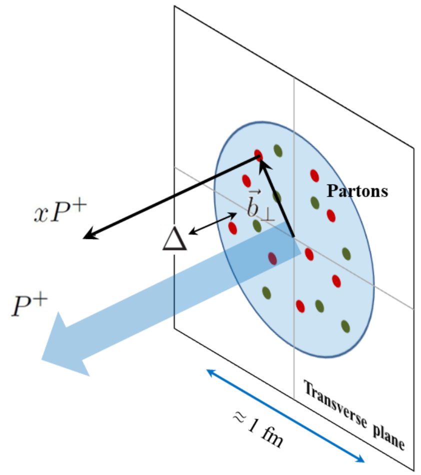
Gauge invariance and spin



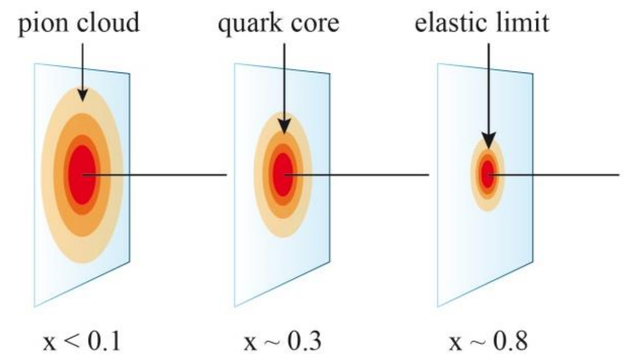
[Soper, PRD15 (1977) 1141]
[Burkardt, PRD62 (2000) 071503]
[Burkardt, IJMPA18 (2003) 2, 173]
[Diehl, Hägler, EPJC44 (2005) 87]

Hadron tomography

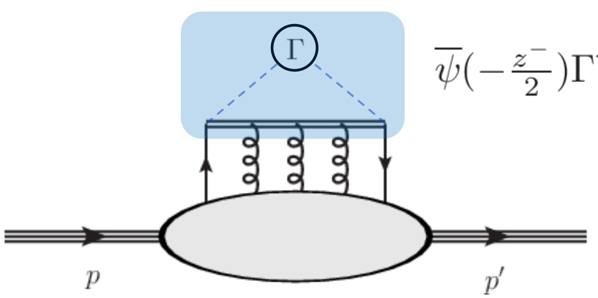
(2+1)D picture $\Delta^+ = 0$



***tomê* = « cut »**
***graphein* = « draw/write »**



Gauge invariance and spin



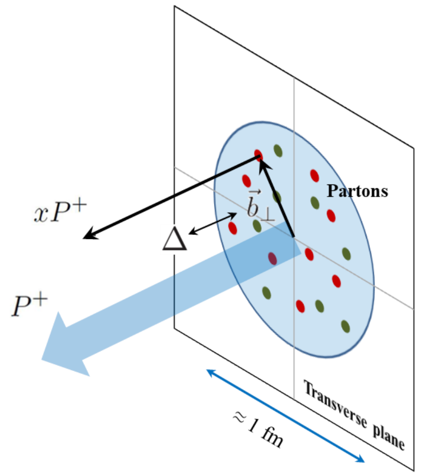
$$\overline{\psi}(-\tfrac{z^-}{2})\Gamma\mathcal{W}\psi(\tfrac{z^-}{2})$$

$$\mathcal{W} = \mathcal{P}\left[\exp\left(ig\int_{\frac{z^-}{2}}^{-\frac{z^-}{2}} \mathrm{d}y^-\, A^+(y)\right)\right] \overset{A^+=0}{\Rightarrow} 1$$

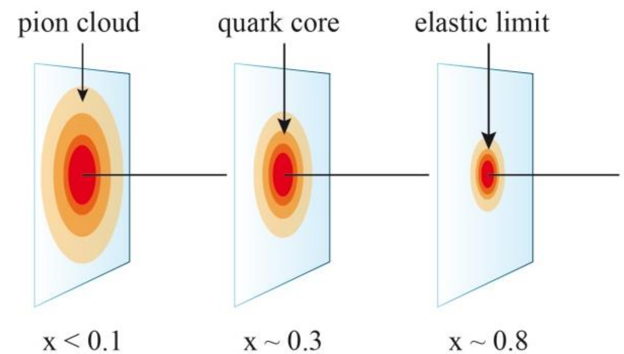
[Soper, PRD15 (1977) 1141]
[Burkardt, PRD62 (2000) 071503]
[Burkardt, IJMPA18 (2003) 2, 173]
[Diehl, Hägler, EPJC44 (2005) 87]

Hadron tomography

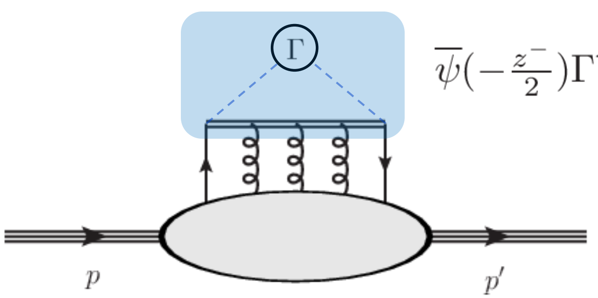
(2+1)D picture $\Delta^+ = 0$



tomê **= « cut »**
graphein **= « draw/write »**



Gauge invariance and spin



$$\overline{\psi}(-\tfrac{z^-}{2})\Gamma\mathcal{W}\psi(\tfrac{z^-}{2})$$

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Leading twist Γ

γ^+

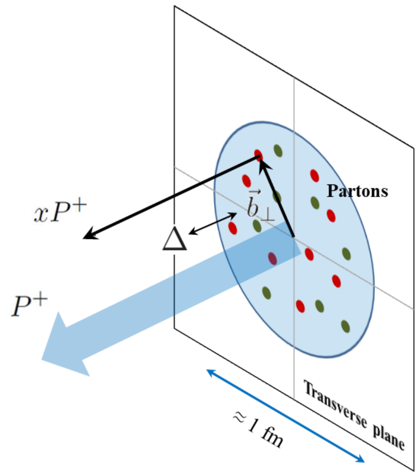
$\gamma^+\gamma_5$

$i\sigma^{j+}\gamma_5$

[Soper, PRD15 (1977) 1141]
[Burkardt, PRD62 (2000) 071503]
[Burkardt, IJMPA18 (2003) 2, 173]
[Diehl, Hägler, EPJC44 (2005) 87]

Hadron tomography

(2+1)D picture $\Delta^+ = 0$



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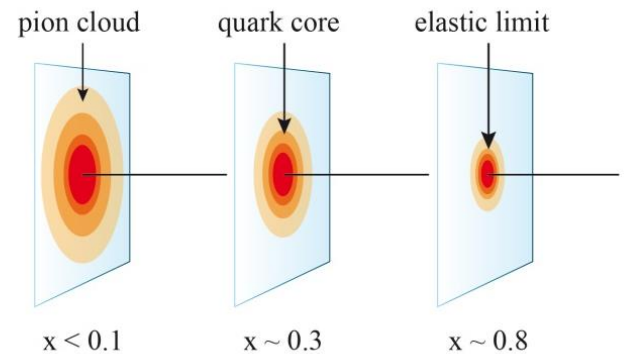
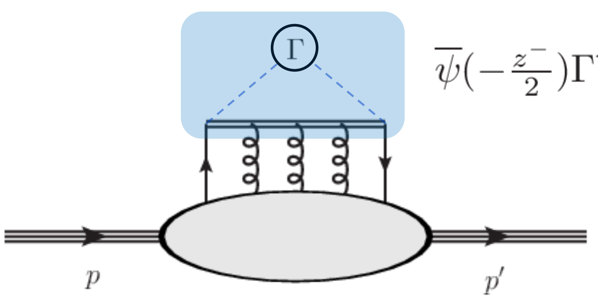


Gauge invariance and spin



$$\overline{\psi}(-\tfrac{z^-}{2})\Gamma\mathcal{W}\psi(\tfrac{z^-}{2})$$

$$\mathcal{W} = \mathcal{P}\left[\exp\left(ig\int_{\frac{z^-}{2}}^{-\frac{z^-}{2}} \mathrm{d}y^- \, A^+(y)\right)\right] \overset{A^+=0}{\Rightarrow} 1$$

Leading twist Γ

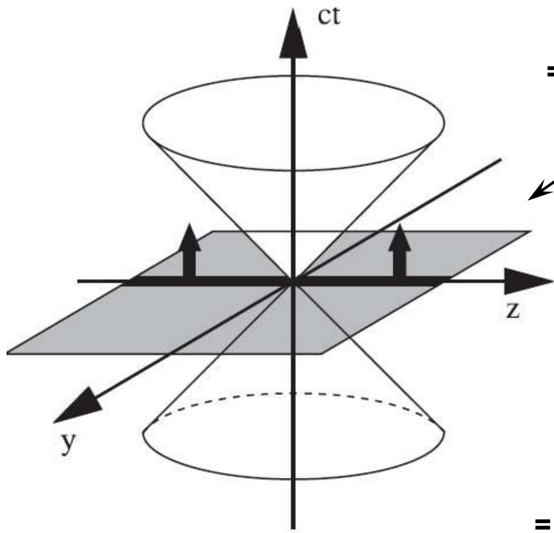
$\gamma^+ \sim \delta_{\lambda'_q\lambda_q}$	**Unpolarized quark**
$\gamma^+\gamma_5 \sim (\sigma_3)_{\lambda'_q\lambda_q}$	**Longitudinally polarized quark**
$i\sigma^{j+}\gamma_5 \sim (\sigma_j)_{\lambda'_q\lambda_q}$	**Tranversely polarized quark**

[Soper, PRD15 (1977) 1141]
[Burkardt, PRD62 (2000) 071503]
[Burkardt, IJMPA18 (2003)2, 173]
[Diehl, Hägler, EPJC44 (2005) 87]

Light-front picture

Forms of dynamics

Space-time foliation



« Space »
= 3D hypersurface

« Time »
= hypersurface label

Instant form dynamics

Time	x^0
Space	$\vec{x}$
Energy	p^0
Momentum	$\vec{p}$

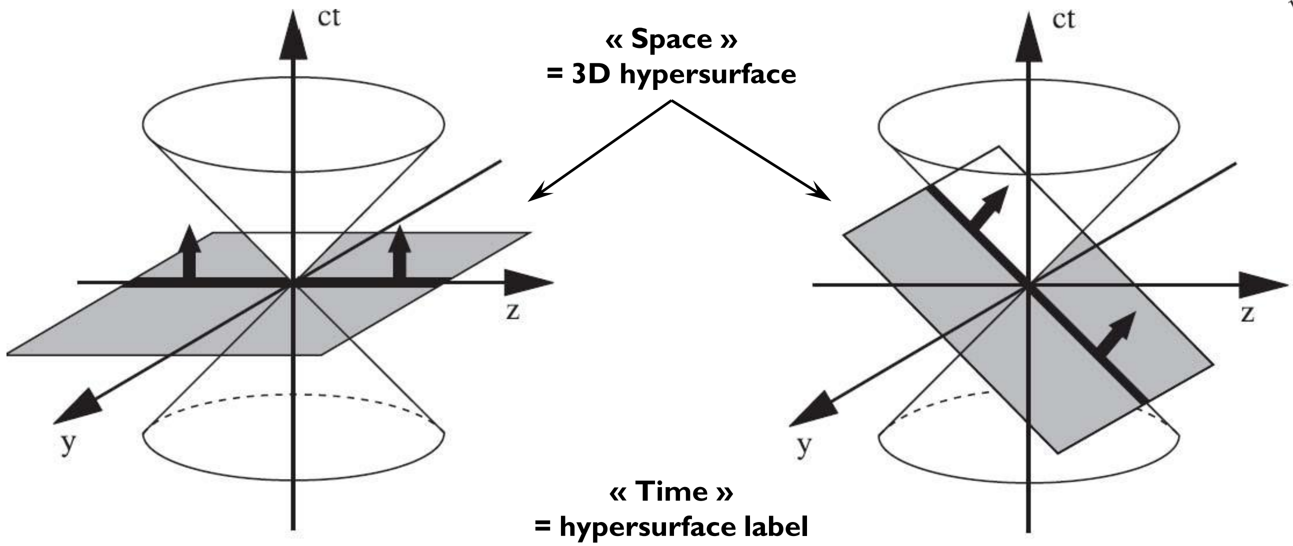
[Dirac, RMP21 (1949) 292]
[Brodsky, Pauli, Pinsky, PR301 (1998) 299]

Forms of dynamics

Space-time foliation

Light-front components

$$a^{\pm} = \frac{1}{\sqrt{2}}(a^0 \pm a^3)$$

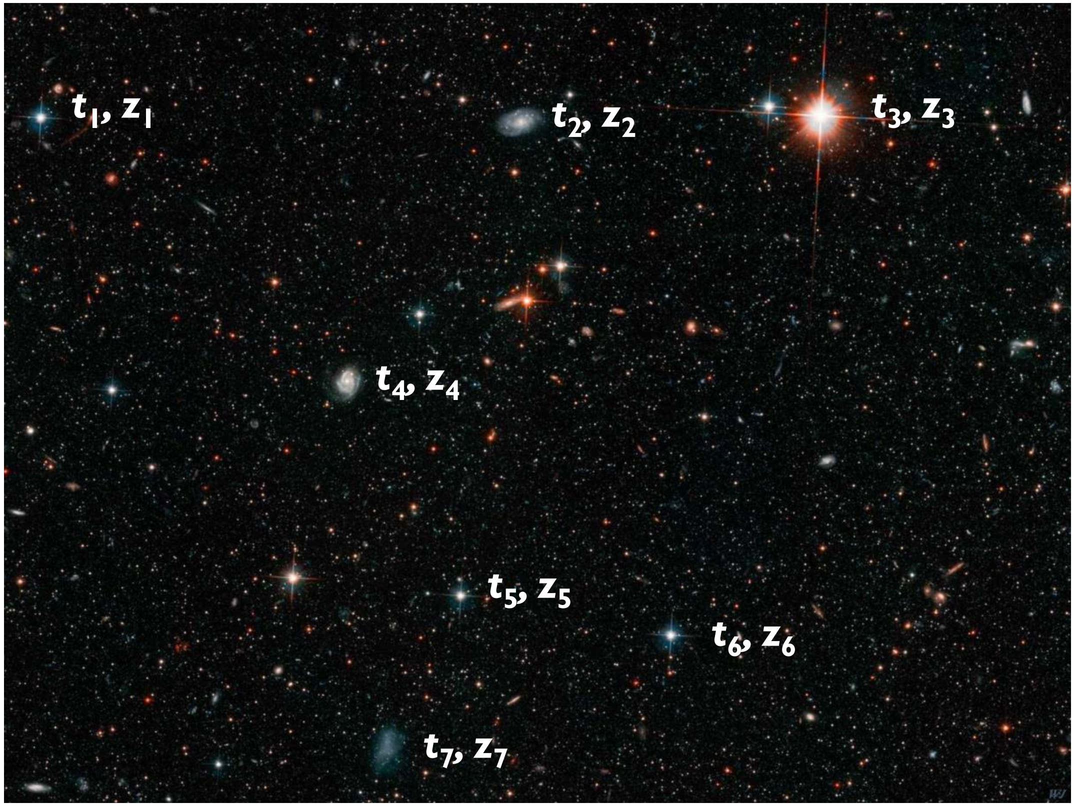


	Instant form dynamics	**Light-front form** dynamics
Time	x^0	x^+
Space	$\vec{x}$	$\vec{x}_\perp, x^-$
Energy	p^0	p^-
Momentum	$\vec{p}$	$\vec{p}_\perp, p^+$

[Dirac, RMP21 (1949) 292]
[Brodsky, Pauli, Pinsky, PR301 (1998) 299]

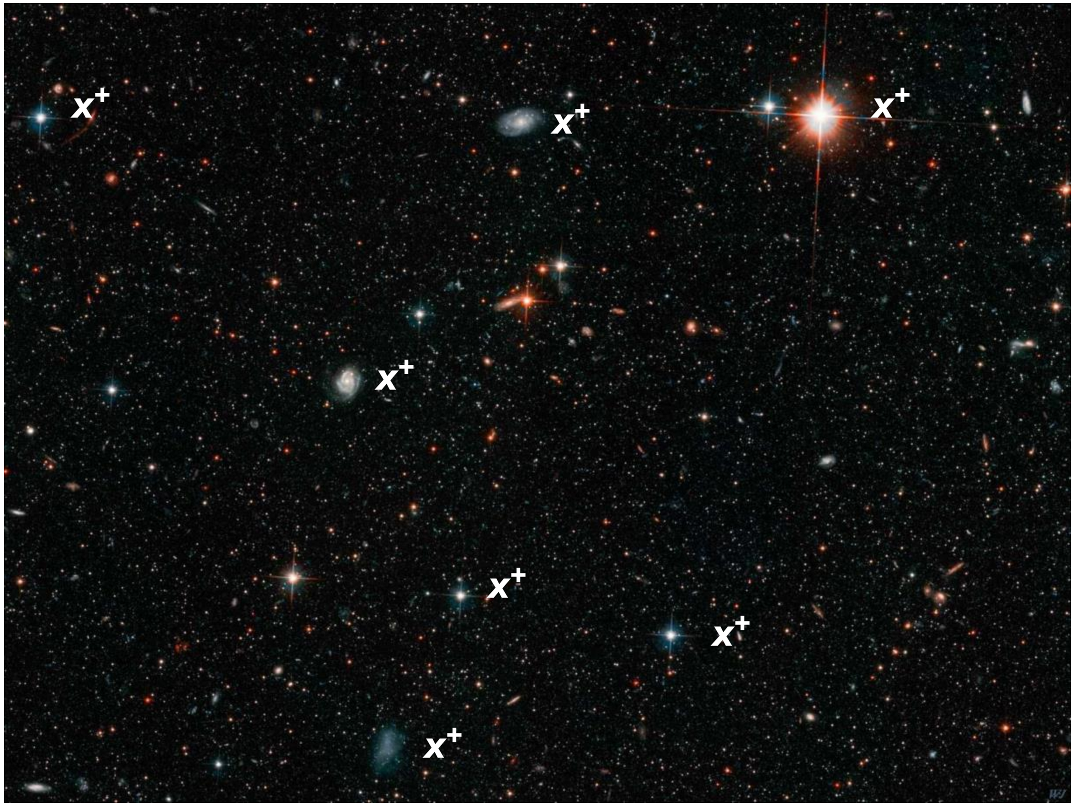
Instant form *vs* light-front form

Ordinary point of view



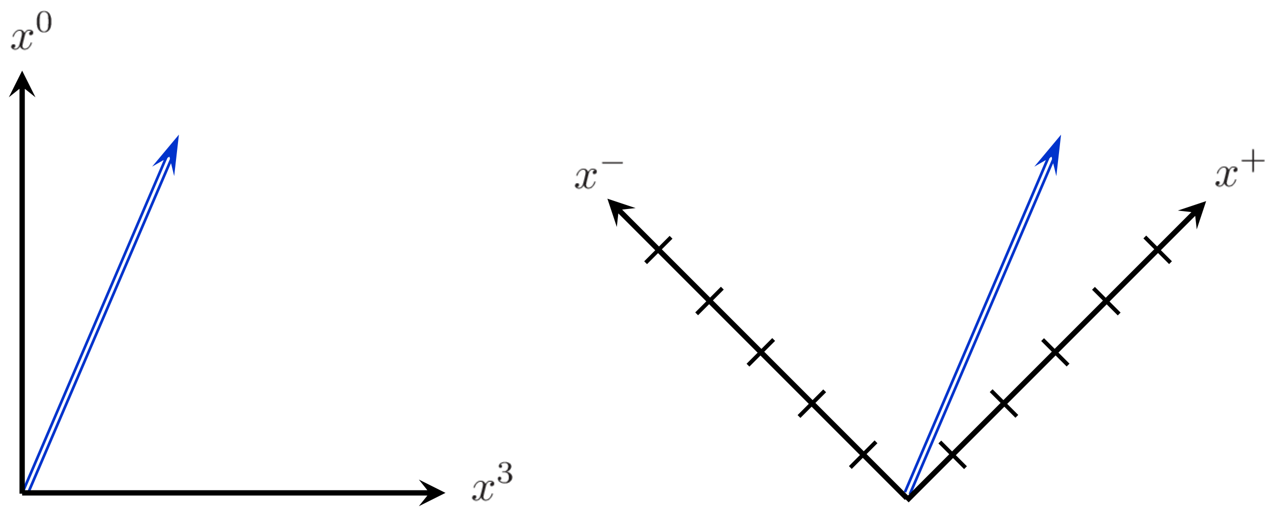
Instant form *vs* light-front form

Light front point of view $x^+ = \frac{1}{\sqrt{2}}(t+z)$



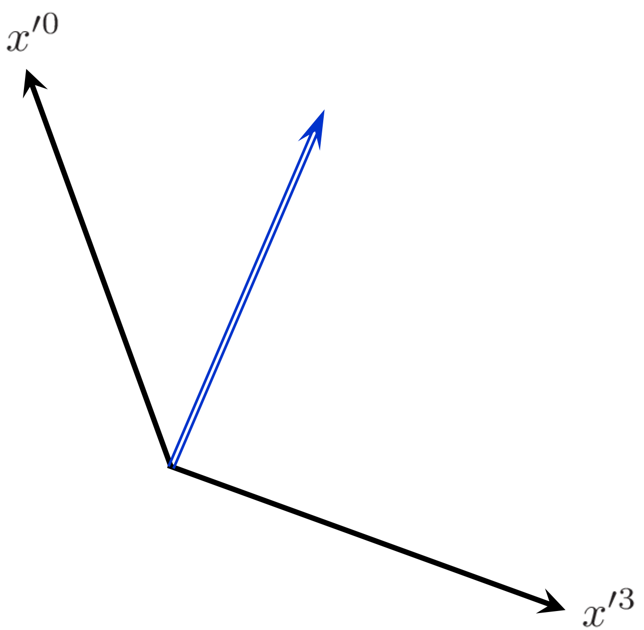
Instant form *vs* light-front form

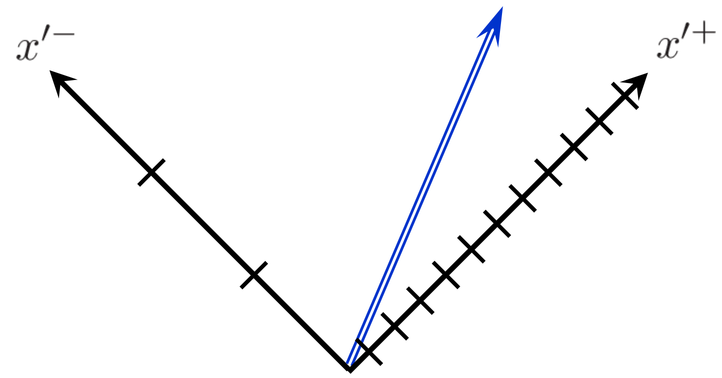
Initial frame



Instant form *vs* light-front form

Boosted frame





$$x'^0 = \frac{x^0 + \beta x^3}{\sqrt{1-\beta^2}}$$

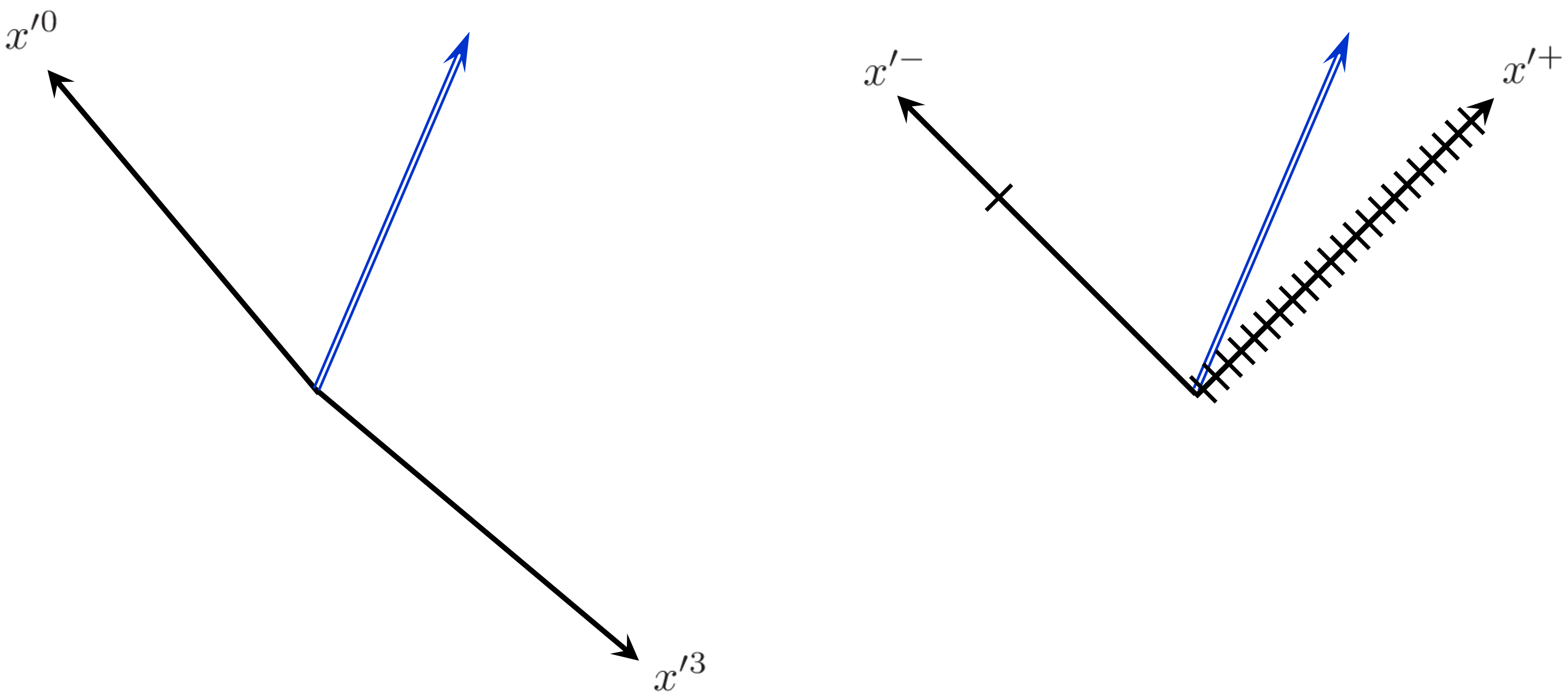
$$x'^3 = \frac{x^3 + \beta x^0}{\sqrt{1-\beta^2}}$$

$$x'^+ = \sqrt{\frac{1+\beta}{1-\beta}}\, x^+ = e^{\eta}\, x^+$$

$$x'^- = \sqrt{\frac{1-\beta}{1+\beta}}\, x^- = e^{-\eta}\, x^-$$

Instant form *vs* light-front form

Infinite-momentum frame $\beta \to 1$



$$x'^0 \approx x'^3 \approx \frac{x'^+}{\sqrt{2}}$$

$$x'^- \approx 0$$

Light-front Poincaré algebra

$$B_\perp^1 = \frac{1}{\sqrt{2}}(K^1 + J^2) \qquad \mathcal{J}_\perp^1 = \frac{1}{\sqrt{2}}(J^1 + K^2)$$
$$B_\perp^2 = \frac{1}{\sqrt{2}}(K^2 - J^1) \qquad \mathcal{J}_\perp^2 = \frac{1}{\sqrt{2}}(J^2 - K^1)$$

Transverse space-time symmetry

$$[J^3, \mathcal{J}_\perp^i] = \qquad [B_\perp^i, P_\perp^j] =$$
$$[J^3, B_\perp^i] = \qquad [B_\perp^i, P^-] =$$
$$[B_\perp^i, B_\perp^j] = \qquad [B_\perp^i, P^+] =$$

[Kogut, Soper, PRD1 (1970) 2901]
[Burkardt, IJMPA18 (2003) 2, 173]
[C.L., EPJC78 (2018) 9, 785]

Light-front Poincaré algebra

$$B_\perp^1 = \frac{1}{\sqrt{2}}(K^1 + J^2) \qquad \mathcal{J}_\perp^1 = \frac{1}{\sqrt{2}}(J^1 + K^2)$$
$$B_\perp^2 = \frac{1}{\sqrt{2}}(K^2 - J^1) \qquad \mathcal{J}_\perp^2 = \frac{1}{\sqrt{2}}(J^2 - K^1)$$

Transverse space-time symmetry

$$[J^3, \mathcal{J}_\perp^i] = i\epsilon^{3ij}\mathcal{J}_\perp^j \qquad [B_\perp^i, P_\perp^j] = -i\delta_\perp^{ij}P^+$$
$$[J^3, B_\perp^i] = i\epsilon^{3ij}B_\perp^j \qquad [B_\perp^i, P^-] = -iP_\perp^i$$
$$[B_\perp^i, B_\perp^j] = 0 \qquad [B_\perp^i, P^+] = [J^3, P^+] = [J^3, P^-] = 0$$

[Kogut, Soper, PRD1 (1970) 2901]
[Burkardt, IJMPA18 (2003) 2, 173]
[C.L., EPJC78 (2018) 9, 785]

Galilean/non-relativistic algebra

Galilean symmetry

$$[J^i, J^j] = i\epsilon^{ijk} J^k \qquad [B^i, P^j] = -i\delta^{ij} M$$

$$[J^i, B^j] = i\epsilon^{ijk} B^k \qquad [B^i, H] = -iP^i$$

$$[B^i, B^j] = 0 \qquad [B^i, M] = [J^3, M] = [J^3, H] = 0$$

[Kogut, Soper, PRD1 (1970) 2901]
[Burkardt, IJMPA18 (2003) 2, 173]
[C.L., EPJC78 (2018) 9, 785]

Galilean/non-relativistic algebra

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$$[J^i, J^j] = i\epsilon^{ijk} J^k$$
$$[J^i, B^j] = i\epsilon^{ijk} B^k$$
$$[B^i, B^j] = 0$$

$$[B^i, P^j] = -i\delta^{ij} M$$
$$[B^i, H] = -iP^i$$
$$[B^i, M] = [J^3, M] = [J^3, H] = 0$$

Position operator

$$[R^i, P^j] = i\delta^{ij}\mathbb{1}$$
$$[R^i, R^j] = 0$$
$$[J^i, R^j] = i\epsilon^{ijk} R^k$$

[Kogut, Soper, PRD1 (1970) 2901]
[Burkardt, IJMPA18 (2003) 2, 173]
[C.L., EPJC78 (2018) 9, 785]

Galilean/non-relativistic algebra

Galilean symmetry

$$[J^i, J^j] = i\epsilon^{ijk} J^k \qquad [B^i, P^j] = -i\delta^{ij} M$$

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$$[B^i, B^j] = 0 \qquad [B^i, M] = [J^3, M] = [J^3, H] = 0$$

Position operator (center of inertia)

$$\boxed{B^i = -MR^i} \quad \Rightarrow \quad \begin{aligned} [R^i, P^j] &= i\delta^{ij}\mathbb{1} \\ [R^i, R^j] &= 0 \\ [J^i, R^j] &= i\epsilon^{ijk} R^k \end{aligned}$$

[Kogut, Soper, PRD1 (1970) 2901]
[Burkardt, IJMPA18 (2003) 2, 173]
[C.L., EPJC78 (2018) 9, 785]

Galilean/non-relativistic algebra

Galilean symmetry

$$[J^i, J^j] = i\epsilon^{ijk} J^k \qquad [B^i, P^j] = -i\delta^{ij} M$$

$$[J^i, B^j] = i\epsilon^{ijk} B^k \qquad [B^i, H] = -iP^i$$

$$[B^i, B^j] = 0 \qquad [B^i, M] = [J^3, M] = [J^3, H] = 0$$

Position operator (center of inertia)

$$\boxed{B^i = -MR^i} \quad \Rightarrow \quad \begin{aligned} [R^i, P^j] &= i\delta^{ij}\mathbb{1} \\ [R^i, R^j] &= 0 \\ [J^i, R^j] &= i\epsilon^{ijk} R^k \end{aligned}$$

Galilean boost $\quad M' = M \qquad \vec{p}\,' = \vec{p} + M\vec{v}$

[Kogut, Soper, PRD1 (1970) 2901]
[Burkardt, IJMPA18 (2003) 2, 173]
[C.L., EPJC78 (2018) 9, 785]

Light-front Poincaré algebra

$$B_\perp^1 = \frac{1}{\sqrt{2}}(K^1 + J^2) \qquad \mathcal{J}_\perp^1 = \frac{1}{\sqrt{2}}(J^1 + K^2)$$
$$B_\perp^2 = \frac{1}{\sqrt{2}}(K^2 - J^1) \qquad \mathcal{J}_\perp^2 = \frac{1}{\sqrt{2}}(J^2 - K^1)$$

Transverse space-time symmetry

$$[J^3, \mathcal{J}_\perp^i] = i\epsilon^{3ij}\mathcal{J}_\perp^j \qquad [B_\perp^i, P_\perp^j] = -i\delta_\perp^{ij}P^+$$
$$[J^3, B_\perp^i] = i\epsilon^{3ij}B_\perp^j \qquad [B_\perp^i, P^-] = -iP_\perp^i$$
$$[B_\perp^i, B_\perp^j] = 0 \qquad [B_\perp^i, P^+] = [J^3, P^+] = [J^3, P^-] = 0$$

Transverse position operator (center of P^+)

$$\boxed{B_\perp^i = -P^+R_\perp^i} \quad \Rightarrow \quad \begin{aligned} &[R_\perp^i, P_\perp^j] = i\delta_\perp^{ij}\mathbb{1} \\ &[R_\perp^i, R_\perp^j] = 0 \\ &[J^3, R_\perp^i] = i\epsilon^{3ij}R_\perp^j \end{aligned}$$

Transverse boost

$$p'^+ = p^+ \qquad \vec{p}\,'_\perp = \vec{p}_\perp + p^+\vec{v}_\perp$$

[Kogut, Soper, PRD1 (1970) 2901]
[Burkardt, IJMPA18 (2003) 2, 173]
[C.L., EPJC78 (2018) 9, 785]

Light-front densities

Normalizations

$$\langle p'^+, \vec{p}'_\perp | p^+, \vec{p}_\perp \rangle = 2p^+ \, (2\pi)^3 \, \delta(p'^+ - p^+) \, \delta^{(2)}(\vec{p}'_\perp - \vec{p}_\perp)$$

$$\langle p'^+, \vec{r}'_\perp | p^+, \vec{r}_\perp \rangle = 2p^+ \, 2\pi \, \delta(p'^+ - p^+) \, \delta^{(2)}(\vec{r}'_\perp - \vec{r}_\perp)$$

Localized states

momentum space $|p^+, \vec{p}_\perp\rangle$

mixed space $|p^+, \vec{r}_\perp\rangle = \displaystyle\int \frac{\mathrm{d}^2 p_\perp}{(2\pi)^2} \, e^{-i\vec{p}_\perp \cdot \vec{r}_\perp} \, |p^+, \vec{p}_\perp\rangle$

[Soper, PRD15 (1977) 1141]
[Burkardt, PRD62 (2000) 071503]
[Diehl, EPJC25 (2002) 223]
[Burkardt, IJMPA18 (2003) 2, 173]

Light-front densities

Normalizations

$$\langle p'^+, \vec{p}'_\perp | p^+, \vec{p}_\perp \rangle = 2p^+ (2\pi)^3 \, \delta(p'^+ - p^+) \, \delta^{(2)}(\vec{p}'_\perp - \vec{p}_\perp)$$

$$\langle p'^+, \vec{r}'_\perp | p^+, \vec{r}_\perp \rangle = 2p^+ \, 2\pi \, \delta(p'^+ - p^+) \, \delta^{(2)}(\vec{r}'_\perp - \vec{r}_\perp)$$

Localized states

momentum space $|p^+, \vec{p}_\perp\rangle$

mixed space $$|p^+, \vec{r}_\perp\rangle = \int \frac{\mathrm{d}^2 p_\perp}{(2\pi)^2} \, e^{-i\vec{p}_\perp \cdot \vec{r}_\perp} \, |p^+, \vec{p}_\perp\rangle$$

Impact-parameter dependent distributions (IPDs)

$x^+ = 0$

$$\langle p'^+, \vec{r}'_\perp | \int \mathrm{d}x^- \, J^+(x) | p^+, \vec{r}_\perp \rangle =$$

[Soper, PRD15 (1977) 1141]
[Burkardt, PRD62 (2000) 071503]
[Diehl, EPJC25 (2002) 223]
[Burkardt, IJMPA18 (2003) 2, 173]

Light-front densities

Normalizations $\langle p'^+, \vec{p}\,'_\perp | p^+, \vec{p}_\perp \rangle =$
$2p^+ (2\pi)^3 \, \delta(p'^+ - p^+) \, \delta^{(2)}(\vec{p}\,'_\perp - \vec{p}_\perp)$
$\langle p'^+, \vec{r}\,'_\perp | p^+, \vec{r}_\perp \rangle =$
$2p^+ \, 2\pi \, \delta(p'^+ - p^+) \, \delta^{(2)}(\vec{r}\,'_\perp - \vec{r}_\perp)$

Localized states

momentum space $|p^+, \vec{p}_\perp\rangle$

mixed space $$|p^+, \vec{r}_\perp\rangle = \int \frac{\mathrm{d}^2 p_\perp}{(2\pi)^2} \, e^{-i\vec{p}_\perp \cdot \vec{r}_\perp} \, |p^+, \vec{p}_\perp\rangle$$

Impact-parameter dependent distributions (IPDs) $x^+ = 0$

$$\langle p'^+, \vec{r}\,'_\perp | \int \mathrm{d}x^- \, J^+(x) | p^+, \vec{r}_\perp \rangle = 2p^+ \, 2\pi \, \delta(p'^+ - p^+) \, \delta^{(2)}(\vec{r}\,'_\perp - \vec{r}_\perp)$$
$$\times \underbrace{\int \frac{\mathrm{d}^2 \Delta_\perp}{(2\pi)^2} \, e^{-i\vec{\Delta}_\perp \cdot (\vec{x}_\perp - \vec{r}_\perp)} \, \frac{\langle P^+, \frac{\vec{\Delta}_\perp}{2} | J^+(0) | P^+, -\frac{\Delta_\perp}{2} \rangle}{2P^+}}_{\text{Internal distribution}}$$

Internal distribution

[Soper, PRD15 (1977) 1141]
[Burkardt, PRD62 (2000) 071503]
[Diehl, EPJC25 (2002) 223]
[Burkardt, IJMPA18 (2003) 2, 173]

Light-front densities

Normalizations $\langle p'^+, \vec{p}'_\perp | p^+, \vec{p}_\perp \rangle = 2p^+ (2\pi)^3 \, \delta(p'^+ - p^+) \, \delta^{(2)}(\vec{p}'_\perp - \vec{p}_\perp)$

$\langle p'^+, \vec{r}'_\perp | p^+, \vec{r}_\perp \rangle = 2p^+ \, 2\pi \, \delta(p'^+ - p^+) \, \delta^{(2)}(\vec{r}'_\perp - \vec{r}_\perp)$

Localized states

momentum space $|p^+, \vec{p}_\perp\rangle$

mixed space $$|p^+, \vec{r}_\perp\rangle = \int \frac{\mathrm{d}^2 p_\perp}{(2\pi)^2} \, e^{-i\vec{p}_\perp \cdot \vec{r}_\perp} \, |p^+, \vec{p}_\perp\rangle$$

Impact-parameter dependent distributions (IPDs) $x^+ = 0$

$$\langle p'^+, \vec{r}'_\perp | \int \mathrm{d}x^- \, J^+(x) | p^+, \vec{r}_\perp \rangle = 2p^+ \, 2\pi \, \delta(p'^+ - p^+) \, \delta^{(2)}(\vec{r}'_\perp - \vec{r}_\perp)$$

$$\times \underbrace{\int \frac{\mathrm{d}^2 \Delta_\perp}{(2\pi)^2} \, e^{-i\vec{\Delta}_\perp \cdot (\vec{x}_\perp - \vec{r}_\perp)} \, \frac{\langle P^+, \frac{\vec{\Delta}_\perp}{2} | J^+(0) | P^+, -\frac{\vec{\Delta}_\perp}{2} \rangle}{2P^+}}_{\text{Internal distribution}}$$

Internal distribution

<u>NB:</u> $\int \mathrm{d}x^- \Leftrightarrow \Delta^+ = 0$ (**Drell-Yan frame**) **is essential to ensure probabilistic interpretation !**

[Soper, PRD15 (1977) 1141]
[Burkardt, PRD62 (2000) 071503]
[Diehl, EPJC25 (2002) 223]
[Burkardt, IJMPA18 (2003) 2, 173]

Light-front densities

Unpolarized quark IPD $\vec{b}_\perp = \vec{x}_\perp - \vec{r}_\perp$

$$H_q(x,\xi,t)$$

$$\xi = -\frac{\Delta^+}{2P^+}$$

$$t = \Delta^2 = 2\Delta^+\Delta^- - \vec{\Delta}_\perp^2$$

$$q(x,\vec{b}_\perp) = \int \frac{\mathrm{d}^2\Delta_\perp}{(2\pi)^2}\, e^{-i\vec{\Delta}_\perp\cdot\vec{b}_\perp}\, H_q(x,0,-\vec{\Delta}_\perp^2)$$

[Burkardt, IJMPA18 (2003) 2, 173]

Light-front densities

Unpolarized quark IPD $\vec{b}_\perp = \vec{x}_\perp - \vec{r}_\perp$

$$q(x, \vec{b}_\perp) = \int \frac{\mathrm{d}^2\Delta_\perp}{(2\pi)^2}\, e^{-i\vec{\Delta}_\perp \cdot \vec{b}_\perp}\, H_q(x, 0, -\vec{\Delta}_\perp^2)$$

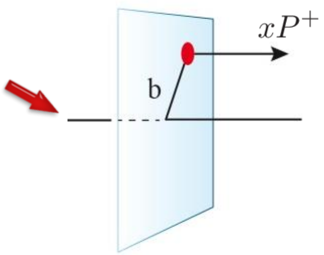
$H_q(x, \xi, t)$

$\xi = -\dfrac{\Delta^+}{2P^+}$

$t = \Delta^2 = 2\Delta^+\Delta^- - \vec{\Delta}_\perp^2$

Center of P^+ (light-front inertia) $B^i_\perp = M^{+i} = \int \mathrm{d}x^-\, \mathrm{d}^2x_\perp\, (x^+ T^{+i} - x^i_\perp T^{++})$

$$\vec{R}_\perp = \frac{1}{P^+} \int \mathrm{d}x^-\, \mathrm{d}^2 x_\perp\, \vec{x}_\perp\, T^{++}(x)$$
$$= \frac{1}{P^+} \sum_n \vec{r}_{\perp,n}\, p_n^+ = \sum_n x_n\, \vec{r}_{\perp,n}$$



[Burkardt, IJMPA18 (2003) 2, 173]

Light-front densities

Unpolarized quark IPD

$$\vec{b}_\perp = \vec{x}_\perp - \vec{r}_\perp$$

$$q(x, \vec{b}_\perp) = \int \frac{\mathrm{d}^2\Delta_\perp}{(2\pi)^2}\, e^{-i\vec{\Delta}_\perp \cdot \vec{b}_\perp}\, H_q(x, 0, -\vec{\Delta}_\perp^2)$$

$$H_q(x, \xi, t)$$

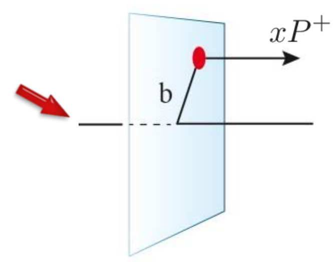
$$\xi = -\frac{\Delta^+}{2P^+}$$

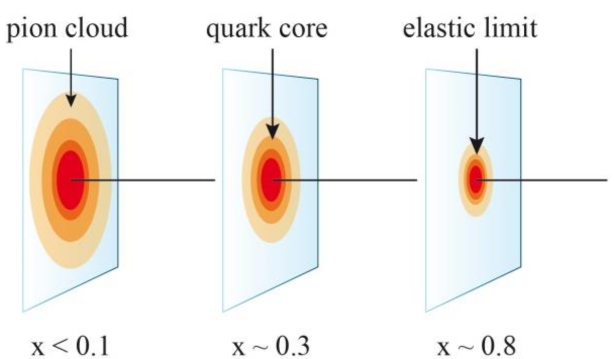
$$t = \Delta^2 = 2\Delta^+\Delta^- - \vec{\Delta}_\perp^2$$

Center of P^+ (light-front inertia)

$$B^i_\perp = M^{+i} = \int \mathrm{d}x^-\, \mathrm{d}^2x_\perp\, \left(x^+ T^{+i} - x^i_\perp T^{++}\right)$$

$$\begin{aligned}\vec{R}_\perp &= \frac{1}{P^+}\int \mathrm{d}x^-\, \mathrm{d}^2x_\perp\, \vec{x}_\perp\, T^{++}(x)\\ &= \frac{1}{P^+}\sum_n \vec{r}_{\perp,n}\, p_n^+ = \sum_n x_n\, \vec{r}_{\perp,n}\end{aligned}$$





$$\langle \vec{b}_\perp^2 \rangle(x) = \frac{\int \mathrm{d}^2b_\perp\, \vec{b}_\perp^2\, q(x, \vec{b}_\perp)}{\int \mathrm{d}^2b_\perp\, q(x, \vec{b}_\perp)}$$

[Burkardt, IJMPA18 (2003) 2, 173]

Light-front densities

Unpolarized quark IPD $\vec{b}_\perp = \vec{x}_\perp - \vec{r}_\perp$

$$q(x,\vec{b}_\perp) = \int \frac{\mathrm{d}^2\Delta_\perp}{(2\pi)^2}\, e^{-i\vec{\Delta}_\perp\cdot\vec{b}_\perp}\, H_q(x,0,-\vec{\Delta}_\perp^2)$$

$H_q(x,\xi,t)$

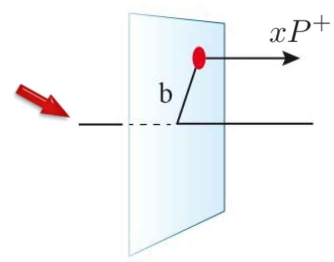
$$\xi = -\frac{\Delta^+}{2P^+}$$

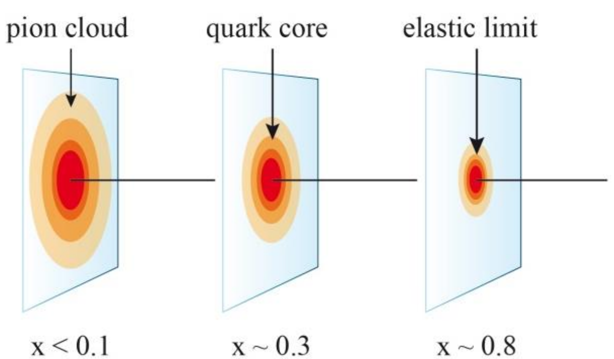
$$t = \Delta^2 = 2\Delta^+\Delta^- - \vec{\Delta}_\perp^2$$

Center of *P*⁺ (light-front inertia) $B^i_\perp = M^{+i} = \int \mathrm{d}x^-\, \mathrm{d}^2x_\perp\, (x^+T^{+i} - x^i_\perp T^{++})$

$$\vec{R}_\perp = \frac{1}{P^+}\int \mathrm{d}x^-\,\mathrm{d}^2x_\perp\,\vec{x}_\perp\,T^{++}(x)$$

$$= \frac{1}{P^+}\sum_n \vec{r}_{\perp,n}\,p_n^+ = \sum_n x_n\,\vec{r}_{\perp,n}$$



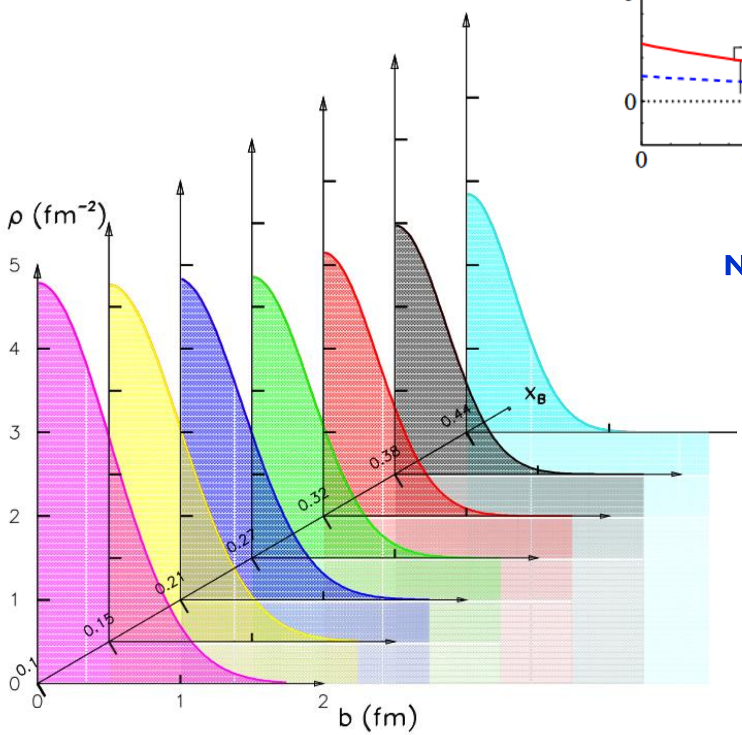


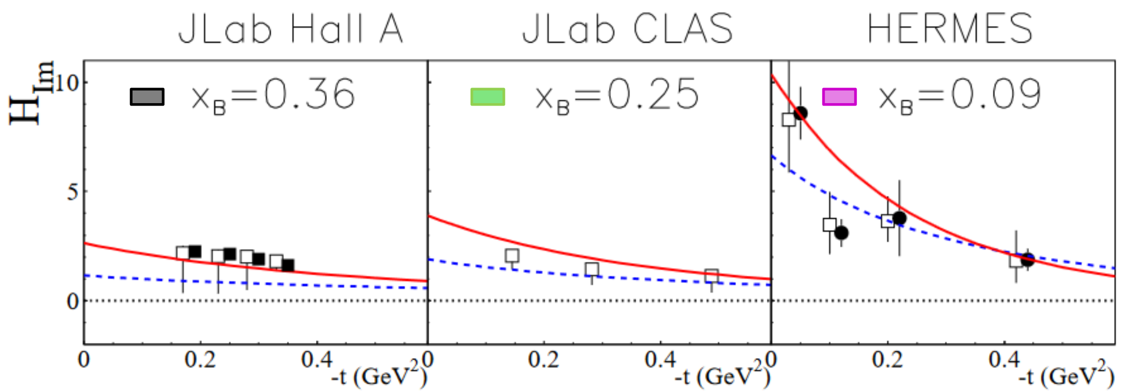
$$\langle\vec{b}_\perp^2\rangle(x) = \frac{\int \mathrm{d}^2b_\perp\,\vec{b}_\perp^2\,q(x,\vec{b}_\perp)}{\int \mathrm{d}^2b_\perp\,q(x,\vec{b}_\perp)} \xrightarrow{x\to 1} 0$$

as a result of $x_n \geq 0, \quad \sum_n x_n = 1$

[Burkardt, IJMPA18 (2003) 2, 173]

Light-front densities





Flat in *t*

Narrow in *b*

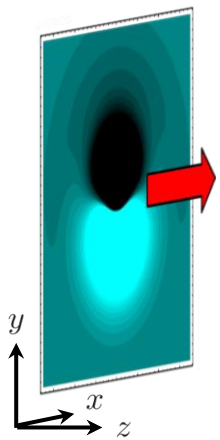
Steep in *t*

Wide in *b*

[Guidal, Moutarde Vanderhaghen, RPP76 (2013) 066202]

Light-front artifacts

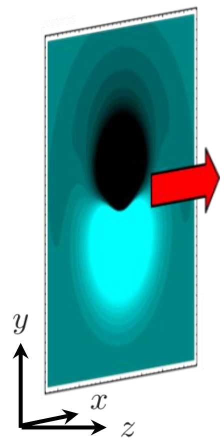
$$J^+ = \frac{J^0 + J^3}{\sqrt{2}}$$



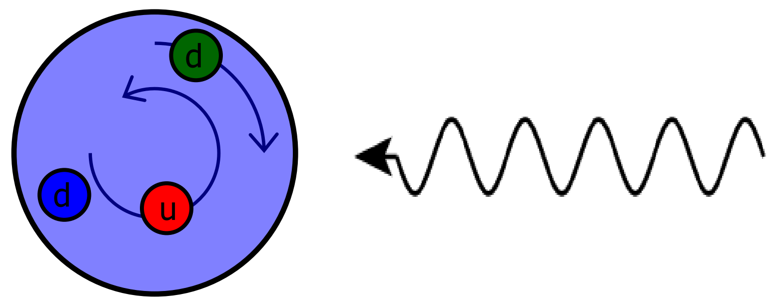
[Burkardt, IJMPA18 (2003) 2, 173]
[Carlson, Vanderhaeghen, PRL100 (2008) 032004]

Light-front artifacts

$$J^+ = \frac{J^0 + J^3}{\sqrt{2}}$$



Neutron
at rest



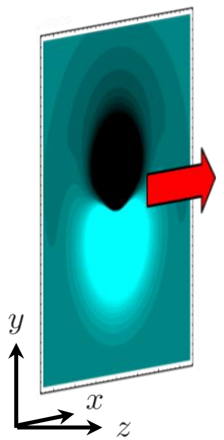
$\vec{S} \otimes$ $\quad$ $\mu_n = -1.91$

Magnetic dipole moment

[Burkardt, IJMPA18 (2003) 2, 173]
[Carlson, Vanderhaeghen, PRL100 (2008) 032004]

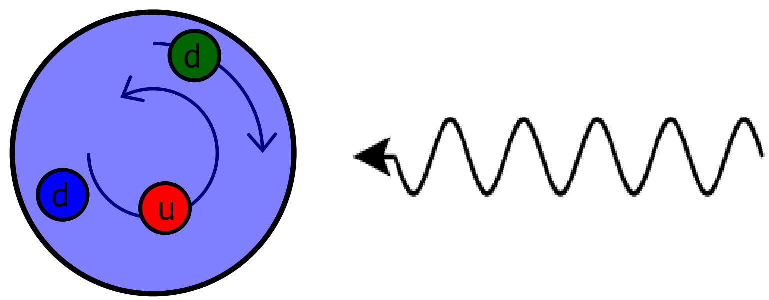
Light-front artifacts

$$J^+ = \frac{J^0 + J^3}{\sqrt{2}}$$



Neutron
at rest

$J^3_{\mathrm{rest}} < 0$



$J^3_{\mathrm{rest}} > 0$

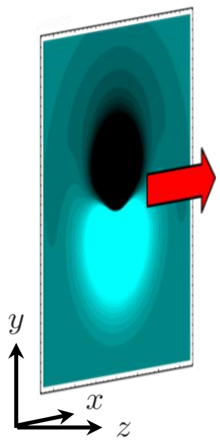
$\vec{S} \otimes$ $\qquad \mu_n = -1.91$

Magnetic dipole moment

[Burkardt, IJMPA18 (2003) 2, 173]
[Carlson, Vanderhaeghen, PRL100 (2008) 032004]

Light-front artifacts

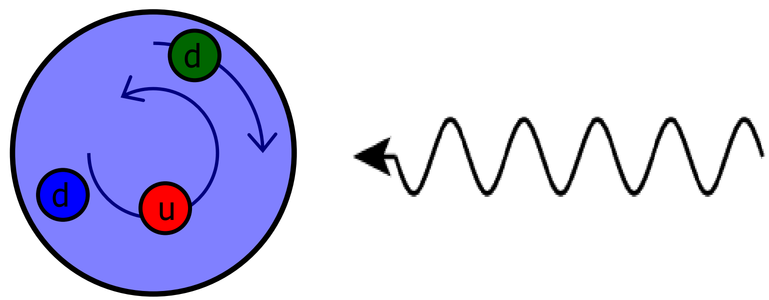
$$J^+ = \frac{J^0 + J^3}{\sqrt{2}}$$



Neutron at rest

$J^3_{\mathrm{rest}} < 0$

$J^3_{\mathrm{rest}} > 0$



$\vec{S} \otimes$

$\mu_n = -1.91$

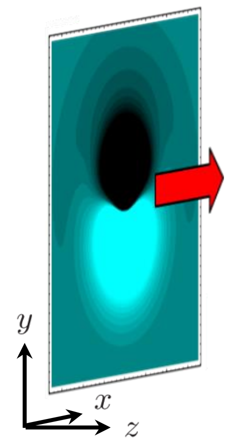
Magnetic dipole moment

$$\vec{E}'_\perp = \gamma\,(\vec{E}_\perp + \vec{v} \times \vec{B}_\perp)$$

[Burkardt, IJMPA18 (2003) 2, 173]
[Carlson, Vanderhaeghen, PRL100 (2008) 032004]

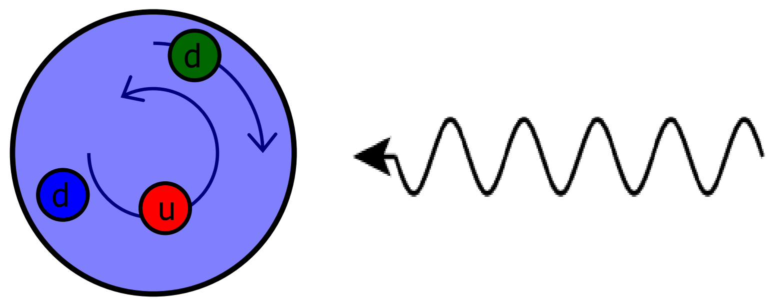
Light-front artifacts

$$J^+ = \frac{J^0 + J^3}{\sqrt{2}}$$



Neutron at rest

$J^3_{\text{rest}} < 0$



$J^3_{\text{rest}} > 0$

$\vec{S} \otimes$ $\quad \mu_n = -1.91$

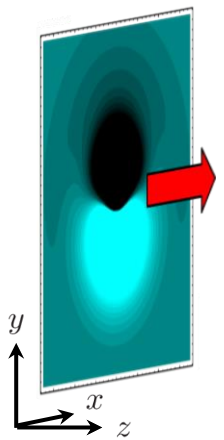
Magnetic dipole moment

$$\vec{E}'_\perp = \gamma\,(\vec{E}_\perp + \vec{v} \times \vec{B}_\perp) \quad \rightarrow \quad \vec{d}'_\perp(\vec{r}\,') = \gamma\,\vec{v} \times \vec{\mu}_\perp(\vec{r})$$

[Burkardt, IJMPA18 (2003) 2, 173]
[Carlson, Vanderhaeghen, PRL100 (2008) 032004]

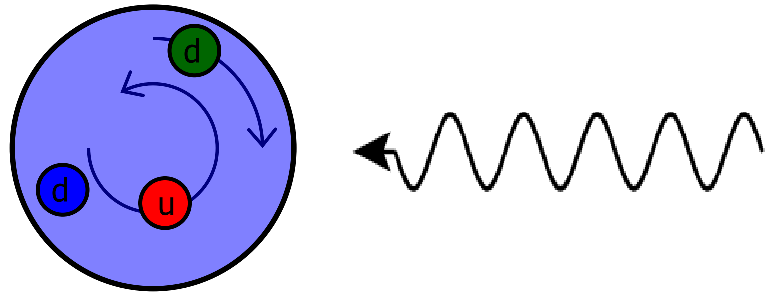
Light-front artifacts

$$J^+ = \frac{J^0 + J^3}{\sqrt{2}}$$



Neutron at rest

$J^3_{\text{rest}} < 0$



$J^3_{\text{rest}} > 0$

$\vec{S} \otimes$ $\quad \mu_n = -1.91$

Magnetic dipole moment

$$\vec{E}'_\perp = \gamma\,(\vec{E}_\perp + \vec{v} \times \vec{B}_\perp) \;\rightarrow\; \vec{d}'_\perp(\vec{r}\,') = \gamma\,\vec{v} \times \vec{\mu}_\perp(\vec{r})$$

$$\boxed{\vec{d}'_\perp = \vec{v} \times \vec{\mu}_\perp} \qquad \int \mathrm{d}^3 r' = \int \frac{\mathrm{d}^3 r}{\gamma}$$

Induced electric dipole moment

[Burkardt, IJMPA18 (2003) 2, 173]
[Carlson, Vanderhaeghen, PRL100 (2008) 032004]

Light-front artifacts

Transverse electric dipole moment

$$\langle \vec{d}_\perp \rangle = \int \mathrm{d}^2 b_\perp \, \vec{b}_\perp \, \rho_{\mathrm{LF}}(\vec{b}_\perp) =$$

[Burkardt, IJMPA18 (2003) 2, 173]
[C.L., EPJC78 (2018) 785]
[Chen, C.L., PRD107 (2023) 9, 096003]

Light-front artifacts

Transverse electric dipole moment $\hat{S} = \vec{S}/|\vec{S}|$ $\vec{e}_z \Leftrightarrow \vec{v}_{\text{IMF}}$

$$\boxed{\langle \vec{d}_\perp \rangle = \int \mathrm{d}^2 b_\perp \, \vec{b}_\perp \, \rho_{\text{LF}}(\vec{b}_\perp) = \frac{\vec{e}_z \times \hat{S}_\perp}{2M_N} \, \kappa_N}$$

[Burkardt, IJMPA18 (2003) 2, 173]
[C.L., EPJC78 (2018) 785]
[Chen, C.L., PRD107 (2023) 9, 096003]

Light-front artifacts

Transverse electric dipole moment $\hat{S} = \vec{S}/|\vec{S}|$ $\vec{e}_z \Leftrightarrow \vec{v}_{\mathrm{IMF}}$

$$\boxed{\langle \vec{d}_\perp \rangle = \int \mathrm{d}^2 b_\perp \, \vec{b}_\perp \, \rho_{\mathrm{LF}}(\vec{b}_\perp) = \frac{\vec{e}_z \times \hat{S}_\perp}{2M_N} \, \kappa_N}$$

$$= \underbrace{\frac{\vec{e}_z \times \hat{S}_\perp}{2M_N} \, G_M(0)} - \frac{\vec{e}_z \times \hat{S}_\perp}{2M_N} \, G_E(0)$$

Expected from Lorentz transformation

[Burkardt, IJMPA18 (2003) 2, 173]
[C.L., EPJC78 (2018) 785]
[Chen, C.L., PRD107 (2023) 9, 096003]

Light-front artifacts

Transverse electric dipole moment $\hat{S} = \vec{S}/|\vec{S}|$ $\vec{e}_z \Leftrightarrow \vec{v}_{\text{IMF}}$

$$\boxed{\langle \vec{d}_\perp \rangle = \int \mathrm{d}^2 b_\perp \, \vec{b}_\perp \, \rho_{\text{LF}}(\vec{b}_\perp) = \frac{\vec{e}_z \times \hat{S}_\perp}{2M_N} \, \kappa_N}$$

$$= \underbrace{\frac{\vec{e}_z \times \hat{S}_\perp}{2M_N} G_M(0)}_{\text{Expected from Lorentz transformation}} - \frac{\vec{e}_z \times \hat{S}_\perp}{2M_N} \overbrace{G_E(0)}^{\text{Electric charge}}$$

[Burkardt, IJMPA18 (2003) 2, 173]
[C.L., EPJC78 (2018) 785]
[Chen, C.L., PRD107 (2023) 9, 096003]

Light-front artifacts

Transverse electric dipole moment

$\hat{S} = \vec{S}/|\vec{S}|$ $\quad$ $\vec{e}_z \Leftrightarrow \vec{v}_{\text{IMF}}$

$$\boxed{\langle \vec{d}_\perp \rangle = \int \mathrm{d}^2 b_\perp \, \vec{b}_\perp \, \rho_{\text{LF}}(\vec{b}_\perp) = \frac{\vec{e}_z \times \hat{S}_\perp}{2M_N} \, \kappa_N}$$

$$= \underbrace{\frac{\vec{e}_z \times \hat{S}_\perp}{2M_N} G_M(0)}_{\text{Expected from Lorentz transformation}} - \underbrace{\frac{\vec{e}_z \times \hat{S}_\perp}{2M_N}}_{\text{Transverse shift of center of } P^+} \overbrace{G_E(0)}^{\text{Electric charge}}$$

[Burkardt, IJMPA18 (2003) 2, 173]
[C.L., EPJC78 (2018) 785]
[Chen, C.L., PRD107 (2023) 9, 096003]

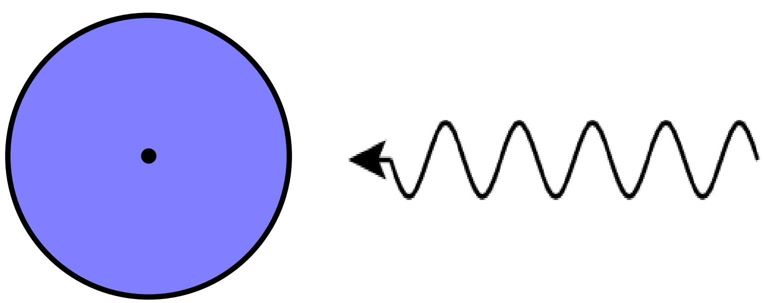
Light-front artifacts

Transverse electric dipole moment $\hat{S} = \vec{S}/|\vec{S}|$ $\vec{e}_z \Leftrightarrow \vec{v}_{\text{IMF}}$

$$\boxed{\langle \vec{d}_\perp \rangle = \int \mathrm{d}^2 b_\perp \, \vec{b}_\perp \, \rho_{\text{LF}}(\vec{b}_\perp) = \frac{\vec{e}_z \times \hat{S}_\perp}{2M_N} \, \kappa_N}$$

$$= \underbrace{\frac{\vec{e}_z \times \hat{S}_\perp}{2M_N} \, G_M(0)}_{\text{Expected from Lorentz transformation}} - \underbrace{\frac{\vec{e}_z \times \hat{S}_\perp}{2M_N}}_{\text{Transverse shift of center of } P^+} \overbrace{G_E(0)}^{\text{Electric charge}}$$

Nucleon at rest



[Burkardt, IJMPA18 (2003) 2, 173]
[C.L., EPJC78 (2018) 785]
[Chen, C.L., PRD107 (2023) 9, 096003]

Light-front artifacts

Transverse electric dipole moment $\hat{S} = \vec{S}/|\vec{S}|$ $\vec{e}_z \Leftrightarrow \vec{v}_{\text{IMF}}$

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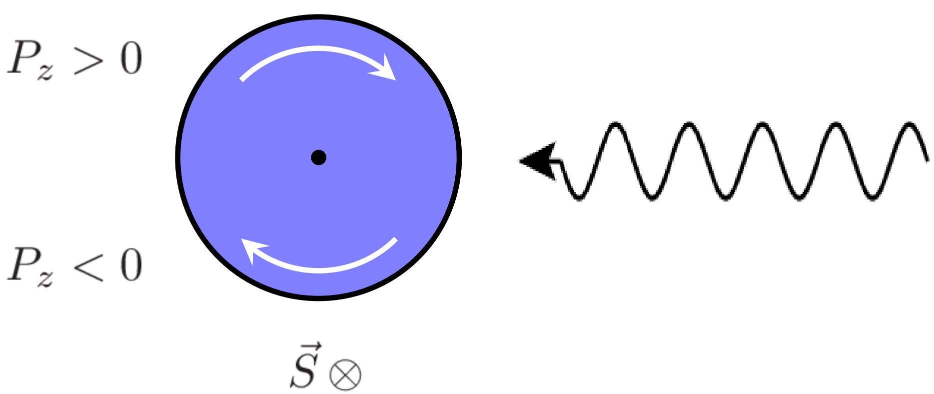
Nucleon at rest



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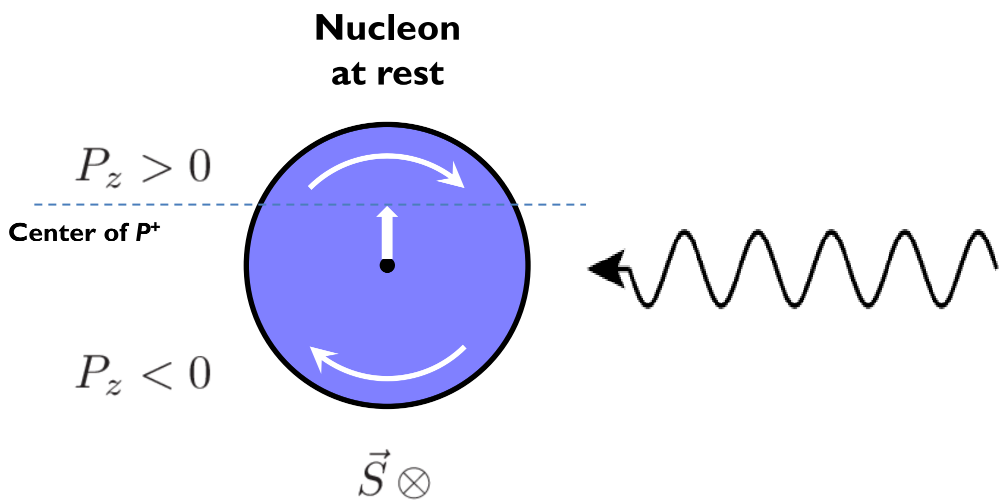

[Burkardt, IJMPA18 (2003) 2, 173]
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Phase-space picture

Position operator in QFT

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Position operator	Canonical relation	Vector under rotations	Compatibility of components	Four-vector transformation
	$[R^i, P^j] = i\delta^{ij}$	$[J^i, R^j] = i\epsilon^{ijk} R^k$	$[R^i, R^j] = 0$	$R'^\mu = \Lambda^\mu{}_\nu R^\nu$

[Pryce, PRSLA195 (1948) 62]
[Newton, Wigner, RMP21 (1949) 3, 400]
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***P*⁺**	$R^i_\perp = \frac{1}{P^+} \int \mathrm{d}x^- \, \mathrm{d}^2x_\perp \, x^i_\perp T^{++} \overset{x^+=0}{=} -\frac{B^i_\perp}{P^+}$	✓	✓	✓	✗	**2D**

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Energy	$R^i_E = \frac{1}{P^0}\int \mathrm{d}^3x \, x^i T^{00} \overset{x^0=0}{=} -\frac{K^i}{P^0}$	✓	✓	✗	✗	**3D**

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Mass	$R^\mu_M = \Lambda^\mu{}_\nu R^\nu_E\big\vert_{\text{rest}}$	✓	✓	✗	✓	

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Mass	$R^\mu_M = \Lambda^\mu{}_\nu R^\nu_E\big\vert_{\text{rest}}$	✓	✓	✗	✓	3D
Canonical	$R^i_c = \frac{P^0 R^i_E + M R^i_M}{P^0 + M}$	✓	✓	✓	✗	

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Localized states on spacelike hypersurface $x^0 = 0$

$$\vec{R}_c\,|\vec{r}\rangle = \vec{r}\,|\vec{r}\rangle$$

$$\boxed{|\vec{r}\rangle = \int \frac{\mathrm{d}^3p}{(2\pi)^3\sqrt{2p^0}}\, e^{-i\vec{p}\cdot\vec{r}}\,|p\rangle}$$

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Spatial distributions (general formalism)

Phase-space representation

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Nucleon Wigner distribution

$$\rho_\psi(\vec{R},\vec{P}) = \int \mathrm{d}^3z\,e^{-i\vec{P}\cdot\vec{z}}\,\psi^*(\vec{R}-\tfrac{\vec{z}}{2})\psi(\vec{R}+\tfrac{\vec{z}}{2})$$
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$$\psi(\vec{r}) = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\,e^{i\vec{p}\cdot\vec{r}}\,\tilde{\psi}(\vec{p})$$
$$\tilde{\psi}(\vec{p}) = \frac{\langle p|\psi\rangle}{\sqrt{2p^0}}$$

[Wigner, PR40 (1932) 749]
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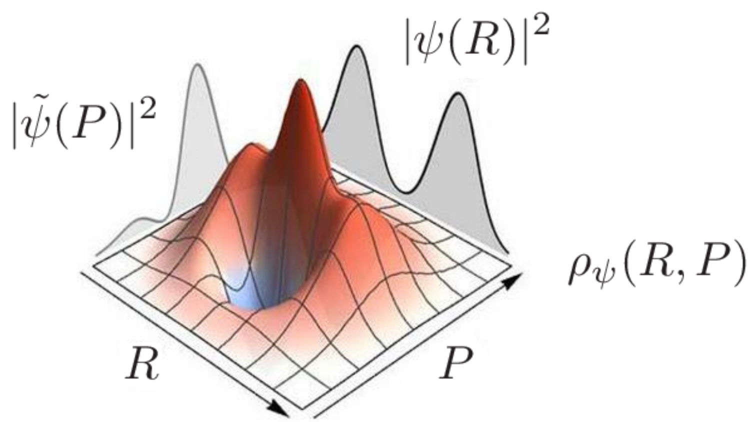
Quasi-probabilistic interpretation

$$\int \mathrm{d}^3R\,\rho_\psi(\vec{R},\vec{P}) = |\tilde{\psi}(\vec{P})|^2$$
$$\int \frac{\mathrm{d}^3P}{(2\pi)^3}\,\rho_\psi(\vec{R},\vec{P}) = |\psi(\vec{R})|^2$$



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Relativistic spatial distributions

Internal distribution (for a state « localized » in phase-space) $x^0 = 0$

$$\langle O\rangle_{\vec R,\vec P}(\vec x) = \int \frac{\mathrm{d}^3\Delta}{(2\pi)^3}\, e^{-i\vec\Delta\cdot(\vec x-\vec R)}\, \frac{\langle P+\frac{\Delta}{2}|O(0)|P-\frac{\Delta}{2}\rangle}{\sqrt{4(P^0)^2-(\Delta^0)^2}}$$

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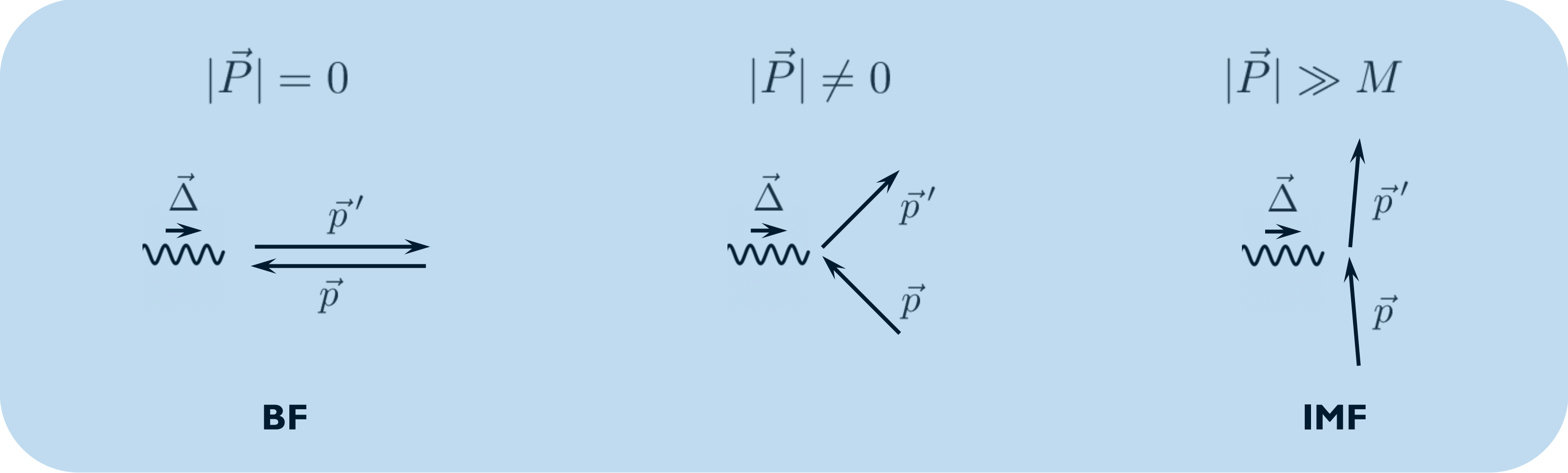
Elastic frames $\Delta^0 = \dfrac{\vec{P}\cdot\vec{\Delta}}{P^0} \stackrel{!}{=} 0$ (no energy transfer ➡ same initial and final boost factor)



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Fundamental features

1) **The notion of spatial distribution relies on simultaneity**
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***Traditional perspective*:** **maintain strict probabilistic interpretation by**

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***Phase-space perspective*:** **relax to quasi-probabilistic interpretation but fully account for frame dependence !**

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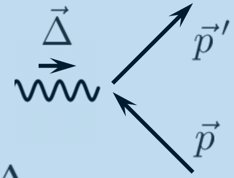
Relativistic spatial distributions

Elastic frame



$$\vec{P} = P_z \vec{e}_z \quad \Rightarrow \quad \Delta^0 = \frac{P_z \Delta_z}{P^0}$$

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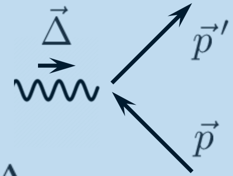

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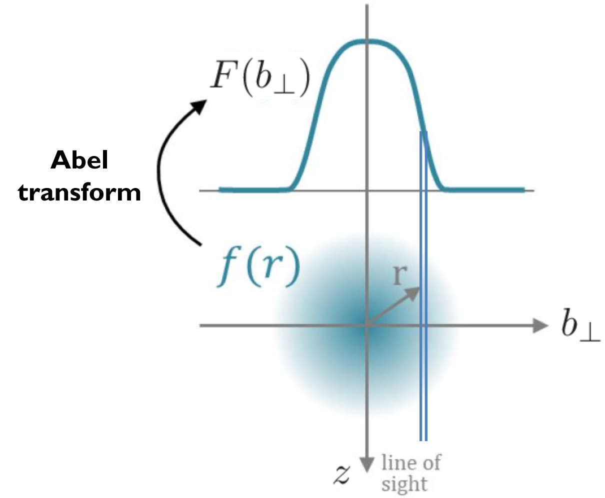

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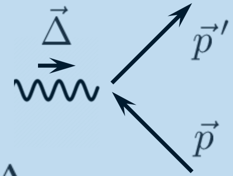

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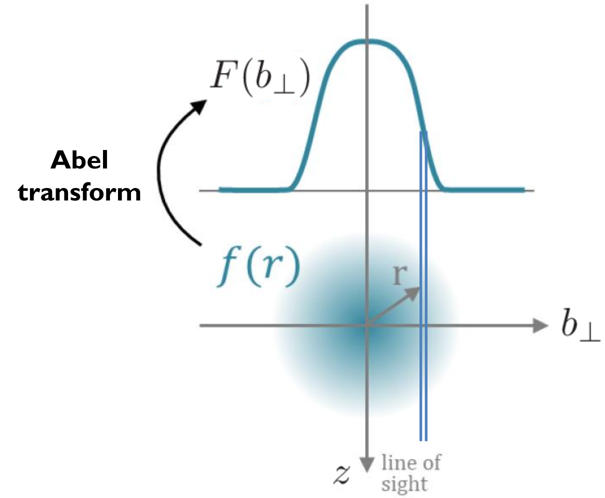

2D charge distribution

$$\boxed{\begin{aligned} \rho_E^{\mathrm{EF}}(\vec{b}_\perp; P_z) &\equiv \int \mathrm{d}z \, \langle J^0 \rangle_{\vec{R}, P_z \vec{e}_z}(\vec{x}) \\ &= \int \frac{\mathrm{d}^2 \Delta_\perp}{(2\pi)^2} \, e^{-i \vec{\Delta}_\perp \cdot \vec{b}_\perp} \left. \frac{\langle p', s' | J^0(0) | p, s \rangle}{2P^0} \right|_{\mathrm{EF}} \end{aligned}}$$

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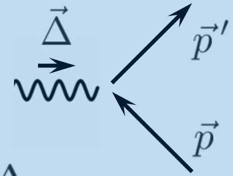

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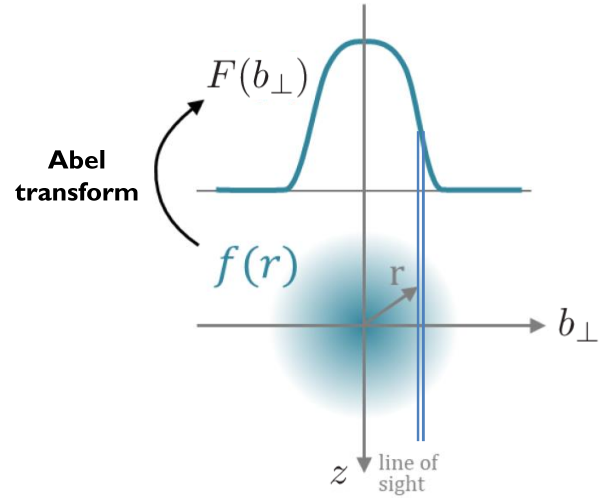

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Interpolates between BF and IMF

$$\rho_E^{\mathrm{EF}}(\vec{b}_\perp; 0) = \int \mathrm{d}z\, \rho_E^{\mathrm{BF}}(\vec{r})$$

$$\rho_E^{\mathrm{EF}}(\vec{b}_\perp; \infty) = \rho_E^{\mathrm{IMF}}(\vec{b}_\perp)$$

$$\vec{b}_\perp = \vec{x}_\perp - \vec{R}_\perp$$

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Four-current amplitude

Lorentz transformation of an off-forward amplitude

$$\langle p', s' | J^\mu(0) | p, s\rangle \neq \Lambda^\mu{}_\nu \langle p'_B, s' | J^\nu(0) | p_B, s\rangle$$

$$p^\mu = \Lambda^\mu{}_\nu \, p_B^\nu$$
$$p'^\mu = \Lambda^\mu{}_\nu \, p_B'^\nu$$

Thomas-Wigner rotation

Relativistic boosts do not commute ! $[K^i, K^j] = -i\epsilon^{ijk} J^k$

[Thomas, Nature 117 (1926) 514]
[Wigner, ZP124 (1948) 665]
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$$|p_{\rm rest}, \sigma\rangle \xrightarrow{\Lambda_{\rm boost}(p_{\rm rest} \to p_B)} |p_B, s_B = \sigma\rangle$$

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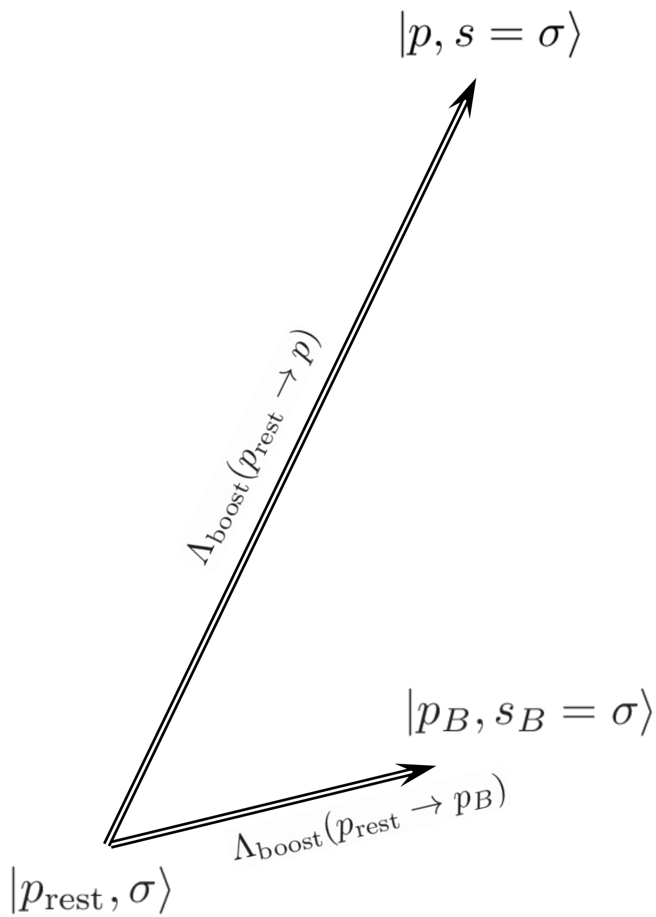

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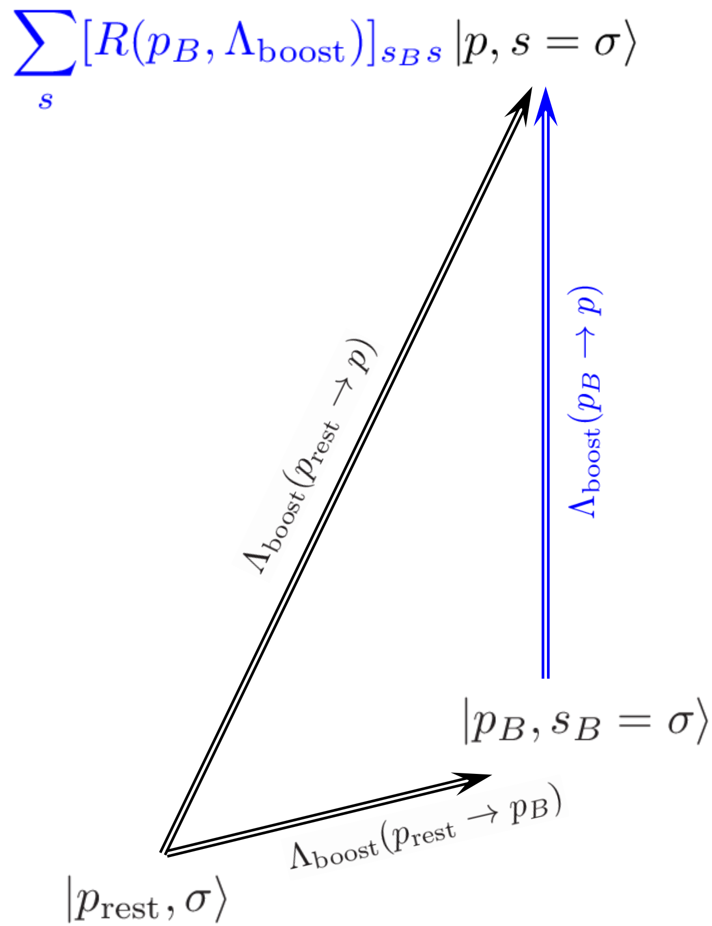

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[Wigner, RMP29 (1957) 255]

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Lorentz transformation of an off-forward amplitude

$$\langle p', s' | J^\mu(0) | p, s\rangle = \sum_{s'_B, s_B} D^{*(j)}_{s'_B s'}(p'_B, \Lambda) D^{(j)}_{s_B s}(p_B, \Lambda)\, \Lambda^\mu{}_\nu \, \langle p'_B, s'_B | J^\nu(0) | p_B, s_B\rangle$$

[Durand, De Celles, Marr, PR126 (1962) 1882]

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Confirmed by explicit evaluation of spinors $\hat{\Delta} = \vec{\Delta}/|\vec{\Delta}|$ $\quad Q^2|_{\text{EF}} = \vec{\Delta}_\perp^2$

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$$\langle p', s'|J^i_\perp(0)|p, s\rangle\big|_{\text{EF}} = 2M_N \qquad \sqrt{\tau} G_M(Q^2)$$

[Chung, Polyzou, Coester, Keister, PRC37 (1988) 2000]
[Rinehimer, Miller, PRC80 (2009) 015201]
[C.L., Wang, PRD105 (2022) 9, 096032]
[Chen, C.L., PRD106 (2022) 11, 116024]

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Boost parameters

$$\beta = \frac{P_z}{P^0} \qquad \gamma = \frac{P^0}{\sqrt{P^2}}$$

$P^2 = M_N^2(1+\tau)$

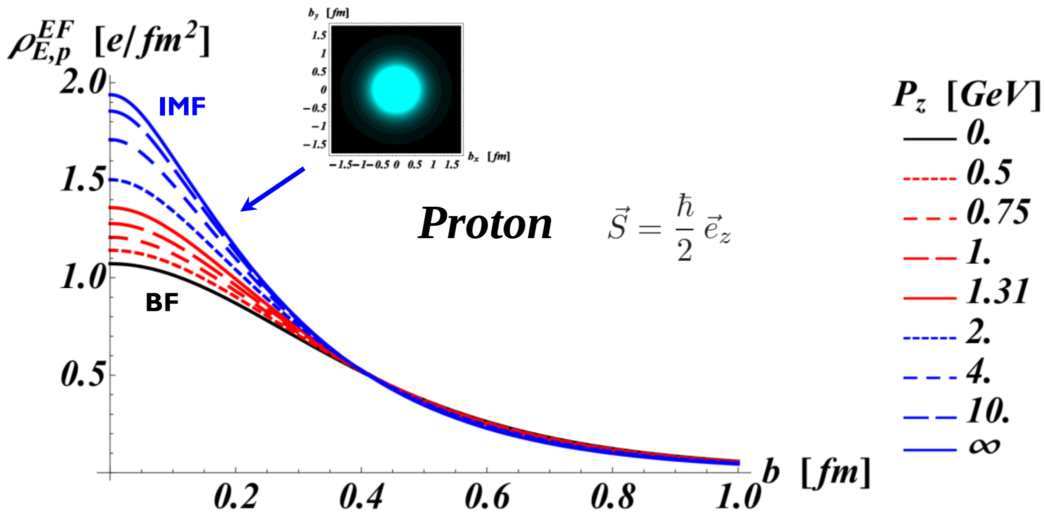
Wigner rotation

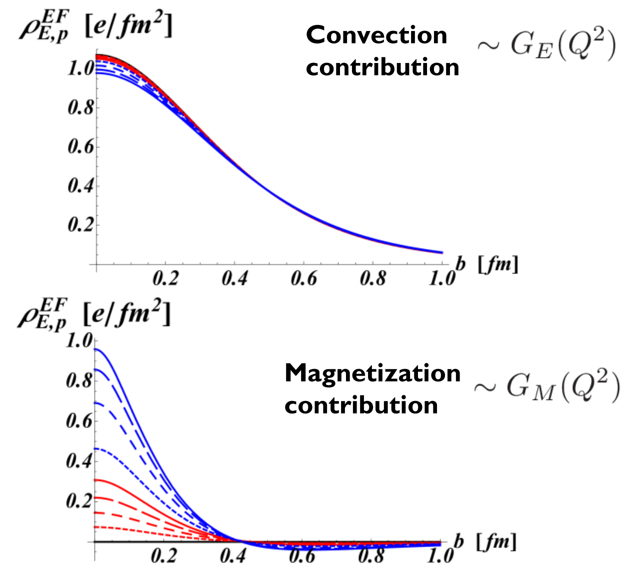
$$\cos\theta = \frac{P^0 + M_N(1+\tau)}{(P^0+M_N)\sqrt{1+\tau}}$$

$$\sin\theta = -\frac{\sqrt{\tau}P_z}{(P^0+M_N)\sqrt{1+\tau}}$$

[Chung, Polyzou, Coester, Keister, PRC37 (1988) 2000]
[Rinehimer, Miller, PRC80 (2009) 015201]
[C.L., Wang, PRD105 (2022) 9, 096032]
[Chen, C.L., PRD106 (2022) 11, 116024]

EF charge distributions (longitudinal polarization)





[C.L., PRL125 (2020) 232002]

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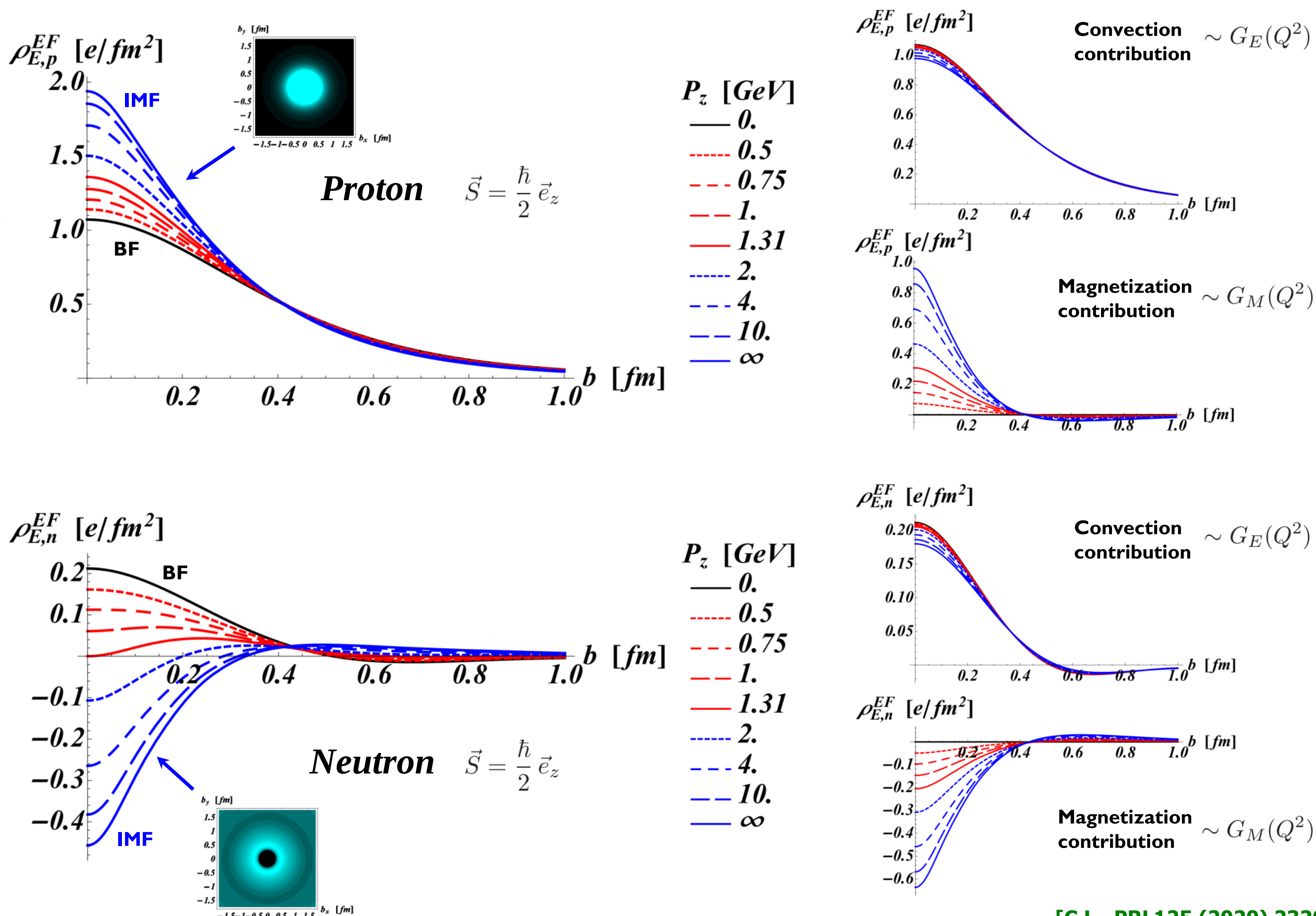
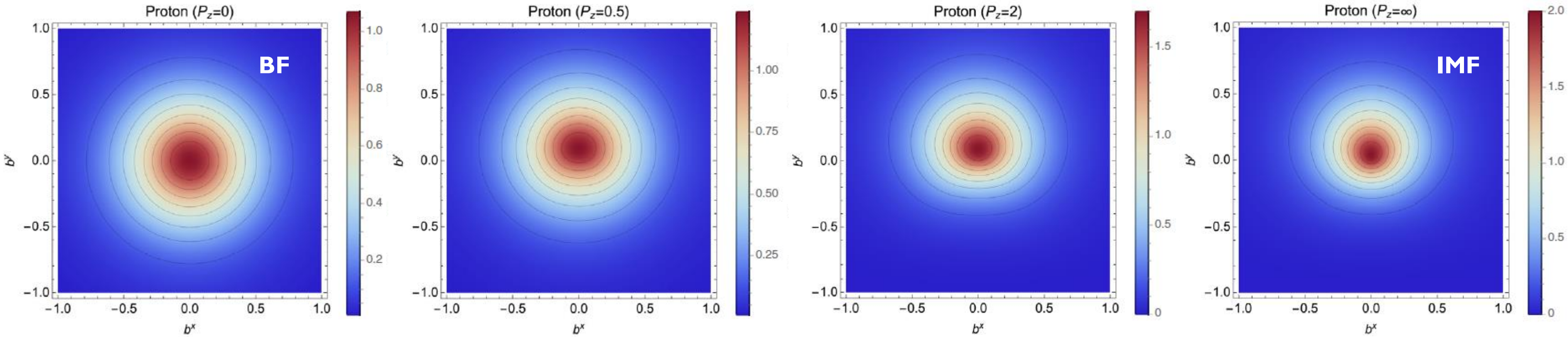


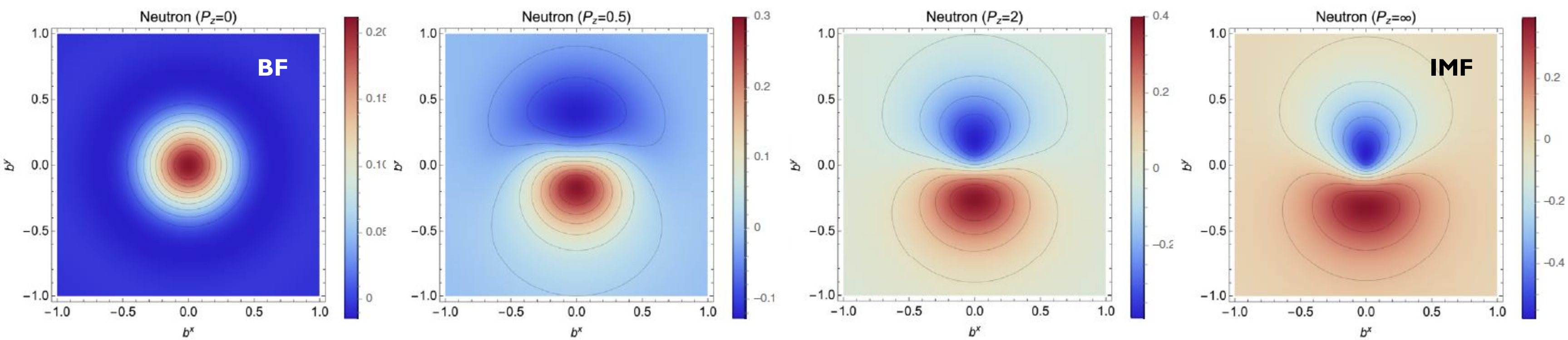
[C.L., PRL125 (2020) 232002]

EF charge distributions (transverse polarization)

Proton $\vec{S} = \frac{\hbar}{2}\,\vec{e}_x$



Neutron $\vec{S} = \frac{\hbar}{2}\,\vec{e}_x$

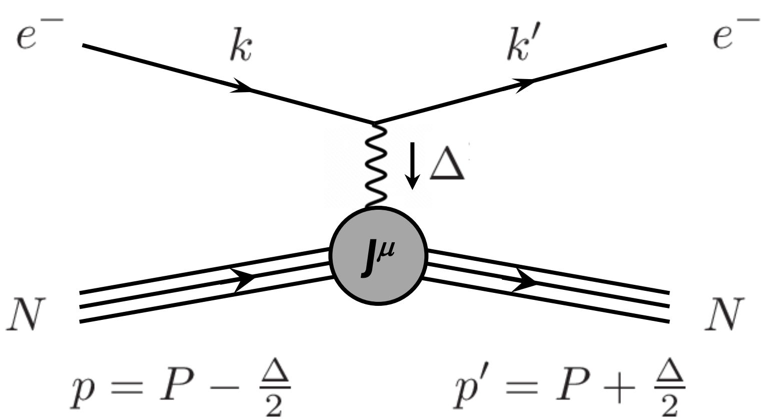


[Kim, Kim, PRD104 (2021) 7, 074003]

Energy-momentum tensor

Local probes

Photon exchange
(~1)



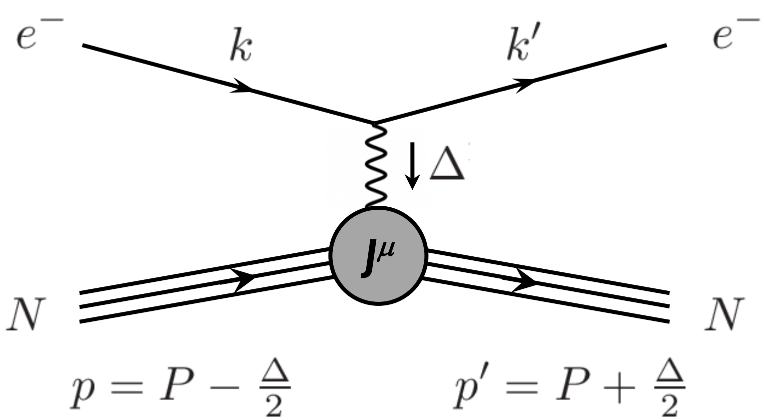
Electromagnetic current

Electromagnetic form factors

Local probes

Photon exchange

(~1)

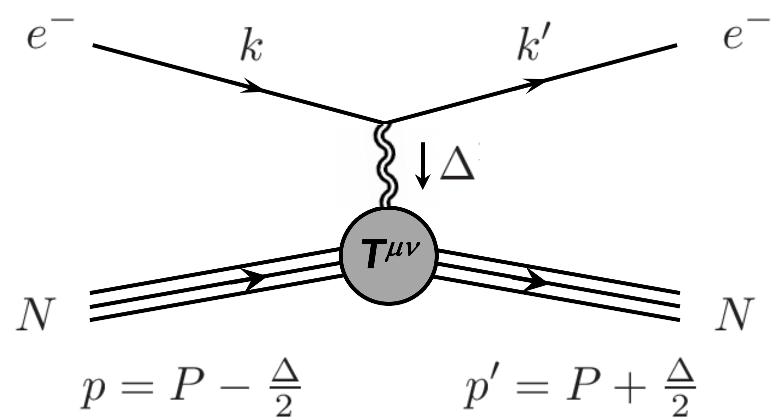


Electromagnetic current

Electromagnetic form factors

Graviton exchange

($\sim 10^{-36}$)

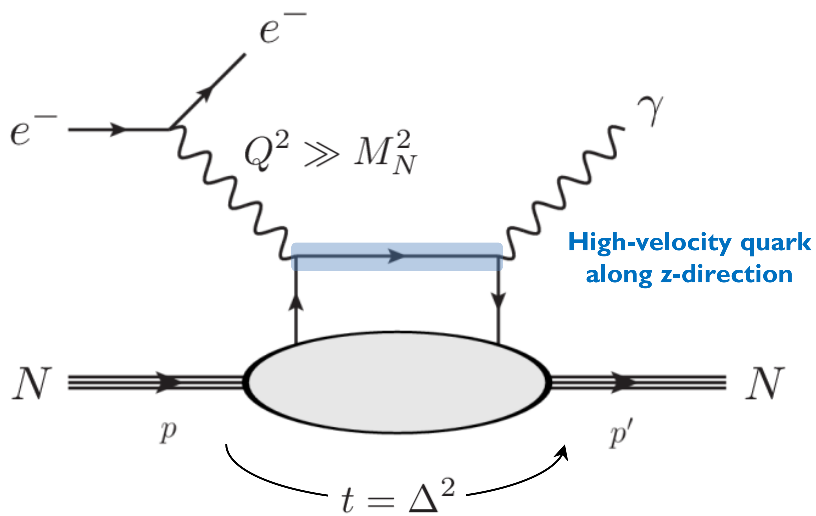


Energy-momentum tensor

Gravitational form factors

A non-local probe

Deeply virtual Compton scattering (DVCS)

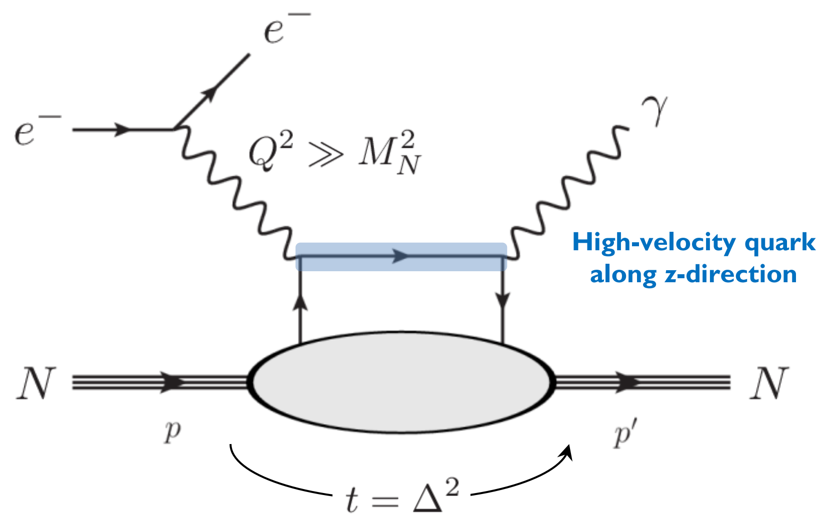


$$\overline{\psi}(-\tfrac{z^-}{2})\gamma^+\mathcal{W}\psi(\tfrac{z^-}{2})$$

[Ji, PRL78 (1997) 610]
[Ji, JPG24 (1998) 1181]
[Diehl, PR388 (2003) 41]
[Belitsky, Radyushkin, PR418 (2005) 1]

A non-local probe

Deeply virtual Compton scattering (DVCS)



$$\overline{\psi}(-\tfrac{z^-}{2})\gamma^+\mathcal{W}\psi(\tfrac{z^-}{2}) \quad \approx \quad \overline{\psi}(0)\gamma^+\psi(0) \quad + \quad \cdots$$

$$J^+ \propto J^0 + J^3$$

Electromagnetic current

[Ji, PRL78 (1997) 610]
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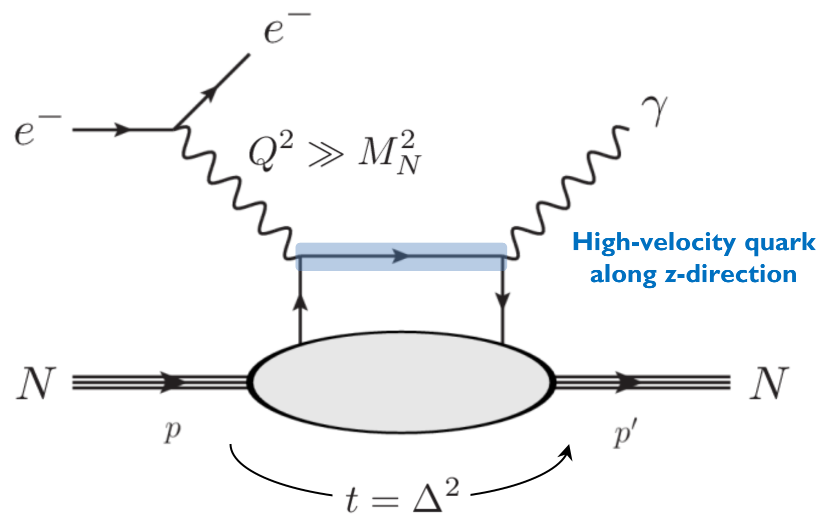

$$\overline{\psi}(-\tfrac{z^-}{2})\gamma^+\mathcal{W}\psi(\tfrac{z^-}{2}) \approx \overline{\psi}(0)\gamma^+\psi(0) + z^-\,\overline{\psi}(0)\gamma^+\tfrac{i}{2}\overset{\leftrightarrow}{D}{}^+\psi(0) + \cdots$$

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Electromagnetic current

$$T^{++} \propto T^{00} + T^{03} + T^{30} + T^{33}$$

Energy-momentum tensor

[Ji, PRL78 (1997) 610]
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Mellin moments

Local LF operators

$$\int \mathrm{d}x\, x^n \int \frac{\mathrm{d}z^-}{2\pi}\, e^{ixP^+z^-}\, \overline{\psi}(-\tfrac{z^-}{2})\Gamma\mathcal{W}\psi(\tfrac{z^-}{2}) =$$

[Ji, PRL78 (1997) 610]
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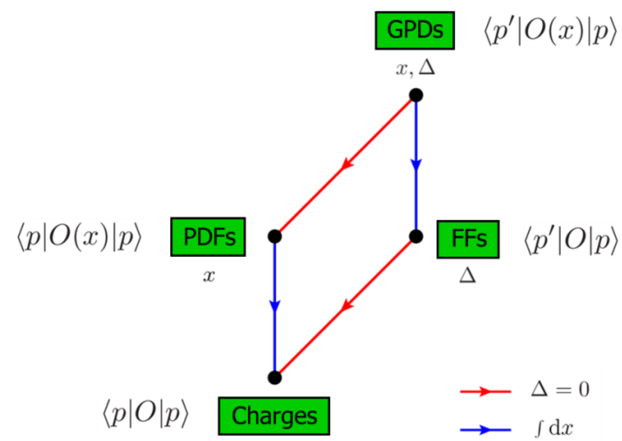
First moment $\Gamma = \gamma^+$

$$\boxed{\begin{aligned}\int \mathrm{d}x\, H_q(x,\xi,t) &= F_1^q(t)\\ \int \mathrm{d}x\, E_q(x,\xi,t) &= F_2^q(t)\end{aligned}}$$

Electromagnetic form factors



[Ji, PRL78 (1997) 610]
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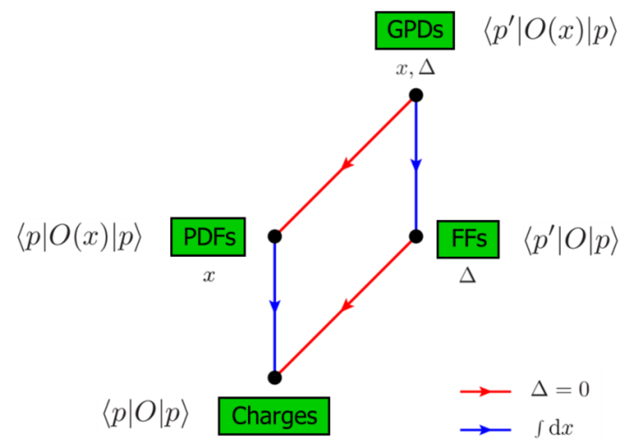

Second moment $\Gamma = \gamma^+$

$$\boxed{\begin{aligned} \int \mathrm{d}x\, x\, H_q(x,\xi,t) &= A_q(t) + 4\xi^2\, C_q(t) \\ \int \mathrm{d}x\, x\, \frac{1}{2}\,[H_q + E_q](x,\xi,t) &= J_q(t) \end{aligned}}$$

Gravitational form factors

[Ji, PRL78 (1997) 610]
[Ji, JPG24 (1998) 1181]
[Diehl, PR388 (2003) 41]
[Belitsky, Radyushkin, PR418 (2005) 1]

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Spin-dependent operators

$\Gamma = \gamma^+\gamma_5$ → **Longitudinal polarization and spin-orbit correlation**

[C.L., Pasquini, PRD84 (2011) 014015]
[C.L., PLB735 (2014) 344]

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[C.L., PLB735 (2014) 344]

$\Gamma = \gamma^i\gamma_5$ ➡ **Transverse spin and color Lorentz force**

[Burkardt, PRD88 (2013) 114502]
[Aslan, Burkardt, Schlegel, PRD100 (2019) 9, 096021]

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[Aslan, Burkardt, Schlegel, PRD100 (2019) 9, 096021]

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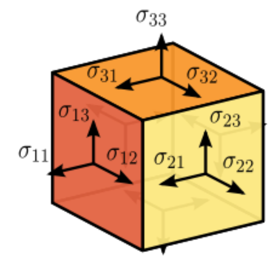
[C.L., Pasquini, PRD93 (2016) 3, 034040]
[Bhoonah, C.L., PLB774 (2017) 435]

Energy-momentum tensor (EMT)

Mass, spin and pressure are all encoded in the EMT



Energy density | Momentum density

$$T^{\mu\nu} = \begin{bmatrix} T^{00} & T^{01} & T^{02} & T^{03} \\ T^{10} & T^{11} & T^{12} & T^{13} \\ T^{20} & T^{21} & T^{22} & T^{23} \\ T^{30} & T^{31} & T^{32} & T^{33} \end{bmatrix}$$

Shear stress

Normal stress (pressure)

Energy flux | Momentum flux

Energy-momentum tensor (EMT)

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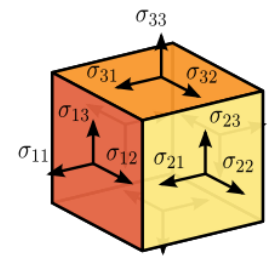

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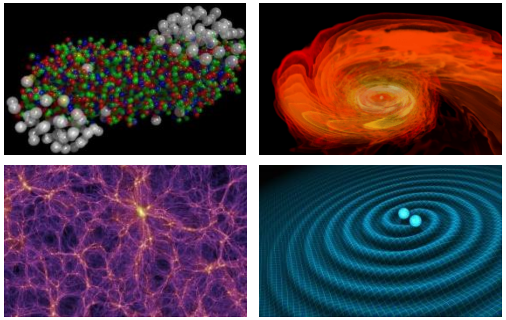
Shear stress

Normal stress (pressure)

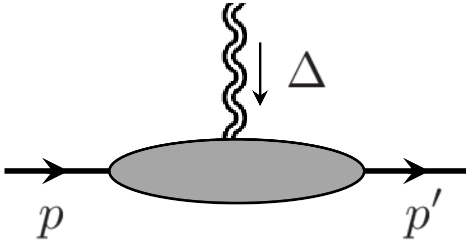
Energy flux | Momentum flux

Central object for

- Nucleon mechanical properties
- Quark-gluon plasma
- Relativistic hydrodynamics
- Stellar structure and dynamics
- Cosmology
- Gravitational waves
- Modified theories of gravitation
- …



Gravitational form factors (GFFs)

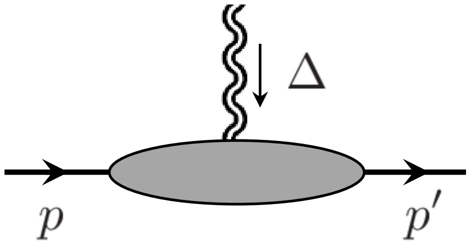


Poincaré symmetry constrains the form of the EMT matrix elements

Symmetrized variables $P = \frac{p' + p}{2}, \qquad \Delta = p' - p, \qquad t = \Delta^2$

$$p'^2 = p^2 = M^2 \Rightarrow \begin{cases} P \cdot \Delta = 0 \\ P^2 = M^2 - \frac{\Delta^2}{4} \end{cases}$$

Gravitational form factors (GFFs)



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Spin-0 target $T^{\mu\nu} = \sum_{a=q,g} T_a^{\mu\nu}$

$$\langle p'|T_a^{\mu\nu}(0)|p\rangle = 2M\left[\frac{P^\mu P^\nu}{M} A_a(t) + \frac{\Delta^\mu\Delta^\nu - g^{\mu\nu}\Delta^2}{M} C_a(t) + M g^{\mu\nu}\bar{C}_a(t)\right]$$

[Kobzarev, Okun, JETP16 (1963) 5, 1343]
[Pagels, PR144 (1966) 4, 1250]
[Ji, PRL78 (1997) 610]

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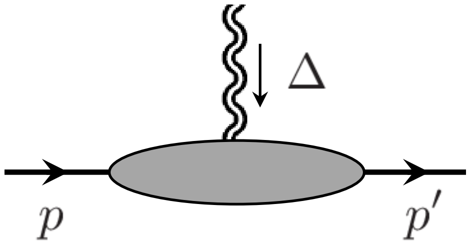

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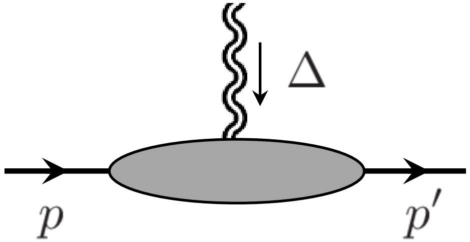

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[Kobzarev, Okun, JETP16 (1963) 5, 1343]
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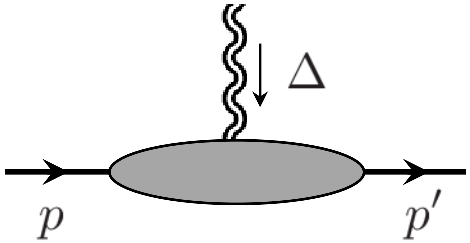

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$$\langle p'|T_a^{\mu\nu}(0)|p\rangle = 2M\left[\frac{P^\mu P^\nu}{M}A_a(t) + \frac{\Delta^\mu\Delta^\nu - g^{\mu\nu}\Delta^2}{M}C_a(t) + Mg^{\mu\nu}\bar{C}_a(t)\right]$$

Non-conserved

$$0 = \langle p'|\partial_\mu T^{\mu\nu}(x)|p\rangle = i\Delta_\mu\langle p'|T^{\mu\nu}(x)|p\rangle \Rightarrow \boxed{\sum_a \bar{C}_a(t) = 0}$$

[Kobzarev, Okun, JETP16 (1963) 5, 1343]
[Pagels, PR144 (1966) 4, 1250]
[Ji, PRL78 (1997) 610]

Gravitational form factors (GFFs)

Spin-1/2 target

$$\langle p', s' | T_a^{\mu\nu}(0) | p, s\rangle = \overline{u}(p', s')\Gamma_a^{\mu\nu}(P, \Delta)u(p, s)$$

[Pagels, PR144 (1966) 4, 1250]
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$$\langle p', s' | T_a^{\mu\nu}(0) | p, s \rangle = \overline{u}(p', s') \Gamma_a^{\mu\nu}(P, \Delta) u(p, s)$$

$$\Gamma_a^{\mu\nu}(P,\Delta) = \frac{P^\mu P^\nu}{M} A_a(t) + \frac{\Delta^\mu \Delta^\nu - g^{\mu\nu}\Delta^2}{M} C_a(t) + M g^{\mu\nu} \bar{C}_a(t)$$
$$+ \frac{P^{\{\mu} i\sigma^{\nu\}\lambda}\Delta_\lambda}{2M} J_a(t) - \frac{P^{[\mu} i\sigma^{\nu]\lambda}\Delta_\lambda}{2M} S_a(t)$$

$$x^{\{\mu} y^{\nu\}} = x^\mu y^\nu + x^\nu y^\mu$$
$$x^{[\mu} y^{\nu]} = x^\mu y^\nu - x^\nu y^\mu$$

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$$x^{\{\mu}y^{\nu\}} = x^\mu y^\nu + x^\nu y^\mu$$
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NB: Because of the Dirac equation, alternative but equivalent parametrizations may *look* quite different !

Gordon identity

$$\overline{u}(p', s')\gamma^\mu u(p, s) = \overline{u}(p', s')\left[\frac{P^\mu}{M} + \frac{i\sigma^{\mu\nu}\Delta_\nu}{2M}\right] u(p, s)$$

[Pagels, PR144 (1966) 4, 1250]
[Ji, PRL78 (1997) 610]
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Four-momentum conservation

Expectation value $\langle P_a^\mu \rangle = \dfrac{\langle p | P_a^\mu | p \rangle}{\langle p | p \rangle} =$

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Not a four-vector !
(unless state is massless)

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Light-front version

$$\langle P_{a,\mathrm{LF}}^\mu \rangle = p^\mu A_a(0) + \frac{M^2}{p^+}\, g^{+\mu}\bar{C}_a(0)$$

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Deep-inelastic scattering

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$$p^\mu = \sum_a \langle P_a^\mu \rangle \quad \Rightarrow \quad \boxed{\begin{aligned} \sum_a A_a(0) &= 1 \\ \sum_a \bar{C}_a(0) &= 0 \end{aligned}}$$

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$$\langle P^\mu_a\rangle = \frac{\langle p|P^\mu_a|p\rangle}{\langle p|p\rangle} = \frac{\langle P|T^{0\mu}_a(0)|p\rangle}{2p^0}$$

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Why two sum rules ?

What is the meaning of $\bar{C}_a(0)$?

Mechanical equilibrium

Physical interpretation is simpler in target rest frame

$$\frac{\langle p_{\text{rest}}|\int \mathrm{d}^3x\, T_a^{\mu\nu}(x)|p_{\text{rest}}\rangle}{\langle p_{\text{rest}}|p_{\text{rest}}\rangle} = M \left(\begin{array}{c|ccc} A_a(0)+\bar{C}_a(0) & 0 & 0 & 0 \\ \hline 0 & -\bar{C}_a(0) & 0 & 0 \\ 0 & 0 & -\bar{C}_a(0) & 0 \\ 0 & 0 & 0 & -\bar{C}_a(0) \end{array}\right)$$

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$$\Leftrightarrow \quad \left(\begin{array}{c|ccc} \varepsilon_a & 0 & 0 & 0 \\ \hline 0 & p_a & 0 & 0 \\ 0 & 0 & p_a & 0 \\ 0 & 0 & 0 & p_a \end{array}\right) V$$

➡ $-\bar{C}_a(0)$ **measures the average stress (or pressure) exerted by subsystem** a

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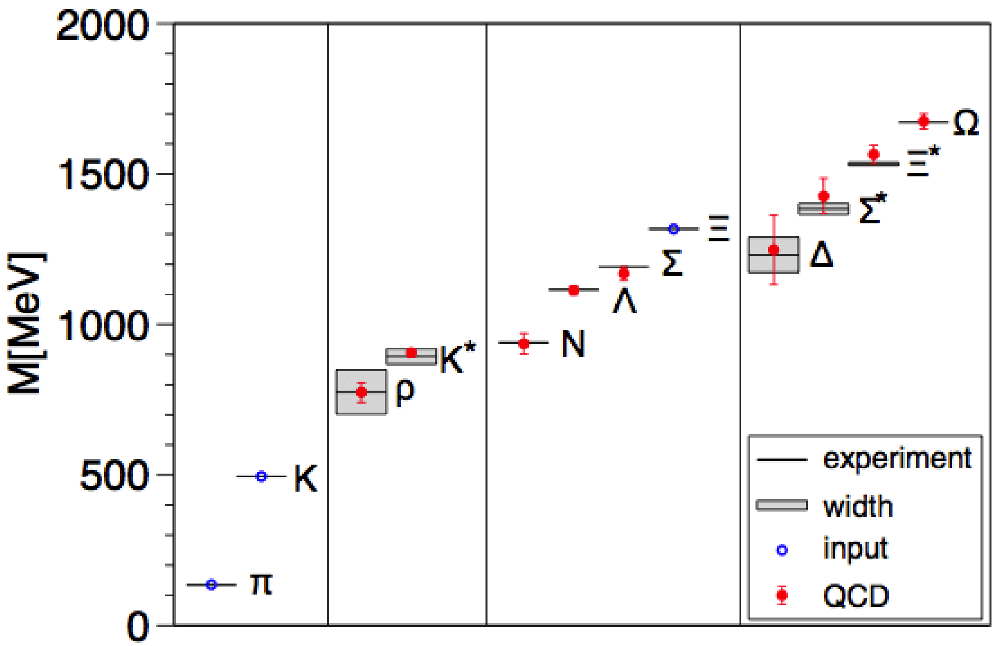
Mechanical equilibrium implies $\sum_a p_a = 0 \quad \Rightarrow \quad \sum_a \bar{C}_a(0) = 0$

Mass decomposition

Hadron spectroscopy

Lattice QCD reproduces very well the light hadron spectrum



[Durr *et al.*, Science 322 (2008) 1224]

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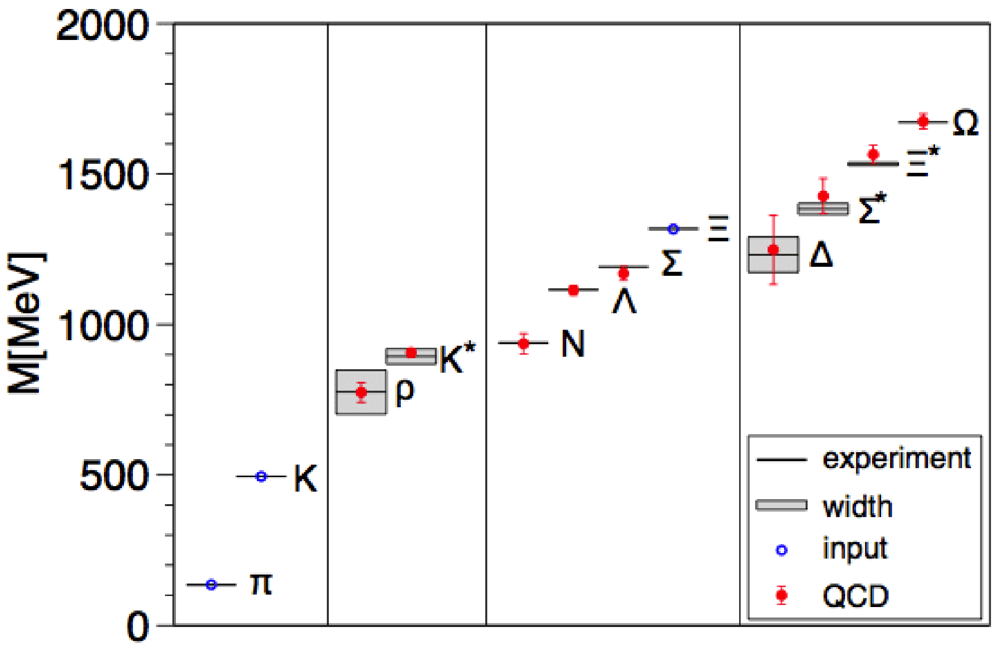
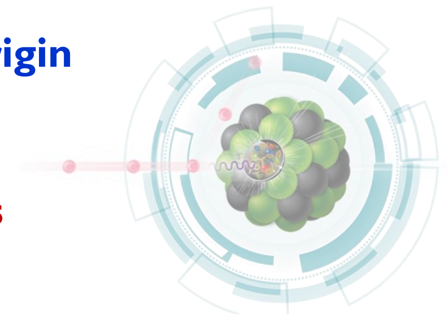


[Durr *et al.*, Science 322 (2008) 1224]

… but this does not tell us much about the origin of the hadron masses

One of the goals of the EIC is to provide clues to this fundamental question



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« Formal » definition $\boxed{p^\mu p_\mu = M^2}$ A global Lorentz-invariant quantity characterizing the physical system

Not additive ! $p^\mu = p_q^\mu + p_g^\mu \quad \Rightarrow \quad M^2 = p_q^2 + p_g^2 + 2\,p_q \cdot p_g$

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« Physical » definition $\boxed{p^\mu u_\mu = M}$

Proper inertia (i.e. rest-frame energy) of the system

CM four-velocity $u^\mu = p^\mu / M$

Additive $p^\mu = p^\mu_q + p^\mu_g \quad \Rightarrow \quad M = p_q \cdot u + p_g \cdot u$

$$= \quad p_q^0 \quad + \quad p_g^0$$ **Rest frame**

What is mass ?

Poincaré symmetry tells us that $\boxed{\langle p|T^{\mu\nu}(0)|p\rangle = 2p^\mu p^\nu}$ $\langle p'|p\rangle = 2p^0(2\pi)^3\delta^{(3)}(\vec{p}\,' - \vec{p})$

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$$\langle H\rangle\big|_{\text{rest}} = \frac{\langle p_{\text{rest}}|\int \mathrm{d}^3x\, T^{00}(x)|p_{\text{rest}}\rangle}{\langle p_{\text{rest}}|p_{\text{rest}}\rangle} = M$$

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$u^\mu = p^\mu/M$

$$\langle P^\mu\rangle u_\mu = \frac{\langle p|\int \mathrm{d}^3x\, T^{0\mu}(x)|p\rangle}{\langle p|p\rangle}\, u_\mu = M$$

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Quark mass and quantum corrections break conformal symmetry

$$\boxed{T^\mu{}_\mu = \underbrace{\left[\frac{\beta(g)}{2g}\, G^2 + \gamma_m\, \overline{\psi} m \psi\right]}_{\text{Trace anomaly}} + \overline{\psi} m \psi}$$

Quark mass matrix

[Crewther, PRL28 (1972) 1421]
[Chanowitz, Ellis, PRD7 (1972) 2490]
[Adler, Collins, Duncan, PRD15 (1977) 1712]
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$$M = \langle \textstyle\int \mathrm{d}\mathcal{V} \left(\frac{\beta(g)}{2g}\, G^2 + \gamma_m\, \overline{\psi} m \psi \right) \rangle + \langle \textstyle\int \mathrm{d}\mathcal{V}\, \overline{\psi} m \psi \rangle$$

$\langle \overline{\psi} m \psi \rangle$ **Nucleon-meson scattering**

$\langle G^2 \rangle$ **Near-threshold heavy meson production**

[Shifman, Vainshtein, Zakharov, PLB78 (1978) 443]
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$$M = \underbrace{\langle \textstyle\int \mathrm{d}\mathcal{V} \left(\frac{\beta(g)}{2g}\, G^2 + \gamma_m\, \overline{\psi} m \psi\right)\rangle}_{\sim 92\%} + \underbrace{\langle \textstyle\int \mathrm{d}\mathcal{V}\, \overline{\psi} m \psi\rangle}_{\sim 8\%}$$

$\mu = 2\,\mathrm{GeV}$

~ 92% (to be measured)

~ 8% (measurement to be improved)

$\langle \overline{\psi} m \psi \rangle$ **Nucleon-meson scattering**

$\langle G^2 \rangle$ **Near-threshold heavy meson production**

Based on this picture, one often concludes that most of the hadron mass comes from gluons !

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The physical interpretation of the « quark » and « gluon » contributions is however not so clear …

[C.L., EPJC78 (2018) 2, 120]

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Rest frame

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Mechanical equilibrium
(virial theorem)

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$$T^{\mu\nu} = \sum_a T_a^{\mu\nu}$$

$$\underbrace{\langle \textstyle\int \mathrm{d}^3x\, T^\mu_{a\mu}\rangle}_{\text{Can be negative !}} = \underbrace{\langle \textstyle\int \mathrm{d}^3x\, T_a^{00}\rangle}_{=\langle H_a\rangle} - \sum_i \underbrace{\langle \textstyle\int \mathrm{d}^3x\, T_a^{ii}\rangle}_{\neq 0} \qquad a = q, g$$

Partial pressure-volume work

[C.L., EPJC78 (2018) 2, 120]

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Partial pressure-volume work

The « gluon » contribution is amplified because the gluon pressure-volume work is negative (attractive forces)

[C.L., EPJC78 (2018) 2, 120]

Energy decomposition

Energy decomposition

Renormalized QCD operators $T^{\mu\nu} = T^{\mu\nu}_q + T^{\mu\nu}_g$

$$T^{\mu\nu}_q = \overline{\psi}\gamma^\mu \tfrac{i}{2}\overleftrightarrow{D}^\nu\psi$$
$$T^{\mu\nu}_g = -G^{\mu\lambda}G^\nu{}_\lambda + \tfrac{1}{4}\,g^{\mu\nu}\,G^2$$

[C.L., EPJC78 (2018) 2, 120]
[Metz, Pasquini, Rodini, PRD102 (2020) 114042]
[C.L., Metz, Pasquini, Rodini, JHEP11 (2021) 121]

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Renormalized QCD operators $T^{\mu\nu} = T^{\mu\nu}_q + T^{\mu\nu}_g$

$$T^{\mu\nu}_q = \overline{\psi}\gamma^\mu \tfrac{i}{2}\overleftrightarrow{D}^\nu\psi$$
$$T^{\mu\nu}_g = -G^{\mu\lambda}G^\nu{}_\lambda + \tfrac{1}{4}\,g^{\mu\nu}\,G^2$$

Rest-frame energy

$$\begin{aligned} M &= \langle H_q\rangle + \langle H_g\rangle \\ &= \langle \textstyle\int \mathrm{d}^3x\,\overline{\psi}\gamma^0 iD^0\psi\rangle + \langle \textstyle\int \mathrm{d}^3x\,\tfrac{1}{2}(\vec{E}^2+\vec{B}^2)\rangle \end{aligned}$$

[C.L., EPJC78 (2018) 2, 120]
[Metz, Pasquini, Rodini, PRD102 (2020) 114042]
[C.L., Metz, Pasquini, Rodini, JHEP11 (2021) 121]

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$$A_a(0) = \langle x\rangle_a$$
$$\bar{C}_a(0) = f_a(\langle x\rangle_q, \langle\overline{\psi}m\psi\rangle)$$

Known but scheme and scale-dependent !

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Known but scheme and scale-dependent !

Refinement

$$M = \langle\textstyle\int \mathrm{d}^3x\,\overline{\psi}\vec{\gamma}\cdot i\vec{D}\psi\rangle + \langle\textstyle\int \mathrm{d}^3x\,\overline{\psi}m\psi\rangle + \langle\textstyle\int \mathrm{d}^3x\,\tfrac{1}{2}(\vec{E}^2+\vec{B}^2)\rangle$$

[C.L., EPJC78 (2018) 2, 120]
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Known but scheme and scale-dependent !

Refinement

$$M = \underbrace{\langle \textstyle\int \mathrm{d}^3x\,\overline{\psi}\vec{\gamma}\cdot i\vec{D}\psi\rangle}_{\sim 21\%\ (\overline{\text{MS}}),\ \sim 38\%\ (\text{D2})} + \underbrace{\langle \textstyle\int \mathrm{d}^3x\,\overline{\psi}m\psi\rangle}_{\sim 8\%\ (\overline{\text{MS}}),\ \sim 8\%\ (\text{D2})} + \underbrace{\langle \textstyle\int \mathrm{d}^3x\,\tfrac{1}{2}(\vec{E}^2+\vec{B}^2)\rangle}_{\sim 71\%\ (\overline{\text{MS}}),\ \sim 54\%\ (\text{D2})}$$

$\mu = 2\,\text{GeV}$

[C.L., EPJC78 (2018) 2, 120]
[Metz, Pasquini, Rodini, PRD102 (2020) 114042]
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Ji's decomposition

Combination of features from *both* trace and energy decompositions

[Ji, PRL74 (1995) 7, 1071]
[Ji, PRD52 (1995) 1, 271]
[Ji, FP16 (2021) 6, 64601]

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Combination of features from *both* trace and energy decompositions

Step 1 $$T^{\mu\nu} = \bar{T}^{\mu\nu} + \hat{T}^{\mu\nu}$$

$$\bar{T}^{\mu\nu} = T^{\mu\nu} - \frac{1}{4}\, g^{\mu\nu} T^{\alpha}{}_{\alpha} \quad \text{Twist-2}$$

$$\hat{T}^{\mu\nu} = \frac{1}{4}\, g^{\mu\nu} T^{\alpha}{}_{\alpha} \quad \text{Twist-4}$$

Poincaré symmetry ensures that this separation is scheme and scale-independent !

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Step 2

$$\bar{T}^{\mu\nu} = \bar{T}^{\mu\nu}_{q} + \bar{T}^{\mu\nu}_{g}$$

$$\hat{T}^{\mu\nu} = \hat{T}^{\mu\nu}_{m} + \hat{T}^{\mu\nu}_{\mathsf{a}}$$

$$\hat{T}^{\mu\nu}_{m} = \frac{1}{4}\, g^{\mu\nu}\, \overline{\psi} m \psi$$

$$\hat{T}^{\mu\nu}_{\mathsf{a}} = \frac{1}{4}\, g^{\mu\nu} \left[\frac{\beta(g)}{2g}\, G^2 + \gamma_m\, \overline{\psi} m \psi\right]$$

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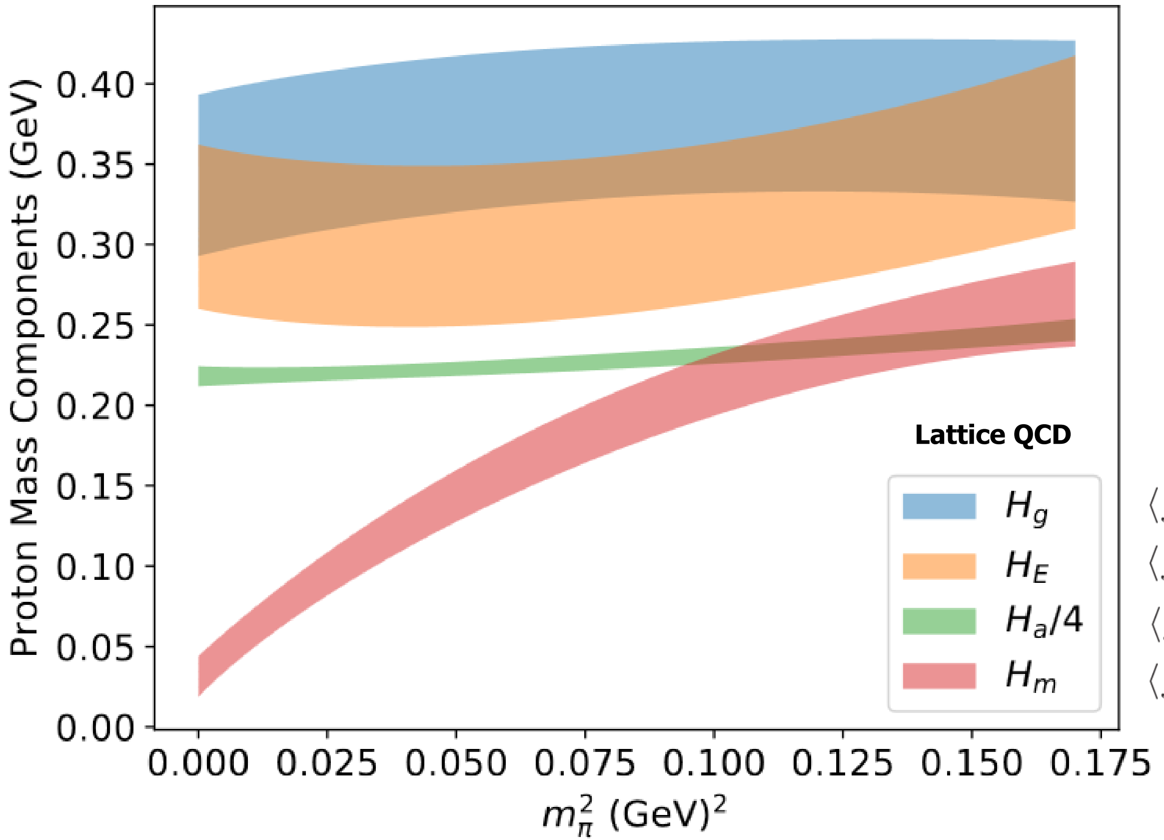
Rest-frame energy

$$\Rightarrow \quad M = \langle \textstyle\int \mathrm{d}^3x\, (\bar{T}^{00}_q - \tfrac{3}{4}\, \overline{\psi} m \psi) \rangle + \langle \textstyle\int \mathrm{d}^3x\, \overline{\psi} m \psi \rangle + \langle \textstyle\int \mathrm{d}^3x\, \bar{T}^{00}_g \rangle + \boxed{\langle \textstyle\int \mathrm{d}^3x\, \hat{T}^{00}_{\mathrm{a}} \rangle}$$

« Quantum anomalous energy »

[Ji, PRL74 (1995) 7, 1071]
[Ji, PRD52 (1995) 1, 271]
[Ji, FP16 (2021) 6, 64601]

Ji's decomposition



$\langle \int d^3x\, \bar{T}_g^{00} \rangle$

$\langle \int d^3x\, (\bar{T}_q^{00} - \frac{3}{4}\, \overline{\psi} m \psi) \rangle$

$\langle \int d^3x\, \hat{T}_a^{00} \rangle$ Determined by mass sum rule

$\langle \int d^3x\, \overline{\psi} m \psi \rangle$

[Yang *et al.*, PRL121 (2018) 21, 212001]

Ji's decomposition

The physical interpretation of the « quark » and « gluon » contributions is however not so clear …

[C.L., EPJC78 (2018) 2, 120]
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$$T_a^{00} = \underbrace{\bar{T}_a^{00}}_{= \frac{3}{4} T_a^{00} + \boxed{\frac{1}{4}\sum_i T_a^{ii}}} + \underbrace{\hat{T}_a^{00}}_{= \frac{1}{4} T_a^{00} \boxed{- \frac{1}{4}\sum_i T_a^{ii}}} \qquad a = q, g$$

$$\sum_i \langle \int \mathrm{d}^3x \, T_a^{ii} \rangle \neq 0$$

Partial pressure-volume work

[C.L., EPJC78 (2018) 2, 120]
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Partial pressure-volume work

Also, it is tempting to write the classical relations

$$\bar{T}_q^{00} - \frac{3}{4}\,\overline{\psi} m \psi \stackrel{?}{=} \overline{\psi} \vec{\gamma} \cdot i\vec{D}\psi$$

$$\bar{T}_g^{00} \stackrel{?}{=} \frac{1}{2}(\vec{E}^2 + \vec{B}^2)$$

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… but there is no scheme where both are simultaneously valid !

[C.L., EPJC78 (2018) 2, 120]
[Metz, Pasquini, Rodini, PRD102 (2020) 114042]
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Mass decompositions (in D2 scheme $g_{\mu\nu}\langle T_q^{\mu\nu}\rangle = [A_q(0) + 4\bar{C}_q(0)]M = \sigma_q$)

Trace decomposition

$$M = \langle \int \mathrm{d}^3x\, \overline{\psi} m\psi \rangle + \langle \int \mathrm{d}^3x\, (\tfrac{\beta(g)}{2g}\, G^2 + \gamma_m \overline{\psi} m\psi) \rangle$$

Energy decomposition

$$M = \langle \int \mathrm{d}^3x\, (T_q^{00} - \overline{\psi} m\psi) \rangle + \langle \int \mathrm{d}^3x\, \overline{\psi} m\psi \rangle + \langle \int \mathrm{d}^3x\, T_g^{00} \rangle$$

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1 independent input

$$M = \underbrace{\langle \int \mathrm{d}^3x\, \overline{\psi} m \psi \rangle}_{\sigma_q} + \underbrace{\langle \int \mathrm{d}^3x\, (\tfrac{\beta(g)}{2g}\, G^2 + \gamma_m \overline{\psi} m \psi) \rangle}_{M - \sigma_q}$$

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Energy decomposition

2 independent inputs

$$M = \underbrace{\langle \int \mathrm{d}^3x\,(T_q^{00} - \overline{\psi}m\psi)\rangle}_{[A_q(0)+\bar{C}_q(0)]\,M-\sigma_q} + \underbrace{\langle \int \mathrm{d}^3x\, \overline{\psi}m\psi\rangle}_{\sigma_q} + \underbrace{\langle \int \mathrm{d}^3x\, T_g^{00}\rangle}_{[A_g(0)+\bar{C}_g(0)]\,M}$$

$$A_q(0) + A_g(0) = 1$$
$$\bar{C}_q(0) + \bar{C}_g(0) = 0$$

Ji's decomposition

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Energy decomposition

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Energy decomposition

2 independent inputs

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Ji's decomposition

2 independent inputs

$$M = \underbrace{\langle \int \mathrm{d}^3x\, (\bar{T}_q^{00} - \tfrac{3}{4}\, \overline{\psi} m \psi)\rangle}_{\frac{3}{4}[A_q(0)\, M - \sigma_q]} + \underbrace{\langle \int \mathrm{d}^3x\, \overline{\psi} m \psi \rangle}_{\sigma_q} + \underbrace{\langle \int \mathrm{d}^3x\, \bar{T}_g^{00}\rangle}_{\frac{3}{4} A_g(0)\, M} + \underbrace{\langle \int \mathrm{d}^3x\, \tfrac{1}{4}(\tfrac{\beta(g)}{2g}\, G^2 + \gamma_m \overline{\psi} m \psi)\rangle}_{\frac{1}{4}(M - \sigma_q)}$$

Mechanical equilibrium
(virial theotem)

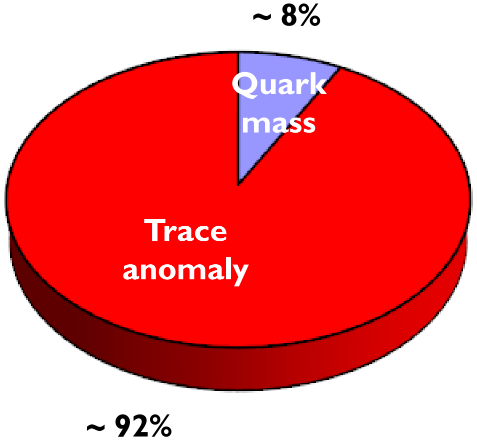
$$\langle \int \mathrm{d}^3x\, T^{ii}\rangle = 0$$

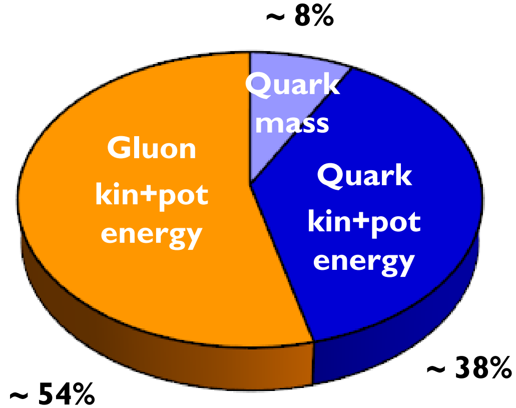
$$\boxed{\langle \int \mathrm{d}^3x\, (\bar{T}_q^{00} - \tfrac{3}{4}\, \overline{\psi} m \psi)\rangle + \langle \int \mathrm{d}^3x\, \bar{T}_g^{00}\rangle = 3\, \langle \int \mathrm{d}^3x\, \hat{T}_{\mathrm{a}}^{00}\rangle}$$

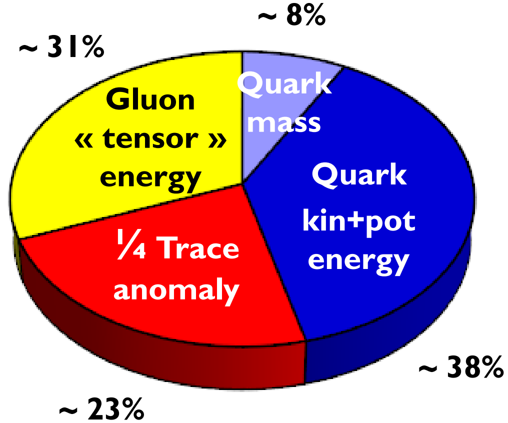
Mass decompositions (in D2 scheme $g_{\mu\nu}\langle T_q^{\mu\nu}\rangle = [A_q(0) + 4\bar{C}_q(0)]M = \sigma_q$)

Trace decomposition | **Energy decomposition** | **Ji's decomposition**

$\mu = 2\,\mathrm{GeV}$







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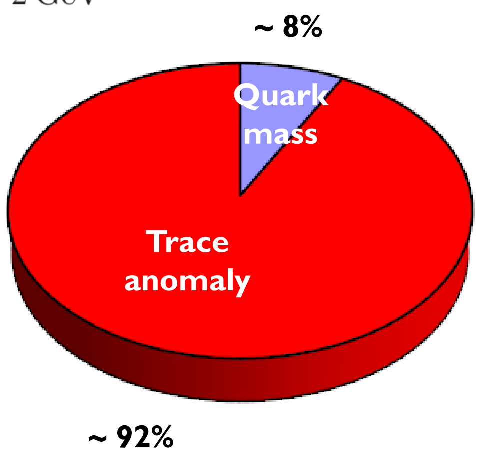
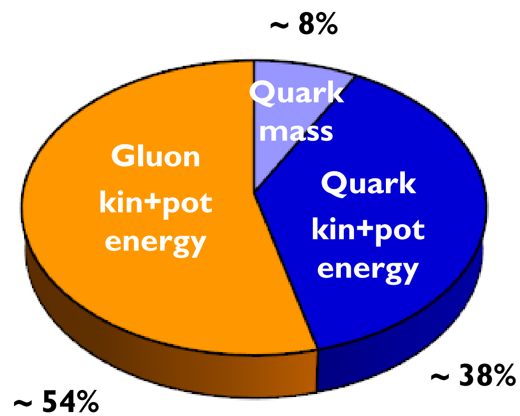
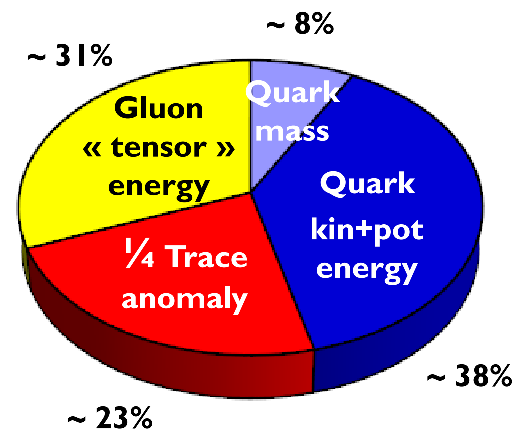
Trace decomposition	Energy decomposition	Ji's decomposition
		

Scale dependence	No	Yes	Yes

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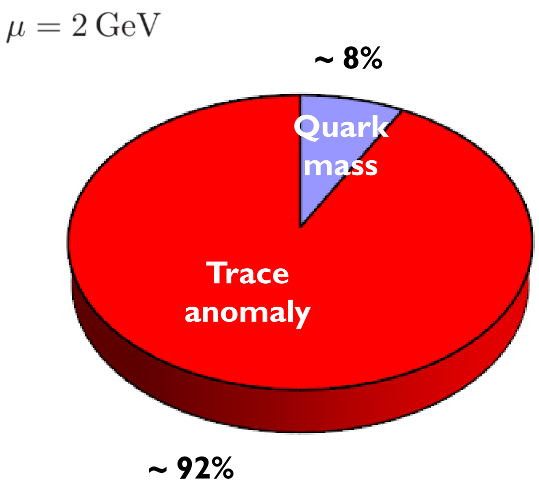
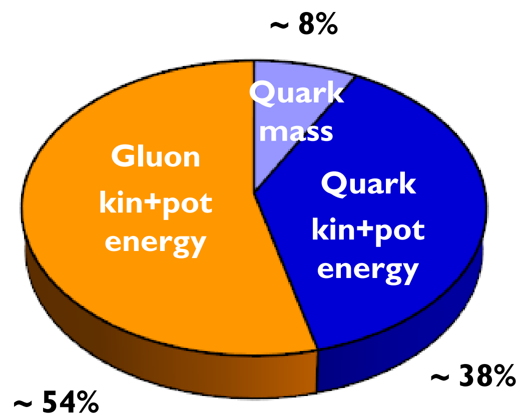
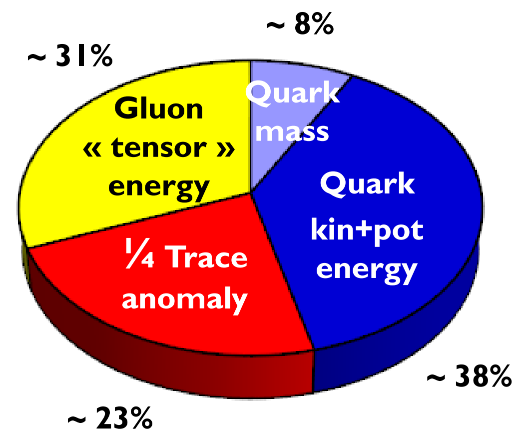
Trace decomposition



Energy decomposition



Ji's decomposition



	Trace decomposition	Energy decomposition	Ji's decomposition
Scale dependence	No	Yes	Yes
Relation to mass	Amplitude level	Operator level	Operator level

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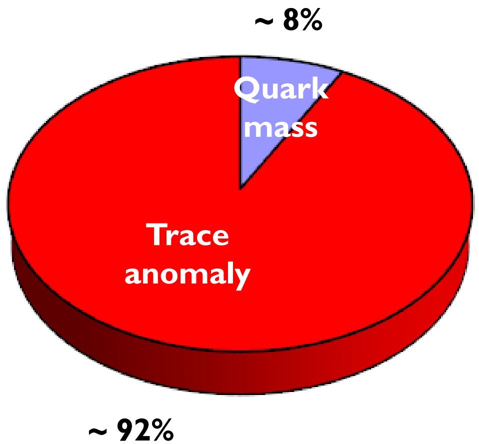

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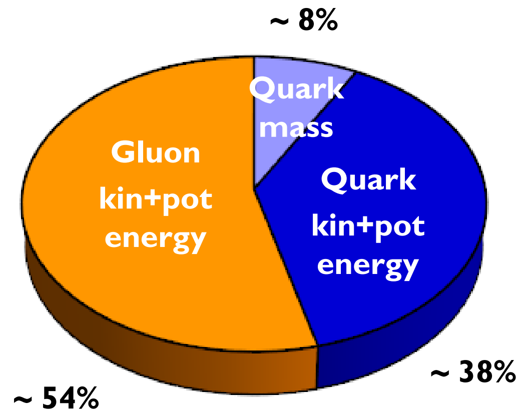
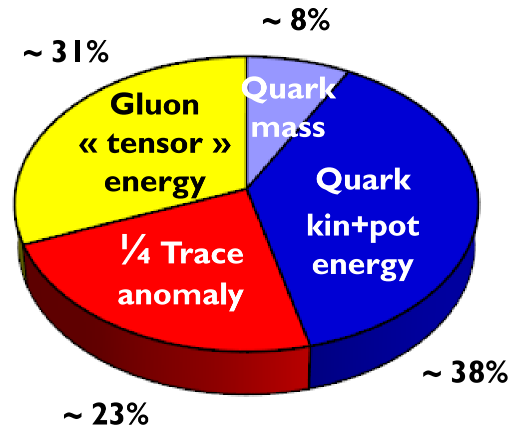


Ji's decomposition



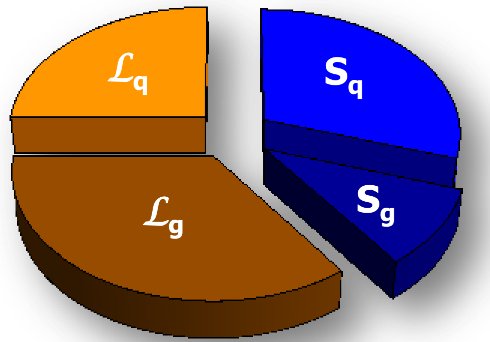
	Trace decomposition	Energy decomposition	Ji's decomposition
Scale dependence	No	Yes	Yes
Relation to mass	Amplitude level	Operator level	Operator level
Virial theorem	Dependent	Independent	Involved

Spin decomposition

Spin sum rules (1990-2008)

Canonical

$$\vec{p} = \frac{\partial L}{\partial \vec{v}}$$



$$\vec{S}_q = \int \mathrm{d}^3r\, \psi^\dagger \tfrac{1}{2}\vec{\Sigma}\psi$$

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$$\vec{S}_g = \int \mathrm{d}^3r\, \vec{E}^a \times \vec{A}^a$$

$$\vec{\mathcal{L}}_g = \int \mathrm{d}^3r\, E^{ai}(\vec{r} \times \vec{\nabla})A^{ai}$$

Gauge fixed ! $\boxed{A^+ = 0}$

[Jaffe, Manohar, NPB337 (1990) 509]

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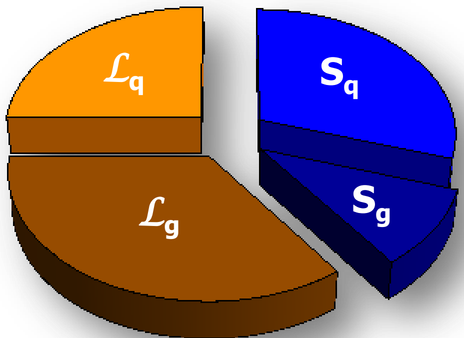

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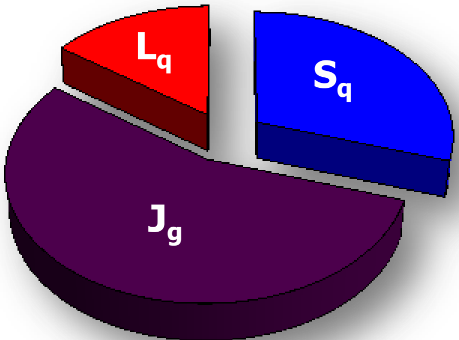
Kinetic

$$\vec{\pi} = m\vec{v} = \vec{p} + g\vec{A}$$



$$\boxed{\vec{D} = -\vec{\nabla} - ig\vec{A}}$$

$$\vec{S}_q = \int \mathrm{d}^3r\, \psi^\dagger \tfrac{1}{2}\vec{\Sigma}\psi$$

$$\vec{L}_q = \int \mathrm{d}^3r\, \psi^\dagger \vec{r} \times (i\vec{D})\psi$$

$$\vec{J}_g = \int \mathrm{d}^3r\, \vec{r} \times (\vec{E}^a \times \vec{B}^a)$$

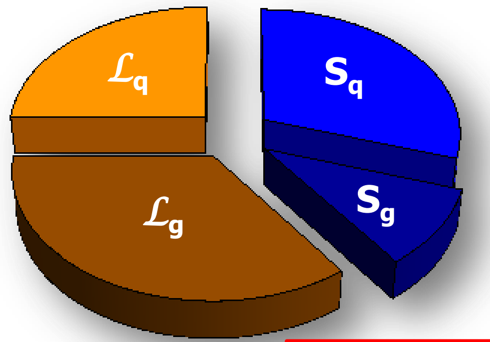
« Incomplete » ΔG **?**

[Ji, PRL78 (1997) 610]

Spin sum rules (2008-now)

$$\vec{A} = \vec{A}_{\text{pure}} + \vec{A}_{\text{phys}}$$

Canonical



$$\boxed{\vec{D}_{\text{pure}} = -\vec{\nabla} - ig\vec{A}_{\text{pure}}}$$

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Gauge invariant !

[Chen, Lu, Sun, Wang, Goldstein, PRL100 (2008) 232002]

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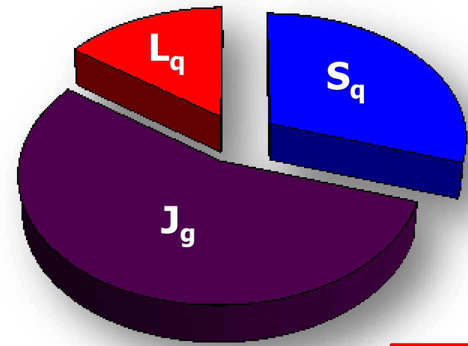

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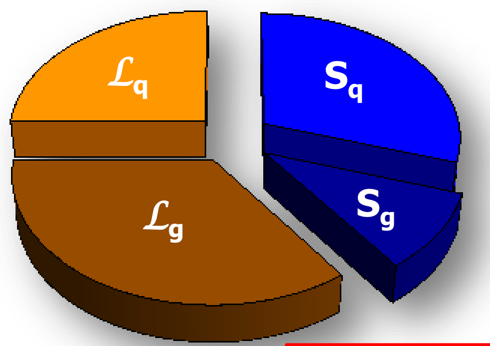

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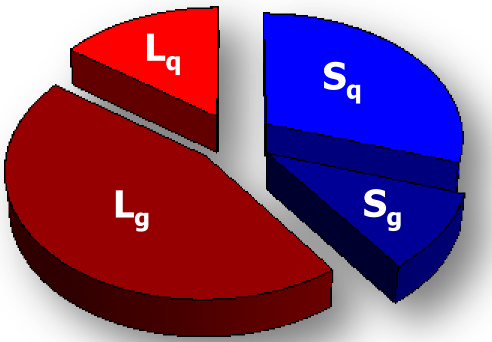

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$$\vec{L}_g = \vec{\mathcal{L}}_g - \int \mathrm{d}^3r\, \rho^a \vec{r} \times \vec{A}^a_{\text{phys}}$$

« Complete »

[Wakamatsu, PRD81 (2010) 114010]

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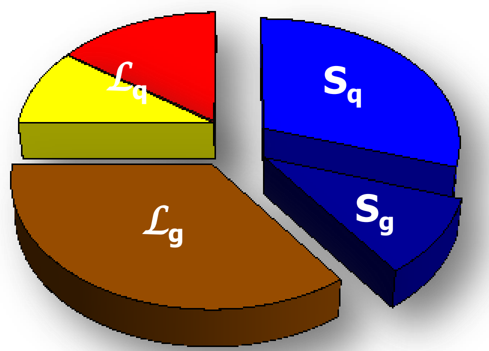
Canonical



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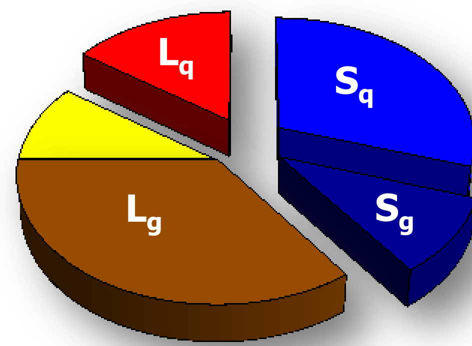

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« Potential » angular momentum

$$\vec{L}_{\text{pot}} = -\int d^3r\, \rho^a \vec{r} \times \vec{A}^a_{\text{phys}}$$

[Wakamatsu, PRD81 (2010) 114010]
[C.L., Leader, PR541 (2014) 163]

Ambiguity

$$\boxed{A^\mu = A^\mu_{\text{pure}} + A^\mu_{\text{phys}}}$$

Pure gauge field $A^\mu_{\text{pure}} \equiv U\,\frac{i}{g}\partial^\mu U^{-1} \quad \Leftrightarrow \quad F^{\mu\nu}_{\text{pure}} = 0$

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[Hatta, PLB708 (2012) 186]
[C.L., PLB719 (2013) 185]
[C.L., PRD187 (2013) 034031]

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⚠ **The split is not unique and reflects the original gauge symmetry**

[Hatta, PLB708 (2012) 186]
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In general, A^μ_{pure} and A^μ_{phys} are non-local functionals of A^μ

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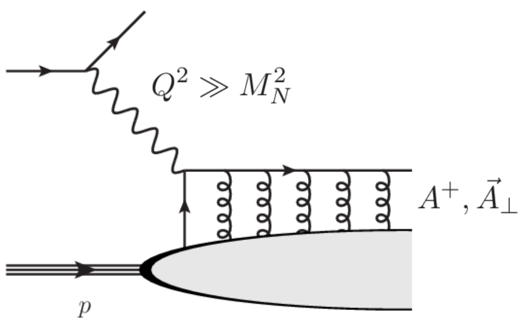
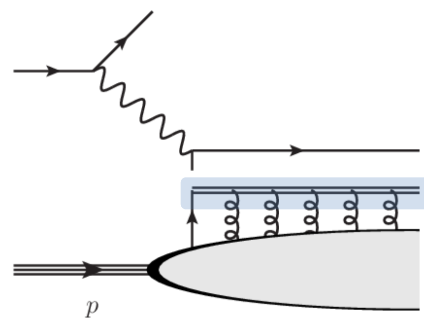
Factorization in high-energy scattering



$\approx$



Non-local

$+$ **higher twists**

$$\mathcal{W} = \mathcal{P}\left[\exp\left(ig\int \mathrm{d}y^- A^+\right)\right]$$

[Hatta, PLB708 (2012) 186]
[C.L., PLB719 (2013) 185]
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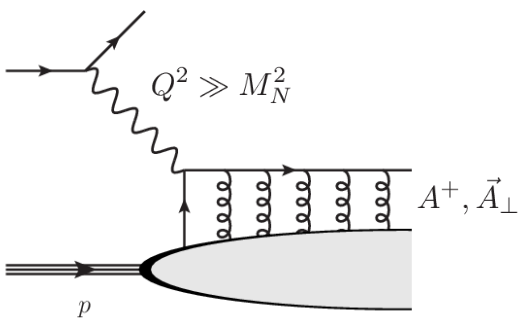

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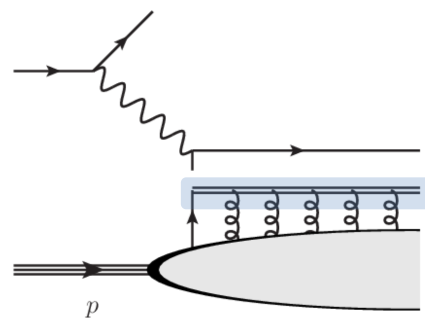

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[Hatta, PLB708 (2012) 186]
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Generalized angular momentum

Poincaré symmetry implies that the following current is conserved

$$J^{\mu\alpha\beta} = \underbrace{x^\alpha T^{\mu\beta} - x^\beta T^{\mu\alpha}}_{\textbf{Orbital}} + \underbrace{S^{\mu\alpha\beta}}_{\textbf{Intrinsic}}$$

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$$\begin{aligned} \Gamma^{\mu\nu}_a(P,\Delta) = \frac{P^\mu P^\nu}{M}\, A_a(t) + \frac{\Delta^\mu\Delta^\nu - g^{\mu\nu}\Delta^2}{M}\, C_a(t) + M g^{\mu\nu} \bar{C}_a(t) \\ + \frac{P^{\{\mu} i\sigma^{\nu\}\lambda}\Delta_\lambda}{2M}\, J_a(t) - \frac{P^{[\mu} i\sigma^{\nu]\lambda}\Delta_\lambda}{2M}\, S_a(t) \end{aligned}$$

$$\rightarrow \boxed{\begin{aligned} S_q(t) &= \tfrac{1}{2}\, G^q_A(t) \\ S_g(t) &= 0 \end{aligned}}$$

[Shore, White, NPB581 (2000) 409]
[Bakker, Leader, Trueman, PRD70 (2004) 114001]
[C.L., Leader, PR541 (2014) 163]

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Forward matrix elements of angular momentum are ill-defined

$$x^0 = 0 \qquad \langle p|\int \mathrm{d}^3x\, x^i O(x)|p\rangle = \langle p|O(0)|p\rangle \underbrace{\int \mathrm{d}^3x\, x^i}_{\textbf{Ambiguous !}}$$

Amplitude for compound operators

Standard prescription

$$\langle \int \mathrm{d}^3x\, x^i O(x)\rangle \equiv \lim_{\Delta\to 0} \frac{\langle P+\frac{\Delta}{2}|\int \mathrm{d}^3x\, x^i O(x)|P-\frac{\Delta}{2}\rangle}{\langle p|p\rangle}$$

$$=$$

[Jaffe, Manohar, NPB337 (1990) 509]
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$$= \frac{\langle p|O(0)|p\rangle}{\langle p|p\rangle}(2\pi)^3 i\nabla^i\delta^{(3)}(\vec{0}) + \frac{1}{2p^0}\left[-i\nabla^i_\Delta\langle P+\tfrac{\Delta}{2}|O(0)|P-\tfrac{\Delta}{2}\rangle\right]_{\Delta=0}$$

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Contribution from the « center » of the wave packet

Internal contribution

[Jaffe, Manohar, NPB337 (1990) 509]
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Phase-space approach

$$\langle \int d^3x \, x^i O(x) \rangle_{\vec{R}, \vec{P}} =$$

[C.L., EPJC78 (2018) 9, 785]
[C.L., EPJC81 (2021) 5, 431]

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Phase-space approach

$$\langle \int \mathrm{d}^3x\, x^i O(x)\rangle_{\vec{R},\vec{P}} = R^i \frac{\langle p|O(0)|p\rangle}{2p^0} + \frac{1}{2p^0}\left[-i\nabla^i_\Delta\langle P+\tfrac{\Delta}{2}|O(0)|P-\tfrac{\Delta}{2}\rangle\right]_{\Delta=0}$$

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Amplitude for compound operators

Standard prescription

$$\langle \int \mathrm{d}^3x\, x^i O(x) \rangle \equiv \lim_{\Delta\to 0} \frac{\langle P+\frac{\Delta}{2}| \int \mathrm{d}^3x\, x^i O(x) |P-\frac{\Delta}{2}\rangle}{\langle p|p\rangle}$$

$$= \underbrace{\frac{\langle p|O(0)|p\rangle}{\langle p|p\rangle}(2\pi)^3 i\nabla^i \delta^{(3)}(\vec{0})}_{\text{Contribution from the « center » of the wave packet}} + \underbrace{\frac{1}{2p^0}\left[-i\nabla^i_\Delta \langle P+\tfrac{\Delta}{2}|O(0)|P-\tfrac{\Delta}{2}\rangle\right]_{\Delta=0}}_{\text{Internal contribution}}$$

Contribution from the « center » of the wave packet

Internal contribution

[Jaffe, Manohar, NPB337 (1990) 509]
[Bakker, Leader, Trueman, PRD70 (2004) 114001]

Phase-space approach

$$\langle \int \mathrm{d}^3x\, x^i O(x) \rangle_{\vec{R},\vec{P}} = R^i \frac{\langle p|O(0)|p\rangle}{2p^0} + \underbrace{\frac{1}{2p^0}\left[-i\nabla^i_\Delta \langle P+\tfrac{\Delta}{2}|O(0)|P-\tfrac{\Delta}{2}\rangle\right]_{\Delta=0}}_{=\langle \int \mathrm{d}^3r\, r^i O(r)\rangle_{\vec{0},\vec{P}}}$$

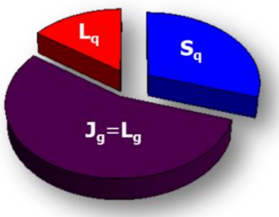
Internal contribution

Remark: $\langle O\rangle_{\text{standard}} = \int \frac{\mathrm{d}^3R}{(2\pi)^3\,\delta^{(3)}(\vec{0})} \langle O\rangle_{\vec{R},\vec{P}}$ explains the origin of the delta contribution in standard approach

[C.L., EPJC78 (2018) 9, 785]
[C.L., EPJC81 (2021) 5, 431]

Angular momentum conservation *(local kinetic operators)*

Longitudinally polarized target $\langle J_z \rangle = \langle L_z^q \rangle + \langle S_z^q \rangle + \langle L_z^g \rangle$



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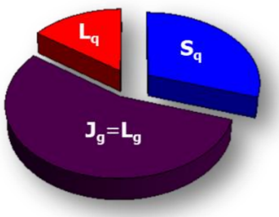

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For a wave-packet centered at the origin, one finds

$$\langle L_z^q\rangle = \langle \int \mathrm{d}^3x \, \left(x^1 T_q^{02} - x^2 T_q^{01}\right)\rangle =$$

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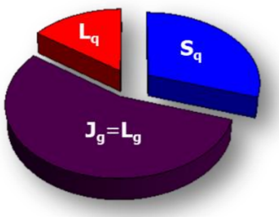

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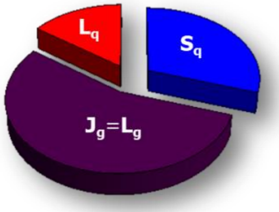

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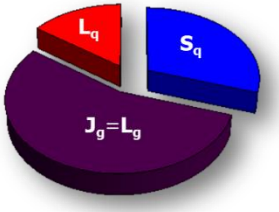

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$$\boxed{S_g(t) = 0}$$

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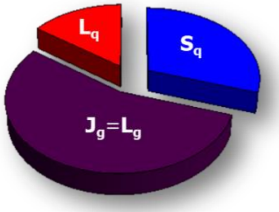

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Angular momentum (or spin) sum rule

$$\boxed{\sum_a J_a(0) = \frac{1}{2}}$$

[Ji, PRL78 (1997) 610]
[Bakker, Leader, Trueman, PRD70 (2004) 114001]

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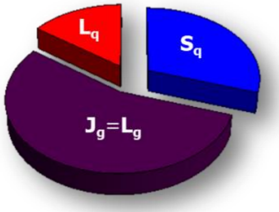

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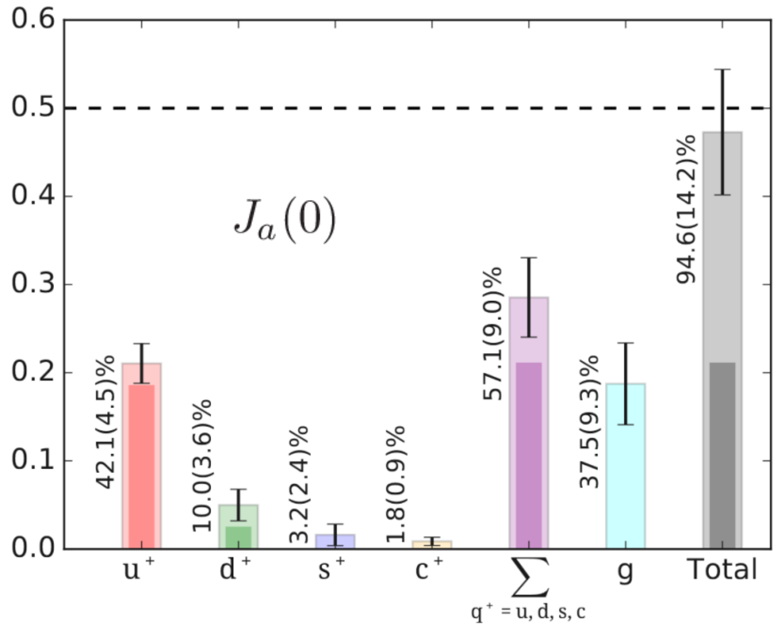
The same sum rule is obtained with a transversely polarized target

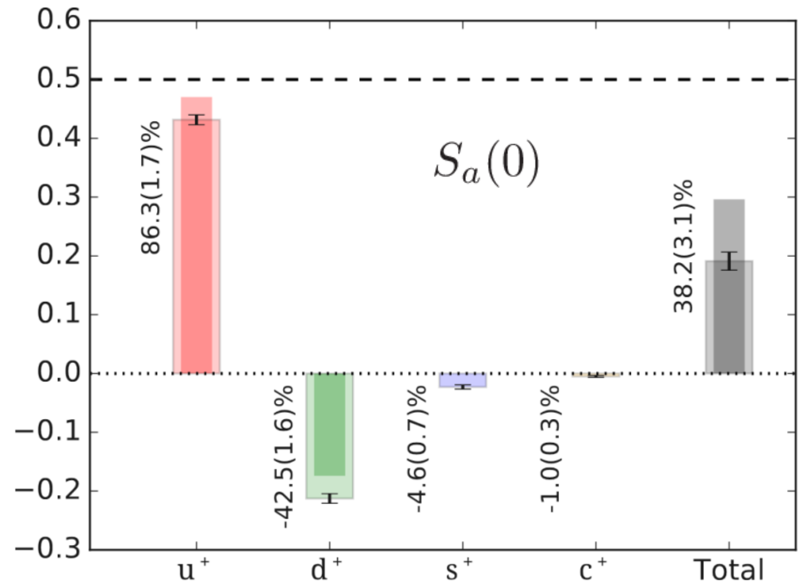
(but more complicated analysis due to non-commutativity of boosts)

[C.L., EPJC81 (2021) 5, 431]

Angular momentum conservation *(local kinetic operators)*

Lattice QCD has made a lot of progress recently in computing the nucleon spin contributions





[Alexandrou *et al.*, PRD101 (2020) 094513]

Anomalous gravitomagnetic moment

Mellin moments $\xi = 0$

$$\int \mathrm{d}x\, H_q(x,0,t) = F_1^q(t)$$

$$\int \mathrm{d}x\, E_q(x,0,t) = F_2^q(t)$$

Anomalous magnetic moment

$$\boxed{F_2^q(0) = \kappa_q}$$

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[Ji, PRL78 (1997) 610]
[Teryaev, hep-ph/9904376]
[Brodsky, Hwang, Ma, Schmidt, NPB593 (2001) 311]
[Burkardt, IJMPA18 (2003) 2, 173]
[Lowdon, Chiu, Brodsky, PLB774 (2017) 1]
[C.L., EPJC78 (2018) 9, 785]
[Cotogno, C.L., Lowdon, PRD100 (2019) 045003]

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$$\left.\begin{array}{l} \sum_a A_a(0) = 1 \\ \sum_a J_a(0) = \frac{1}{2} \end{array}\right\} \Rightarrow \boxed{\sum_a B_a(0) = 0}$$

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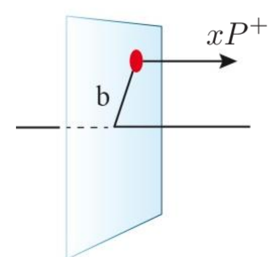

$$\vec{R}_\perp = \frac{1}{P^+} \int \mathrm{d}x^- \,\mathrm{d}^2 x_\perp\, \vec{x}_\perp\, T^{++}(x)$$

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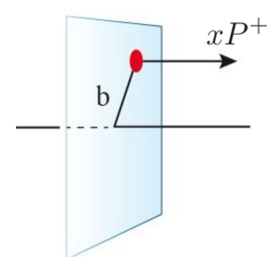

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Equivalence principle for spinning targets

→ **No intrinsic energy dipole moment !**

[Ji, PRL78 (1997) 610]
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Generalizations

Distribution in momentum space

$$\langle J_z^a \rangle = \int \mathrm{d}x \, \frac{x}{2} \left[H_a(x,0,0) + E_a(x,0,0) \right]$$

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[Hoodbhoy, Ji, Lu, PRD 59 (1999) 014013]
[Hägler, Mukherjee, Schäfer, PLB582 (2004) 55]
[Ji, Xiong, Yuan, PRL109 (2012) 152005]
[Ji, Xiong, Yuan, PLB717 (2012) 214]
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[Liu, C.L., EPJA52 (2016) 6, 160]

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$[D^\mu, \mathcal{W}] \neq 0$ ➡ **Non-local version of covariant derivative is ambiguous and involves higher-twist distributions !**

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[Polyakov, PLB555 (2003) 57]
[Goeke *et al.*, PRD75 (2007) 094021]
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[C.L., Mantovani, Pasquini, PLB776 (2018) 38]

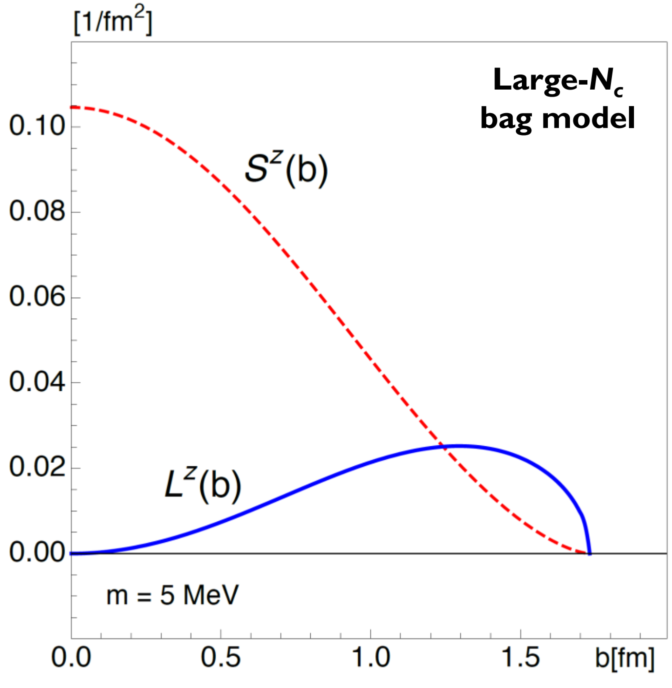
$\vec{\Delta}_\perp^2 \dfrac{\mathrm{d}J_a(t)}{\mathrm{d}t}$ **does not contribute to the integral but affects the spatial distribution !**

Angular momentum distributions

Orbital *vs* intrinsic

$$L^i(\vec{r}) = \epsilon^{ijk} r^j \langle T^{0k} \rangle_{\vec{0},\vec{0}}(\vec{r})$$

$$S^i(\vec{r}) = \frac{1}{2} \langle \overline{\psi} \gamma^i \gamma_5 \psi \rangle_{\vec{0},\vec{0}}(\vec{r})$$



Impact parameter $b = \sqrt{x^2 + y^2} = \sqrt{r^2 - z^2}$

[C.L., Mantovani, Pasquini, PLB776 (2018) 38]
[C.L., Schweitzer, Tezgin, PRD106 (2022) 1, 014012]

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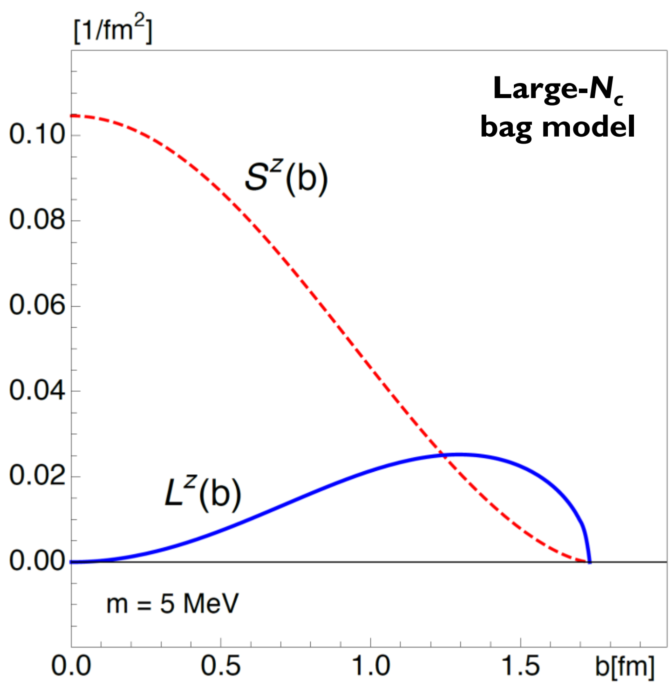
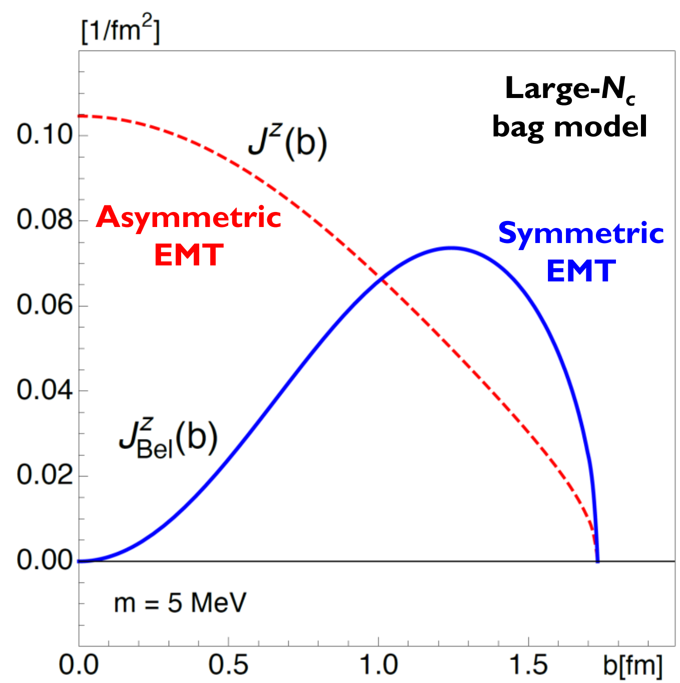


Kinetic *vs* Belinfante

$$J^i(\vec{r}) = L^i(\vec{r}) + S^i(\vec{r})$$

$$J^i_{\text{Bel}}(\vec{r}) = \epsilon^{ijk} r^j \langle \tfrac{1}{2}(T^{0k} + T^{k0}) \rangle_{\vec{0},\vec{0}}(\vec{r})$$



Impact parameter $b = \sqrt{x^2 + y^2} = \sqrt{r^2 - z^2}$

[C.L., Mantovani, Pasquini, PLB776 (2018) 38]
[C.L., Schweitzer, Tezgin, PRD106 (2022) 1, 014012]

Mechanical properties

Let us recap

$$\langle p', s' | T_a^{\mu\nu}(0) | p, s \rangle = \overline{u}(p', s') \Gamma_a^{\mu\nu}(P, \Delta) u(p, s)$$

$$\Gamma_a^{\mu\nu}(P, \Delta) = \frac{P^\mu P^\nu}{M} A_a(t) + \frac{\Delta^\mu \Delta^\nu - g^{\mu\nu}\Delta^2}{M} C_a(t) + M g^{\mu\nu} \bar{C}_a(t) + \frac{P^{\{\mu} i\sigma^{\nu\}\lambda}\Delta_\lambda}{2M} J_a(t) - \frac{P^{[\mu} i\sigma^{\nu]\lambda}\Delta_\lambda}{2M} S_a(t)$$

$A_a(0) \leftrightarrow$ **Momentum**

$\bar{C}_a(0) \leftrightarrow$ **Pressure**

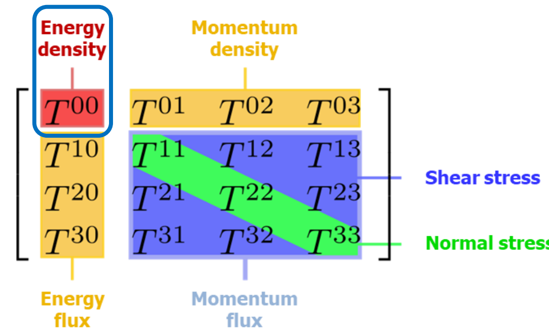
$J_a(0) \leftrightarrow$ **Total angular momentum**

$S_q(0) \leftrightarrow$ **Intrinsic angular momentum**

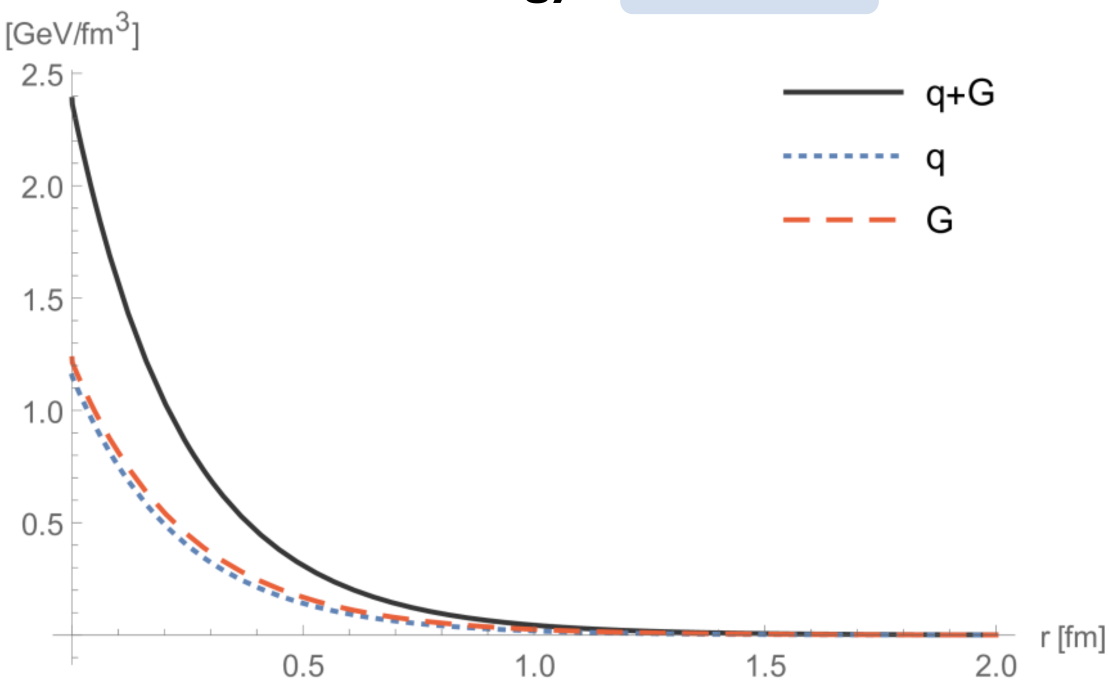
$C_a(0) \leftrightarrow$ **?**

Rest frame energy (or mass) distribution

$$\langle T^{00}\rangle(\vec{r})$$



Energy $A_a, C_a, \bar{C}_a, J_a$



Multipole model for the gravitational form factors

$$F(t) = \frac{F(0)}{(1 + t/\Lambda^2)^n}$$

[Polyakov, PLB555 (2003) 57]
[Polyakov, Schweitzer, IJMPA33 (2018) 26, 1830025]
[C.L., Moutarde, Trawinski, EPJC79 (2019) 1, 89]

Rest frame energy (or mass) distribution

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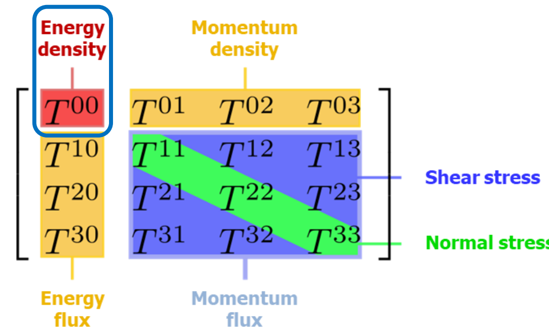

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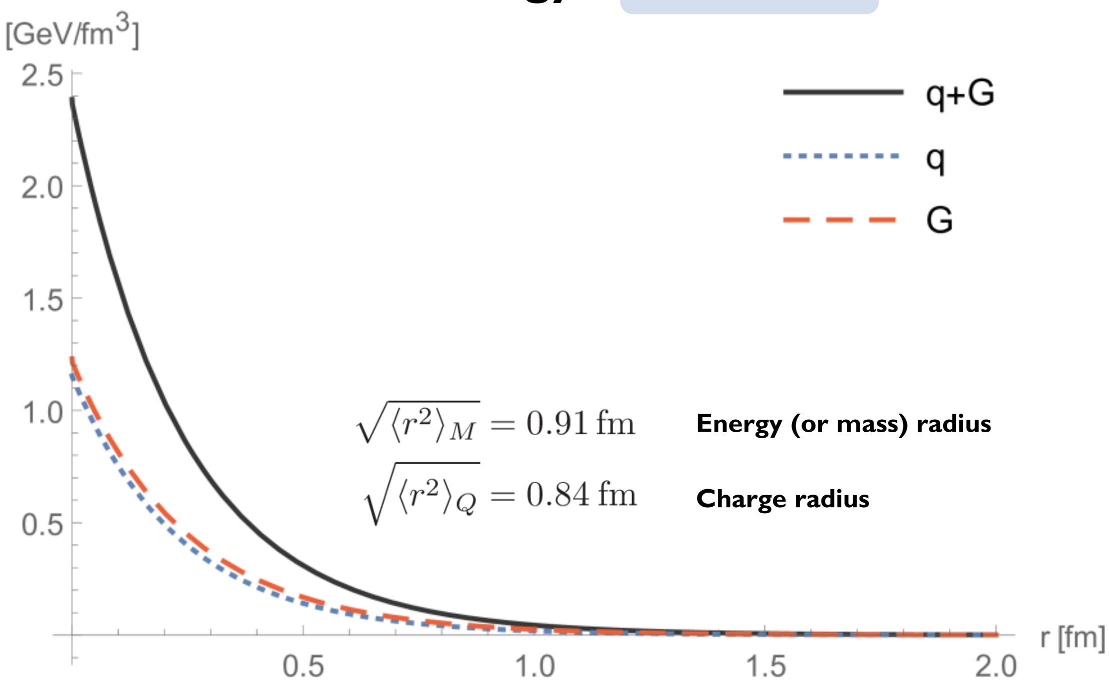

$$\sqrt{\langle r^2 \rangle_M} = 0.91\,\mathrm{fm}$$ **Energy (or mass) radius**

$$\sqrt{\langle r^2 \rangle_Q} = 0.84\,\mathrm{fm}$$ **Charge radius**

Multipole model for the gravitational form factors

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[Polyakov, PLB555 (2003) 57]
[Polyakov, Schweitzer, IJMPA33 (2018) 26, 1830025]
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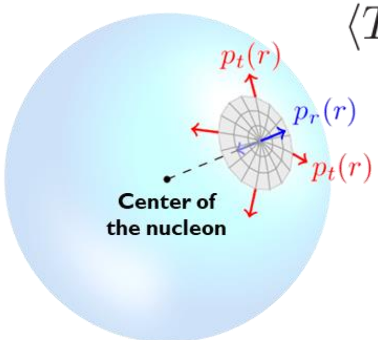
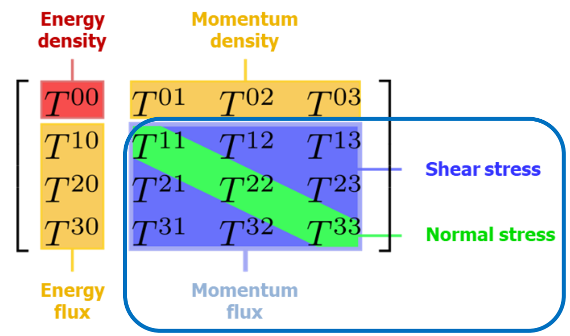
Rest frame pressure distributions



$$\langle T^{ij}\rangle(\vec{r}) = \delta^{ij}\, p(r) + \left(\frac{r^i r^j}{r^2} - \frac{1}{3}\,\delta^{ij}\right) s(r)$$



[Polyakov, PLB555 (2003) 57]
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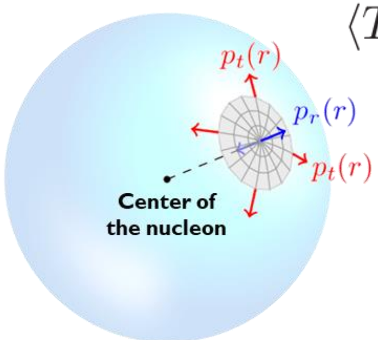

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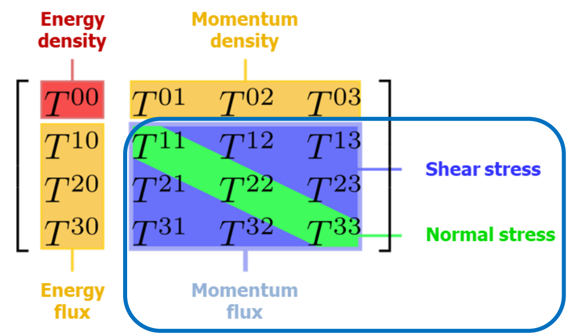
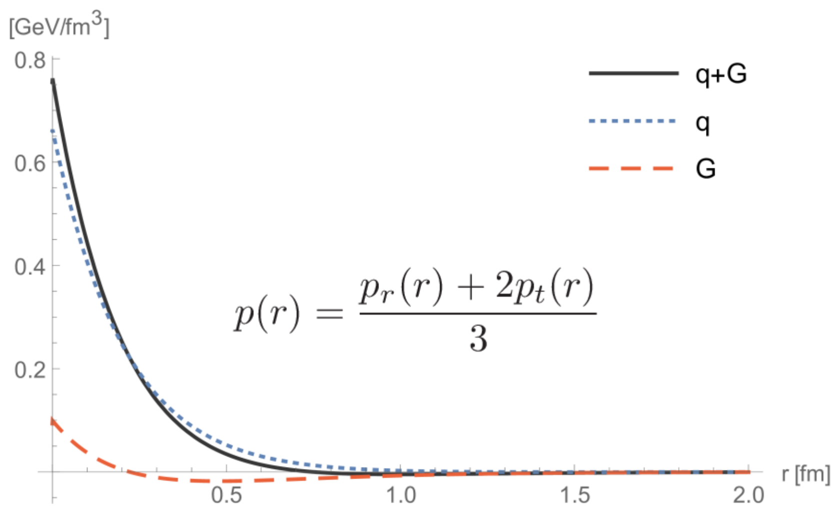


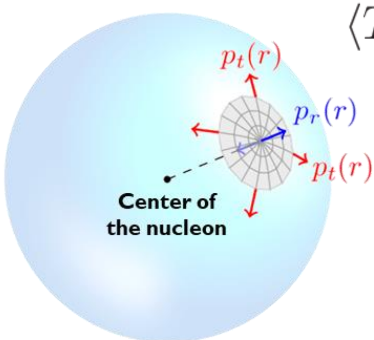
Isotropic pressure $C_a, \bar{C}_a$



$$p(r) = \frac{p_r(r) + 2p_t(r)}{3}$$

[Polyakov, PLB555 (2003) 57]
[Polyakov, Schweitzer, IJMPA33 (2018) 26, 1830025]
[C.L., Moutarde, Trawinski, EPJC79 (2019) 1, 89]

Rest frame pressure distributions



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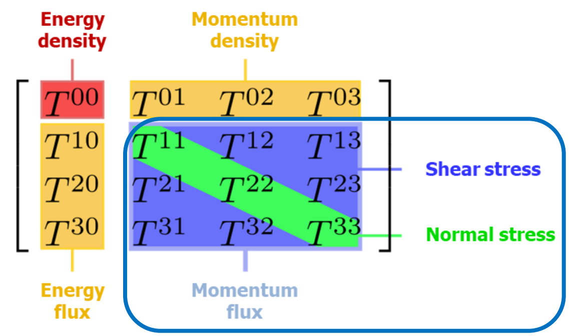

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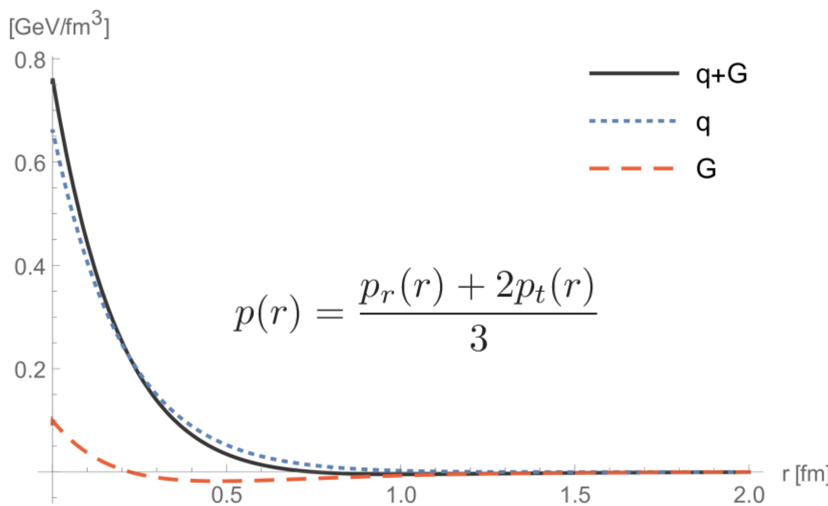
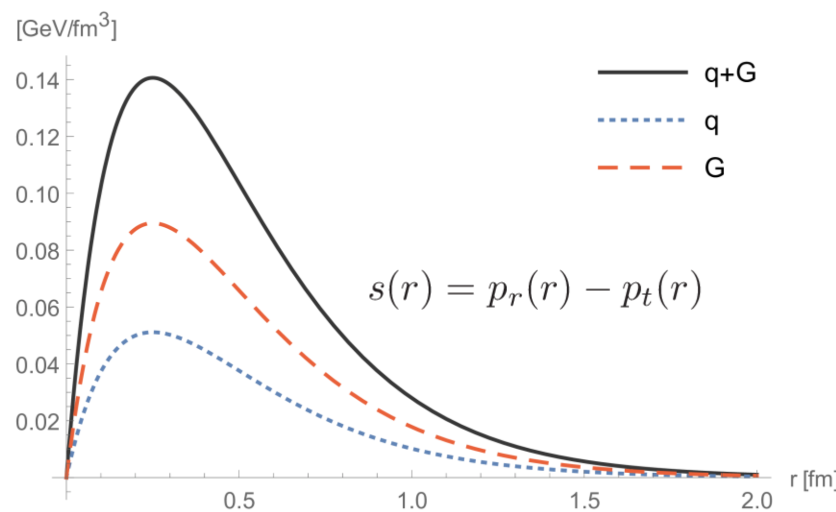


Pressure anisotropy C_a



[Polyakov, PLB555 (2003) 57]
[Polyakov, Schweitzer, IJMPA33 (2018) 26, 1830025]
[C.L., Moutarde, Trawinski, EPJC79 (2019) 1, 89]

Some figures

	Density (kg/m^3)	**Pressure (Pa or N/m^2)**
Atmosphere at sea level	≈ 1.2	$\approx 10^5$

Some figures

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PhD student a month before graduation	$\approx 10^3$	$> 10^{42}$

Mechanical equilibrium

$$\nabla^i \langle T^{ij} \rangle(\vec{r}) = 0$$

von Laue relation

$$\boxed{\int_0^\infty \mathrm{d}r\, r^2\, p(r) = 0}$$

[Laue, AP340 (1911) 8, 524]

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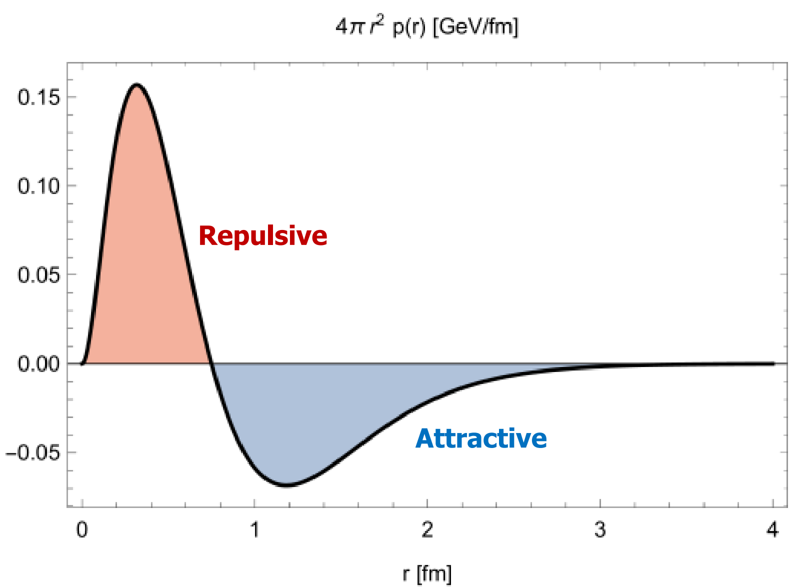

Mechanical equilibrium

$$\nabla^i \langle T^{ij} \rangle(\vec{r}) = 0 \quad \Rightarrow \quad \frac{\mathrm{d}p_r(r)}{\mathrm{d}r} = -\frac{2s(r)}{r}$$

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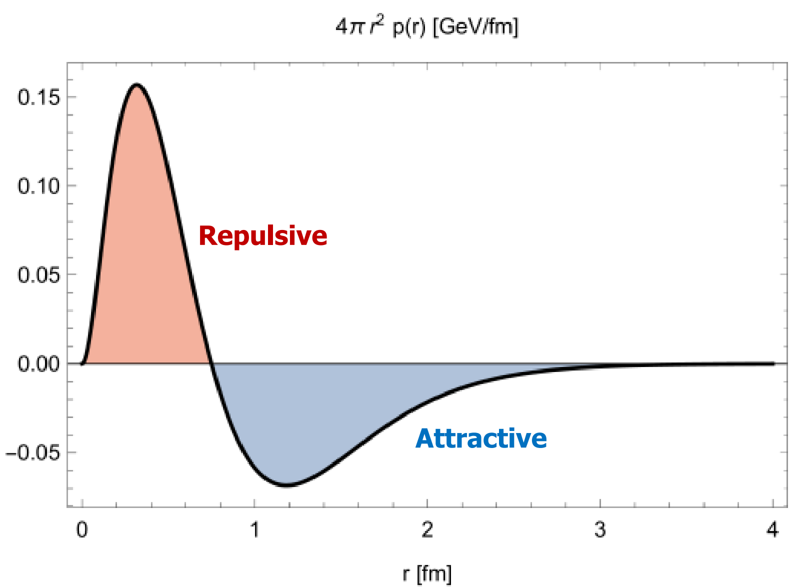

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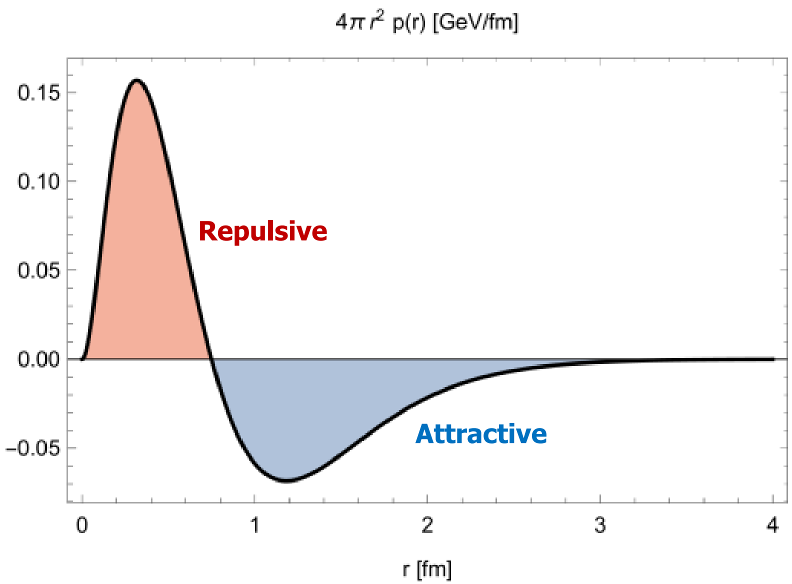

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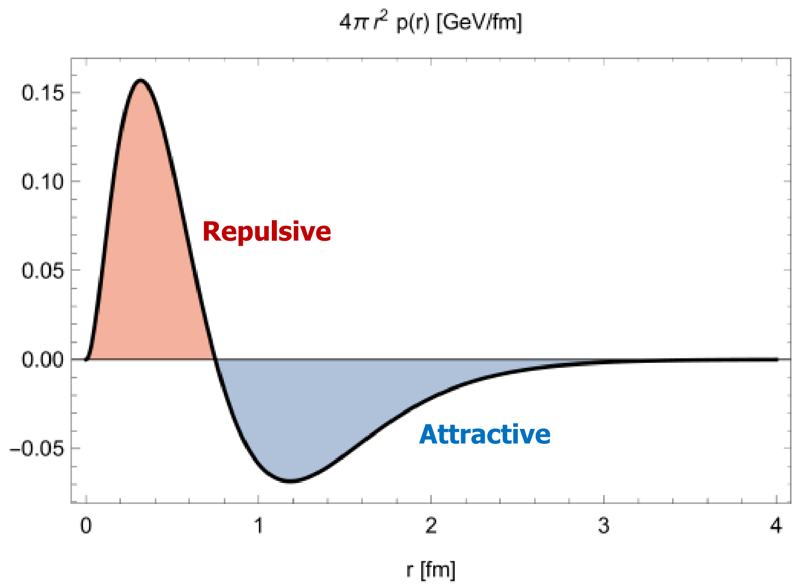
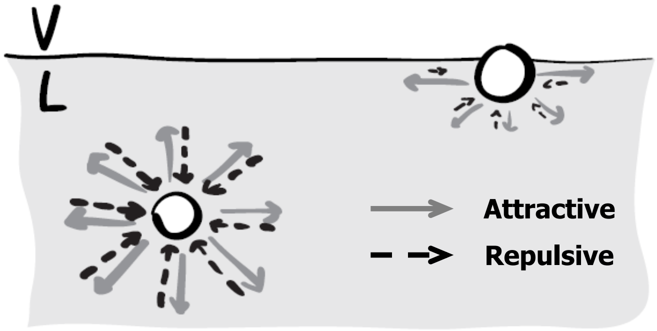


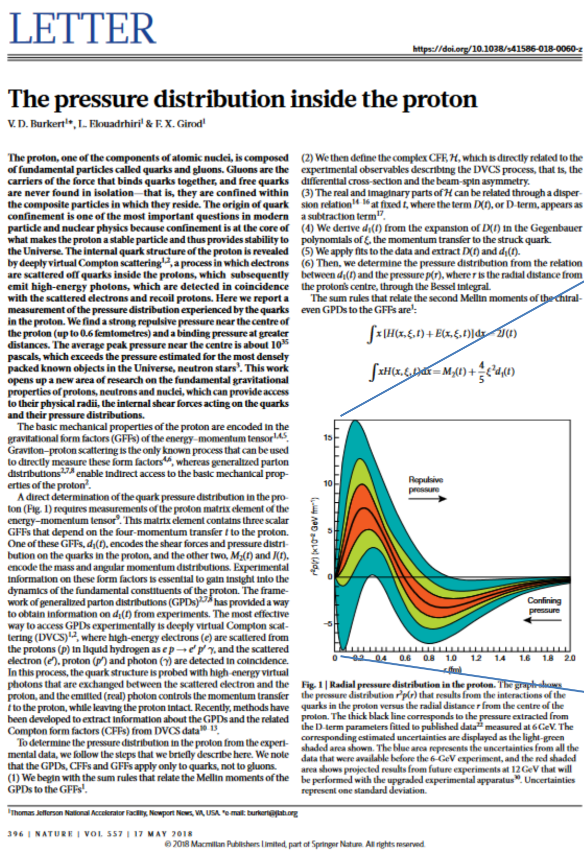
Surface tension

$$\gamma = \int \mathrm{d}r\, s(r)$$



[Bakker, *Kapillarität und Oberflächenspannung* (1928)]
[Kirkwood, Buff, JCP17 (1949) 338]
[Marchand, Weijs, Snoeijer, Andreotti, AJP79 (2011) 999]

First experimental extractions



LETTER

https://doi.org/10.1038/s41586-018-0060-z

The pressure distribution inside the proton

V. D. Burkert[1*], L. Elouadrhiri[1] & F. X. Girod[1]

The proton, one of the components of atomic nuclei, is composed of fundamental particles called quarks and gluons. Gluons are the carriers of the force that binds quarks together, and free quarks are never found in isolation—that is, they are confined within the composite particles in which they reside. The origin of quark confinement is one of the most important questions in modern particle and nuclear physics because confinement is at the core of what makes the proton a stable particle and thus provides stability to the Universe. The internal quark structure of the proton is revealed by deeply virtual Compton scattering[1,2], a process in which electrons are scattered off quarks inside the protons, which subsequently emit high-energy photons, which are detected in coincidence with the scattered electrons and recoil protons. Here we report a measurement of the pressure distribution experienced by the quarks in the proton. We find a strong repulsive pressure near the centre of the proton (up to 0.6 femtometres) and a binding pressure at greater distances. The average peak pressure near the centre is about 10^{35} pascals, which exceeds the pressure estimated for the most densely packed known objects in the Universe, neutron stars[3]. This work opens up a new area of research on the fundamental gravitational properties of protons, neutrons and nuclei, which can provide access to their physical radii, the internal shear forces acting on the quarks and their pressure distributions.

The basic mechanical properties of the proton are encoded in the gravitational form factors (GFFs) of the energy–momentum tensor[1,4,5]. Graviton–proton scattering is the only known process that can be used to directly measure these form factors[4,6], whereas generalized parton distributions[2,7,8] enable indirect access to the basic mechanical properties of the proton[2].

A direct determination of the quark pressure distribution in the proton (Fig. 1) requires measurements of the proton matrix element of the energy–momentum tensor[9]. This matrix element contains three scalar GFFs that depend on the four-momentum transfer t to the proton. One of these GFFs, $d_1(t)$, encodes the shear forces and pressure distribution on the quarks in the proton, and the other two, $M_2(t)$ and $J(t)$, encode the mass and angular momentum distributions. Experimental information on these form factors is essential to gain insight into the dynamics of the fundamental constituents of the proton. The framework of generalized parton distributions (GPDs)[2,7,8] has provided a way to obtain information on $d_1(t)$ from experiments. The most effective way to access GPDs experimentally is deeply virtual Compton scattering (DVCS)[1,2], where high-energy electrons (e) are scattered from the protons (p) in liquid hydrogen as $e\,p \to e'\,p'\,\gamma$, and the scattered electron (e'), proton (p') and photon (γ) are detected in coincidence. In this process, the quark structure is probed with high-energy virtual photons that are exchanged between the scattered electron and the proton, and the emitted (real) photon controls the momentum transfer t to the proton, while leaving the proton intact. Recently, methods have been developed to extract information about the GPDs and the related Compton form factors (CFFs) from DVCS data[10–13].

To determine the pressure distribution in the proton from the experimental data, we follow the steps that we briefly describe here. We note that the GPDs, CFFs and GFFs apply only to quarks, not to gluons.
(1) We begin with the sum rules that relate the Mellin moments of the GPDs to the GFFs[1].
(2) We then define the complex CFF, $\mathcal{H}$, which is directly related to the experimental observables describing the DVCS process, that is, the differential cross-section and the beam-spin asymmetry.
(3) The real and imaginary parts of $\mathcal{H}$ can be related through a dispersion relation[14–16] at fixed t, where the term $D(t)$, or D-term, appears as a subtraction term[17].
(4) We derive $d_1(t)$ from the expansion of $D(t)$ in the Gegenbauer polynomials of ξ, the momentum transfer to the struck quark.
(5) We apply fits to the data and extract $D(t)$ and $d_1(t)$.
(6) Then, we determine the pressure distribution from the relation between $d_1(t)$ and the pressure $p(r)$, where r is the radial distance from the proton's centre, through the Bessel integral.

The sum rules that relate the second Mellin moments of the chiral-even GPDs to the GFFs are[1]:

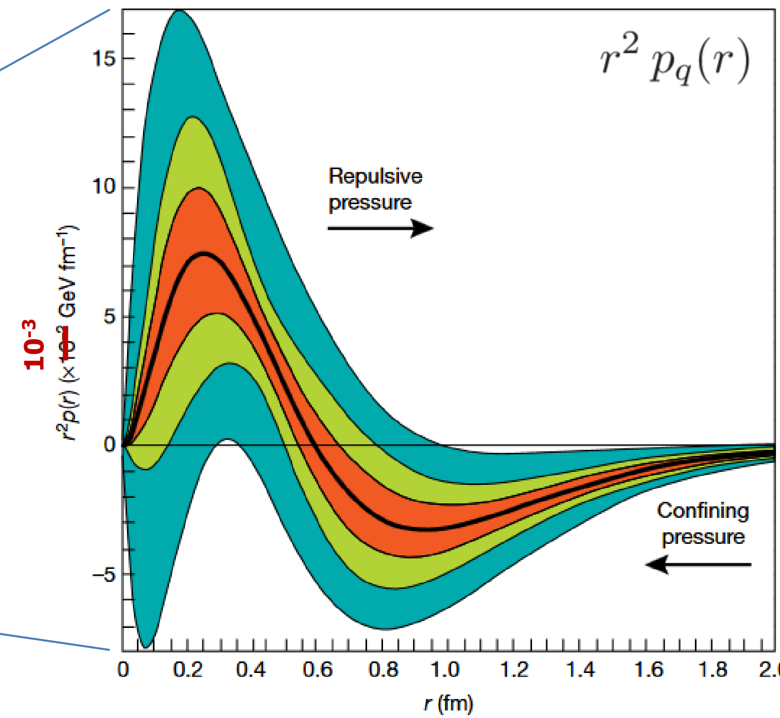
$$\int x\,[H(x,\xi,t)+E(x,\xi,t)]\,\mathrm{d}x = 2J(t)$$

$$\int xH(x,\xi,t)\,\mathrm{d}x = M_2(t) + \frac{4}{5}\xi^2 d_1(t)$$

Fig. 1 | Radial pressure distribution in the proton. The graph shows the pressure distribution $r^2p(r)$ that results from the interactions of the quarks in the proton versus the radial distance r from the centre of the proton. The thick black line corresponds to the pressure extracted from the D-term parameters fitted to published data[22] measured at 6 GeV. The corresponding estimated uncertainties are displayed as the light-green shaded area shown. The blue area represents the uncertainties from all the data that were available before the 6-GeV experiment, and the red shaded area shows projected results from future experiments at 12 GeV that will be performed with the upgraded experimental apparatus[30]. Uncertainties represent one standard deviation.

[1]Thomas Jefferson National Accelerator Facility, Newport News, VA, USA. *e-mail: burkert@jlab.org

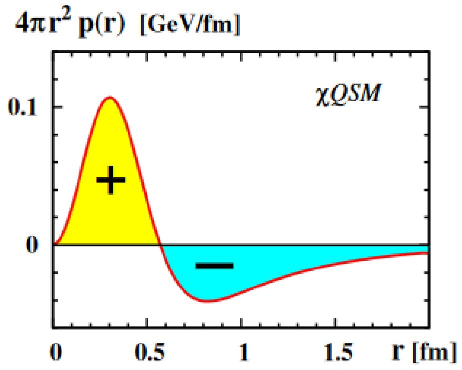
396 | NATURE | VOL 557 | 17 MAY 2018



Contribution from $\bar{C}_q(t)$ still missing !

[Burkert, Elouadrhiri, Girod, Nature557 (2018) 7705, 396]
[Kumericki, Nature570 (2019) 7759, E1]
[Dutrieux, C.L., Moutarde, Sznajder, Trawinski, EPJC81 (2021) 4, 300]

D-term and stability

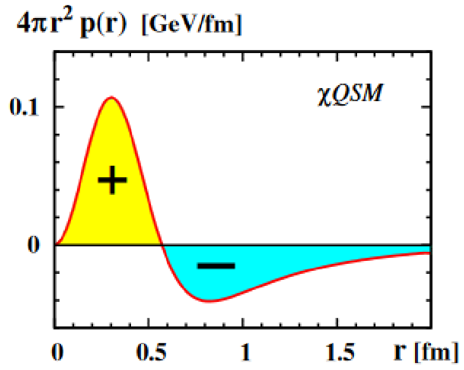


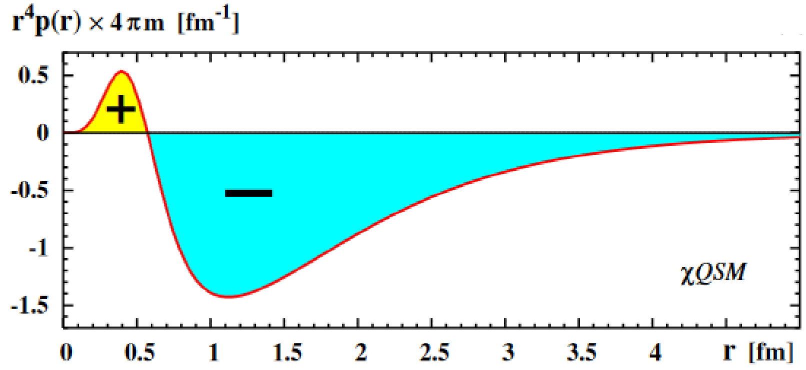
$$\int_0^\infty \mathrm{d}r\, r^2\, p(r) = 0$$

[Laue, AP340 (1911) 8, 524]

Equilibrium

D-term and stability





$$\int_0^\infty \mathrm{d}r\, r^2\, p(r) = 0$$

$$M \int_0^\infty \mathrm{d}r\, r^4\, p(r) = 4C(0) = D(0)$$

Druck **=**
« pressure »

[Laue, AP340 (1911) 8, 524]

[Polyakov, Weiss, PRD60 (1999) 114017]
[Goeke *et al.*, PRD75 (2007) 094021]

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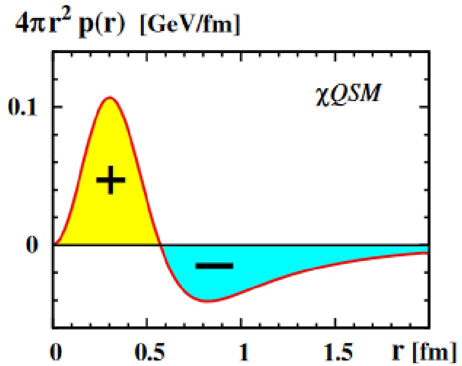

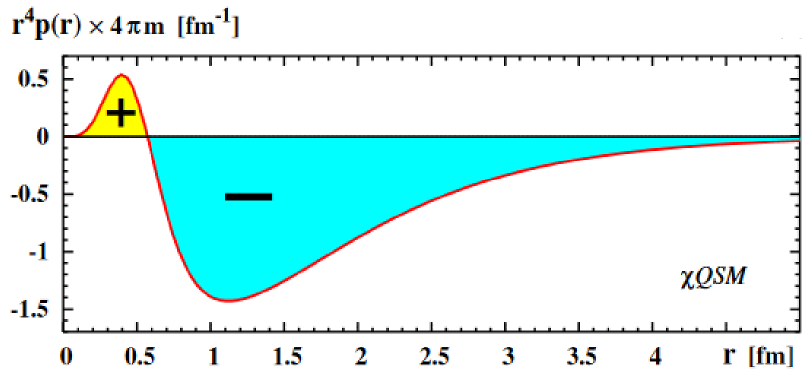

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***Druck* = « pressure »**

[Polyakov, Weiss, PRD60 (1999) 114017]
[Goeke *et al.*, PRD75 (2007) 094021]

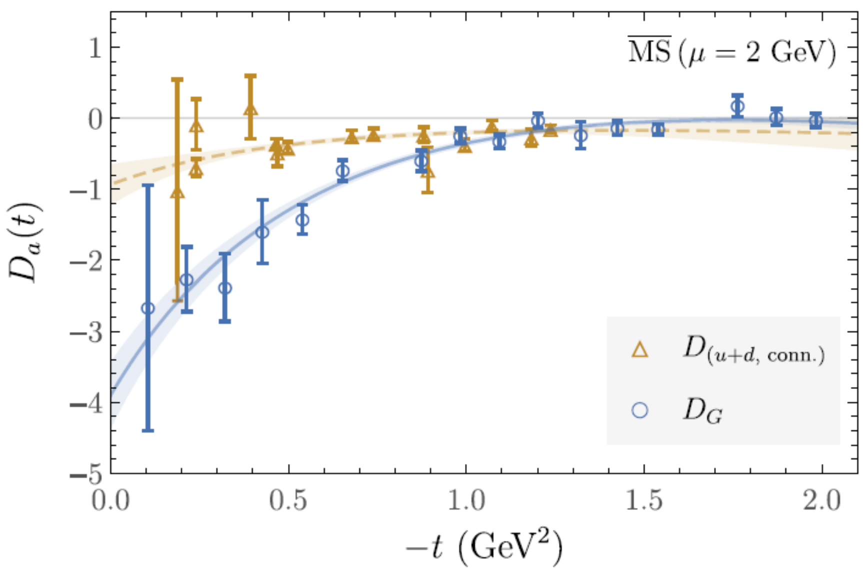
Equilibrium

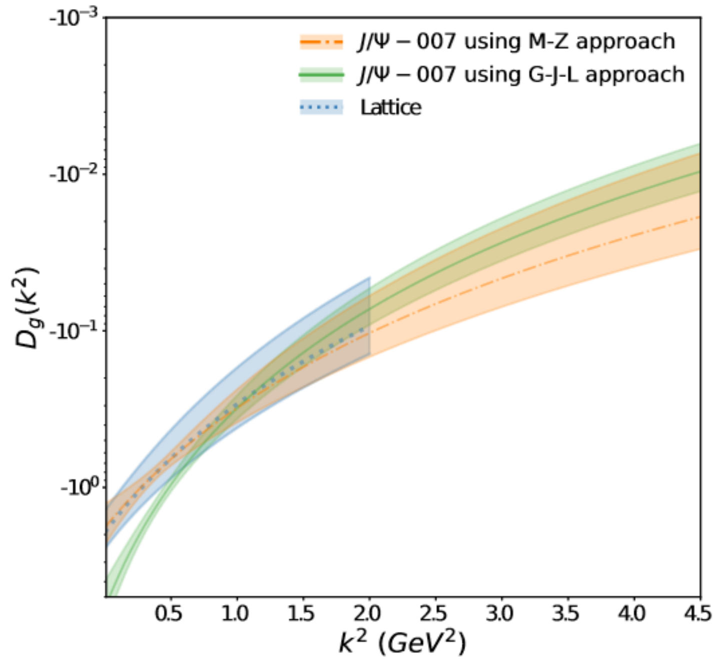
Stability $\overset{?}{\Leftrightarrow}$ $D(0) < 0$

[Perevalova, Polyakov, Schweitzer, PRD94 (2016) 5, 054024]
[Polyakov, Schweitzer, IJMPA33 (2018) 26, 1830025]
[Varma, Schweitzer, PRD102 (2020) 1, 014047]

D-term and stability

This conjecture is supported by both Lattice QCD and experiments



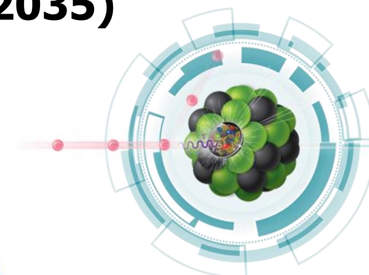


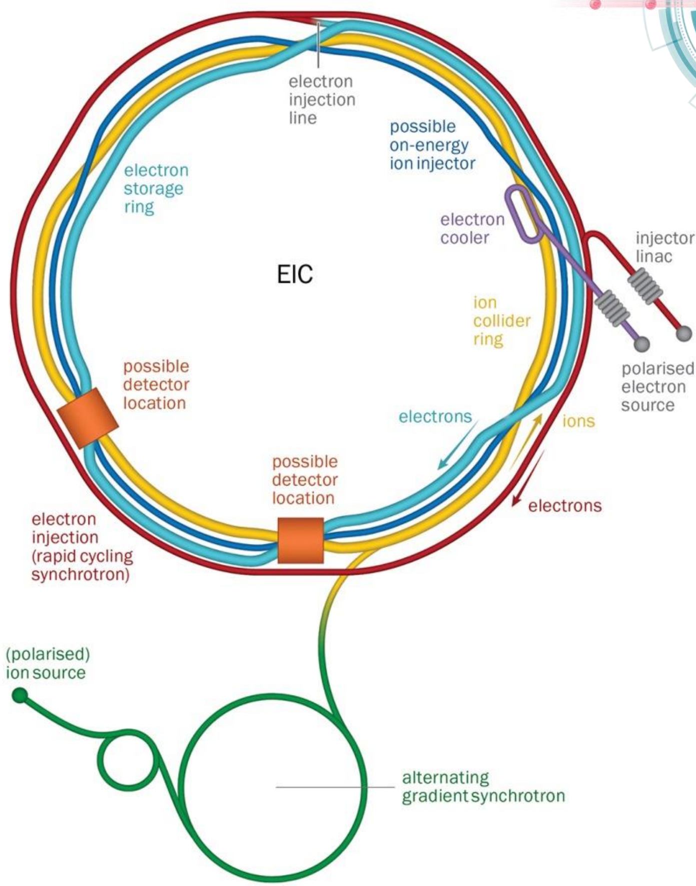
[Hägler *et al.*, PRD77 (2008) 094502]
[Shanahan, Detmold, PRD99 (2019) 4, 014511]
[Shanahan, Detmold, PRL122 (2019) 072003]

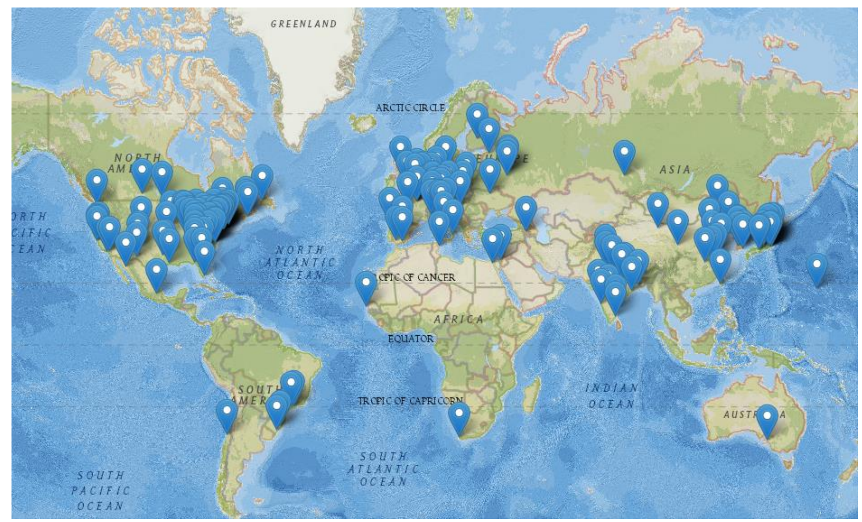
[Duran *et al.*, Nature615 (2023) 7954, 813]
[Meziani, PoS SPIN2023 (2024) 168]

Key players in the near future

Electron-Ion Collider (~2035)
Brookhaven, Long Island







>1500 members from 290 institutions in 40 countries

[Abdul Khalek *et al.*, NPA1026 (2022) 122447]

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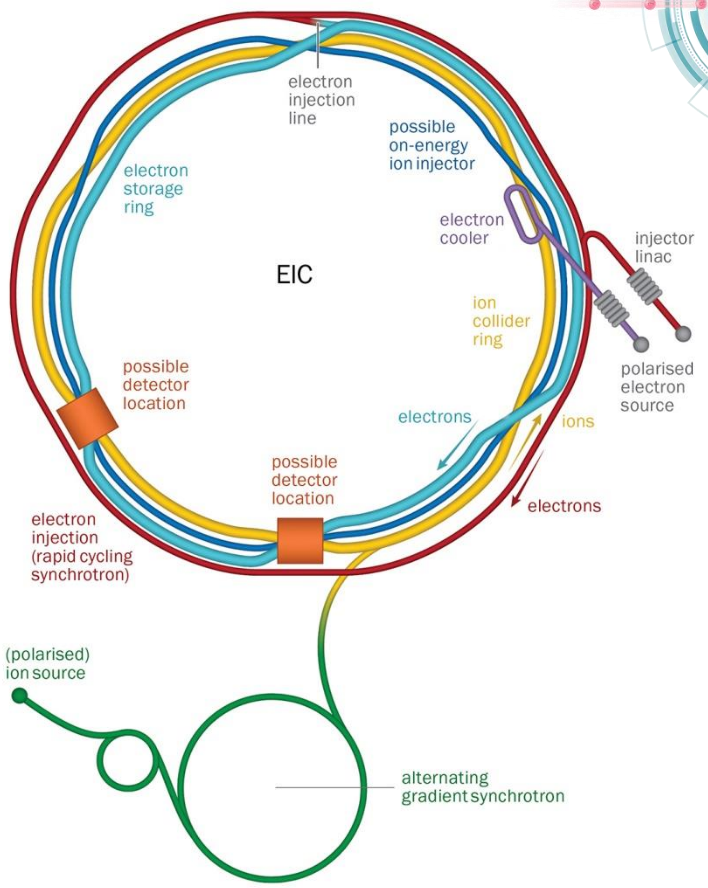
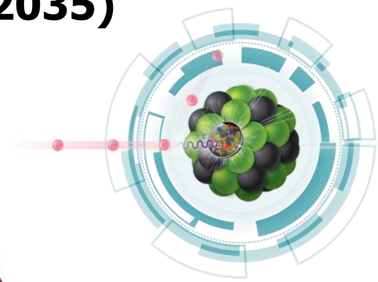
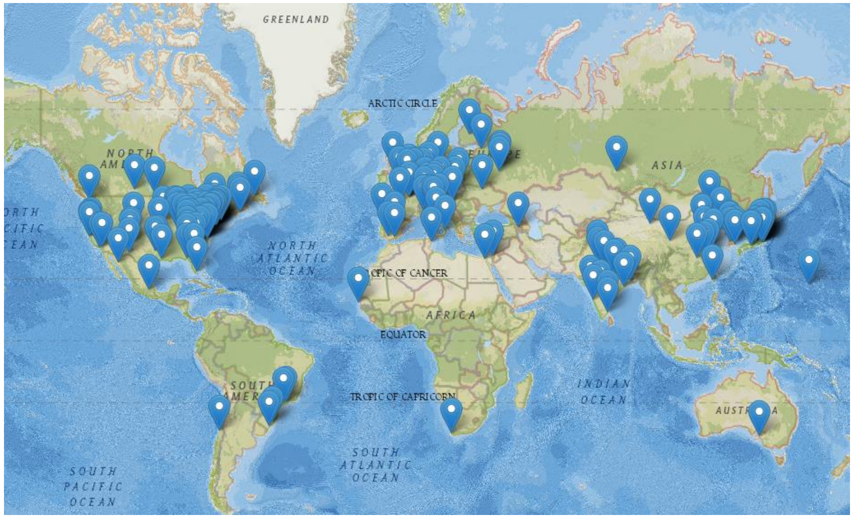






>1500 members from 290 institutions in 40 countries

[Abdul Khalek *et al.*, NPA1026 (2022) 122447]

But also EicC, JLab 20+ GeV, … ?

[Anderle *et al.*, FP16 (2021) 6, 64701]
[Accardi *et al.*, EPJA 60 (2024) 9, 173]

Some reviews

GPDs

[Diehl, PR388 (2003) 41]
[Belitsky, Radyushkin, PR418 (2005) 1]
[Kumericki, Liuti, Moutarde, EPJA52 (2016) 6, 157]

EMT & GFFs

[Polyakov, Schweitzer, IJMPA33 (2018) 26, 1830025]
[Burkert *et al.*, RMP95 (2023) 4, 041002]

Spin decomposition

[Leader, C.L., PR541 (2014) 3, 163]
[Wakamatsu, IJMPA29 (2014) 1430012]
[Liu, C.L., EPJA52 (2016) 6, 379]
[Ji, Yuan, Zhao, NRP3 (2021) 1, 27]

Mass decomposition

[Ji, FP16 (2021) 6, 64601]
[C.L., Metz, Pasquini, Rodini, JHEP11 (2021) 121]

… and references therein !

Backup

In short …

Nucleon spin decomposition $$\langle \vec{J} \rangle = \sum_{a=q,g} \langle \vec{J}_a \rangle$$ **Belinfante form**

Longitudinal spin $$\langle J_a^L \rangle = \frac{1}{2}\left[A_a(0) + B_a(0)\right]$$ **[Ji, PRL78 (1997)]**

$$\boxed{\sum_a A_a(0) = 1 \qquad \sum_a B_a(0) = 0}$$

$$A_q(0) = \int \mathrm{d}x\, x\, H_q(x,\xi,0)$$

$$B_q(0) = \int \mathrm{d}x\, x\, E_q(x,\xi,0)$$

Gravitational form factor **GPD**

Transverse spin $$\langle J_a^T \rangle = \frac{1}{2}\left[A_a(0) + \frac{p^0}{M} B_a(0)\right]$$ **[Leader, PRD85 (2012)]**

$$\langle J_a^T \rangle = \frac{p^0}{M}\,\frac{A_a(0) + B_a(0)}{2} \equiv \frac{p^0}{M}\langle S_a^T \rangle$$ **[Ji-Yuan, PLB810 (2020)]**

Both are correct ! **[C.L., EPJC81 (2021)]**

What do we mean by « spin »?

Originally « spin » was reserved to *intrinsic* angular momentum (AM),
to be distinguished from orbital AM

Nowadays « spin » refers more generally to internal AM,
i.e. AM about the center of the system

$$\vec{J} = \vec{R} \times \vec{P} + \vec{S}$$

Position of the center

Momentum of the system

The key question is: **what is the relativistic center of the system?**

Poorly addressed in the nucleon spin literature …
triggered a review of the subject **[C.L., EPJC78 (2018)]**

Option 1: Relativistic center of energy

Relativistic version of the center of inertial mass

[Fokker, *Relativiteitstheorie* (1929)]
[Born-Infeld, PRSLA150 (1935)]

$$R^\mu_E = \frac{1}{P^0}\int \mathrm{d}^3r\, r^\mu T^{00}$$

$$P^\mu = \int \mathrm{d}^3r\, T^{0\mu}$$

$$\Rightarrow \quad \vec{R}_E = t\,\frac{\vec{P}}{P^0} - \frac{\vec{K}}{P^0}$$

← Lorentz boost generator

Spin operator

$$\boxed{\vec{S}_E \equiv \vec{J} - \vec{R}_E \times \vec{P} = \frac{\vec{W}}{P^0}}$$

Pauli-Lubański pseudo-vector

$$W^\mu = \frac{1}{2}\,\epsilon^{\mu\alpha\beta\lambda} M_{\alpha\beta} P_\lambda$$

$$M^{\alpha\beta} = \int \mathrm{d}^3r\, \left[r^\alpha T^{0\beta} - r^\beta T^{0\alpha}\right]$$

<u>Remark</u>: These definitions coincide in the infinite-momentum frame with the corresponding light-front operators

Option 2: Relativistic center of mass

R^μ_E does not transform as a Lorentz four-vector

[Pryce, PRSLA195 (1948)]
[Møller, CDIASA5 (1949)]
[Fleming, PR137 (1965)]

Covariant four-position operator

$$R^\mu_M = \left(t + \frac{\vec{P}\cdot\vec{K}}{M^2}\right)\frac{P^\mu}{P^0} - \frac{P_\nu M^{\nu\mu}}{M^2}$$

$$\tau = \frac{M}{P^0}\left(t + \frac{\vec{P}\cdot\vec{K}}{M^2}\right)$$

Proper time

$$\Rightarrow \quad \vec{R}_M = \vec{R}_E - \frac{\vec{P}\times\vec{W}}{P^0 M^2}$$

Spin operator

$$\boxed{\vec{S}_M \equiv \vec{J} - \vec{R}_M \times \vec{P} = \frac{P^0\vec{W} - \vec{P}W^0}{M^2}}$$

<u>Remarks:</u>

- $\vec{R}_M = \vec{R}_E$ in the rest frame
- Relativistic center of mass can be considered as a physical point

(i.e. not just as a mere representative point)

Option 3: Relativistic center of spin

Canonical relations $\begin{cases}[R^i_X, R^j_X] = 0\\ [S^i_X, S^j_X] = i\epsilon^{ijk} S^k_X\end{cases}$ not satisfied for $X = E, M$

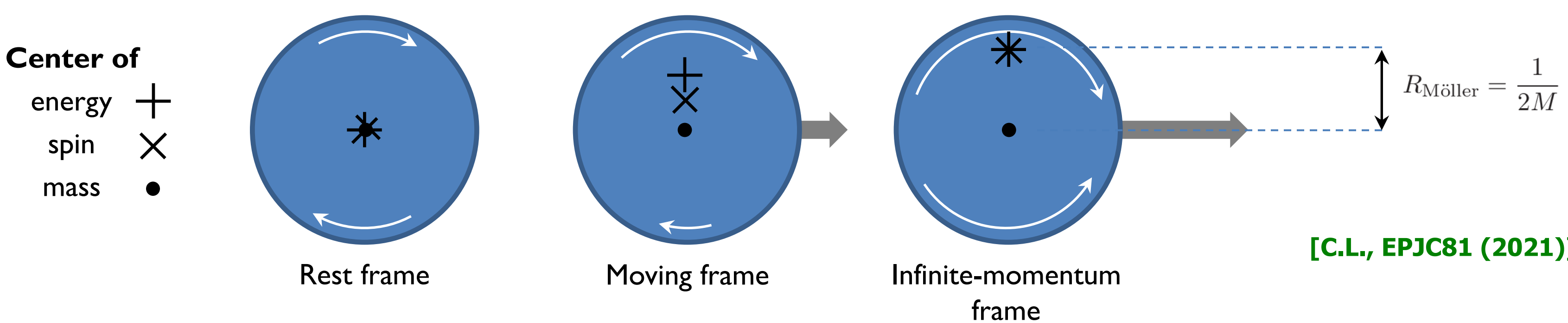
[Pryce, PRSLA180 (1935)]
[Pryce, PRSLA195 (1948)]
[Møller, CDIASA5 (1949)]
[Newton-Wigner, RMP21 (1949)]
[Bogolyubov-Logunov-Todorov, *Introduction to Axomatic Quantum Field Theory* (1975)]

Canonical operator
$$R^\mu_c = \frac{P^0 R^\mu_E + M R^\mu_M}{P^0 + M}$$

$$\Rightarrow \quad \vec{R}_c = \vec{R}_E - \frac{\vec{P} \times \vec{W}}{M P^0 (P^0 + M)} = \vec{R}_M + \frac{\vec{P} \times \vec{W}}{M^2 (P^0 + M)}$$

Spin operator
$$\boxed{\vec{S}_c \equiv \vec{J} - \vec{R}_c \times \vec{P} = \frac{\vec{W}}{M} - \frac{\vec{P} W^0}{M(P^0 + M)}}$$

Transversely polarized nucleon



[C.L., EPJC81 (2021)]

Transverse spin sum rules

$$\mathcal{S}^\mu = \left(\frac{\vec{p}\cdot\vec{s}}{M}, \vec{s} + \frac{\vec{p}(\vec{p}\cdot\vec{s})}{M(p^0+M)}\right)$$

$$\langle \int \mathrm{d}^3r\, r^i T^{0j}_a(r)\rangle_{\mathrm{int}} = -\left(p^0\epsilon^{ij\alpha\beta} + p^i\epsilon^{j0\alpha\beta} + p^j\epsilon^{i0\alpha\beta}\right)\frac{\mathcal{S}_\alpha p_\beta}{2p^0M}\,\frac{A_a(0)+B_a(0)}{2} - \frac{p^j\epsilon^{0i\alpha\beta}\mathcal{S}_\alpha p_\beta}{2M(p^0+M)}\,A_a(0)$$

Blue tems do not contribute to W^μ

Leader sum rule

[Leader, PRD85 (2012)]
[Leader-C.L., PR541 (2014)]

$$\langle J^k_a\rangle_{\mathrm{Leader}} = \epsilon^{kij}\langle \int \mathrm{d}^3r\, r^i T^{0j}_a(r)\rangle_{\mathrm{int}} = \frac{s^k}{2}\,A_a(0) + \frac{p^0}{2M}\left(s^k - \frac{p^k(\vec{p}\cdot\vec{s})}{p^0(p^0+M)}\right)B_a(0)$$

Ji-Yuan sum rule

[Ji-Yuan, PLB810 (2020)]

$$\langle J^k_a\rangle_{\text{Ji-Yuan}} = \epsilon^{kij}\langle \int \mathrm{d}^3r\, r^i T^{0j}_a(r)\rangle_{\mathrm{int}}\Big|_{without\ \text{blue terms}} = \frac{p^0}{M}\left(s^k - \frac{p^k(\vec{p}\cdot\vec{s})}{p^0(p^0+M)}\right)\frac{A_a(0)+B_a(0)}{2}$$

Interpreted as contributions from « center-of-mass motion »

(requires rigorous justification)

Comparison between expectation values

Let us denote rest-frame spin vector by $\frac{1}{2}\vec{s}$ with $\vec{s}^{\,2} = 1$

Longitudinal component

$$\langle S_E^L \rangle = \langle S_M^L \rangle = \langle S_c^L \rangle = \tfrac{1}{2} s_L$$

Explains why the question of the nucleon center did not draw much attention in the past

Transverse component

$$\langle S_E^T \rangle = \gamma^{-1}\, \tfrac{1}{2} s_T$$

Transverse part of a four-vector is subleading $\qquad \gamma = p^0/M$

$$\langle S_M^T \rangle = \gamma\, \tfrac{1}{2} s_T$$

Transverse part of an antisymmetric rank-two tensor is leading

[Landau-Lifshitz, *Classical Theory of Fields* (1951)]

$$\langle S_c^T \rangle = \tfrac{1}{2} s_T$$

Frame-independent!

(simple AM composition crucial for the wavefunction formalism)

Phase-space approach

$$\langle \vec{R}_E\rangle = \vec{\mathcal{R}} + \frac{\vec{p}\times\vec{s}}{2p^0(p^0+M)} \qquad \langle \vec{S}_E\rangle = \frac{M}{2p^0}\left(\vec{s} + \frac{\vec{p}(\vec{p}\cdot\vec{s})}{M(p^0+M)}\right)$$

$$\langle \vec{R}_M\rangle = \vec{\mathcal{R}} - \frac{\vec{p}\times\vec{s}}{2M(p^0+M)} \qquad \langle \vec{S}_M\rangle = \frac{p^0}{2M}\left(\vec{s} - \frac{\vec{p}(\vec{p}\cdot\vec{s})}{p^0(p^0+M)}\right) \qquad \langle O\rangle \equiv \langle O\rangle_{\vec{\mathcal{R}},\vec{p}}$$

$$\langle \vec{R}_c\rangle = \vec{\mathcal{R}} \qquad \langle \vec{S}_c\rangle = \frac{\vec{s}}{2}$$

We define quark and gluon contributions to internal AM operator as

$$S^k_{X,a} \equiv \epsilon^{kij}\int \mathrm{d}^3r\,\left(r^i - \langle R^i_X\rangle\right) T^{0j}_a(r) = J^k_a - \epsilon^{kij}\langle R^i_X\rangle P^j_a \qquad X = E, M, c$$

Relativistic spin sum rules **[C.L., EPJC81 (2021)]**

$$\langle \vec{S}_{X,a}\rangle = \langle \vec{S}_X\rangle\, A_a(0) + \langle \vec{S}_M\rangle\, B_a(0)$$

Leader: $X = c$ canonical spin sum rule

Ji-Yuan: $X = M$ covariant spin sum rule

(When « spin » is properly defined, there is no need to drop terms!)