Practical corollaries of transverse Ward-Green-Takahashi identities

Sixue Qin

Argonne National Laboratory

with Craig D. Roberts

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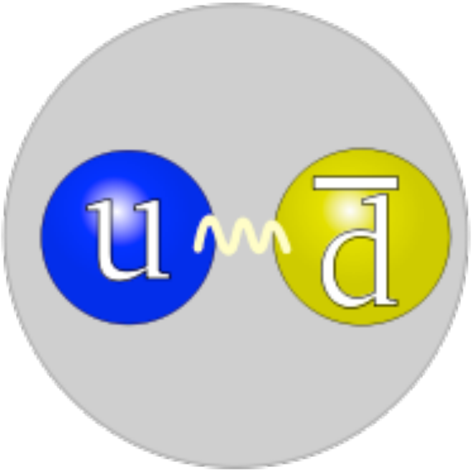
Meson: Two-body bound-state of QCD

QCD

DCSB & Confinement



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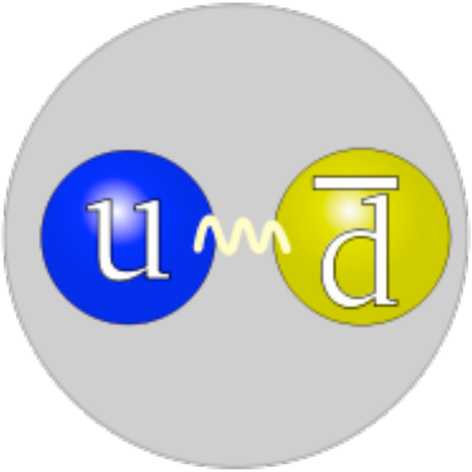
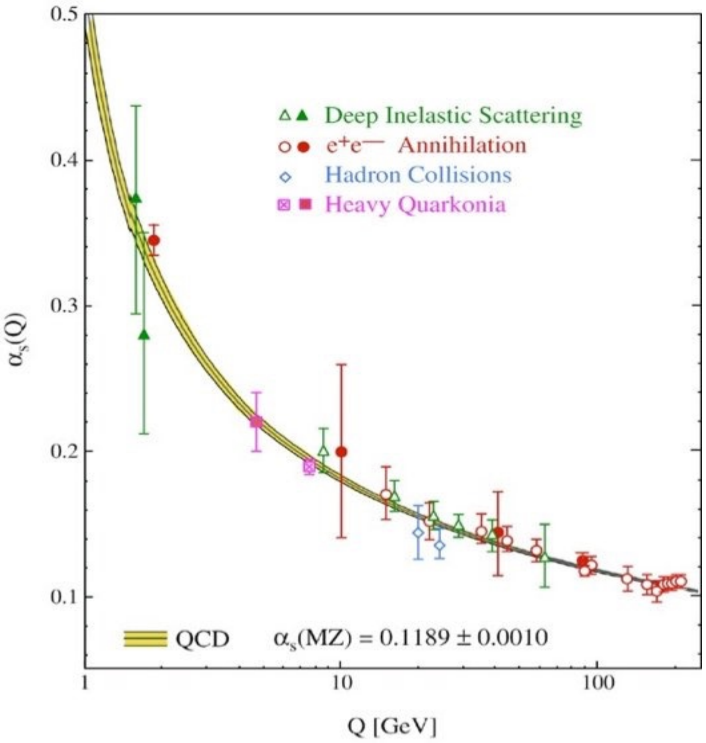


Non-perturbative



QCD running coupling constant

Dyson-Schwinger approach: Equation of motion of QCD

Classical Mechanics | Quantum Field Theory

Principle of Least Action

$$\frac{\delta S[q]}{\delta q} = 0$$

$$\left\langle \frac{\delta S[\phi(x)]}{\delta \phi(x)} \right\rangle = 0$$

Equations of Motion (EoM)

Euler-Lagrange Equation | Dyson-Schwinger Equation

Dyson-Schwinger approach: Equation of motion of QCD

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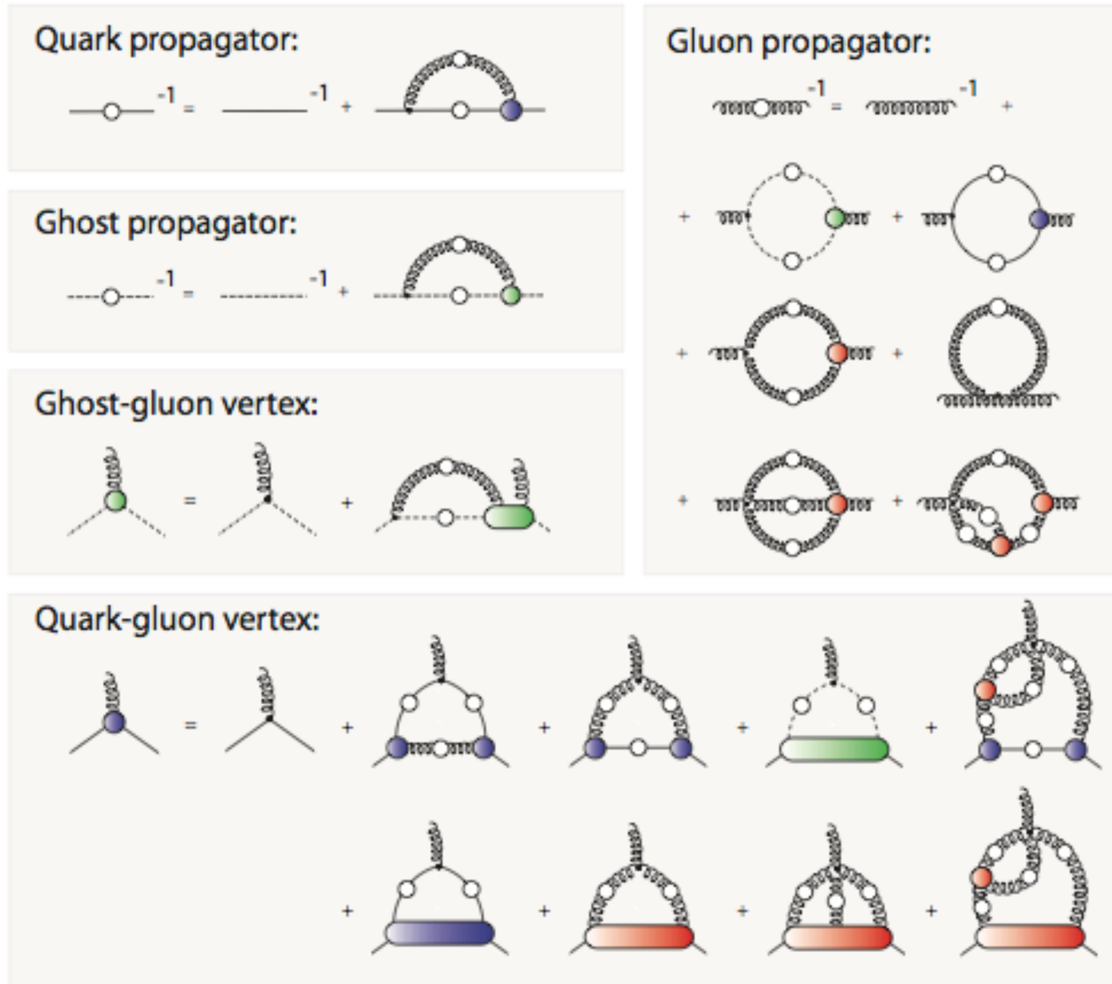
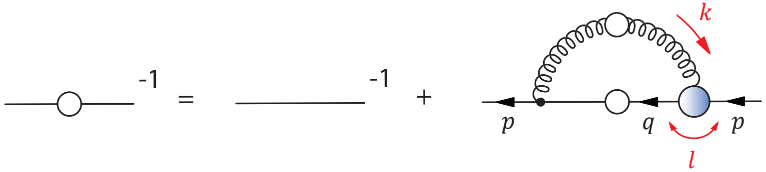
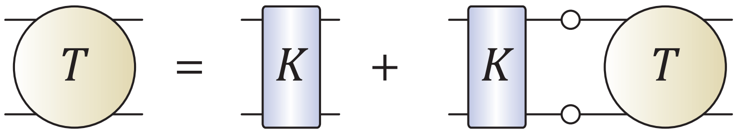
G. Eichmann, arXiv:0909.0703



Two-body problem: low-order Dyson-Schwinger equations





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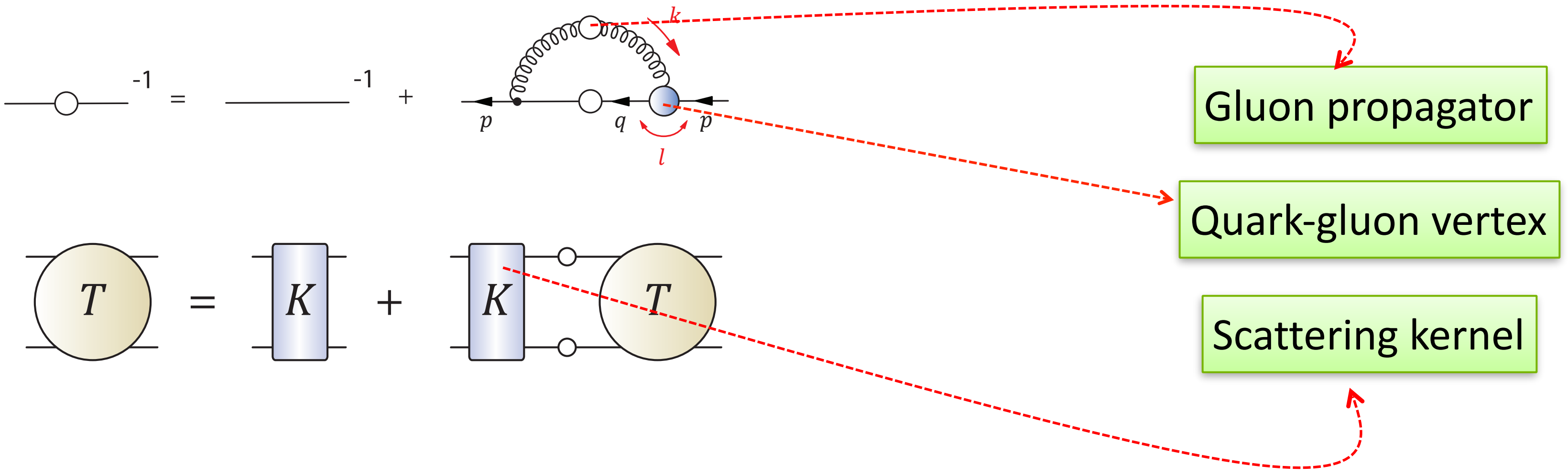

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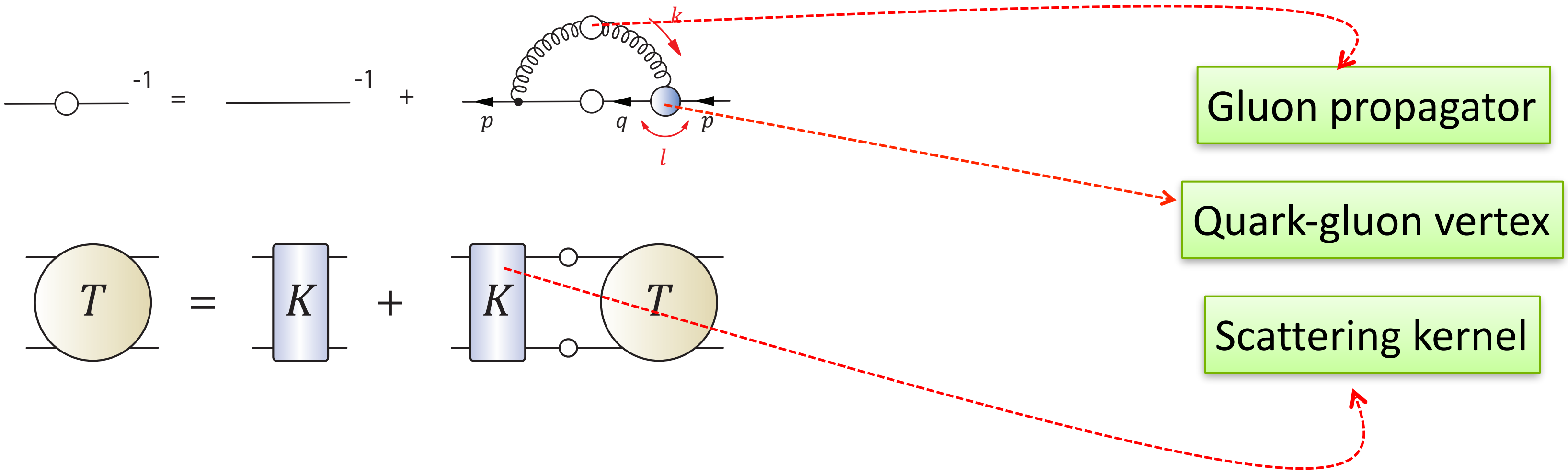
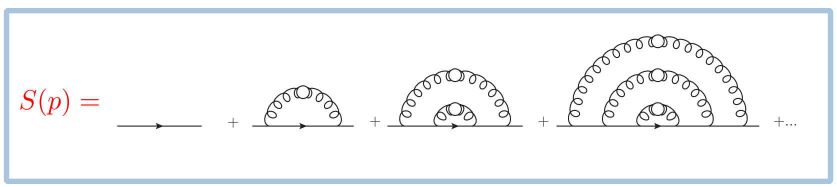
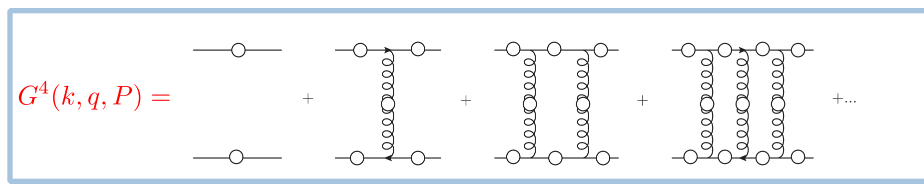


Rainbow-Ladder truncation: The leading symmetry-preserving approximation





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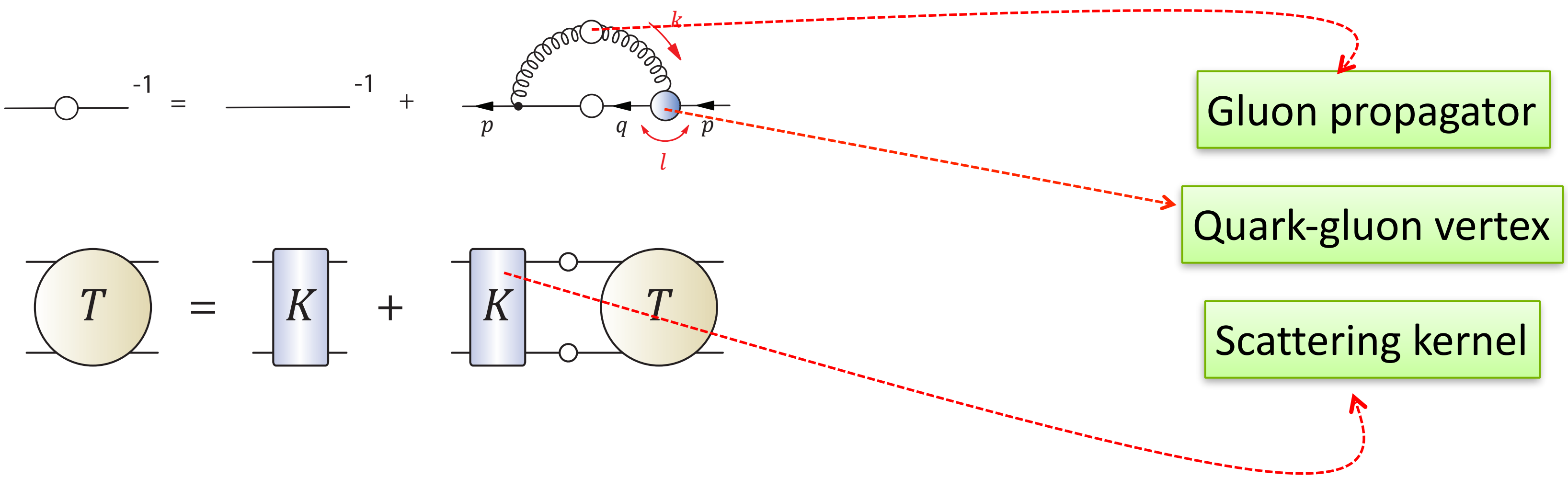

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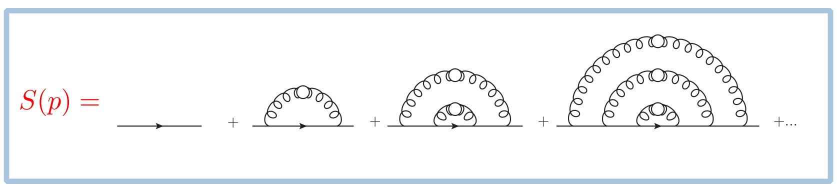

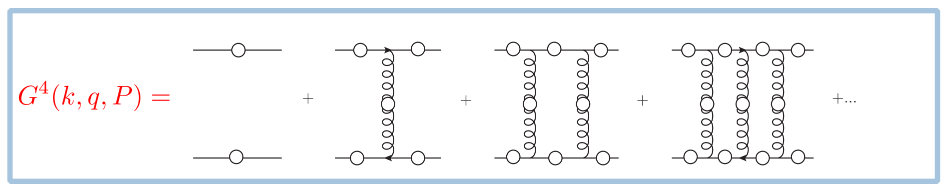

Is there a systematic way to improve the truncation to approach the full QCD?

Ward-Green-Takahashi identities: Lagrangian symmetries

- **Gauge symmetry (vector current conservation):** vector WGTI

$$\psi(x) \to \psi(x) + ig\alpha(x)\psi(x),$$
$$\bar{\psi}(x) \to \bar{\psi}(x) - ig\alpha(x)\bar{\psi}(x)$$

$$iq_\mu \Gamma_\mu(k,p) = S^{-1}(k) - S^{-1}(p)$$

- **Chiral symmetry (axial-vector current conservation):** axial-vector WGTI

$$\psi(x) \to \psi(x) + ig\alpha(x)\gamma^5\psi(x),$$
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- **Lorentz symmetry + (axial-)vector current conservation:** transverse WGTIs

$$\delta_T\phi^a(x) = \delta_{\text{Lorentz}}(\delta\phi^a(x)) = -\frac{i}{2}\epsilon^{\mu\nu}S_{\mu\nu}^{(\delta\phi^a)}(\delta\phi^a(x)).$$
$$S_{\mu\nu}^{(\text{spinor})} = \frac{1}{2}\sigma_{\mu\nu}, \qquad (S_{\mu\nu}^{(\text{vector})})^\alpha_\beta = i(\delta^\alpha_\mu g_{\nu\beta} - \delta^\alpha_\nu g_{\mu\beta});$$

He, PRD, 80, 016004 (2009)

$$\begin{aligned} q_\mu\Gamma_\nu(k,p) - q_\nu\Gamma_\mu(k,p) = {} & S^{-1}(p)\sigma_{\mu\nu} + \sigma_{\mu\nu}S^{-1}(k) \\ & + 2im\Gamma_{\mu\nu}(k,p) + t_\lambda\varepsilon_{\lambda\mu\nu\rho}\Gamma_\rho^A(k,p) \\ & + A_{\mu\nu}^V(k,p), \\ q_\mu\Gamma_\nu^A(k,p) - q_\nu\Gamma_\mu^A(k,p) = {} & S^{-1}(p)\sigma_{\mu\nu}^5 - \sigma_{\mu\nu}^5 S^{-1}(k) \\ & + t_\lambda\varepsilon_{\lambda\mu\nu\rho}\Gamma_\rho(k,p) \\ & + V_{\mu\nu}^A(k,p), \qquad \sigma_{\mu\nu}^5 = \gamma_5\sigma_{\mu\nu} \end{aligned}$$

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The longitudinal and transverse WGTIs express the vertex divergences and curls, respectively.

$$\nabla \cdot \Phi \qquad \nabla \times \Phi$$

Fermion--gauge-boson coupling: Solution of WGTIs

Define two projection tensors and contract them with the transverse WGTIs,

$$T^1_{\mu\nu} = \frac{1}{2}\varepsilon_{\alpha\mu\nu\beta}t_\alpha q_\beta \mathbf{I}_{\rm D}, \qquad T^2_{\mu\nu} = \frac{1}{2}\varepsilon_{\alpha\mu\nu\beta}\gamma_\alpha q_\beta.$$

one can decouple the WGTIs and obtain a group of equations for the vector vertex:

$$q_\mu i\Gamma_\mu(k,p) = S^{-1}(k) - S^{-1}(p),$$

$$\begin{aligned} q\cdot tt\cdot\Gamma(k,p) &= T^1_{\mu\nu}\left[S^{-1}(p)\sigma^5_{\mu\nu} - \sigma^5_{\mu\nu}S^{-1}(k)\right] \\ &\quad + t^2 q\cdot\Gamma(k,p) + T^1_{\mu\nu}V^A_{\mu\nu}(k,p), \\ q\cdot t\gamma\cdot\Gamma(k,p) &= T^2_{\mu\nu}\left[S^{-1}(p)\sigma^5_{\mu\nu} - \sigma^5_{\mu\nu}S^{-1}(k)\right] \\ &\quad + \gamma\cdot tq\cdot\Gamma(k,p) + T^2_{\mu\nu}V^A_{\mu\nu}(k,p). \end{aligned}$$

Qin et. al., PLB 722, 384 (2013)

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They are a group of full-determinant linear equations.
Thus, a unique solution for the vector vertex is exposed:

$$\Gamma^{\rm Full}_\mu(k,p) = \Gamma^{\rm BC}_\mu(k,p) + \Gamma^{\rm T}_\mu(k,p) + \Gamma^{\rm FP}_\mu(k,p).$$

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❖ The quark propagator contributes to the longitudinal and transverse parts. The DCSB-related terms are highlighted.

$$\Gamma^{\mathrm{BC}}_\mu(k,p) = \gamma_\mu\Sigma_A + t_\mu \not{t}\frac{\Delta_A}{2} - it_\mu\Delta_B,$$

$$\Gamma^{\mathrm{T}}_\mu(k,p) = -\sigma_{\mu\nu}q_\nu\Delta_B + \gamma^T_\mu q^2\frac{\Delta_A}{2} - \left(\gamma^T_\mu[\not{q},\not{t}] - 2t^T_\mu\not{q}\right)\frac{\Delta_A}{4}.$$

$$S(p) = \frac{1}{i\gamma\cdot p\,A(p^2) + B(p^2)}$$

$$\Sigma_\phi(x,y) = \frac{1}{2}[\phi(x)+\phi(y)],$$

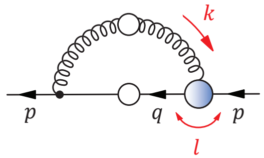
$$\Delta_\phi(x,y) = \frac{\phi(x)-\phi(y)}{x-y}.$$

$$X^T_\mu = X_\mu - \frac{q\cdot X q_\mu}{q^2}$$

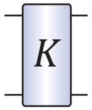
❖ The unknown high-order terms only contribute to the transverse part, i.e., the longitudinal part has been completely determined by the quark propagator.

Qin et. al., PLB 722, 384 (2013)

Scattering kernel: Beyond rainbow-ladder approximation

$$\Sigma(x,y) =$$ 

$\longrightarrow$



$$= \frac{\delta \Sigma(x,y)}{\delta S(x',y')}$$

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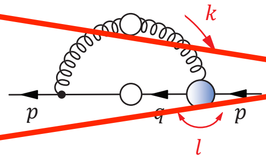
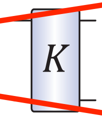
$\Sigma(x, y) =$ 

$\Longrightarrow$



$= \dfrac{\delta\Sigma(x, y)}{\delta S(x', y')}$

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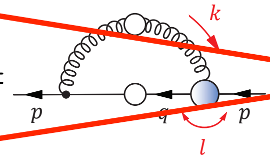
$\Sigma(x,y) =$ 

$$\longrightarrow \quad \boxed{K} = \frac{\delta\Sigma(x,y)}{\delta S(x',y')}$$

The quark equation is specified by the scattering kernel:

$$\Gamma^H_{\alpha\beta}(k,P) = \gamma^H_{\alpha\beta} + \int_q \mathcal{K}(k_\pm,q_\pm)_{\alpha\alpha',\beta'\beta}[S(q_+)\Gamma^H(q,P)S(q_-)]_{\alpha'\beta'}.$$

$$\boxed{S(p) = \frac{1}{i\gamma\cdot p\,A(p^2)+B(p^2)}}$$

$$i\hat{P}_\mu\Gamma_\mu(k,0) = \hat{P}_\mu\frac{\partial S^{-1}(k)}{\partial k_\mu},$$

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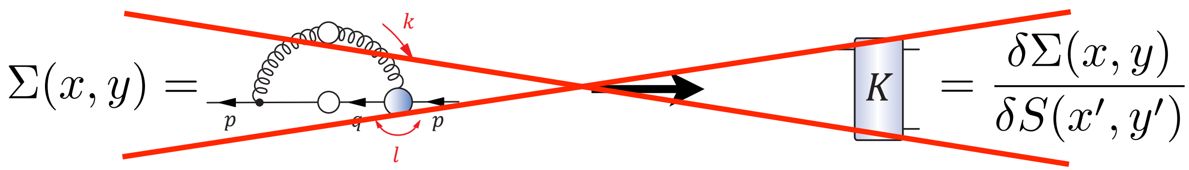

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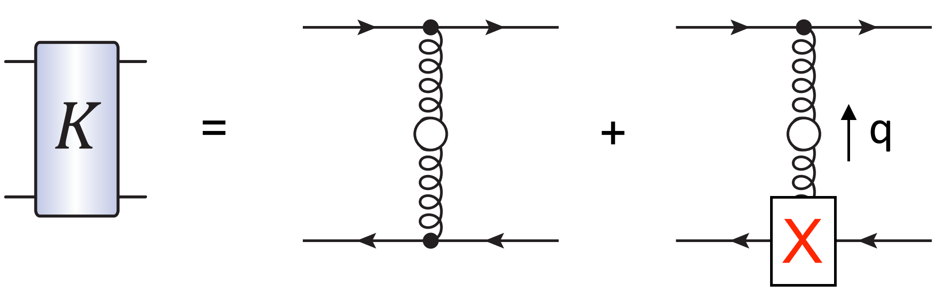
As an example, the kernel is written as two parts (anomalous magnetic moment):



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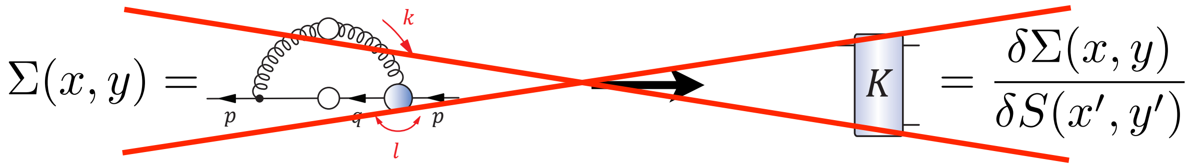

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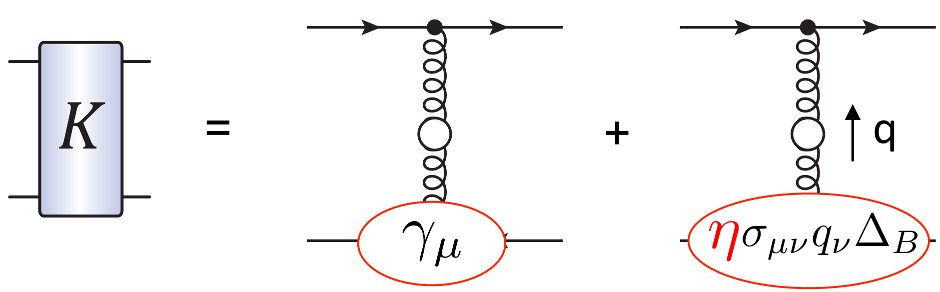
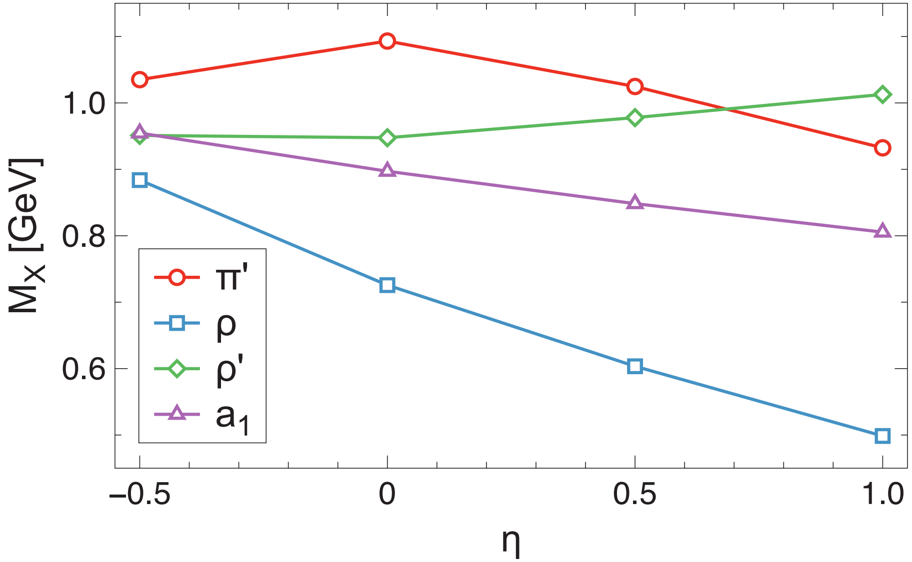
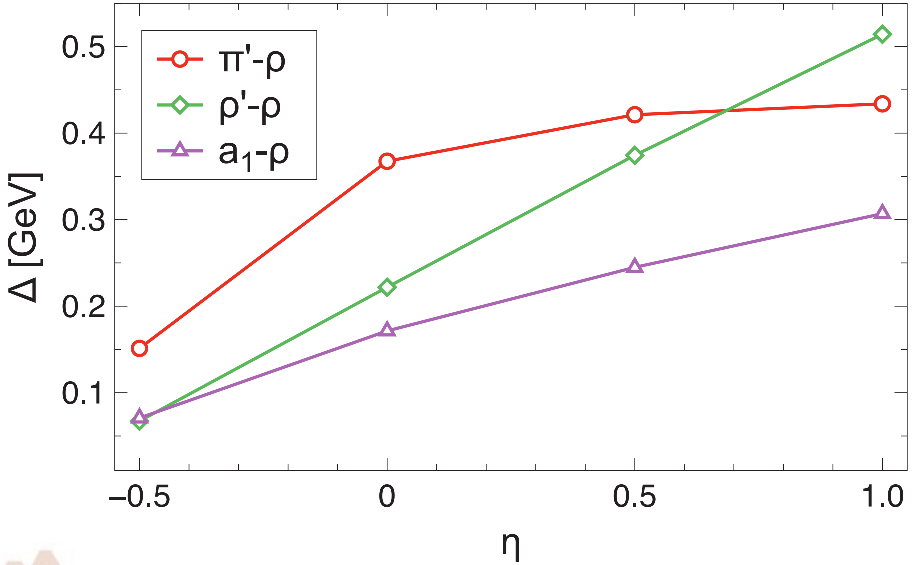


Meson spectrum v.s. AMM strength: mass < 2 GeV



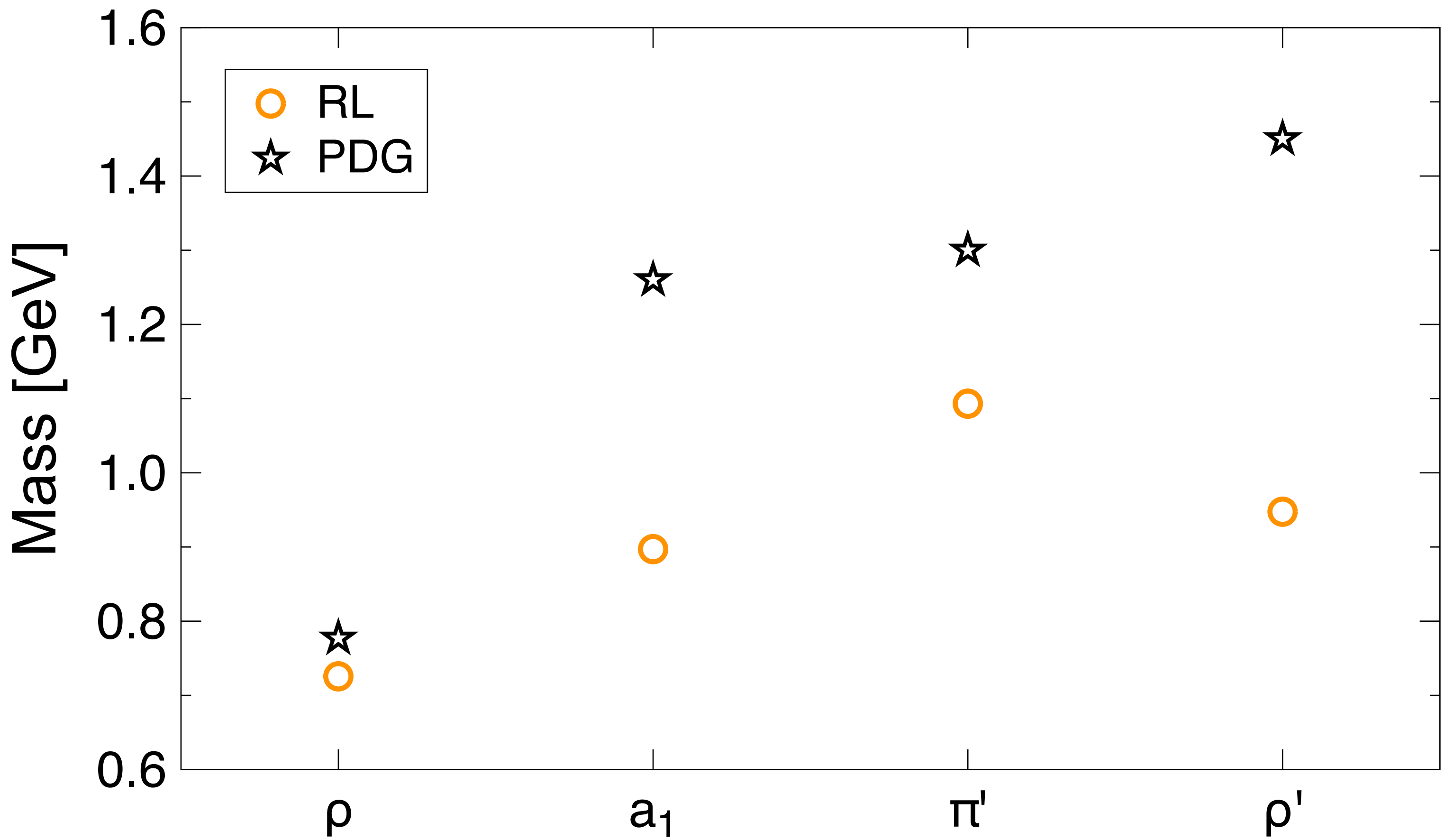


- When the AMM is negative, masses of all states are squeezed together with increasing the strength.

- When the AMM is positive, with increasing the strength, the rho-a1 and rho-rho' mass splittings are enhanced.

- When the AMM is positive, with increasing the strength, the pi'-rho' mass ordering is reversed.

Best meson spectrum: comparison with RL approximation



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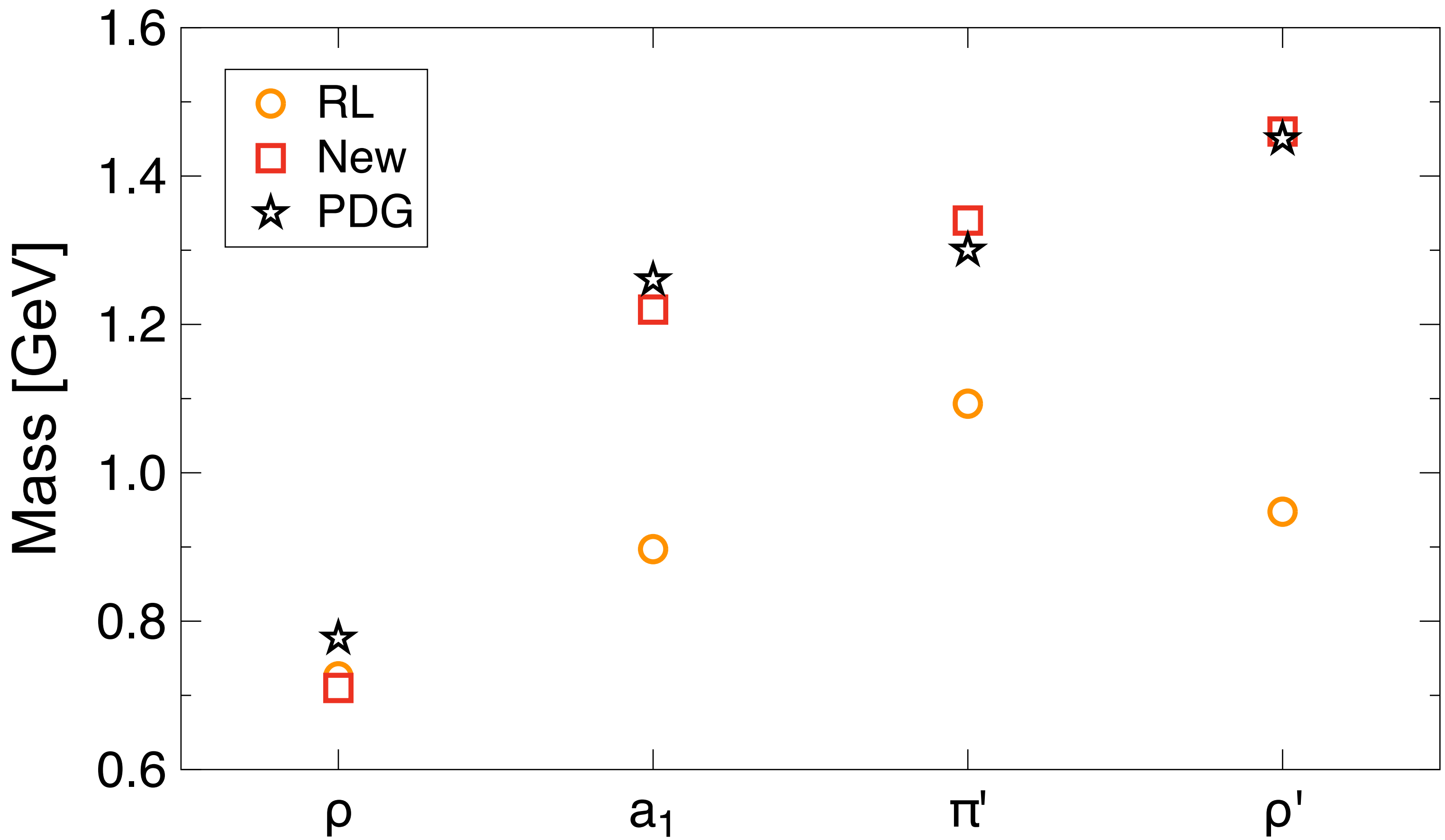


Summary

✦ A **novel structure** for the fermion—gauge-boson vertex is exposed by solving WGTIs

✦ A **new method** beyond rainbow-ladder approximation is proposed based on WGTIs

✦ A **demonstration** applying the new method to light meson spectrum is presented.

Outlook

◆ The WGTIs exposes more structures of the vertex and thus provides more information for truncating the DSEs.

◆ Using more sophisticated scattering kernels, the new method is potentially useful for hadron phenomenology.

Appendices

Appendix 1: modeling gluon propagator

- Using Oliveira's scheme, we can readily parameterize our interaction model as follows,

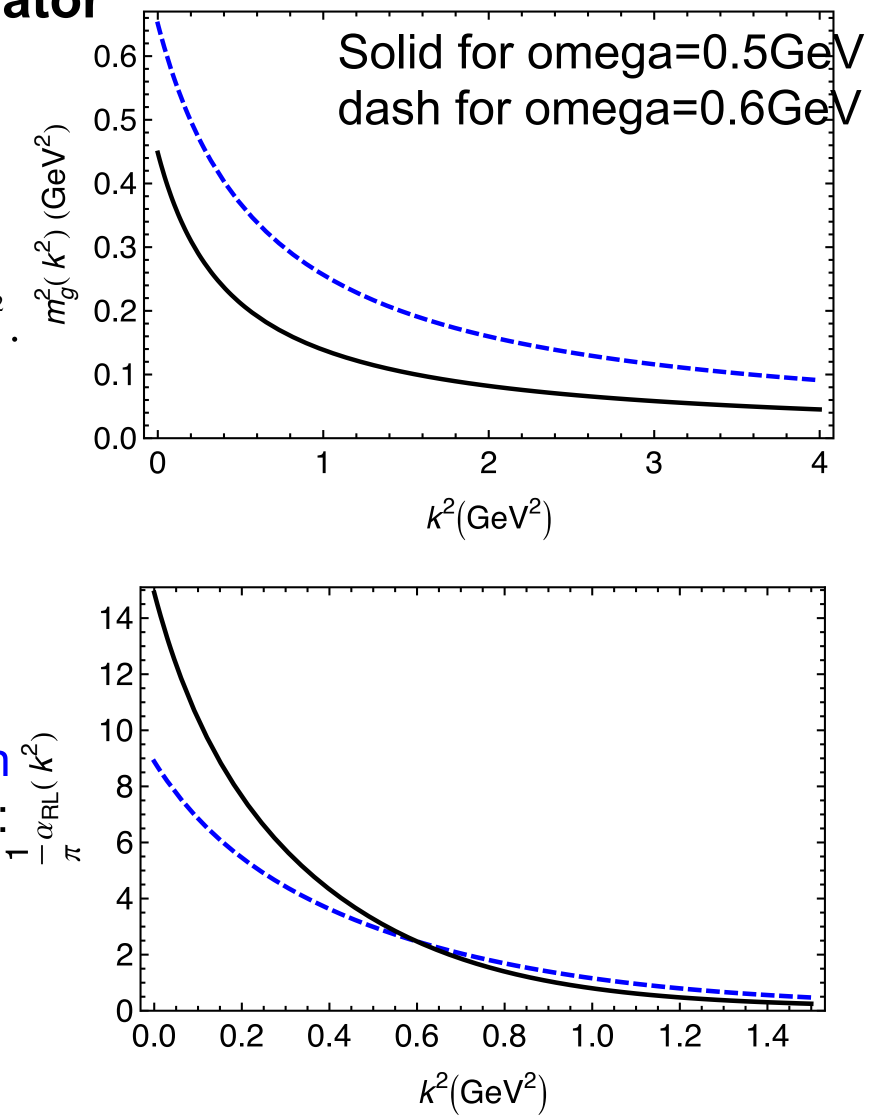
$$\delta^4(k) \overset{\omega\sim 0}{\approx} \frac{1}{\pi^2}\frac{1}{\omega^4}e^{-k^2/\omega^2}, \quad \delta^4(k) \overset{\omega\sim 0}{\approx} \frac{1}{2\pi^2}\frac{1}{\omega^6}k^2 e^{-k^2/\omega^2}.$$

$$\mathcal{G}(s) = \frac{8\pi^2}{\omega^4}De^{-s/\omega^2} + \frac{8\pi^2\gamma_m \mathcal{F}(s)}{\ln\left[\tau + \left(1+s/\Lambda_{\rm QCD}^2\right)^2\right]}$$

$$\mathcal{G}(k^2) \approx \frac{4\pi\alpha_{RL}(k^2)}{k^2+m_g^2(k^2)}, \; m_g^2(k^2) = \frac{M_g^4}{M_g^2+k^2},$$

- The infrared scale for the running gluon mass increases with increasing omega:

$$M_g = 0.67\text{GeV}, \omega = 0.5\text{GeV}$$
$$M_g = 0.81\text{GeV}, \omega = 0.6\text{GeV}$$



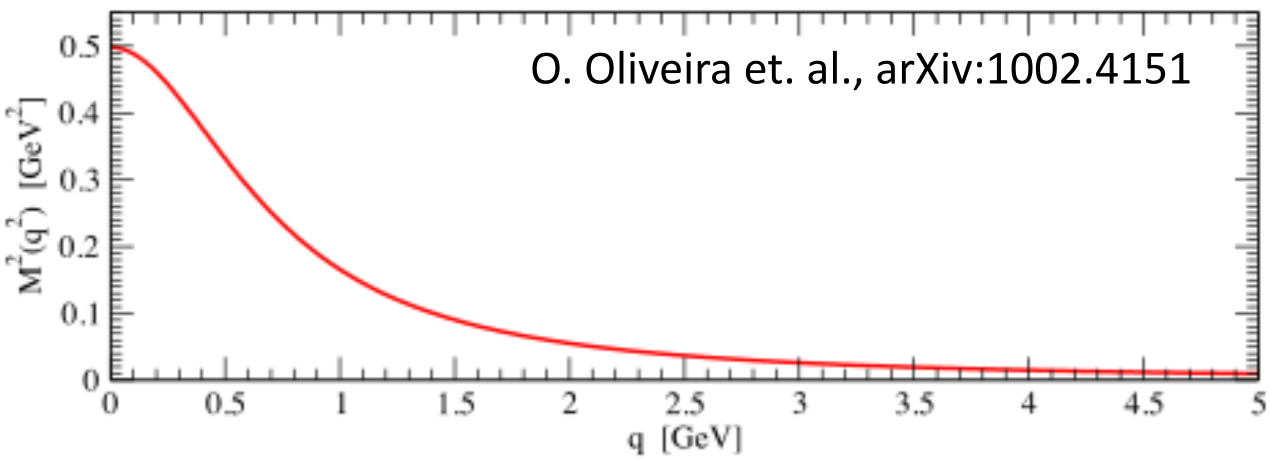
➢ In Landau gauge (a fixed point of the renormalization group):

$$g^2 D_{\mu\nu}(k) = \mathcal{G}(k^2)(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2})$$

➢ Modeling the dress function: ***gluon mass scale*** + ***effective running coupling constant***

$$\mathcal{G}(k^2) \approx \frac{4\pi\alpha_{RL}(k^2)}{k^2 + m_g^2(k^2)}, \quad m_g^2(k^2) = \frac{M_g^4}{M_g^2 + k^2},$$

- The gluon mass scale is *typical lattice QCD values* in our parameter range: *Mg* in [0.6, 0.8] GeV.
- The gluon mass scale is inversely proportional to the confinement length.



Note that the gluon propagator has to support both ***confinement*** and ***DCSB***.

Appendix II: Calculated meson properties in Rainbow-Ladder approximation

$(D\omega)^{1/3}$	0.8	0.8	0.8
ω	0.4	0.5	0.6
$m_{u,d}^{\zeta}$	0.0034	0.0034	0.0034
m_s^{ζ}	0.082	0.082	0.082
$A(0)$	2.07	1.70	1.38
$M(0)$	0.62	0.52	0.42
m_π	0.139	0.134	0.136
f_π	0.094	0.093	0.090
$\rho_\pi^{1/2}$	0.49	0.49	0.49
m_K	0.496	0.495	0.497
f_K	0.11	0.11	0.11
$\rho_K^{1/2}$	0.55	0.55	0.55
m_ρ	0.76	0.74	0.72
f_ρ	0.14	0.15	0.14
m_ϕ	1.09	1.08	1.07
f_ϕ	0.19	0.19	0.19
m_σ	0.67	0.65	0.59
$\rho_\sigma^{1/2}$	0.53	0.53	0.51