Explore Parton Distributions
from
Lattice QCD Calculations

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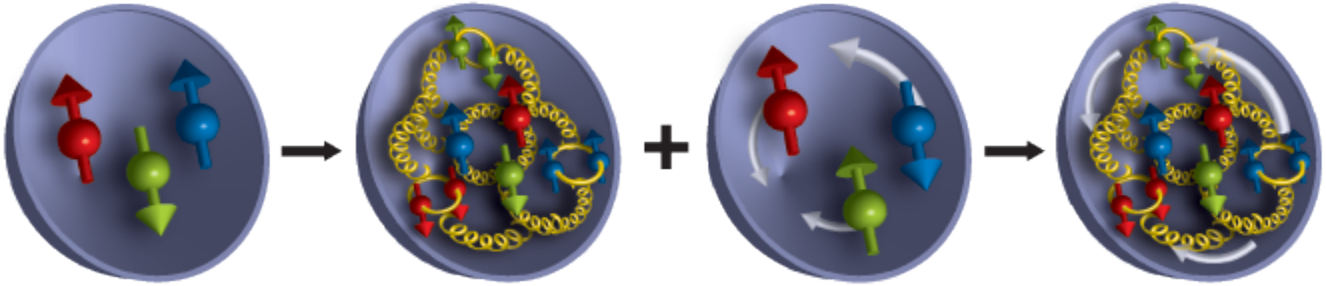
Based on work done with
Tomomi Ishikawa, Yan-Qing Ma, Shinsuke Yoshida, …
arXiv:1404.6860, 1412.2688, …
and work by many others, …

Workshop of the APS Topical Group on Hadronic Physics (GHP2015)
Hilton Baltimore - Johnson, Baltimore, MD, April 8-10, 2015

Nucleon's internal structure

- ☐ Our understanding of the nucleon evolves



1970s 1980s/2000s Now

Nucleon is a strongly interacting, relativistic bound state of quarks and gluons

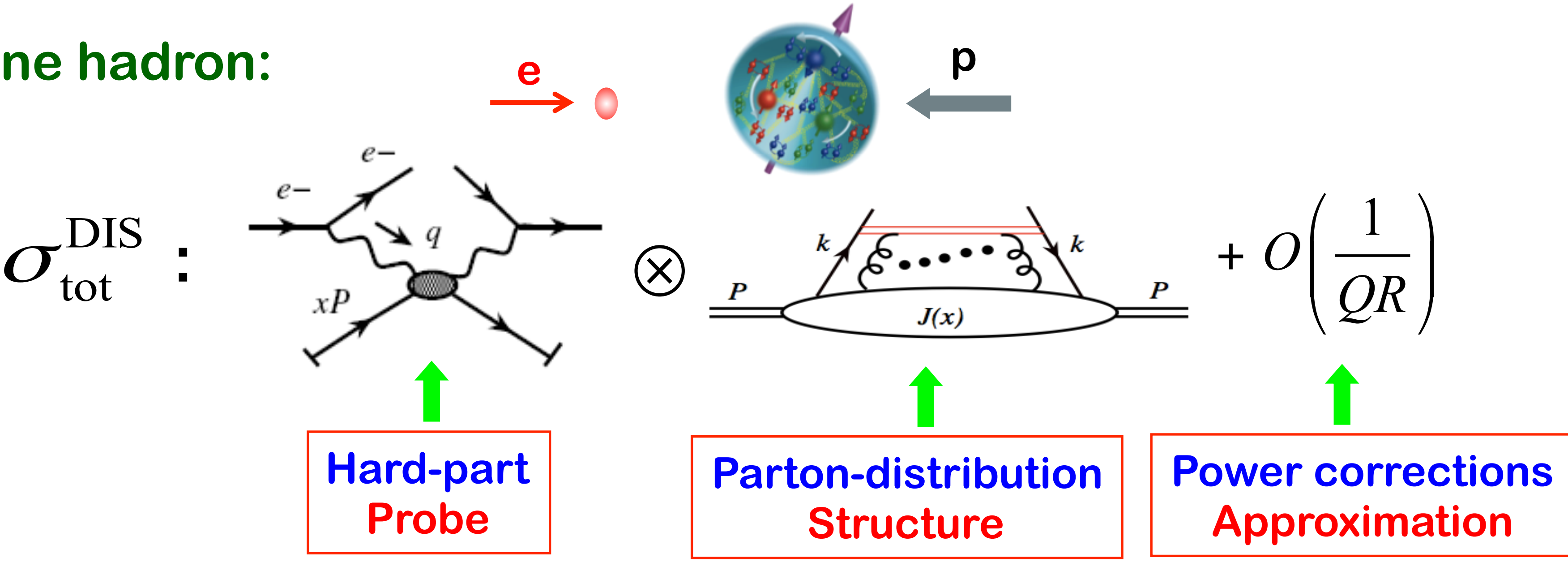
- ☐ QCD bound states:
  - ✧ Neither quarks nor gluons appear in isolation!
  - ✧ Understanding such systems completely is still beyond the capability of the best minds in the world

- ☐ The great intellectual challenge:

Probe nucleon structure without "seeing" quarks and gluons?

Hard probe and QCD factorization

❑ **One hadron:**



$$\sigma_{\text{tot}}^{\text{DIS}} : \quad \otimes \quad + O\left(\frac{1}{QR}\right)$$

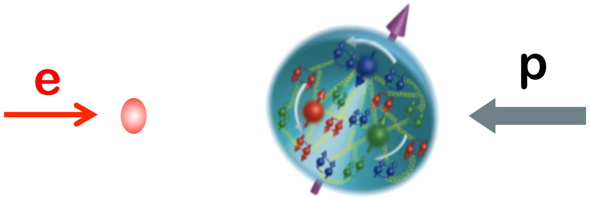
Hard-part
Probe

Parton-distribution
Structure

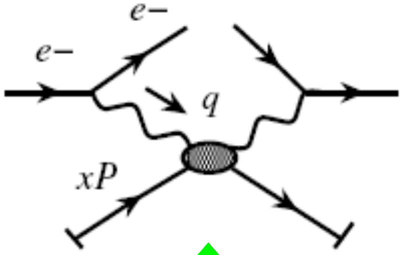
Power corrections
Approximation

Hard probe and QCD factorization

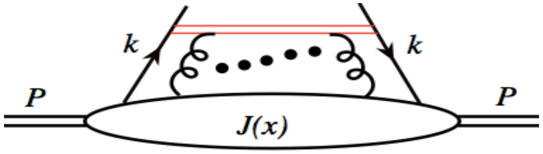
- **One hadron:**



$\sigma_{\mathrm{tot}}^{\mathrm{DIS}}$:



$\otimes$



$+ O\left(\frac{1}{QR}\right)$

Hard-part
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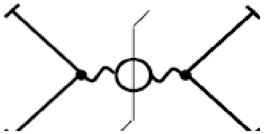
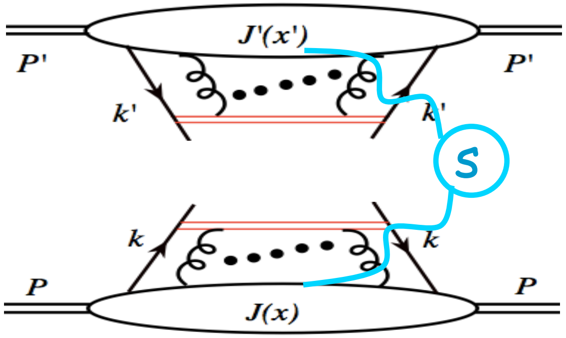
- **Two hadrons:**

$\sigma_{\mathrm{tot}}^{\mathrm{DY}}$:



$\otimes$



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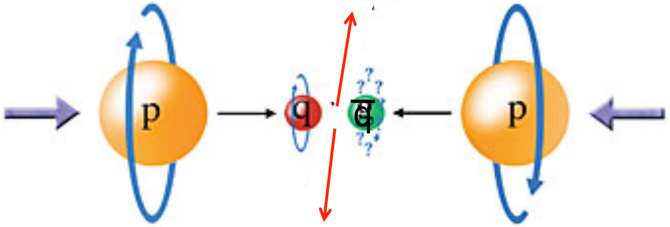


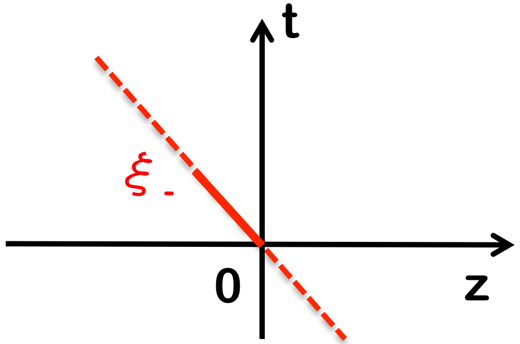
Predictive power:
Universal Parton Distributions

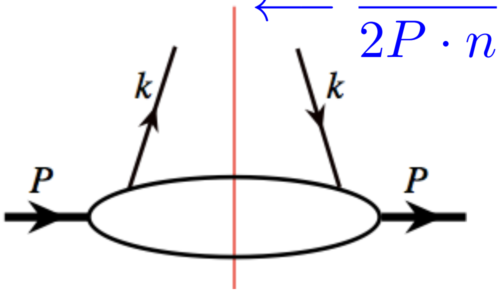
Operator definition of PDFs

- **Quark distribution (spin-averaged):**

$$q(x,\mu^2) = \int \frac{d\xi_-}{4\pi} e^{-ix\xi_- P_+} \langle P|\overline{\psi}(\xi_-)\gamma_+ \exp\left\{-ig\int_0^{\xi_-} d\eta_- A_+(\eta_-)\right\}\psi(0)|P\rangle \;+\mathrm{UVCT}$$

- **Cut-vertex notation:**





$$\longleftarrow \frac{\gamma\cdot n}{2P\cdot n}\,\delta\left(x-\frac{k\cdot n}{P\cdot n}\right)\frac{d^4k}{(2\pi)^4}$$

$$+\,\mathrm{UVCT}$$

PDFs are not direct physical observables, such as cross sections!
But, well-defined in QCD and process independent!

- **Parton interpretation emerges in *n.A = 0* gauge**
- **Independent of hadron momentum *P***
- **Simplest of all parton correlation functions of the hadron**

Global QCD analyses – a successful story

- ❑ World data with "Q" > 2 GeV

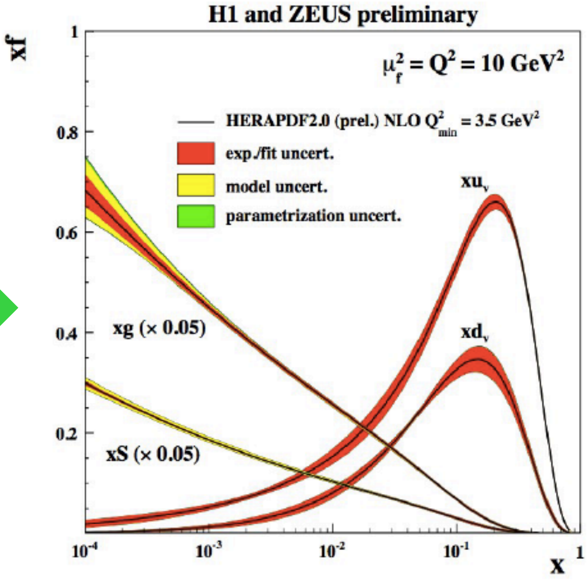
  + Factorization:

DIS: $$F_2(x_B,Q^2) = \Sigma_f C_f(x_B/x,\mu^2/Q^2)\otimes f(x,\mu^2)$$

H-H: $$\frac{d\sigma}{dydp_T^2} = \Sigma_{ff'} f(x)\otimes\frac{d\hat{\sigma}_{ff'}}{dydp_T^2}\otimes f'(x')$$

  + DGLAP Evolution:

$$\frac{\partial f(x,\mu^2)}{\partial\ln\mu^2} = \Sigma_{f'} P_{ff'}(x/x')\otimes f'(x',\mu^2)$$



Global QCD analyses – a successful story

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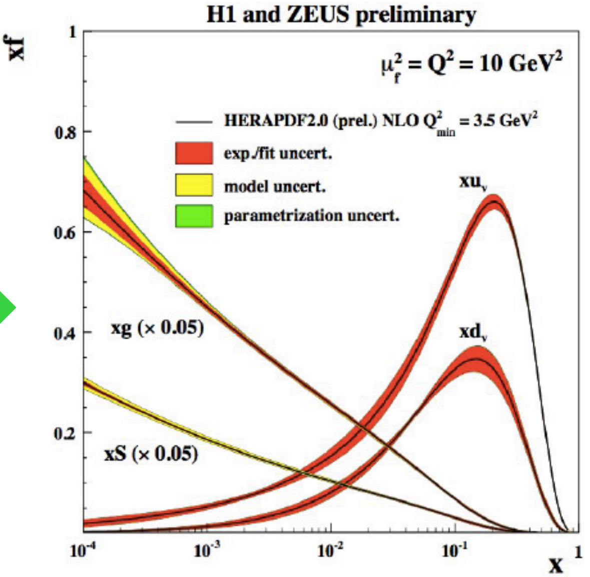
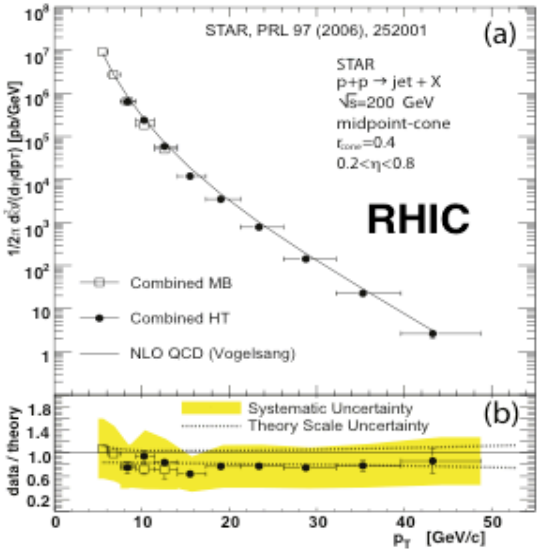
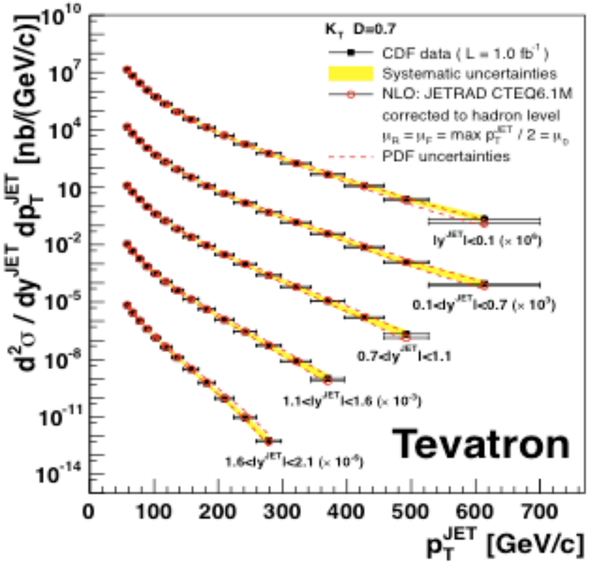
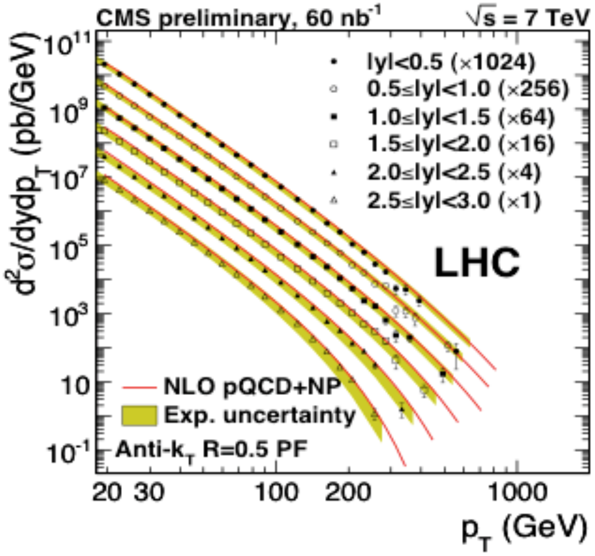


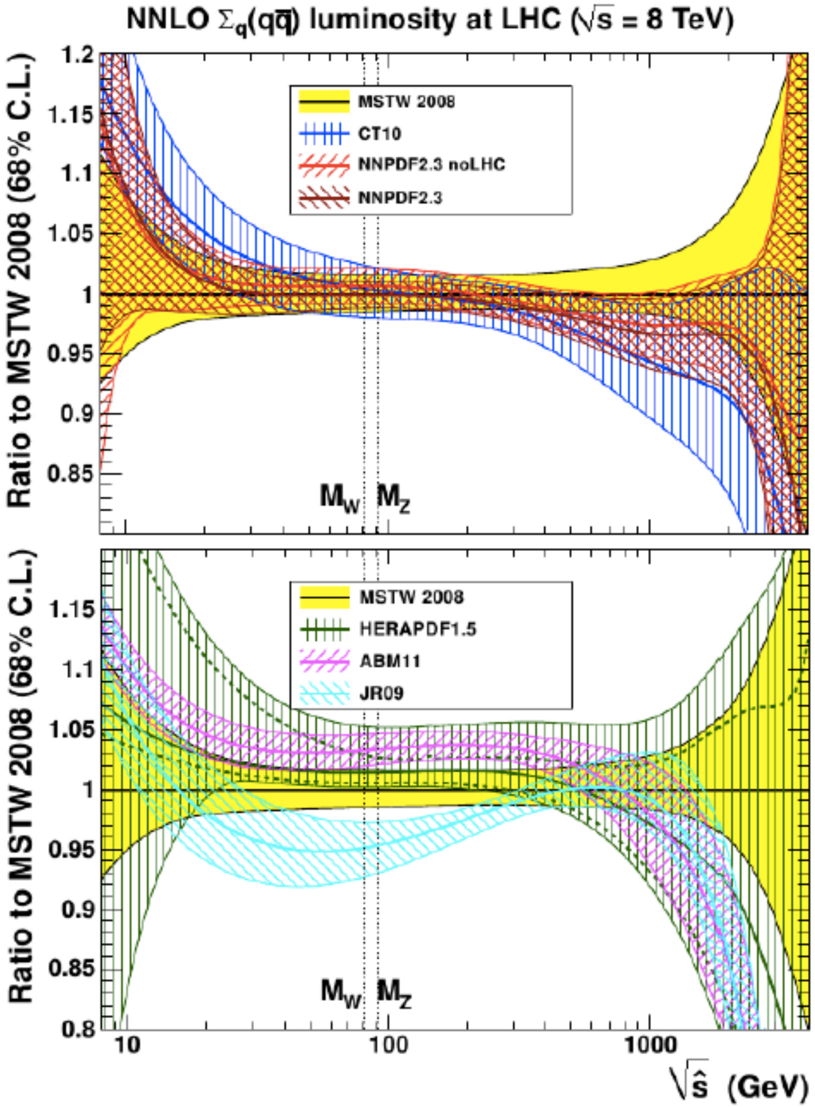




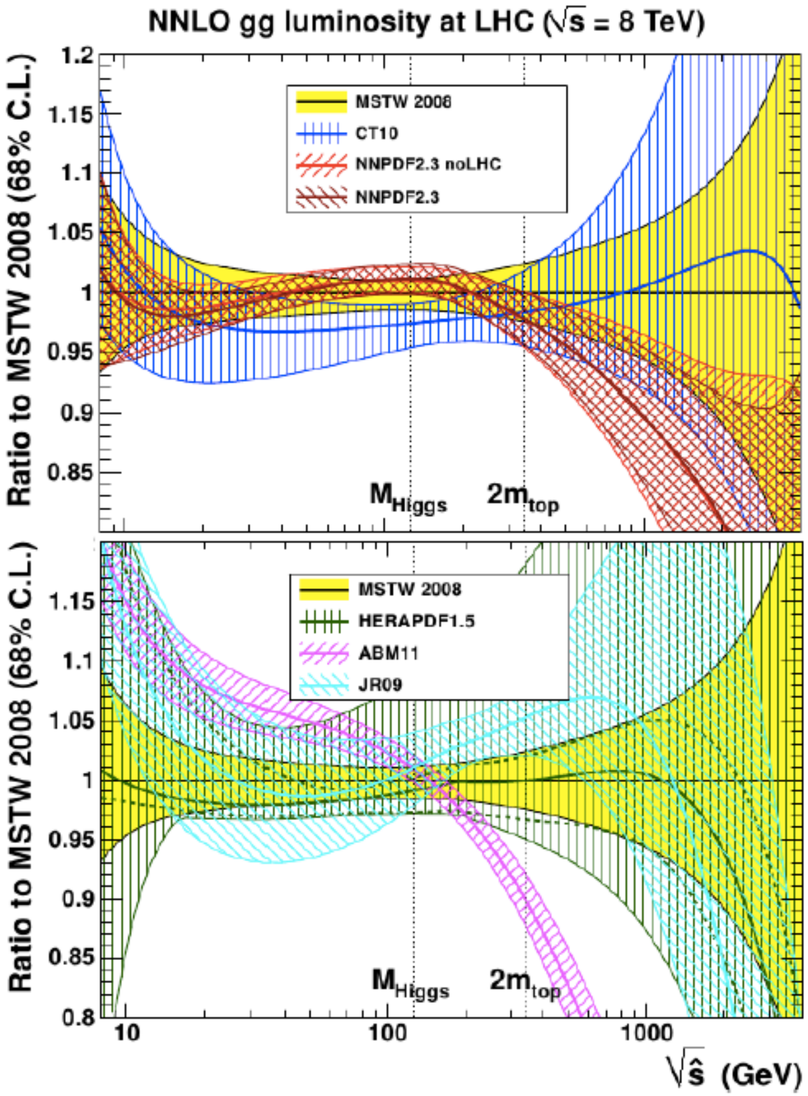


Partonic luminosities

q - qbar



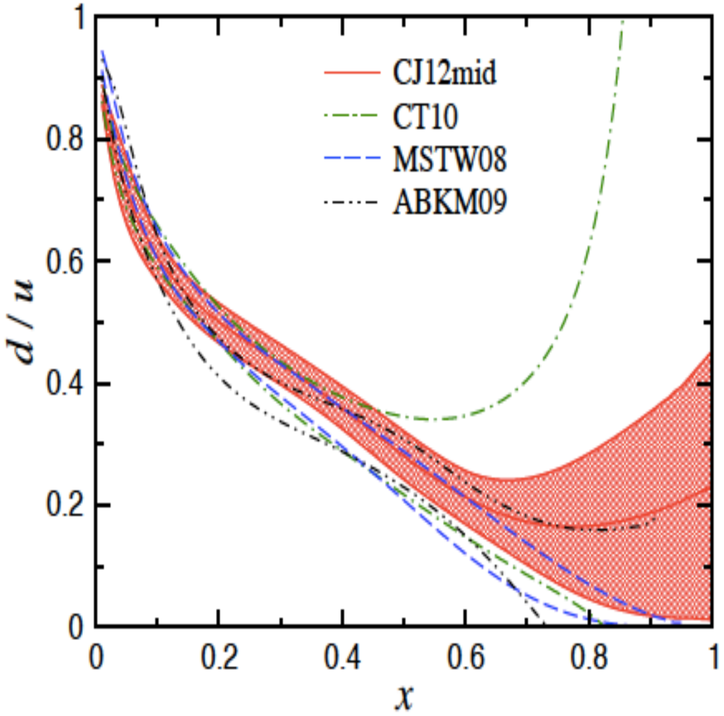
g - g



PDFs at large x

❑ **Testing ground for hadron structure at x →1:**

- ✧ $d/u \to 1/2$ **SU(6) Spin-flavor symmetry**
- ✧ $d/u \to 0$ **Scalar diquark dominance**
- ✧ $d/u \to 1/5$ **pQCD power counting**
- ✧ $d/u \to \frac{4\mu_n^2/\mu_p^2 - 1}{4 - \mu_n^2/\mu_p^2} \approx 0.42$ **Local quark-hadron duality**



PDFs at large x

- **Testing ground for hadron structure at x → 1:**

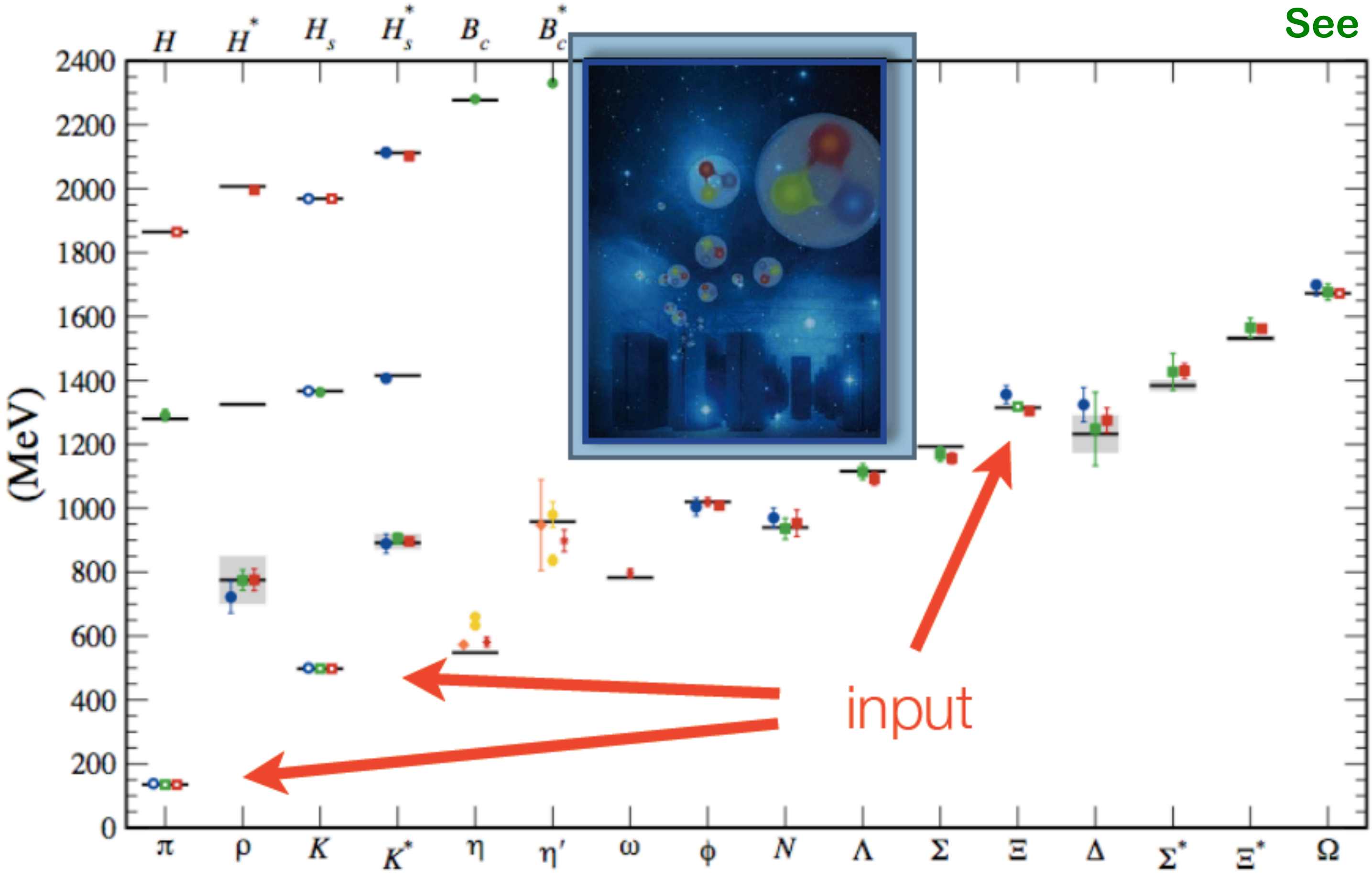
✧ $d/u \to 1/2$	**SU(6) Spin-flavor symmetry**	✧ $\Delta u/u \to 2/3$ $\Delta d/d \to -1/3$
✧ $d/u \to 0$	**Scalar diquark dominance**	✧ $\Delta u/u \to 1$ $\Delta d/d \to -1/3$
✧ $d/u \to 1/5$	**pQCD power counting**	✧ $\Delta u/u \to 1$ $\Delta d/d \to 1$
✧ $d/u \to \frac{4\mu_n^2/\mu_p^2 - 1}{4 - \mu_n^2/\mu_p^2}$ ≈ 0.42	**Local quark-hadron duality**	✧ $\Delta u/u \to 1$ $\Delta d/d \to 1$

Can lattice QCD help?

Lattice QCD

- ☐ **Hadron masses:** Predictions with limited inputs

See Savage’s talk



(Kronfeld)

- ☐ **Lattice “time” is Euclidean:** $\tau = i\,t$

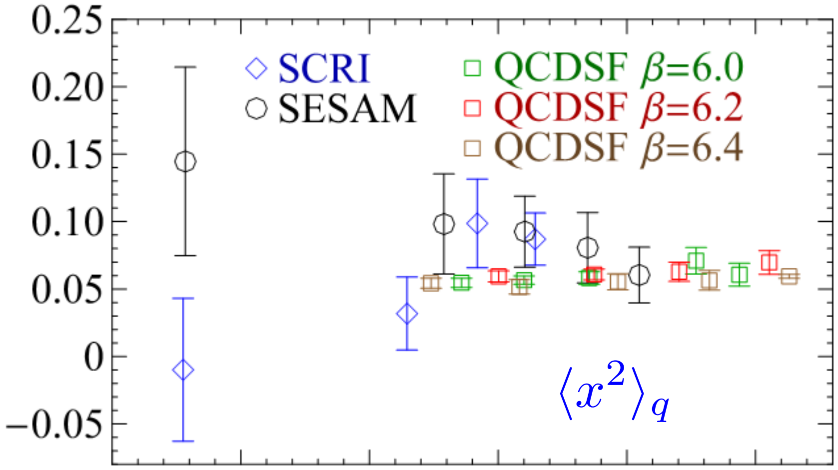
Cannot calculate PDFs directly, whose operators are time-dependent

PDFs from lattice QCD

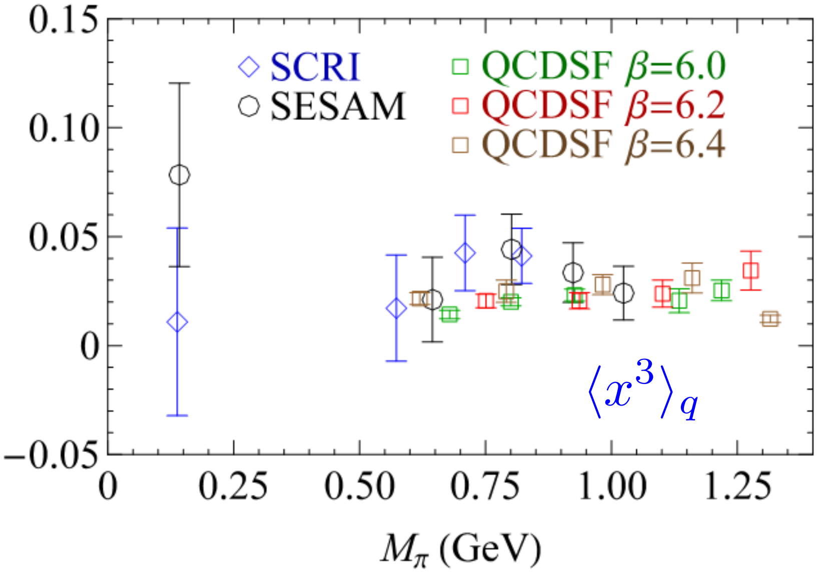
- **Moments of PDFs – matrix elements of local operators**

$$\langle x^n(\mu^2)\rangle_q \equiv \int_0^1 dx\, x^n\, q(x,\mu^2)$$

- **Works, but, hard and limited moments:**



Dolgov et al., hep-lat/0201021



Gockeler et al., hep-ph/0410187

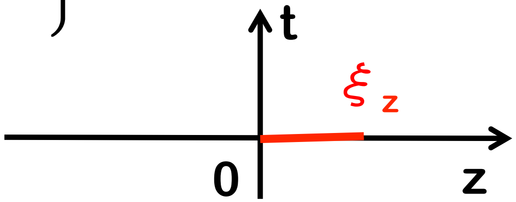
Limited moments – hard to get the full x-dependent distributions!

From quasi-PDFs to PDFs (Ji's idea)

Ji, arXiv:1305.1539

☐ **"Quasi" quark distribution (spin-averaged):**

$$\tilde{q}(x,\mu^2,P_z) \equiv \int \frac{d\xi_z}{4\pi} e^{-ixP_z\xi_z} \langle P|\overline{\psi}(\xi_z)\gamma_z \exp\left\{-ig\int_0^{\xi_z} d\eta_z A_z(\eta_z)\right\}\psi(0)|P\rangle + \text{UVCT}(\mu^2)$$



☐ **Features:**

- Quark fields separated along the z-direction – not boost invariant!
- Perturbatively UV power divergent: $\propto (\mu/P_z)^n$ with $n > 0$ - renormalizable?
- Quasi-PDFs → *Normal PDFs* *when* $P_z \to \infty$
- Quasi-PDFs could be calculated using standard lattice method

☐ **Proposed matching:**

Ji, arXiv:1305. 1539

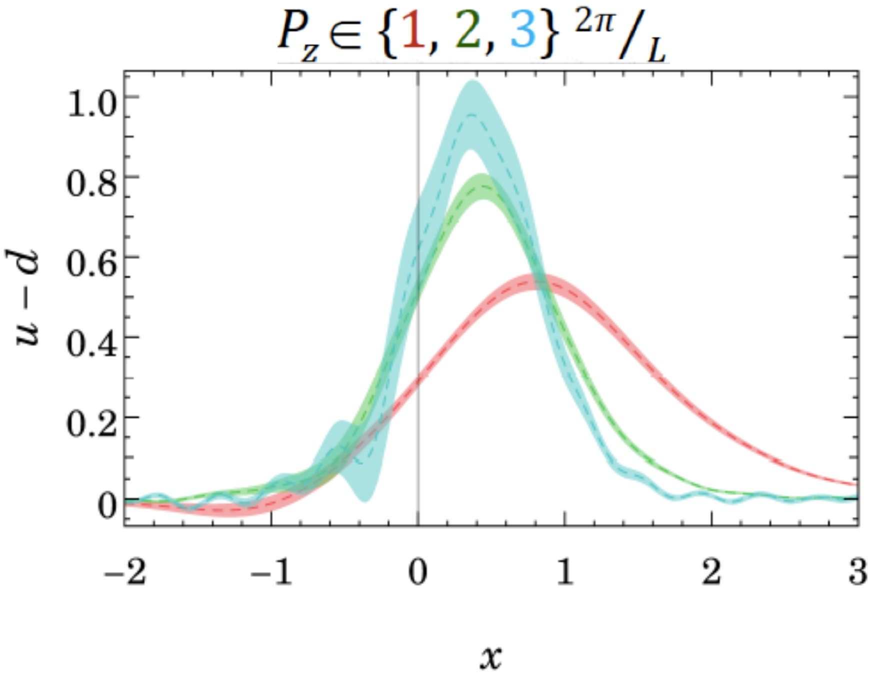
$$\tilde{q}(x,\mu^2,P_z) = \int_x^1 \frac{dy}{y} Z\left(\frac{x}{y},\frac{\mu}{P_z}\right) q(y,\mu^2) + \mathcal{O}\left(\frac{\Lambda^2}{P_z^2},\frac{M^2}{P_z^2}\right)$$

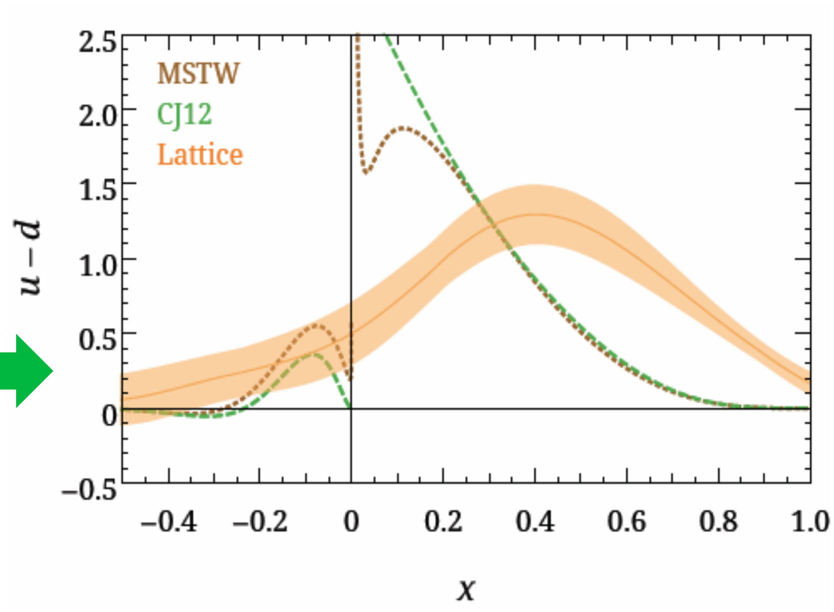
- Size of $O(1/P_z^2)$ terms, non-perturbative subtraction of power divergence
- Mixing with lower dimension operators cannot be treated perturbatively, …

Lattice calculation of quasi-PDFs

Lin *et al.*, arXiv:1402.1462

- Exploratory study:





Quasi-Quark Distribution with different P_z

Predicted quark distribution along with global fitted one

Matching – taking into account:

Target mass: $(M_N/P_z)^2$
High twist: $a+b/P_z^2$

Our observation

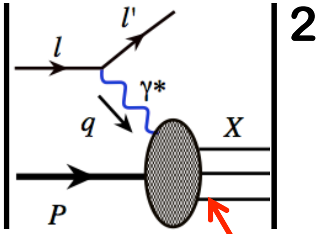
❑ **QCD factorization of single-hadron cross section:**

Momentum transfer

CO Factorization

PDFs

$$\sigma^{\mathrm{DIS}}(x, Q^2; \sqrt{s}) \propto$$



$$\approx \sum_f \mathcal{C}_f(x, \frac{Q^2}{\mu^2}, \sqrt{s}) \otimes f_i(x, \mu^2) + \mathcal{O}\left[\frac{1}{Q^2}\right]$$

Collision energy

Single hadron Matrix elements

Perturbative coefficients

Power corrections

✧ **PDFs are UV and IR finite, but, absorb perturbative CO divergence!**

✧ **With a large momentum transfer, PDFs completely cover all leading power CO divergence of single hadron matrix elements**

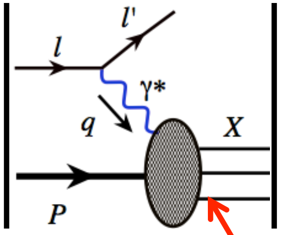
Our observation

- QCD factorization of single-hadron cross section:

Momentum transfer

CO Factorization

PDFs



$$\sigma^{\mathrm{DIS}}(x, Q^2; \sqrt{s}) \propto \Big| \quad \Big|^2 \approx \sum_f \mathcal{C}_f(x, \frac{Q^2}{\mu^2}, \sqrt{s}) \otimes f_i(x, \mu^2) + \mathcal{O}\left[\frac{1}{Q^2}\right]$$

Collision energy

Single hadron Matrix elements

Perturbative coefficients

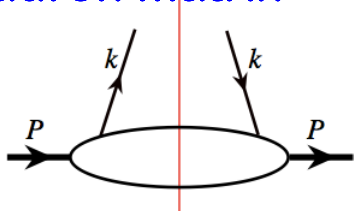
Power corrections

✧ PDFs are UV and IR finite, but, absorb perturbative CO divergence!

✧ With a large momentum transfer, PDFs completely cover all leading power CO divergence of single hadron matrix elements

- Collinear divergences are from the region when k_T ➔0:

Leading power perturbative CO divergences of single hadron matrix elements are logarithmic, $\propto \int dk_T^2/k_T^2$, and are the same for both Minkowski and Euclidean time



Our ideas

❑ **Lattice QCD can calculate "single" hadron matrix elements:**

$$\langle 0|\,\mathcal{O}(\overline{\psi},\psi,A)\,|0\rangle = \frac{1}{Z}\int \mathcal{D}A\mathcal{D}\overline{\psi}\mathcal{D}\psi\; e^{iS(\overline{\psi},\psi,A)}\mathcal{O}(\overline{\psi},\psi,A)$$

$$\overset{\uparrow}{\sum_{P'}|P'\rangle\langle P'|} \quad \overset{\uparrow}{\sum_{P}|P\rangle\langle P|} \quad \Longrightarrow \quad \langle P_z|\mathcal{O}(\overline{\psi},\psi,A)|P_z\rangle_{\mathrm{E}}$$

With an Euclidean time

Off-diagonal for GPDs

❑ **Collinear factorization:**

Ma and Qiu,
arXiv:1404.6860
1412.2688

Normal PDFs

$$\widetilde{\sigma}(\tilde{x},P_z;\mu^2)_{\mathrm{E}} = \Sigma_f \int_0^1 \frac{dx}{x}\,\mathcal{C}_f\left(\frac{\tilde{x}}{x},\frac{\bar{\mu}^2}{\mu^2},\alpha_s;P_z\right) f(x,\bar{\mu}^2) + \mathcal{O}\left[\frac{1}{\mu^{\alpha}}\right]$$

✧ **Perturbatively,** $\widetilde{\sigma}(\tilde{x},P_z;\mu^2)$ **and** $f(x,\bar{\mu}^2)$ **have the same CO divergence**

✧ **Matching coefficients,** $\mathcal{C}_f$**, are IR safe and perturbatively calculable**

✧ $P_z > \mu$ **is finite**

Differences between Ji's approach and ours

☐ For the quasi-PDFs:

✧ **Ji's approach – high P_z effective field theory:**

Ji, arXiv:1305.1539
1404.6680

$$\tilde{q}(x,\mu^2,P_z) = \int_x^1 \frac{dy}{y}\, Z\left(\frac{x}{y},\frac{\mu}{P_z}\right) q(y,\mu^2) + \mathcal{O}\left(\frac{\Lambda^2}{P_z^2},\frac{M^2}{P_z^2}\right)$$

✧ **Our approach – QCD collinear factorization:**

Ma and Qiu,
arXiv:1404.6860
1412.2688

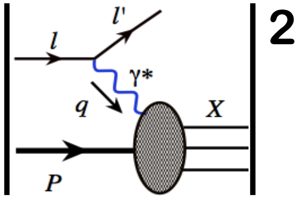
$$\tilde{q}(x,\mu^2,P_z) = \sum_f \int_0^1 \frac{dy}{y}\, \mathcal{C}_f\left(\frac{x}{y},\frac{\mu^2}{\bar{\mu}^2},P_z\right) f(y,\bar{\mu}^2) + \mathcal{O}\left(\frac{1}{\mu^2}\right)$$

Parameter like $\sqrt{s}$

Factorization scale

High twist
Power corrections

$$\sigma^{\rm DIS}(x,Q^2;\sqrt{s}) \propto$$



$$\left|\ \cdots\ \right|^2 \approx \sum_f \mathcal{C}_f(x,\frac{Q^2}{\mu^2},\sqrt{s}) \otimes f_i(x,\mu^2) + \mathcal{O}\left[\frac{1}{Q^2}\right]$$

☐ Our approach goes beyond quasi-PDFs:

**All lattice calculable single hadron matrix elements
with a large momentum transfer – "factorization"**

Extract PDFs from lattice "cross sections"

❑ Lattice "cross section":

$$\tilde{\sigma}_{\rm E}^{\rm Lat}(\tilde{x}, 1/a, P_z) \propto {\rm F.T.\,of}\ \langle P_z|\mathcal{O}(\overline{\psi}, \psi, A)|P_z\rangle + {\rm UVCT}(1/a)$$

- ✧ Its continuum limit is UV renormalizable
- ✧ It is calculable in lattice QCD with an Euclidean time, "E"
- ✧ It is infrared (IR) safe, calculated in lattice perturbation theory
- ✧ All CO divergences of its continuum limit ($a \to 0$) can be factorized into the normal PDFs with perturbatively calculable hard coefficients

"Collision energy" $P_z \sim$ "$\sqrt{s}$" *"rapidity"* $\tilde{x} \sim$ "y"
"Hard momentum transfer" $1/a \sim \tilde{\mu} \sim$ "Q"

❑ UV renormalization:

- ✧ No UVCT needed if $\mathcal{O}(\overline{\psi}, \psi, A)$ is made of conserved currents
- ✧ The quasi-PDFs are not made of conserved currents – UVCT needed

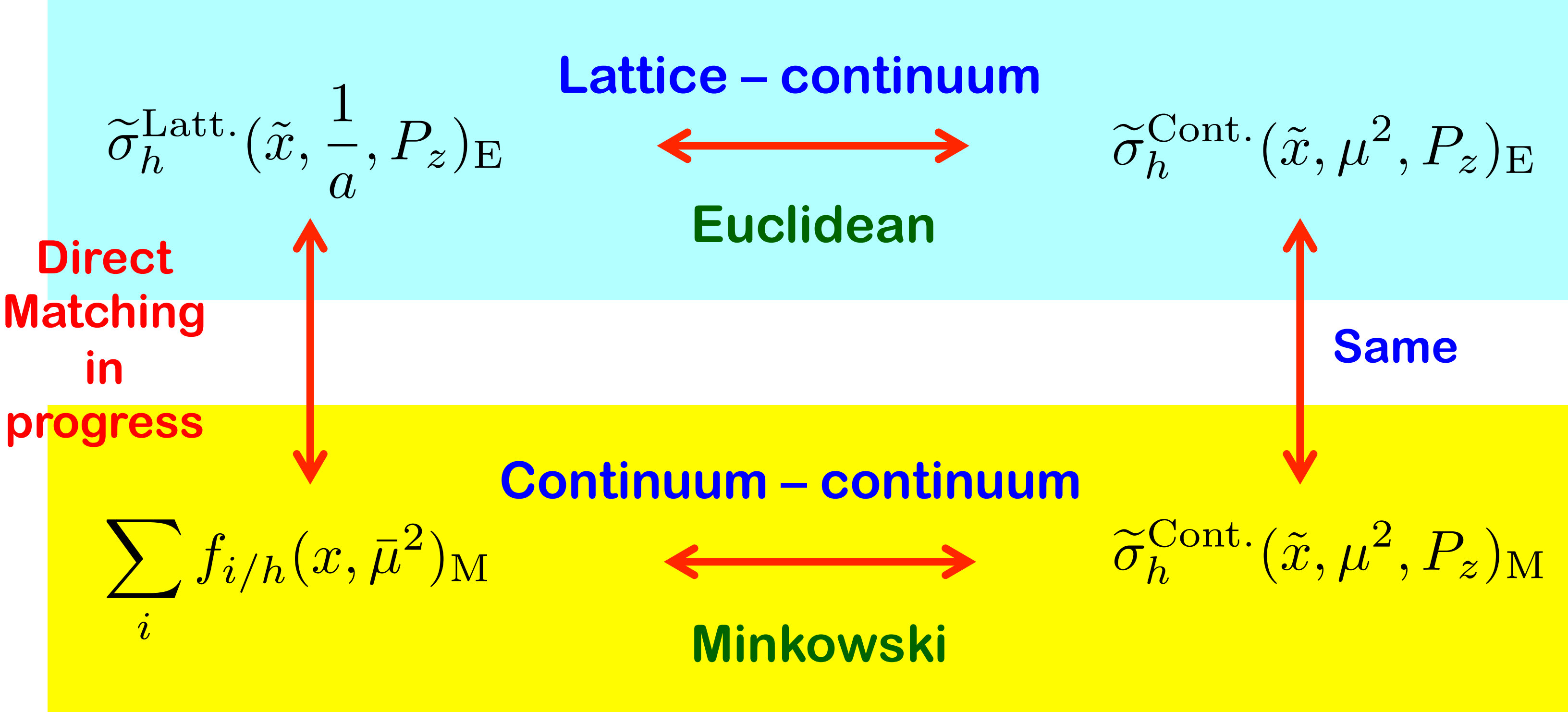
❑ CO Factorization – IR safe matching coefficients:

$$\tilde{\sigma}_{\rm E}^{\rm Lat}(\tilde{x}, \frac{1}{a}, P_z) \approx \sum_i \int_0^1 \frac{dx}{x} f_{i/h}(x, \mu^2)\, \widetilde{\mathcal{C}}_i(\frac{\tilde{x}}{x}, \frac{1}{a}, \mu^2, P_z)$$

QCD Global analysis of lattice data

Matching overview

- ❑ **Goal: Match lattice "cross sections" to normal PDFs**



$$\begin{tikzcd}
\widetilde{\sigma}_h^{\text{Latt.}}\left(\tilde{x}, \frac{1}{a}, P_z\right)_{\text{E}} \arrow[r, leftrightarrow, "\text{Lattice – continuum}", "\text{Euclidean}"'] \arrow[d, leftrightarrow, "\text{Direct Matching in progress}"'] & \widetilde{\sigma}_h^{\text{Cont.}}(\tilde{x}, \mu^2, P_z)_{\text{E}} \arrow[d, leftrightarrow, "\text{Same}"] \\
\sum_i f_{i/h}(x, \bar{\mu}^2)_{\text{M}} \arrow[r, leftrightarrow, "\text{Continuum – continuum}", "\text{Minkowski}"'] & \widetilde{\sigma}_h^{\text{Cont.}}(\tilde{x}, \mu^2, P_z)_{\text{M}}
\end{tikzcd}$$

- ✧ One-loop matching in continuum Minkowski space has been done

  Ji (2013), Xiong et. al. (2013), Ma and Qiu (2014) [all flavors]

- ✧ One-loop matching between lattice and continuum in Euclidean space

  Ishikawa, Qiu and Yoshida (just completed, paper is in preparation)

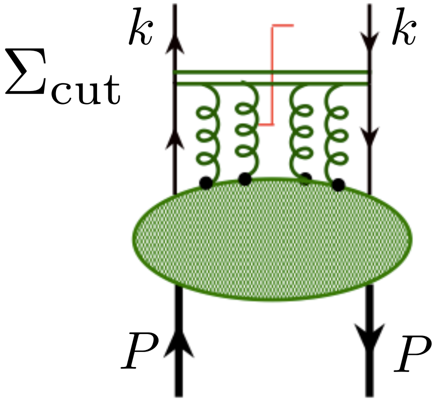
Case study – factorization of quasi-PDFs

Ma and Qiu,
arXiv:1404.6860
1412.2688

❑ The "Quasi-quark" distribution, as an example:

$$\tilde{q}(\tilde{x},\tilde{\mu}^2,P_z)=\int\frac{dy_z}{4\pi}e^{i\tilde{x}P_zy_z}\langle P|\overline{\psi}(y_z)\gamma_z\exp\left\{-ig\int_0^{y_z}dy_z'A_z(y_z')\right\}\psi(0)|P\rangle$$

- ✧ **Feynman diagram representation:** $\Phi_{n_z}^{(f,a)}(\{\xi_z,0\}) = \Phi_{n_z}^{\dagger(f,a)}(\{\infty,\xi_z\})\,\Phi_{n_z}^{(f,a)}(\{\infty,0\})$



$$\longleftarrow\ \frac{\gamma_z}{2P_z}\,\delta\left(\tilde{x}-\frac{k_z}{P_z}\right)$$

$$P_z\sim\text{“}\sqrt{s}\text{”}\sim\tilde{\mu}/\tilde{x}$$ ***Sufficiently large***

$$P^2\sim 0$$

- ✧ **Like PDFs, it is IR finite**

- ✧ **Like PDFs, it is UV divergent, but, worse (linear UV divergence)**

  Potential trouble! - mixing with the Log UV of PDFs?

- ✧ **Like PDFs, it is CO divergent – factorizes CO divergence into PDFs**

  Show to all orders in perturbation theory

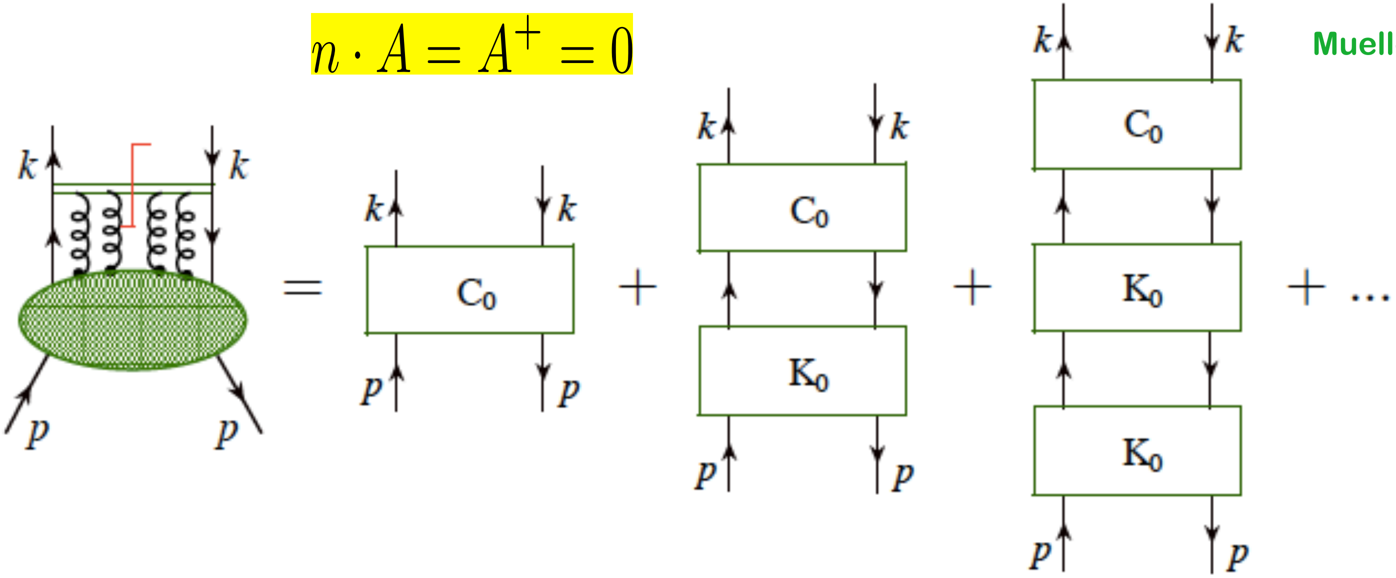
All order QCD factorization of CO divergence

Ma and Qiu, arXiv:1404.6860

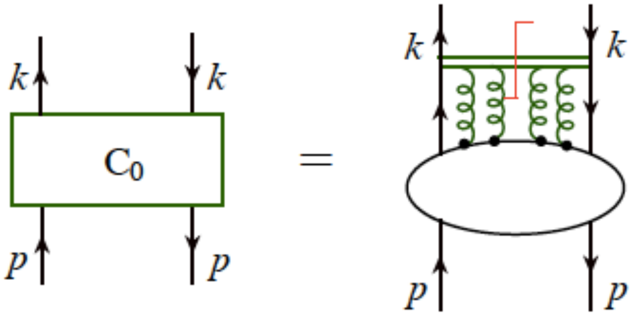
- **Generalized ladder decomposition in a physical gauge**

$n \cdot A = A^+ = 0$

Mueller, PRD 1974



- $C_0, \ K_0$:2PI kernels
  - ✧ Only process dependence:

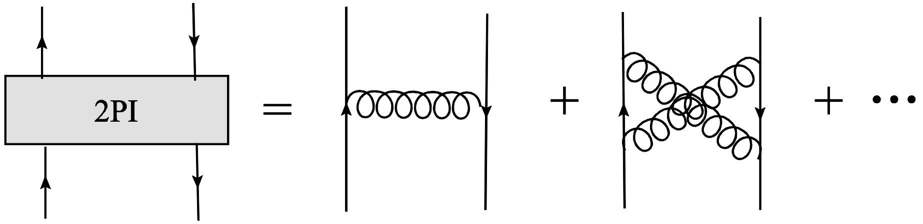


  - ✧ 2PI are finite in a physical gauge for fixed *k* and *p*:

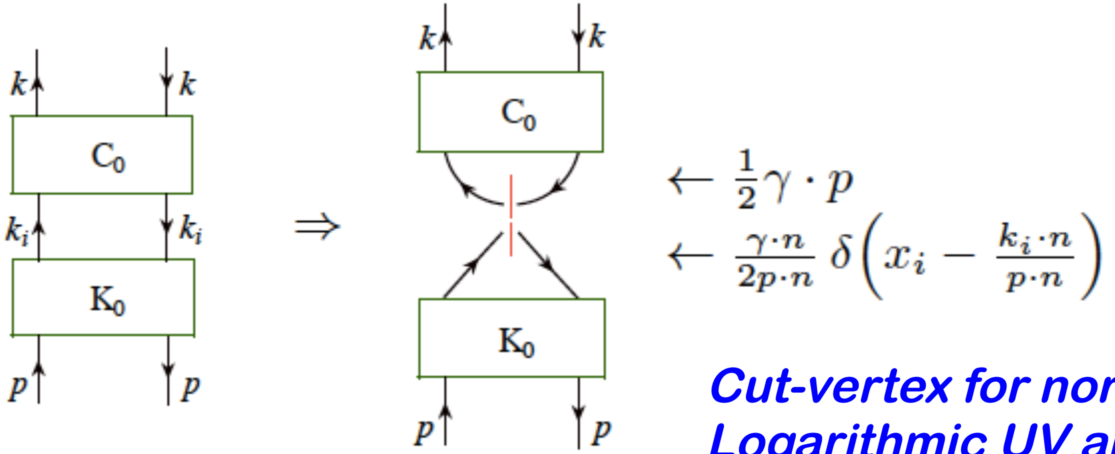
Ellis, Georgi, Machacek, Politzer, Ross, 1978, 1979

All order QCD factorization of CO divergence

❑ 2PI kernels – Diagrams:



❑ Ordering in virtuality: $P^2 \ll k^2 \lesssim \tilde{\mu}^2$ – Leading power in $\frac{1}{\tilde{\mu}}$



$\leftarrow \frac{1}{2}\gamma\cdot p$

$\leftarrow \frac{\gamma\cdot n}{2p\cdot n}\,\delta\left(x_i - \frac{k_i\cdot n}{p\cdot n}\right)$

+ power suppressed

Cut-vertex for normal quark distribution
Logarithmic UV and CO divergence

❑ Renormalized kernel - parton PDF:

$$K \equiv \int d^4k_i\,\delta\left(x_i - \frac{k^+}{p^+}\right)\mathrm{Tr}\left[\frac{\gamma\cdot n}{2p\cdot n}\,K_0\,\frac{\gamma\cdot p}{2}\right] + \mathrm{UVCT}_{\mathrm{Logarithmic}}$$

All order QCD factorization of CO divergence

- **Projection operator for CO divergence:**

$\widehat{\mathcal{P}}\,K$ Pick up the logarithmic CO divergence of *K*

- **Factorization of CO divergence:**

$$\tilde{f}_{q/p} = \lim_{m\to\infty} C_0 \sum_{i=0}^{m} K^i + \mathrm{UVCTs}$$

$$= \lim_{m\to\infty} C_0 \left[1 + \sum_{i=0}^{m-1} K^i (1-\widehat{\mathcal{P}})K\right]_{\mathrm{ren}} + \tilde{f}_{q/p}\,\widehat{\mathcal{P}}\,K$$

$$= \lim_{m\to\infty} C_0 \left[1 + \sum_{i=1}^{m} \left[(1-\widehat{\mathcal{P}})K\right]^i\right]_{\mathrm{ren}} + \tilde{f}_{q/p}\,\widehat{\mathcal{P}}\,K$$

$$\tilde{f}_{q/P} = \left[C_0 \frac{1}{1-(1-\widehat{\mathcal{P}})K}\right]_{\mathrm{ren}} \left[\frac{1}{1-\widehat{\mathcal{P}}K}\right]$$

Normal Quark distribution

CO divergence free

All CO divergence of quasi-quark distribution

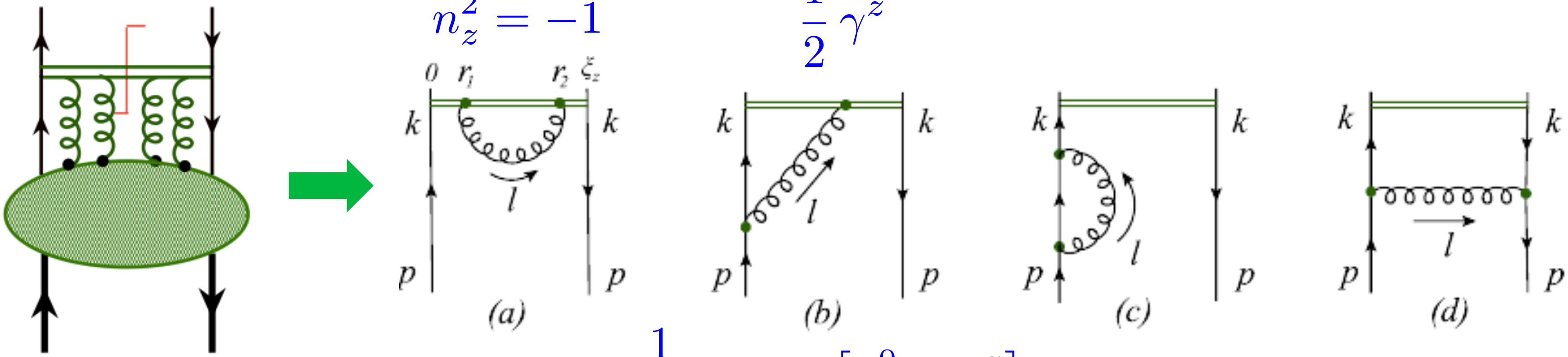
$$\tilde{f}_{i/h}(\tilde{x},\tilde{\mu}^2,P_z) \approx \sum_j \int_0^1 \frac{dx}{x}\, \mathcal{C}_{ij}\left(\frac{\tilde{x}}{x},\tilde{\mu}^2,P_z\right) f_{j/h}(x,\mu^2)$$

UV finite?

UV renormalization

Ma and Qiu, arXiv:1404.6860, …

❑ UV divergences (difference in gauge link):



$$n_z^2 = -1$$

$$\frac{1}{2}\gamma^z$$

$$\frac{1}{2}\gamma \cdot p \propto \left[\gamma^0 - \gamma^z\right]$$

❑ Renormalization:

$$\left[C_0 \frac{1}{1-(1-\hat{\mathcal{P}})K}\right]_{\mathrm{ren}} \equiv C_0 \frac{1}{1-(1-\hat{\mathcal{P}})K} + \mathrm{UVCTs}$$

In coordinate space:
ξ_z
Independence!

- ✧ **Power divergence:** **Diagram (a) – independent of ξ_z**

  Removed by "mass" renormalization of a test particle – the gauge link

- ✧ **Left-over log divergence:**

  Dotsenko and Vergeles NPB, 1980)

  Dimensional regularization – ξ_z independence of $1/\varepsilon$ – finite CTs

- ✧ **Log(ξ_z) – term:** **Artifact of dimensional regularization**

One-loop example: quark →quark

Ma and Qiu, arXiv:1404.6860

- ❑ **Expand the factorization formula:**

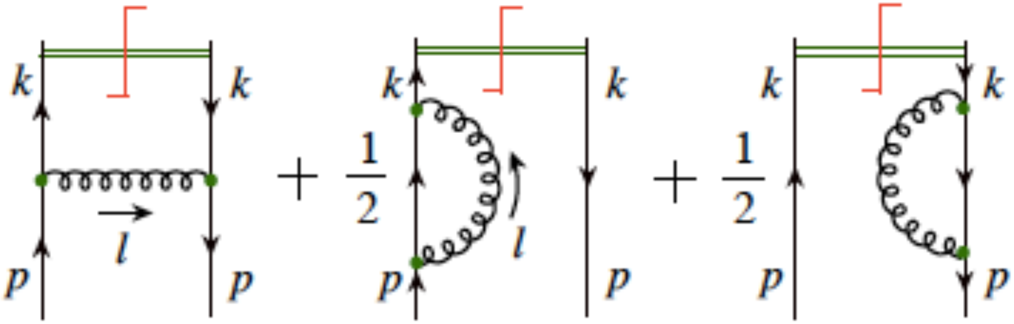
$$\tilde{f}_{i/h}(\tilde{x},\tilde{\mu}^2,P_z) \approx \sum_j \int_0^1 \frac{dx}{x} \mathcal{C}_{ij}(\frac{\tilde{x}}{x},\tilde{\mu}^2,P_z)\, f_{j/h}(x,\mu^2)$$

To order α_s :

$$\tilde{f}^{(1)}_{q/q}(\tilde{x}) = f^{(0)}_{q/q}(x) \otimes \mathcal{C}^{(1)}_{q/q}(\tilde{x}/x) + f^{(1)}_{q/q}(x) \otimes \mathcal{C}^{(0)}_{q/q}(\tilde{x}/x)$$

$$\Rightarrow \quad \mathcal{C}^{(1)}_{q/q}(t,\tilde{\mu}^2,\mu^2,P_z) = \tilde{f}^{(1)}_{q/q}(t,\tilde{\mu}^2,P_z) - f^{(1)}_{q/q}(t,\mu^2)$$

- ❑ **Feynman diagrams:**



Same diagrams for both

$\tilde{f}_{q/q}$ and $f_{q/q}$

But, in different gauge: $n_z \cdot A = 0$ for $\tilde{f}_{q/q}$ $n \cdot A = 0$ for $f_{q/q}$

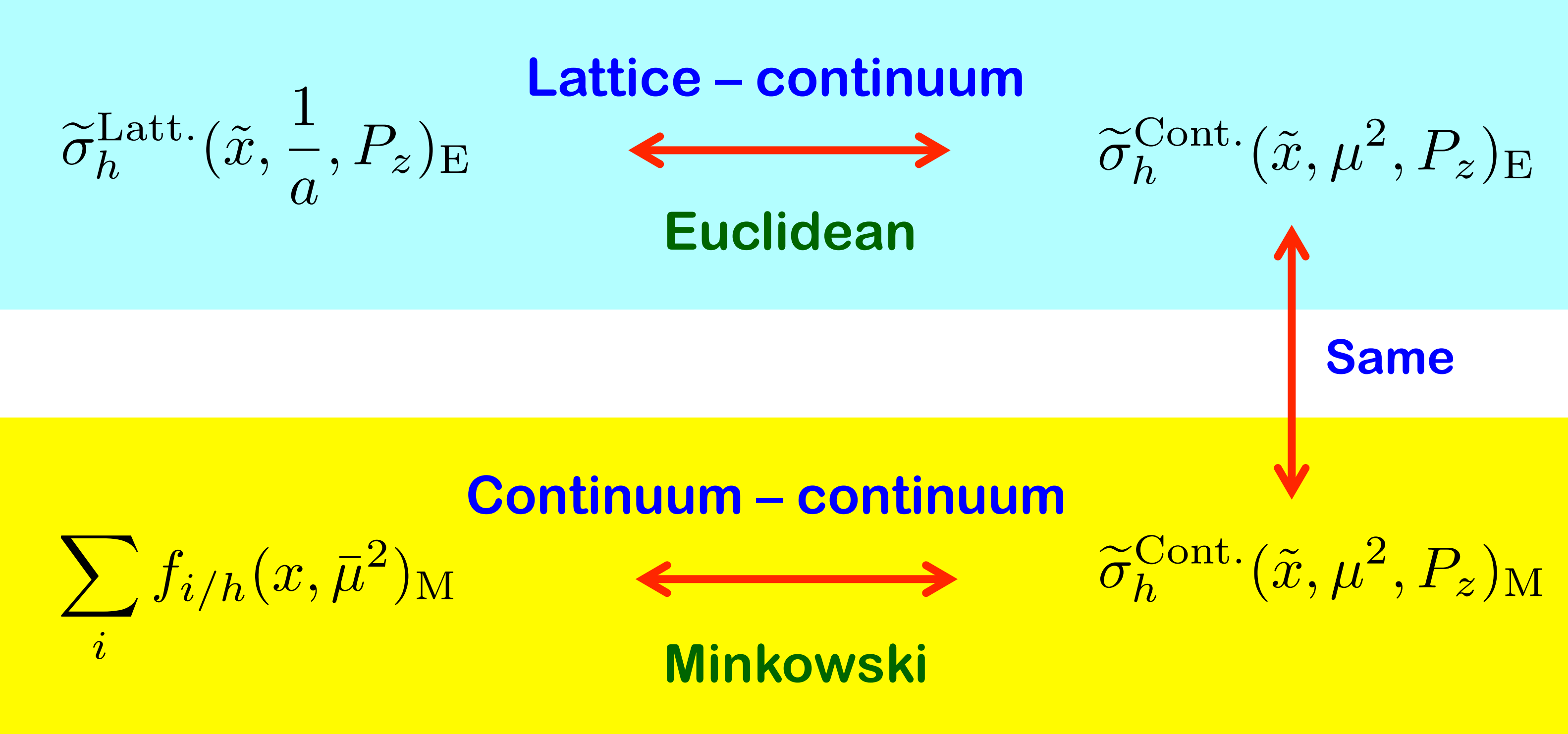
- ❑ **Gluon propagator in n_z . A = 0:**

$$\tilde{d}^{\alpha\beta}(l) = -g^{\alpha\beta} + \frac{l^\alpha n_z^\beta + n_z^\alpha l^\beta}{l_z} - \frac{n_z^2\, l^\alpha l^\beta}{l_z^2}$$

with $n_z^2 = -1$

Matching overview

❑ **Goal: Match lattice "cross sections" to normal PDFs**



$\widetilde{\sigma}_h^{\rm Latt.}(\tilde{x}, \frac{1}{a}, P_z)_{\rm E}$

Lattice – continuum

Euclidean

$\widetilde{\sigma}_h^{\rm Cont.}(\tilde{x}, \mu^2, P_z)_{\rm E}$

Same

Continuum – continuum

$\sum_i f_{i/h}(x, \bar{\mu}^2)_{\rm M}$

Minkowski

$\widetilde{\sigma}_h^{\rm Cont.}(\tilde{x}, \mu^2, P_z)_{\rm M}$

✧ One-loop matching in continuum Minkowski space has been done

Ji (2013), Xiong et. al. (2013), Ma and Qiu (2014) [all flavors]

✧ One-loop matching between lattice and continuum in Euclidean space

Ishikawa, Qiu and Yoshida (just completed, paper is in preparation)

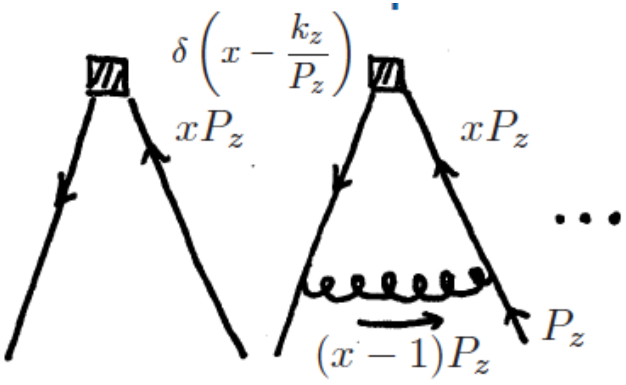
Match lattice to continuum

❑ Momentum space vs. coordinate space:

$$\tilde{q}(\tilde{x},\mu,P_z) = \int \frac{d\delta_z}{2\pi} e^{-i\tilde{x}P_z\delta_z} \langle \mathcal{N}(P_z)|\widetilde{O}(\delta_z)|\mathcal{N}(P_z)\rangle$$
$$\widetilde{O}(\delta_z) = \overline{\psi}(\delta_z)\,\gamma^z\,U_z(\delta_z,0)\,\psi(0)$$

Momentum space

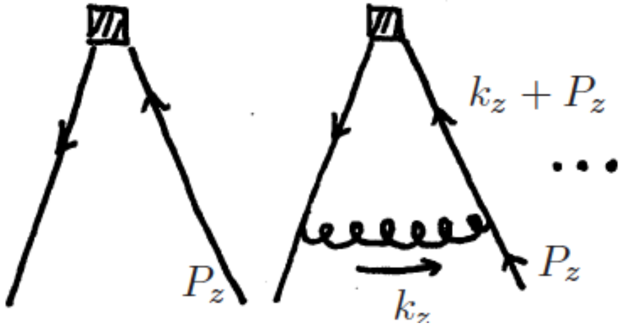
$$\widetilde{q}_{\text{Cont.}}(\tilde{x},\mu,P_z) \Longleftrightarrow \widetilde{q}_{\text{Latt.}}(\tilde{x},a^{-1},P_z)$$



- ✧ z-component of the momentum is restricted to be xP_z.
- ✧ Loop-momentum becomes 3-dimensional

Coordinate space

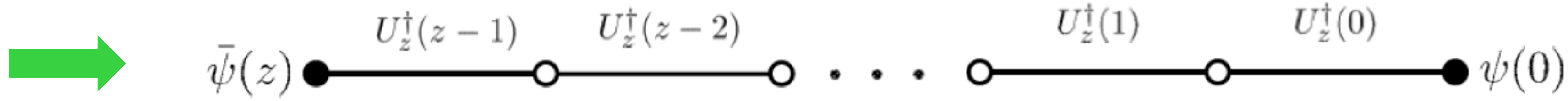
$$\widetilde{O}_{\text{Cont.}}(\delta_z) \Longleftrightarrow \widetilde{O}_{\text{Latt.}}(\delta_z)$$



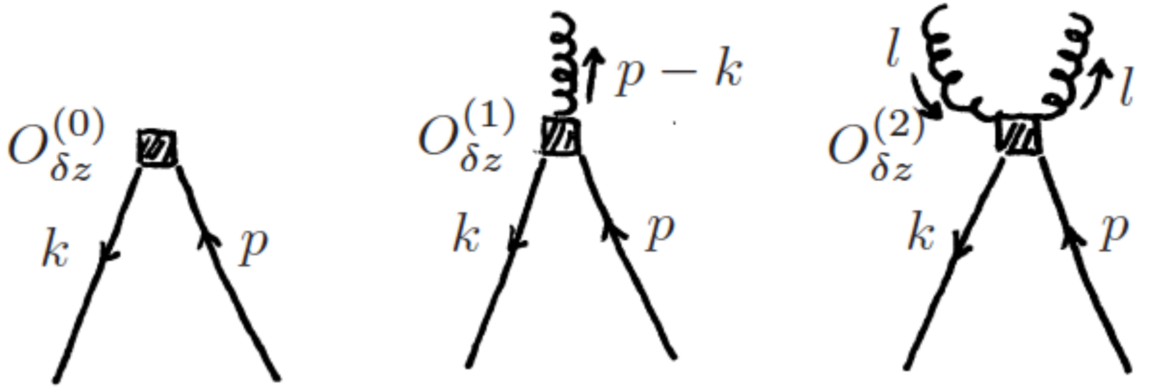
- ✧ No restriction on momentum.
- ✧ Loop-momentum is 4-dimensional.

Feynman rule in a covariant gauge

$$\widetilde{O}(\delta_z) = \overline{\psi}(\delta_z)\,\gamma^z\,U_z(\delta_z,0)\,\psi(0)$$



- **Tree, one-gluon, two-gluon (at one-loop level):**



$$O^{(0)}_{\delta z}(p,k) = \gamma_z\delta(p-k)e^{-ip_z\delta z}$$

$$O^{(1)}_{\delta z}(p,k) = ig\gamma_z\frac{e^{-ip_z\delta z} - e^{-ik_z\delta z}}{i(p-k)_z}$$

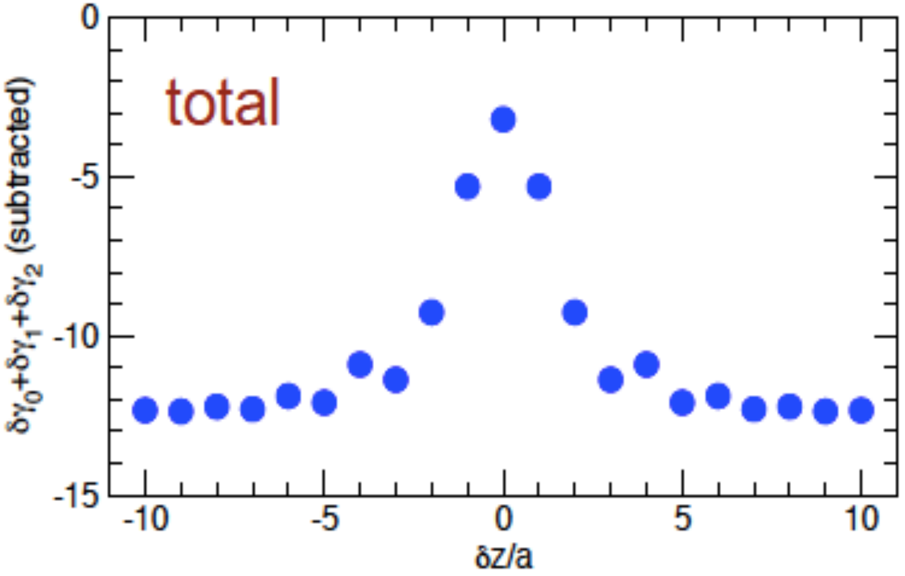
$$O^{(2)}_{\delta z}(p,k,l) = -g^2\gamma_z\delta(p-k)e^{-ip_z\delta z}\left(\frac{1-e^{il_z\delta z}}{l_z^2} - \frac{\delta z}{il_z}\right)$$

There is no pole.

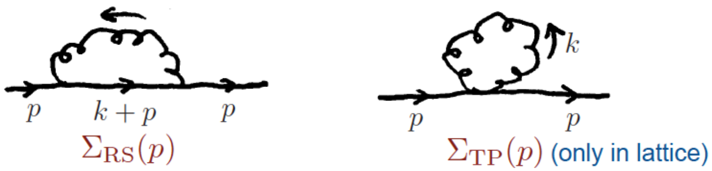
Matching lattice to continuum at one-loop

❑ One-loop matching coefficients:

$$\delta\Gamma_{\rm cont} - \delta\Gamma_{\rm latt} \equiv \frac{g^2}{16\pi^2} C_F \gamma_z \delta\gamma$$



Wave function part is not included:



(It is the same as usual local operator case)

❑ Comments:

- ✧ Realistic lattice fermion should be used in the actual matching factor
- ✧ Other lattice actions and the link smearing can be easily implemented
- ✧ …

Summary and outlook

- ☐ "lattice cross sections" = single hadron matrix elements calculable in Lattice QCD and factorizable in QCD

  Key difference from Ji's idea:

  Expansion in 1/μ instead of that in 1/P_z

- ☐ Extract PDFs by global analysis of data on "Lattice x sections". Same should work for other distributions (TMDs, GPDs)

$$\widetilde{\sigma}_{\rm E}^{\rm Lat}(\tilde{x},\frac{1}{a},P_z) \approx \sum_i \int_0^1 \frac{dx}{x}\, f_{i/h}(x,\mu^2)\,\widetilde{\mathcal{C}}_i(\frac{\tilde{x}}{x},\frac{1}{a},\mu^2,P_z)$$

- ☐ Conservation of difficulties – complementarity:

  High energy scattering experiments

  – less sensitive to large x parton distribution/correlation

  "Lattice factorizable cross sections"

  – more suited for large x PDFs

  Great potential: PDFs of neutron, PDFs of mesons, …

- ☐ Lattice QCD can calculate PDFs, but, more works are needed!

Thank you!

BACKUP SLIDES

Global QCD analysis – the machinery

Input PDFs at Q_0

$\varphi_{f/h}\left(x,\{a_j\}\right)$

Vary $\{a_j\}$

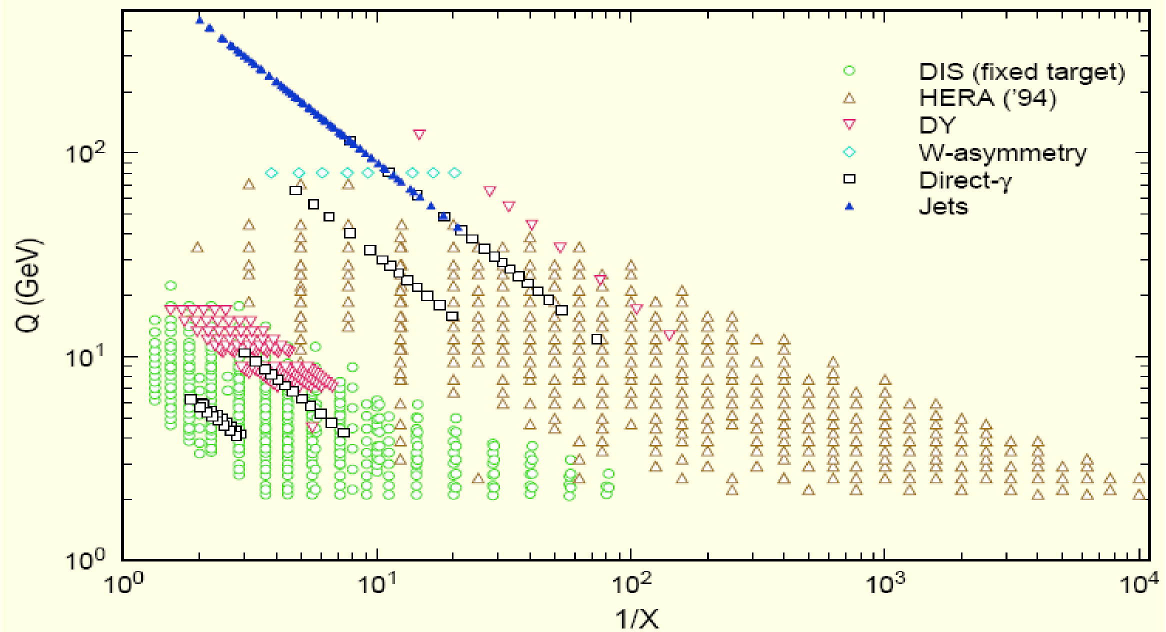
Minimize Chi^2

DGLAP

$\varphi_{f/h}\left(x\right)$ **at $Q>Q_0$**

QCD calculation

Comparison with Data at various x and Q



Procedure: Iterate to find the best set of $\{a_j\}$ for the input DPFs

“Quasi-PDFs” have no parton interpretation

- ☐ **Normal PDFs conserve parton momentum:**

$$\begin{aligned} M &= \sum_q \left[\int_0^1 dx\, x f_q(x) + \int_0^1 dx\, x f_{\bar{q}}(x)\right] + \int_0^1 dx\, x f_g(x) \\ &= \sum_q \int_{-\infty}^{\infty} dx\, x f_q(x) + \frac{1}{2}\int_{-\infty}^{\infty} dx\, x f_g(x) \\ &= \frac{1}{2(P^+)^2}\langle P|T^{++}(0)|P\rangle = \text{constant} \end{aligned}$$

$T^{\mu\nu}$

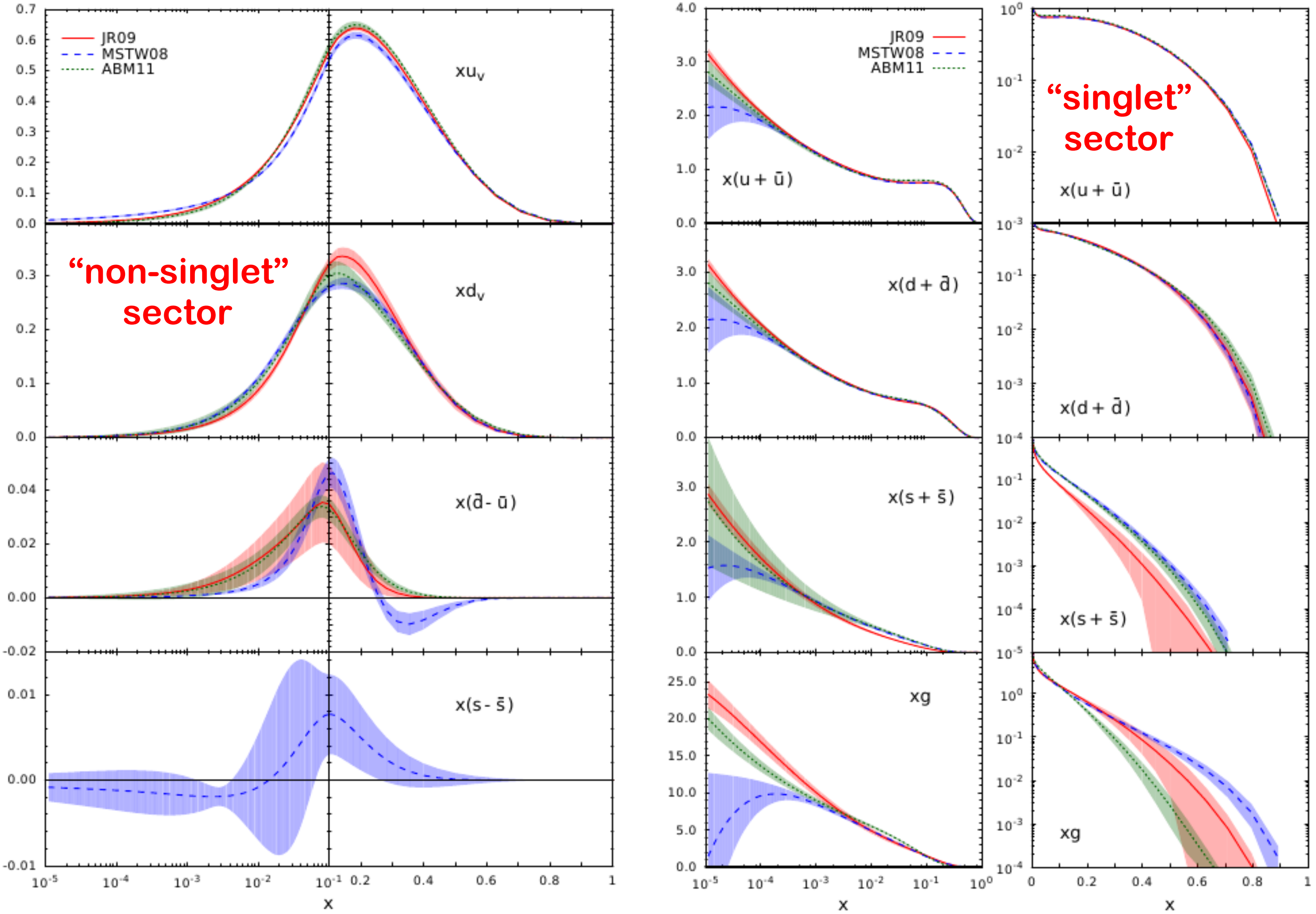
Energy-momentum tensor

- ☐ **“Quasi-PDFs” do not conserve “parton” momentum:**

$$\begin{aligned} \widetilde{\mathcal{M}} &= \sum_q \left[\int_0^\infty d\tilde{x}\, \tilde{x} \tilde{f}_q(\tilde{x}) + \int_0^\infty d\tilde{x}\, \tilde{x} \tilde{f}_{\bar{q}}(\tilde{x})\right] + \int_0^\infty d\tilde{x}\, \tilde{x} \tilde{f}_g(\tilde{x}) \\ &= \sum_q \int_{-\infty}^{\infty} d\tilde{x}\, \tilde{x} \tilde{f}_q(\tilde{x}) + \frac{1}{2}\int_{-\infty}^{\infty} d\tilde{x}\, \tilde{x} \tilde{f}_g(\tilde{x}) \\ &= \frac{1}{2(P_z)^2}\langle P|\left[T^{zz}(0) - g^{zz}(...)\right]|P\rangle \neq \text{constant} \end{aligned}$$

Note: “Quasi-PDFs” are not boost invariant

Uncertainties of PDFs



"non-singlet" sector

"singlet" sector

Lattice QCD

- Formulated in the discretized Euclidean space:

$$S^f = a^4 \sum_x \left[\frac{1}{2a} \sum_\mu [\bar{\psi}(x)\gamma_\mu U_\mu(x)\psi(x+a\hat{\mu}) - \bar{\psi}(x+a\hat{\mu})\gamma_\mu U^\dagger_\mu(x)\psi(x)] + m\bar{\psi}(x)\psi(x) \right]$$

$$S^g = \frac{1}{g_0^2} a^4 \sum_{x,\mu\nu} \left[N_c - \mathrm{ReTr}[U_\mu(x)U_\nu(x+a\hat{\mu})U^\dagger_\mu(x+a\hat{\nu})U^\dagger_\nu(x)] \right]$$

$$U_\mu(x) = e^{-igaT^a A^a_\mu(x+\frac{1}{2})}$$

- Boundary condition is imposed on each field in finite volume:

Momentum space is restricted in finite Brillouin zone: $\{-\frac{\pi}{a}, \frac{\pi}{a}\}$

Lattice QCD is an Ultra-Violet (UV) finite theory

Lattice action is not unique, above action is the simplest one!

Many implementations were proposed to reduce the discretization error

One-loop “quasi-quark” distribution in a quark

Ma and Qiu, arXiv:1404.6860

❑ Real + virtual contribution:

$$\tilde{f}^{(1)}_{q/q}(\tilde{x},\tilde{\mu}^2,P_z)=C_F\frac{\alpha_s}{2\pi}\frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)}\int_0^{\tilde{\mu}^2}\frac{dl_\perp^2}{l_\perp^{2+2\epsilon}}\int_{-\infty}^{+\infty}\frac{dl_z}{P_z}\left[\delta\left(1-\tilde{x}-y\right)-\delta\left(1-\tilde{x}\right)\right]\left\{\frac{1}{y}\left(1-y+\frac{1-\epsilon}{2}y^2\right)\right.$$
$$\left.\times\left[\frac{y}{\sqrt{\lambda^2+y^2}}+\frac{1-y}{\sqrt{\lambda^2+(1-y)^2}}\right]+\frac{(1-y)\lambda^2}{2y^2\sqrt{\lambda^2+y^2}}+\frac{\lambda^2}{2y\sqrt{\lambda^2+(1-y)^2}}+\frac{1-\epsilon}{2}\frac{(1-y)\lambda^2}{[\lambda^2+(1-y)^2]^{3/2}}\right\}$$

where $y=l_z/P_z,\ \lambda^2=l_\perp^2/P_z^2,\ C_F=(N_c^2-1)/(2N_c)$

❑ Cancelation of CO divergence:

$$\frac{y}{\sqrt{\lambda^2+y^2}}+\frac{1-y}{\sqrt{\lambda^2+(1-y)^2}}=2\theta(0<y<1)-\left[\mathrm{Sgn}(y)\frac{\sqrt{\lambda^2+y^2}-|y|}{\sqrt{\lambda^2+y^2}}+\mathrm{Sgn}(1-y)\frac{\sqrt{\lambda^2+(1-y)^2}-|1-y|}{\sqrt{\lambda^2+(1-y)^2}}\right]$$

Only the first term is CO divergent for $0 < y < 1$, which is the same as the divergence of the normal quark distribution – necessary!

❑ UV renormalization:

Different treatment for the upper limit of $l_\perp^2$ integration - “scheme”

Here, a UV cutoff is used – other scheme is discussed in the paper

One-loop coefficient functions

Ma and Qiu, arXiv:1404.6860

- **MS scheme for $f_{q/q}(x,\mu^2)$:**

$$\mathcal{C}^{(1)}_{q/q}(t,\tilde{\mu}^2,\mu^2,P_z) = \tilde{f}^{(1)}_{q/q}(t,\tilde{\mu}^2,P_z) - f^{(1)}_{q/q}(t,\mu^2)$$

$$\frac{\mathcal{C}^{(1)}_{q/q}(t)}{C_F\frac{\alpha_s}{2\pi}} = \left[\frac{1+t^2}{1-t}\ln\frac{\tilde{\mu}^2}{\mu^2}+1-t\right]_+ + \left[\frac{t\Lambda_{1-t}}{(1-t)^2}+\frac{\Lambda_t}{1-t}+\frac{\mathrm{Sgn}(t)\Lambda_t}{\Lambda_t+|t|}\right.$$
$$\left. -\frac{1+t^2}{1-t}\left[\mathrm{Sgn}(t)\ln\left(1+\frac{\Lambda_t}{2|t|}\right)+\mathrm{Sgn}(1-t)\ln\left(1+\frac{\Lambda_{1-t}}{2|1-t|}\right)\right]\right]_N$$

where $\Lambda_t = \sqrt{\tilde{\mu}^2/P_z^2+\underline{t^2}}-|t|$, $\mathrm{Sgn}(t)=1$ if $t\geq 0$, and -1 otherwise.

- **Generalized "+" description:** $t = \tilde{x}/x$

$$\int_{-\infty}^{+\infty} dt\left[g(t)\right]_N h(t) = \int_{-\infty}^{+\infty} dt\, g(t)\,[h(t)-h(1)]$$

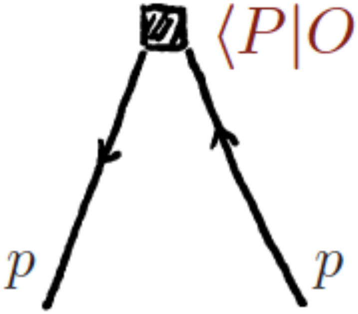
For a testing function $h(t)$

- **Explicit verification of the factorization at one-loop:**

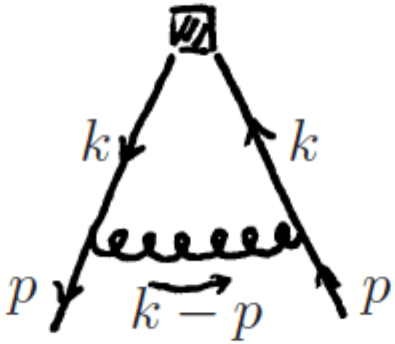
Coefficient functions for all partonic channels are IR safe and finite!

$$\mathcal{C}^{(1)}_{i/j}(t,\tilde{\mu}^2,\mu,P_z) \quad \text{with } i,j = q,\bar{q},g$$

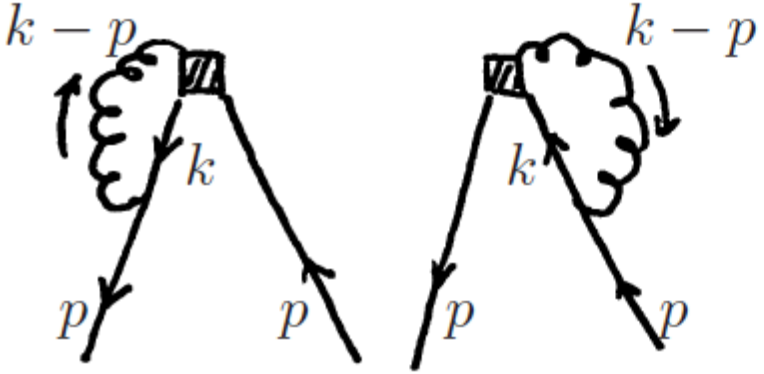
Feynman diagrams at one-loop



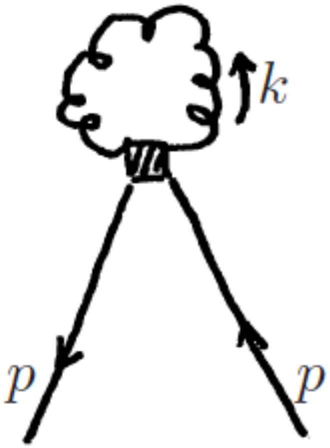
$\langle P|O(\delta z)|P\rangle_0 = e^{-iP_z\delta z}$



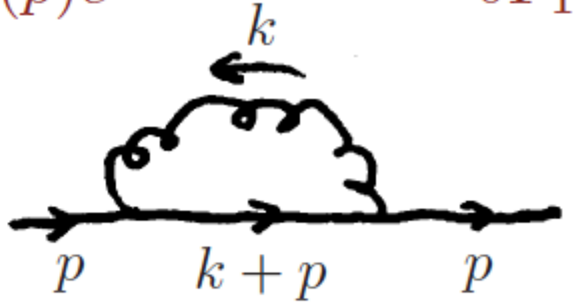
$\delta\Gamma_0(p)e^{-iP_z\delta z}$



$\delta\Gamma_1(p)e^{-iP_z\delta z}$



$\delta\Gamma_2 e^{-iP_z\delta z}$

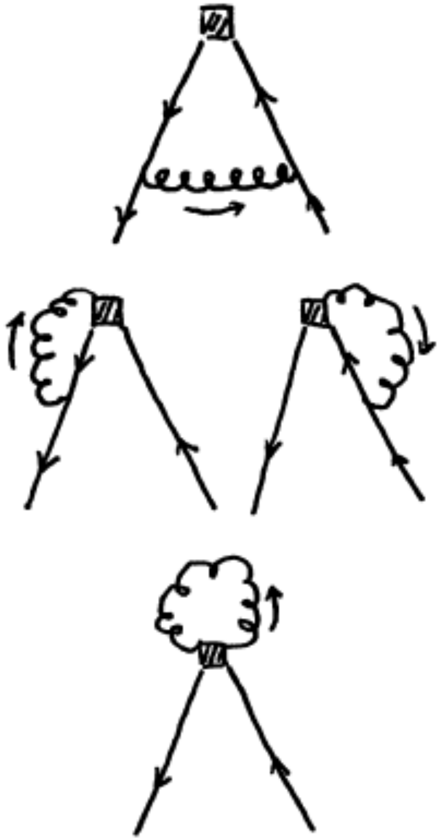


$\Sigma_{\rm RS}(p)$

$\Sigma_{\rm TP}(p)$ (only in lattice)

One-loop in Euclidean continuum

❑ Divergence structure (P=0):

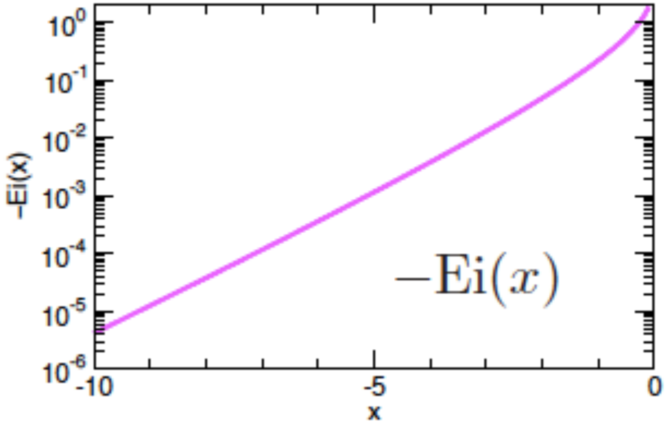


$$\delta\Gamma_0 = \frac{1}{8\pi^2}\left(\mathrm{Ei}(-k_\perp) - (2+k_\perp)e^{-k_\perp}\right)\Big|_{\lambda|\delta z|}^{\mu|\delta z|} \xrightarrow[\delta z \to 0]{} \frac{1}{8\pi^2}\ln\frac{\mu}{\lambda}$$

$$\delta\Gamma_1 = \frac{1}{4\pi^2}\left(\ln(k_\perp) - \mathrm{Ei}(-k_\perp) + e^{-k_\perp}\right)\Big|_{\lambda|\delta z|}^{\mu|\delta z|} \xrightarrow[\delta z \to 0]{} 0$$

$$\delta\Gamma_2 = \frac{1}{4\pi^2}\left(\ln(k_\perp) - \mathrm{Ei}(-k_\perp) - \boxed{k_\perp}\right)\Big|_{\lambda|\delta z|}^{\mu|\delta z|} \xrightarrow[\delta z \to 0]{} 0$$

Linear divergence



$$\mathrm{Ei}(x) = -\int_{-x}^{\infty} dt\,\frac{e^{-t}}{t}$$

: exponential integral

✧ Local case ($\delta_z \to 0$) can be safely reproduced.

✧ Linear divergence from the tad-pole like diagram.

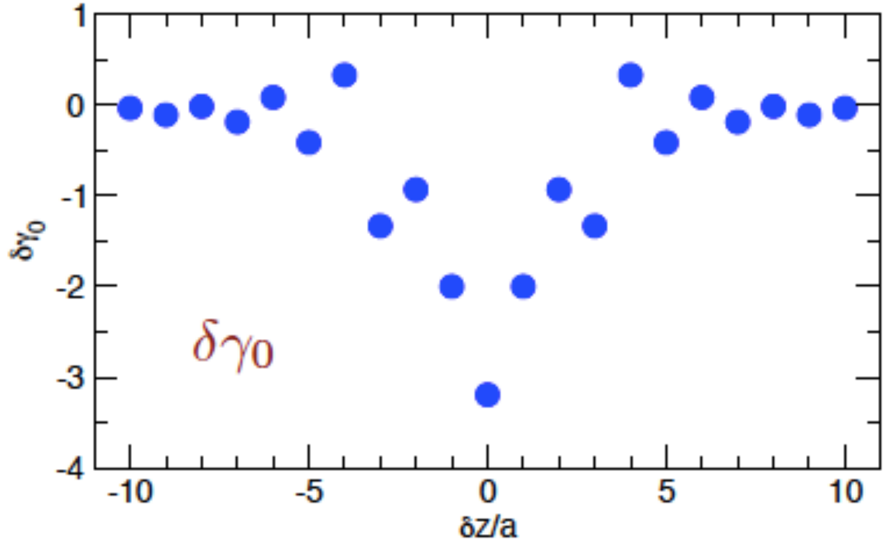
✧ UV(μ) and IR(λ) regulators are introduced in $\perp = (t, x, y)$ direction

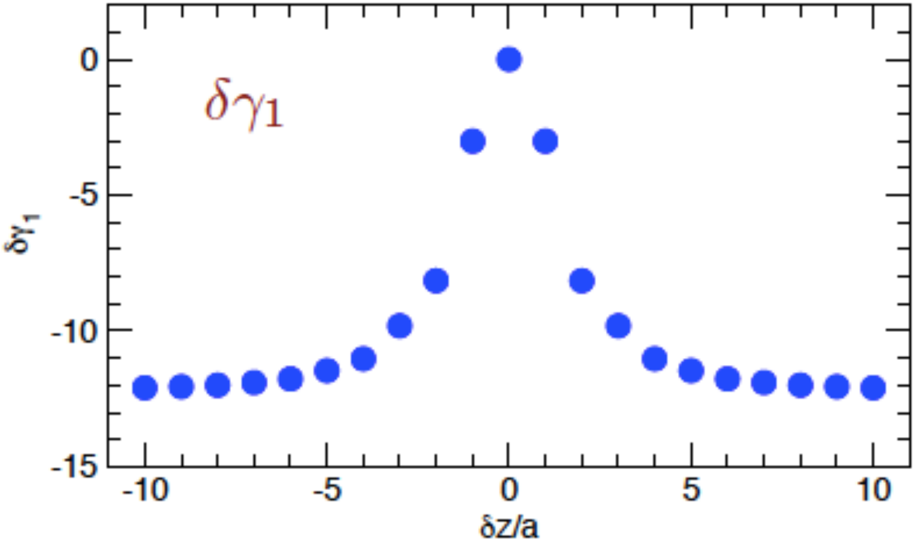
Matching lattice to continuum at one-loop

❑ One-loop matching coefficients:

- ✧ UV cut-off is set to be $\mu = a^{-1}$.
- ✧ Naive fermion is used.
  (not practical, but OK.)

$$\delta\Gamma_{\text{cont}} - \delta\Gamma_{\text{latt}} \equiv \frac{g^2}{16\pi^2} C_F \gamma_z \delta\gamma$$





Matching lattice to continuum at one-loop

❑ One-loop matching coefficients:

$$\delta\Gamma_{\rm cont} - \delta\Gamma_{\rm latt} \equiv \frac{g^2}{16\pi^2} C_F \gamma_z \delta\gamma$$

- ✧ There is a mismatch in linear divergence between continuum and lattice.
- ✧ The linear divergence should be subtracted, otherwise the continuum limit cannot be taken.