EXACT RG APPLIED TO nPI EFFECTIVE THEORIES

in many physical systems of interest can't use standard perturbation theory

- need non-perturbative techniques

different approaches (for example):

- hard thermal loop (HTL) effective theory
- Schwinger-Dyson equations
- n-particle irreducible (nPI) effective theories
- exact renormalization group (eRG)

issues:

- physics
- symmetries (gauge invariance)
- renormalization
- computational advantages

IN COLLABORATION WITH

WEIJIE FU, PAUL MIKULA, DOUG PICKERING AND JOE PULVER

MEC and Wei-Jie Fu, Eur. Phys. J. C **73**, *2399 (2013)*

MEC, Wei-Jie Fu, P. Mikula and D. Pickering, Phys. Rev. D **89**, *025013 (2014)*

MEC, Wei-Jie Fu, D. Pickering and J.W. Pulver, Phys. Rev. D 91, 025003 (2015)

Outline:

- Introduction to nPI
- Compare results from 2pi and 4pi calculations in 3d
- Basics of eRG
- Application of eRG to nPI calculations
- Comparison of 4d 2PI with and without eRG

I consider (symmetric) scalar ϕ^4 *theory*

Introduction to nPI

2PI for Scalar Theories:

generating functional with local and bi-local sources

$$Z[J,K] = e^{iW[J,K]} = \int \mathcal{D}\varphi e^{i(S[\varphi]+J_i\varphi_i+\frac{1}{2}B_{ij}\varphi_i\varphi_j)}$$

short-hand notation:

$$\int dx \int dy\ \varphi(x)B(x,y)\varphi(y) \to \varphi_i B_{ij}\varphi_j \to B\varphi^2$$

Legendre transform:

$$\begin{aligned}\Gamma[\phi, G] &= W[J, K] - J_i\phi_i - \frac{1}{2}B_{ij}\phi_i\phi_j \\ &= S_{\rm cl}[\phi] + \frac{i}{2}{\rm Tr}\,{\rm ln}G^{-1} + \frac{i}{2}{\rm Tr}\,G_0^{-1}(G - G_0) + \Gamma_2[\phi, G]\end{aligned}$$

$\Gamma[\phi, G]$ is a functional of the 1- and 2-point functions

ϕ and G are determined self-consistently from the equations of motion

variational principle (in the absence of sources)

$$\frac{\delta\Gamma}{\delta\phi} = \frac{\delta\Gamma}{\delta G} = 0$$

$$\Rightarrow\ G^{-1} = G_0^{-1} - \Sigma[\phi, G]\,, \qquad \Sigma[\phi, G] := 2\frac{\delta\Phi}{\delta G}\,, \quad \Phi = i\Gamma_2$$

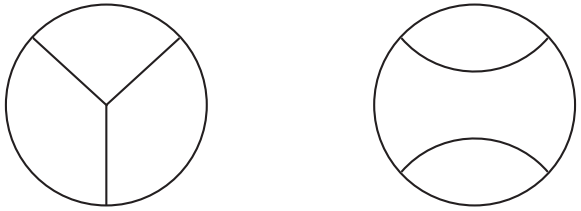
Compare to $\Gamma[\phi] =$ 1PI effective action:

- $\Gamma[\phi, G]$ depends on the self consistent propagator

$\rightarrow$ truncated $\Gamma[\phi, G]$ includes an infinite resummation of diagrams

$\rightarrow$ non-perturbative

- $\Gamma[\phi, G]$ is 2PI - no double counting



$2PI$ $\qquad$ $2PR$

nPI effective action

nPI Γ is a functional of n-point functions

3PI $\Gamma[\phi, G, U]$, 4PI $\Gamma[\phi, G, U, V] \cdots$

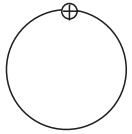
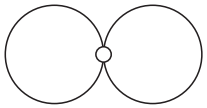
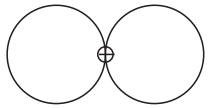
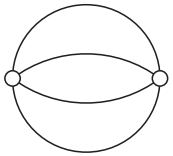
n-point functions determined self-consistently from the equations of motion

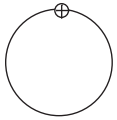
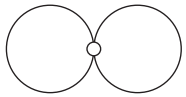
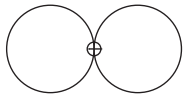
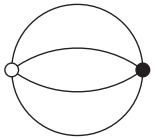
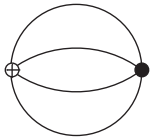
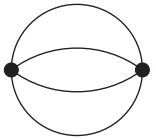
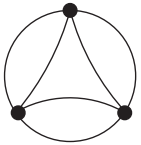
$\Rightarrow$ hierarchy of coupled equations

- no exact solution method is available
- use approximation techniques: truncate the effective action

FIRST CALCULATION:

compare the 2pi and 4pi calculations in 3 dimensions

$\Phi_2^{\text{2pi-3loop}} =$ $-\frac{1}{2}$  $+\frac{1}{8}$  EIGHT $+\frac{1}{8}$  EIGHT_{ct} $+\frac{1}{48}$  BBALL_0

$\Phi_2^{\text{4pi-4loop}} =$ $-\frac{1}{2}$  $+\frac{1}{8}$  EIGHT $+\frac{1}{8}$  EIGHT_{ct} $+\frac{1}{24}$  BBALL_0 $+\frac{1}{24}$  BBALL_{ct} $-\frac{1}{48}$  BBALL $+\frac{1}{48}$  LOOPY

eom for the 2-point function: $\Sigma = 2\delta\Phi/\delta G$

$$\Sigma_{\rm 2pi} \;=\; -\;\underset{\delta G}{\text{—}\oplus\text{—}} \;+\tfrac{1}{2}\;\text{○—○}\;+\tfrac{1}{2}\;\text{○—}\oplus\text{—○}\;+\tfrac{1}{6}\;\text{⊖}$$

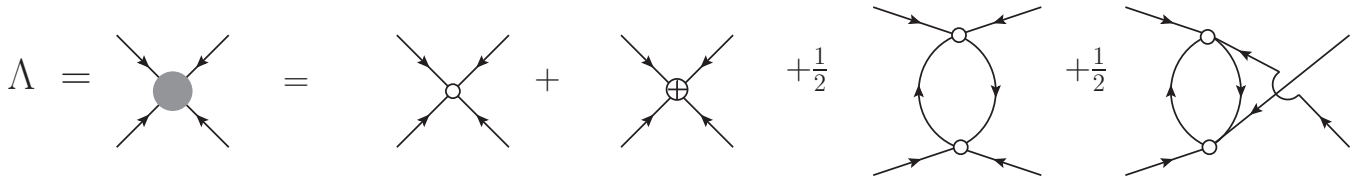
$$\Sigma_{\rm 4pi} \;=\; -\;\underset{\delta G}{\text{—}\oplus\text{—}} \;+\tfrac{1}{2}\;\text{○—○}\;+\tfrac{1}{2}\;\text{○—}\oplus\text{—○}\;+(2)\tfrac{1}{6}\;\text{⊖}\;+(2)\tfrac{1}{6}\;\text{⊖}\;-\tfrac{1}{6}\;\text{●⊖●}\;+\tfrac{1}{4}\;\text{●△●}$$

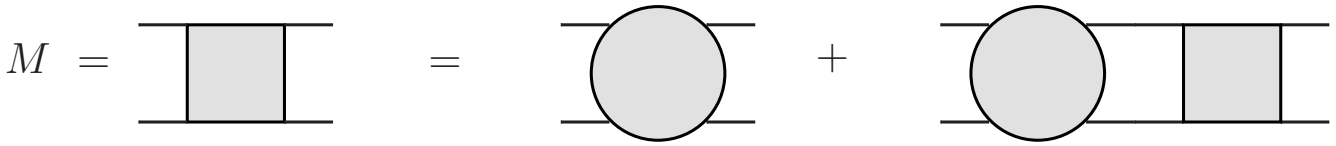
$$=\; -\;\underset{\delta G}{\text{—}\oplus\text{—}} \;+\tfrac{1}{2}\;\text{○—○}\;+\tfrac{1}{2}\;\text{○—}\oplus\text{—○}\;+\tfrac{1}{6}\;\text{⊖}\;+\tfrac{1}{6}\;\text{⊖}$$

eom for the 4-point function:

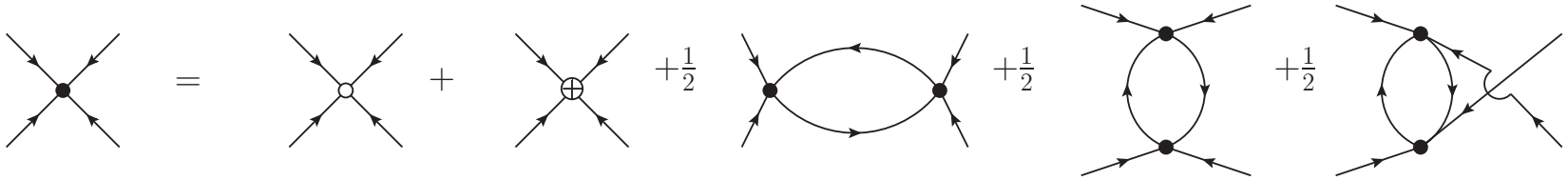
2pi: Bethe-Salpeter (BS) int eqn for self-consistent 4-vertex M from $\frac{\delta^2\Phi}{\delta R_2 \delta G}$

the kernel is the 4-vertex $\Lambda = 4\frac{\delta^2\Phi}{\delta G^2}$





4pi: eom from $\frac{\delta\Phi}{\delta V} = 0$



Solutions:

we have (coupled) self-consistent eom's for the 2- and 4-point functions

i work in 2- and 3-dimensions – no vertex counter-terms

solve using a numerical lattice method

*J. Berges, Sz. Borsányi, U. Reinosa, and J. Serreau, Phys. Rev. D***71**, *105004 (2005)*

- rotate to Euclidean space
- use an N^d symmetric lattice - in 2D $N_{\max} = 16$; in 3D $N_{\max} = 12$

the lattice spacing is $a = 2\pi/(Nm)$

each momentum component is discretized:

$$Q_i = \frac{2\pi}{aN} n_i = m\, n_i\,, \quad n_i = -\frac{N}{2} + 1, ..., \frac{N}{2}$$

- indices outside $\{-N/2+1, N/2\}$ wrapped inside using periodic b.c.
- use a numerical iterative method to solve set of self-consistent equations
  - search for fixed points

Memory constraints:

points in phase space of a vertex is $N^{l \times d}$

- l is the number of independent momenta and d is the dimension

for V: $l = 3$, $d = 3$, $N_{\max} = 12 \quad \Rightarrow \quad 5.16 \times 10^9$ points

trick: reduce the phase space of V using the symmetries of the vertex

- V is symmetric under interchange of legs and directions in momentum space

- don't need to calculate all points

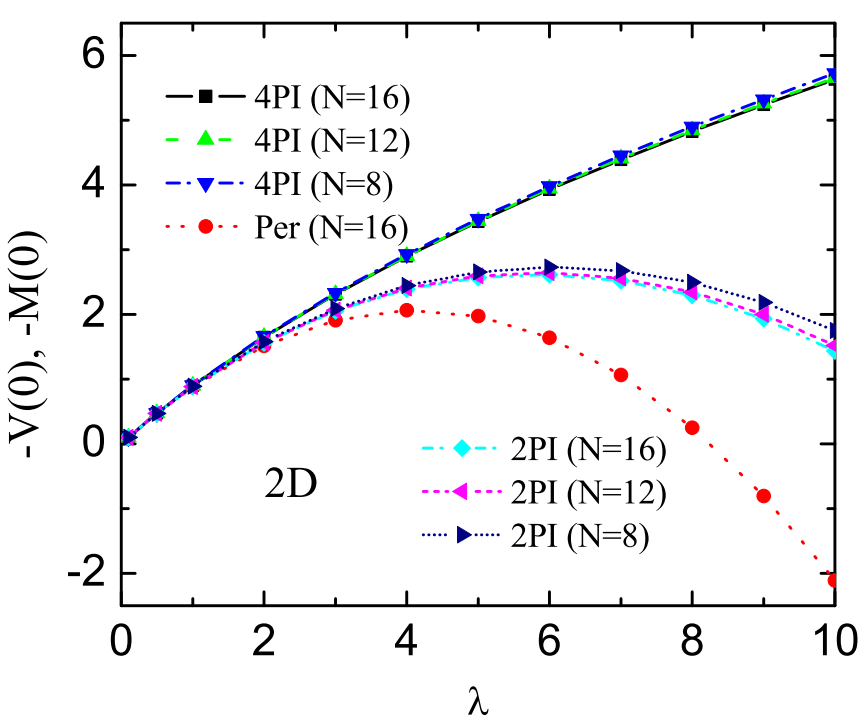
table: size of phase space and number of needed representative points

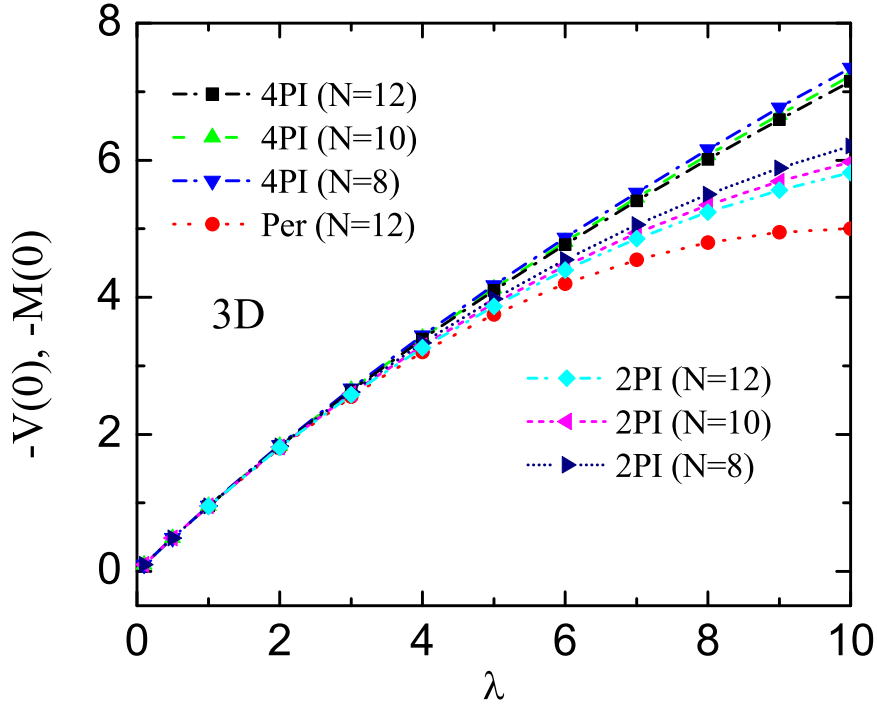
N	$N^{3\cdot(d=3)}$	# of reprs
6	10,077,696	11,424
8	134,217,728	129,502
10	1,000,000,000	913,661
12	5,159,780,352	4,608,136

** the function generates the uncalculated points must be *FAST*

results:

for λ large, the perturbative, 2pi and 4pi vertices are different





for certain momentum configurations M and V are close together

- this happens when s-channel contributions are big

4-dimensions - - - - - - vertex divergences

if you expand nPI eom's you get an infinite set of diagrams

$\Rightarrow$ infinite sets of embedded sub-divergences and counter-terms

need renormalization conditions (RCs) to determine counter-terms

that cancel sub-divergences

compare to perturbative expansion at L loops:

- some diagrams are missing
- some which are present have different coefficients

MEC and Yun Guo, Phys. Rev. D **83**, *016006 (2011); Phys. Rev. D* **85**, *076008 (2012)*

recall: goal of nPI is to (?) resum the physically important contributions

2pi in 4-dimensions: we know how to renormalize

H. van Hees, J. Knoll, Phys. Rev. D **65**, *105005 (2002); Phys. Rev. D* **65**, *025010 (2002)*

J.-P. Blaizot, E. Iancu, U. Reinosa, Nucl. Phys. A **736**, *149 (2004)*

J. Berges, Sz. Borsányi, U. Reinosa, J. Serreau, Annals Phys. **320**, *344 (2005)*

U. Reinosa, J. Serreau, Annals Phys. **325**, *969, (2010)*

trick is to determine vertex counter-term using a RC on the BS 4-point fcn

what to do with 4pi is unclear (more definitions of the 4-point functions)

I want to try a different strategy

Exact Renormalization Group

motivation

will show that one can do the 2pi calculation without using counter-terms

the next step will be to apply the same method to higher npi calculations

exact RG

add to the action a non-local regulator term

$$S_\kappa[\varphi] = S[\varphi] + \Delta S_\kappa[\varphi]\,, \quad \Delta S_\kappa[\varphi] = -\frac{1}{2}\hat{R}_\kappa \varphi^2$$

$$\hat{R}_\kappa(q) = \frac{q^2}{e^{q^2/\kappa^2} - 1} = \begin{cases} 0 \quad \text{for} \quad q \geq \kappa & \sim \quad \text{(unaffected)} \\ \kappa^2 \quad \text{for} \quad q < \kappa & \sim \quad \text{(suppressed)} \end{cases}$$

family of theories indexed by the continuous parameter κ

fluctuations are smoothly taken into account as κ is lowered to zero

$\kappa \to \infty$ regulated action $\to$ classical action

$\kappa \to 0$ (include all fluctuations) regulated action $\to$ full quantum action

J.-P. Blaizot, A. Ipp, N. Wschebor, Nucl. Phys. A **849**, *165 (2011)*

J.-P. Blaizot, J.M. Pawlowski and U. Reinosa, Phys. Lett. B **696**, *523 (2011)*

generating functionals defined in the usual way:

$$Z_\kappa[J, J_2] = \int [d\varphi] \exp\{i(S[\varphi] - \frac{1}{2}\hat{R}_\kappa\varphi^2 + J\varphi + \frac{1}{2}B\varphi^2 + \cdots)\}$$

calculate 1pi, 2pi, $\cdots$ effective action

the flow equation gives the dependence of the action on κ

C. Wetterich, Phys. Lett., B **301**, *90 (1993)*

$$\partial_\kappa \Phi_\kappa = \frac{1}{2}\partial_\kappa R_\kappa \left(\langle\varphi^2\rangle - \phi^2\right)$$

- same form for any nPI effective action
- definition of the expectation values different for different actions

$$\text{1pi}: \quad \partial_\kappa \Phi_{1\text{PI}\cdot\kappa} = -\frac{1}{2}\partial_\kappa R_\kappa \left[\frac{\delta^2 \Phi_{1\text{PI}\cdot\kappa}}{\delta\phi^2} + R_k \right]^{-1}$$

functional derivatives wrt $\phi \rightarrow$ infinite coupled hierarchy of eRG equations

- practical calculations require truncation

regulated 2pi effective action:

- also gives infinite coupled hierarchy of eRG equations
- but they truncate naturally when the action is truncated

advantages to truncating at the level of the action:

- straightforward to systematically extend the order of the approximation
- we expect that the truncation respects the symmetries of the original theory to the order of the approximation

the 1st two flow equations from the regulated 2pi effective action:

$$\partial_\kappa \Sigma_\kappa(P) = \frac{1}{2}\int dQ \partial_\kappa(\Sigma_\kappa(Q) + R_\kappa(Q)) G_\kappa^2(Q)\Lambda_\kappa(Q,P)$$

$$\partial_\kappa \Lambda_\kappa(P,K) = \frac{1}{2}\int dQ\, \partial_\kappa[R_\kappa(Q) + \Sigma_\kappa(Q)] G_\kappa^2(Q)\Lambda_\kappa^{03}(Q,P,K)$$

truncation

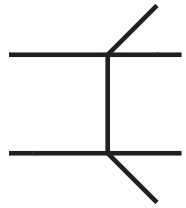
Λ_κ^{03} has its own flow equation of the form $\partial_\kappa \Lambda_\kappa^{03} \sim \int dQ\, \partial_\kappa G_\kappa\, \Lambda_\kappa^{04} \;\; \cdots$

BUT: hierarchy of flow eqns truncates when the action is truncated

- at the level of our approximation (3-loop 2pi) Λ_κ^{04} is a constant
$\rightarrow \partial_\kappa \Lambda_\kappa^{03}$ is an exact differential - can integrate directly
(integration const = zero because no 6-vertex in Lagrangian)

equivalent:
we can simply obtain Λ_κ^{03} directly from the effective action:

$$\begin{aligned}\Lambda^{03}(Q,P,K) \;\; &= -\lambda^2 (G_\kappa(Q+P+K) + G_\kappa(Q+P-K) \\ &\quad + G_\kappa(Q-P+K) + G_\kappa(Q-P-K))\end{aligned}$$



$\Rightarrow$ we have a closed set of integro-differential equations

boundary conditions

to solve flow equations must specify bc's from which flow starts at $\kappa = \mu$

idea is to choose μ large and use the (known) classical solutions as bc's

$\Rightarrow$ solve the equations to obtain the quantum solutions at $\kappa = 0$

must show bc's consistent with RC's (defns of physical parameters) $\kappa = 0$

basic idea

the solution for the 2-point function will look like

$$G_\kappa^{-1} = P^2 + m^2 + \Sigma_\kappa(P) + C$$

C is any κ independent constant

to compare with the standard 2pi calculation we use the same RC's

$$G_0^{-1}(0) = m^2\,, \quad \frac{d}{dP^2} G_0^{-1}|_{P=0} = 1\,, \quad M_0(0) = -\lambda$$

caution: subscripts 0 indicate $\kappa = 0$ not bare/non-interacting quantities

impose RC on the 2-point function $\rightarrow$ choose $C = -(\Sigma_0(0) + P^2\Sigma_0'(0))$

can show that with this choice of C, the limit $\mu \to \infty$ gives

$$G_\mu^{-1} = Z_\mu(P^2 + m_\mu^2)$$

with Z_μ and m_μ momentum independent

for the 4-point function can show:
bc $\Lambda_\mu(P, Q) = -\lambda_\mu$ is consistent with the RC on the 4-point function

equivalently:
momentum integrals in flow eqns are all either finite or $\vec{p}$-independent
$\to$ all divergent contributions can be absorbed into defns of m_μ, λ_μ

a fundamental technical difficulty with the eRG formalism

the RC's (which define the physical parameters)

- are defined in terms of the quantum ($\kappa = 0$) n-point fcns
- these are obtained only after the calculation is finished

we want to specify chosen values for the physical mass and coupling

- but required input is the bare parameters

an arbitrary choice of bare parameters $\not\Rightarrow$ the chosen physical ones

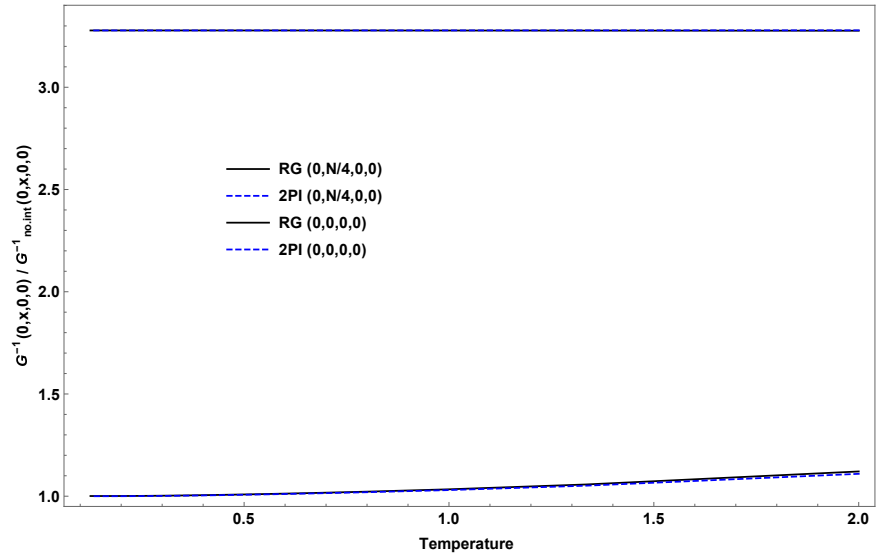
- we do not know in advance which choice of bare parameters will

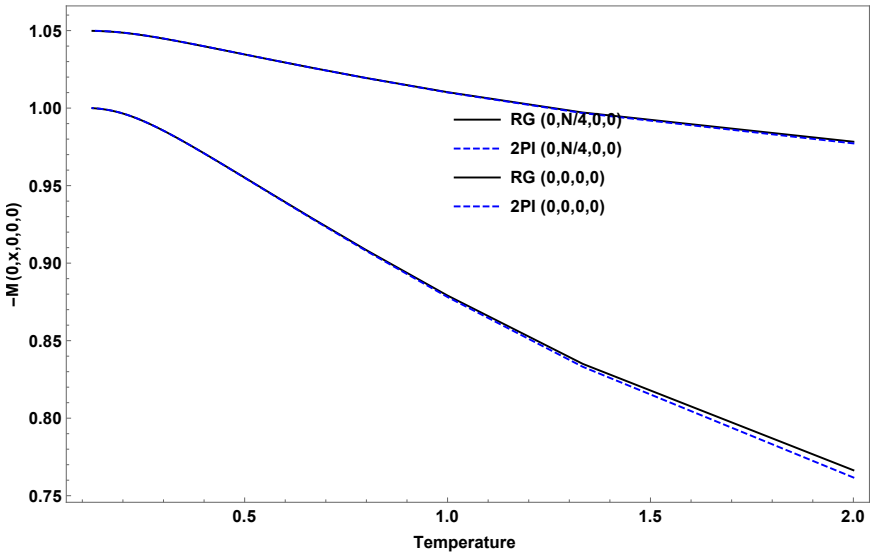
must "tune" the bare parameters to produce physical mass and coupling

calculations at finite temperature are done using these bare parameters

- increase T by decreasing size of euclidean box in the temporal direction

results





eRG method reproduces results of the standard 2pi calculation

- - - without using counter-terms - - -

to test the calculation:

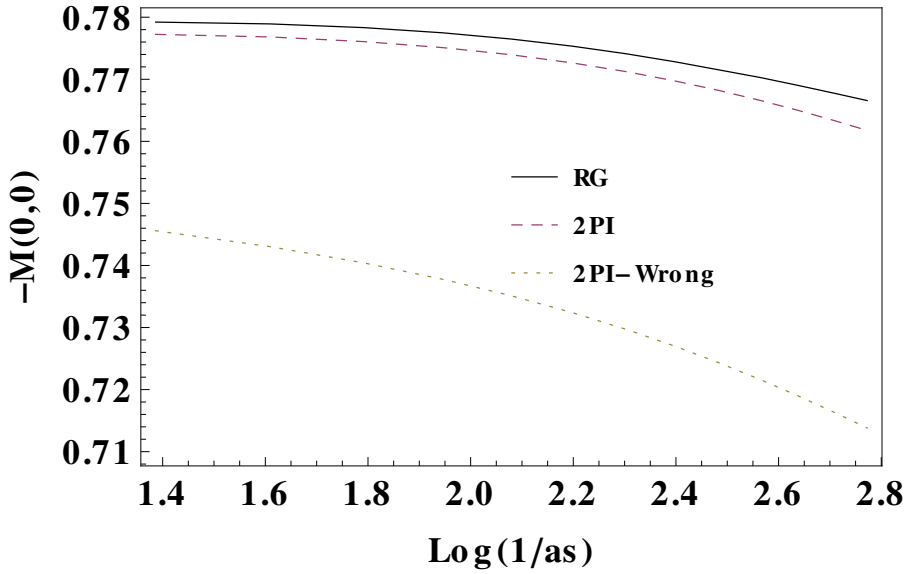
reducing a_s while holding the 3-length of the box $L = a_s N_s$ fixed

compare: 2pi calculation with cts $(\lambda \to \lambda + \delta\lambda)$ on basketball diagram

we plot $M(0,0)$ versus $\log 1/a_s$ at $T = 2m$

- in the incorrect calculation $M(0,0)$ increases when a_s is reduced
- in the correct 2pi and eRG calculations the curve is flat

$\to$ shows that the renormalization is done correctly



Conclusions

- in 3 dimensions the 4pi 2- and 4-point fcns are significantly different from the corresponding 2pi ones when the coupling is large
- the eRG can be used to do 4d 2pi calculations without counter-terms

eRG is a promising method to do higher order 4d $n$pi calculations