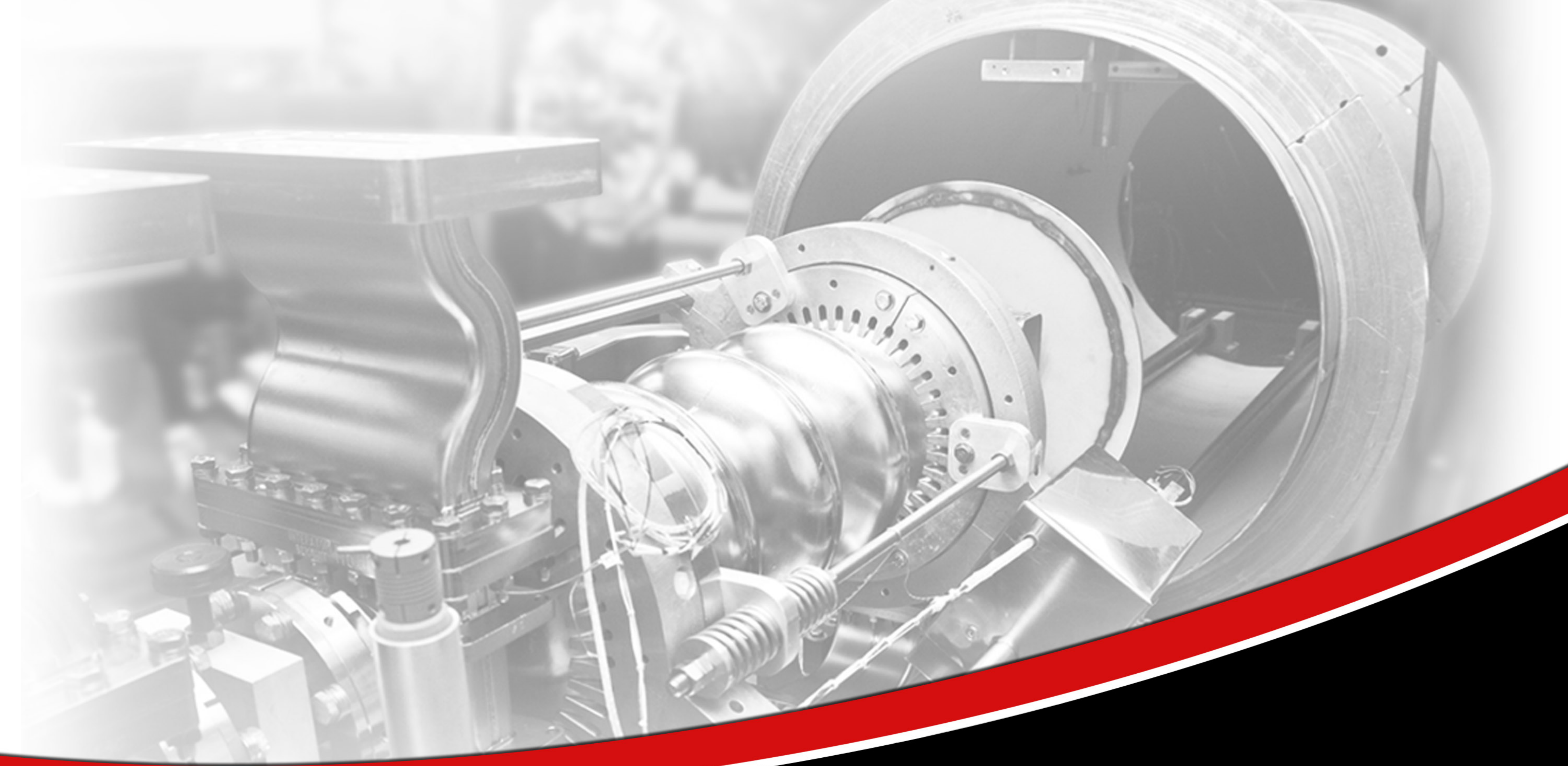


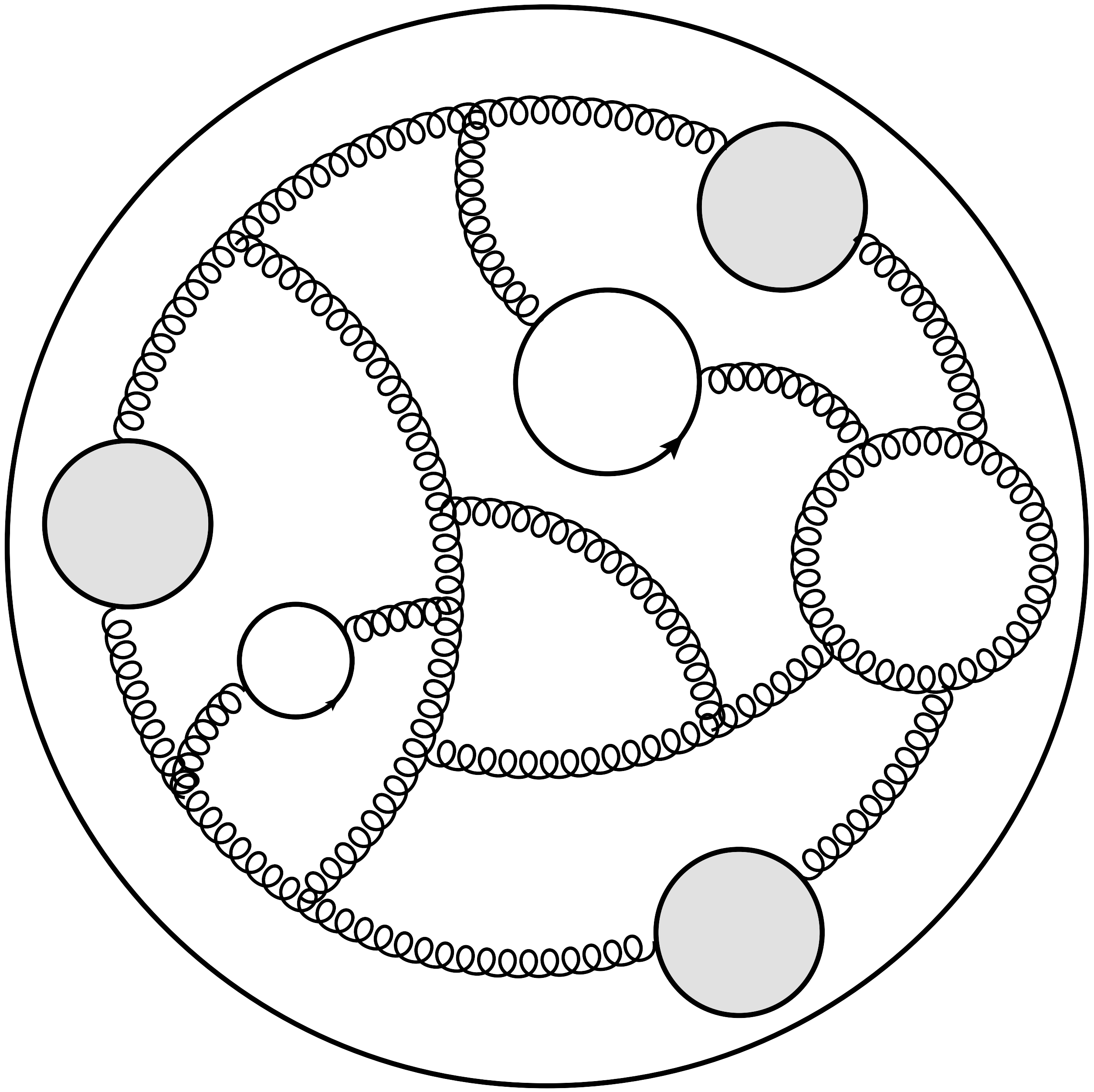
Rapidity evolution of gluon TMD from low to moderate x

Andrey Tarasov

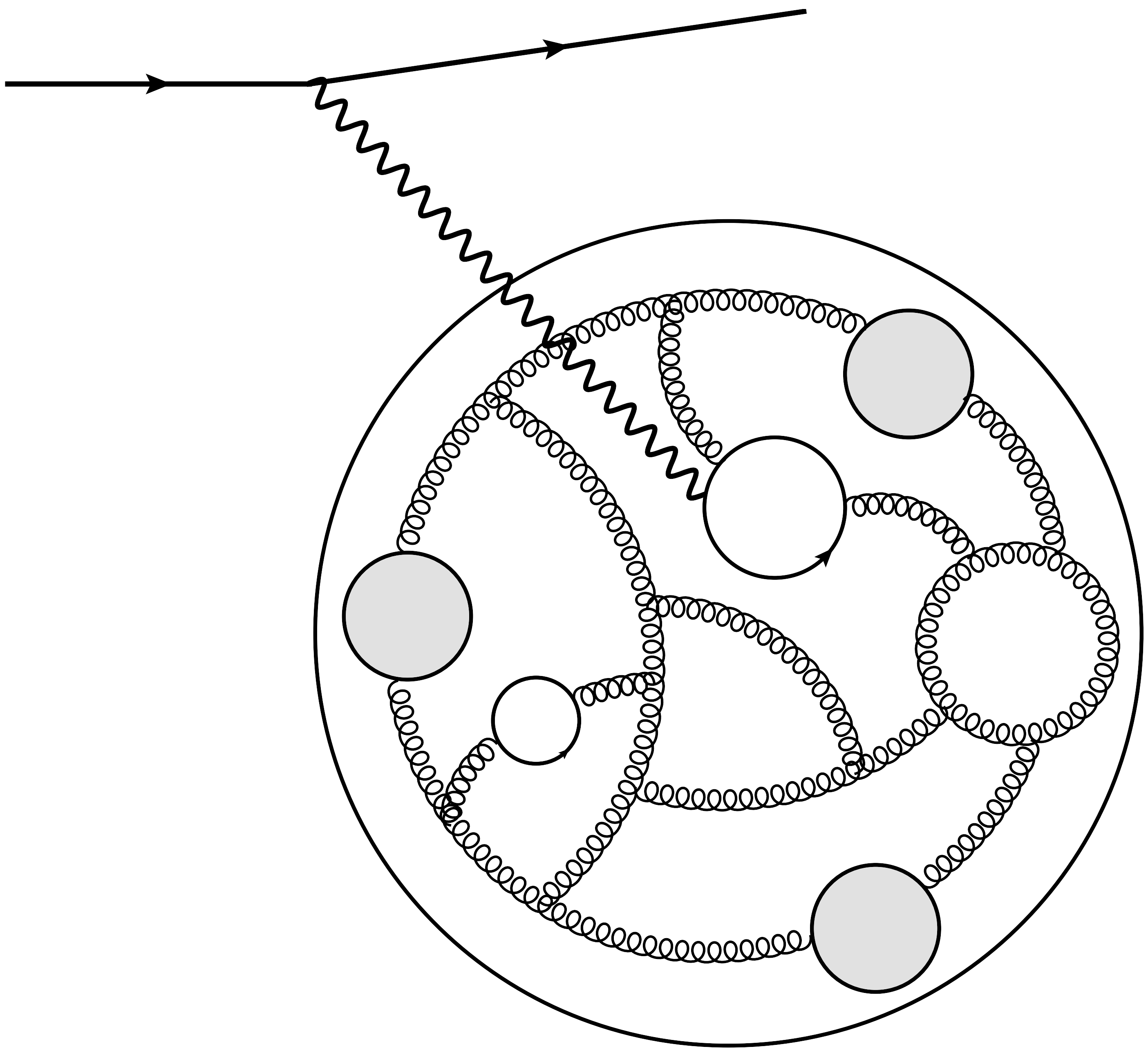
Outline

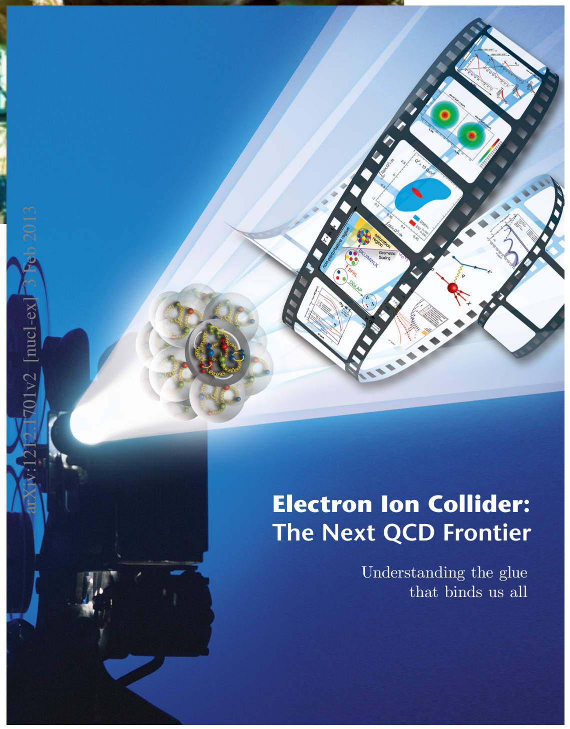
- Definition of gluon TMD
- Light-cone expansion at moderate x
- Rapidity factorization approach at small x
- Generalized evolution at moderate and small x
- BK, DGLAP and Sudakov limits
- Gluon TMD factorization function

Transverse momentum dependent (TMD) distribution

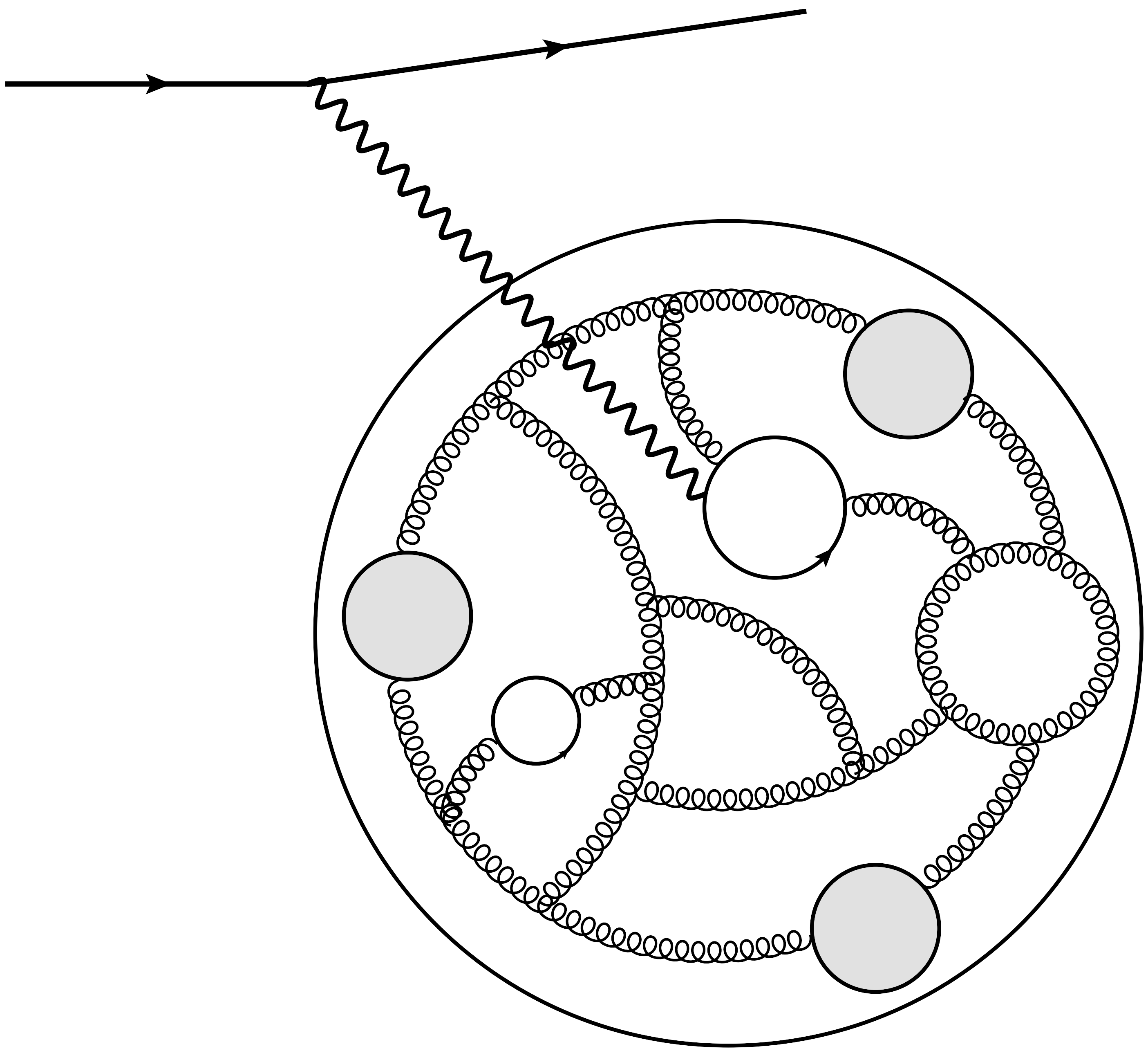


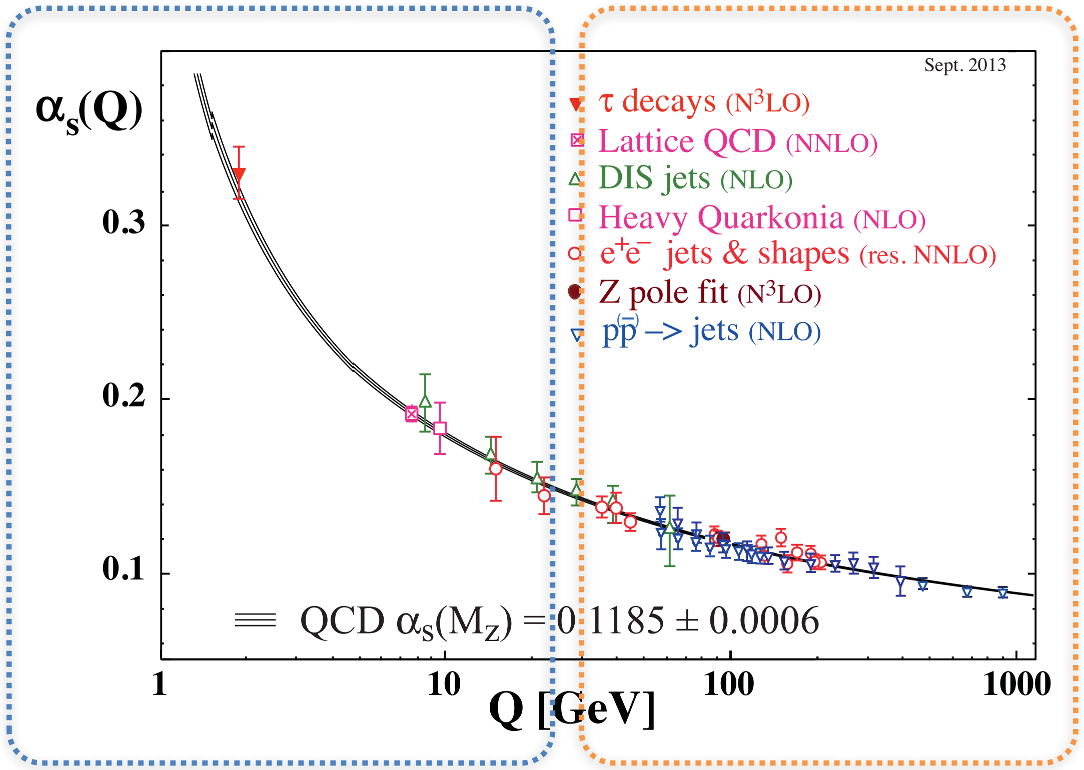
Transverse momentum dependent (TMD) distribution





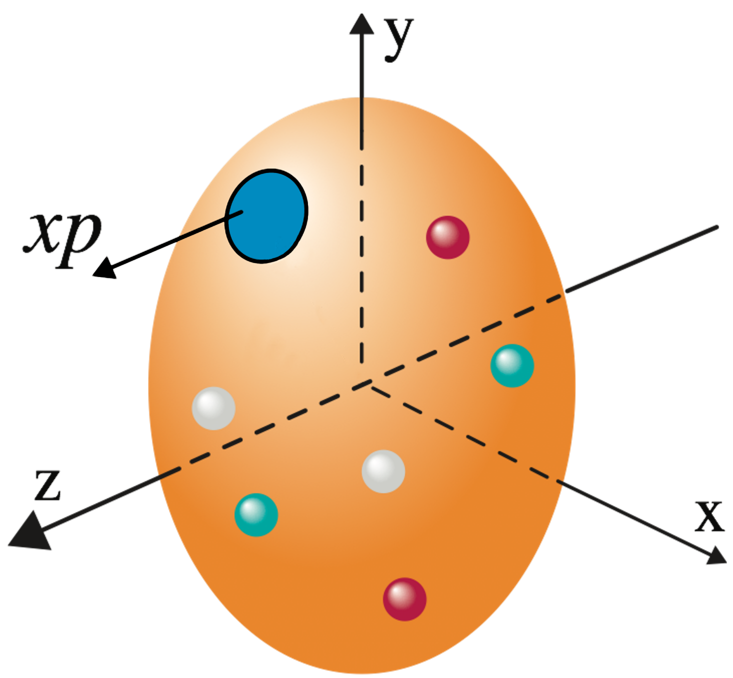
Transverse momentum dependent (TMD) distribution



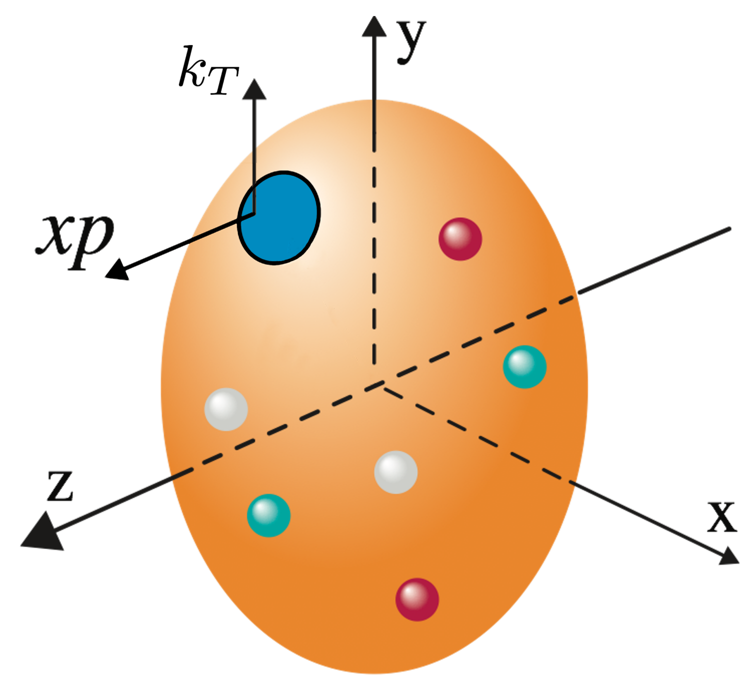


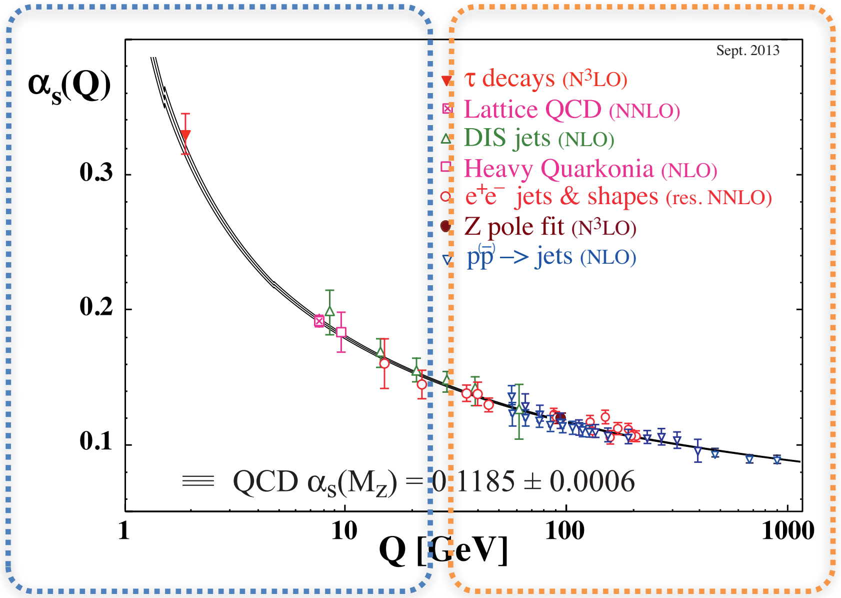
Transverse momentum dependent (TMD) distribution

Collinear distribution:



TMD distribution:

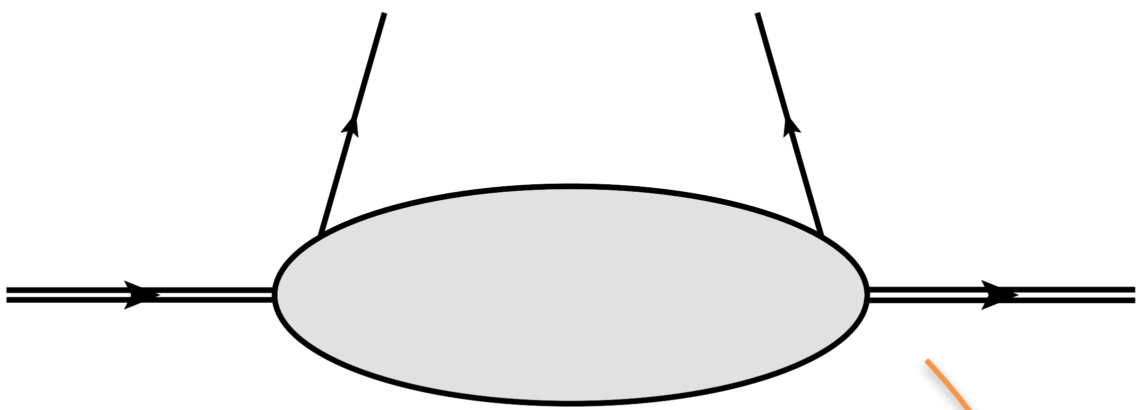


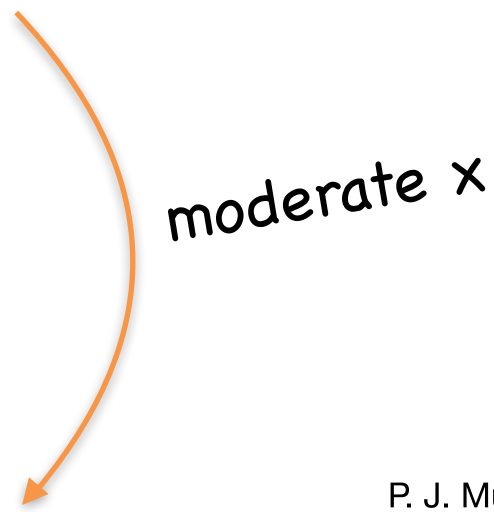


Current status

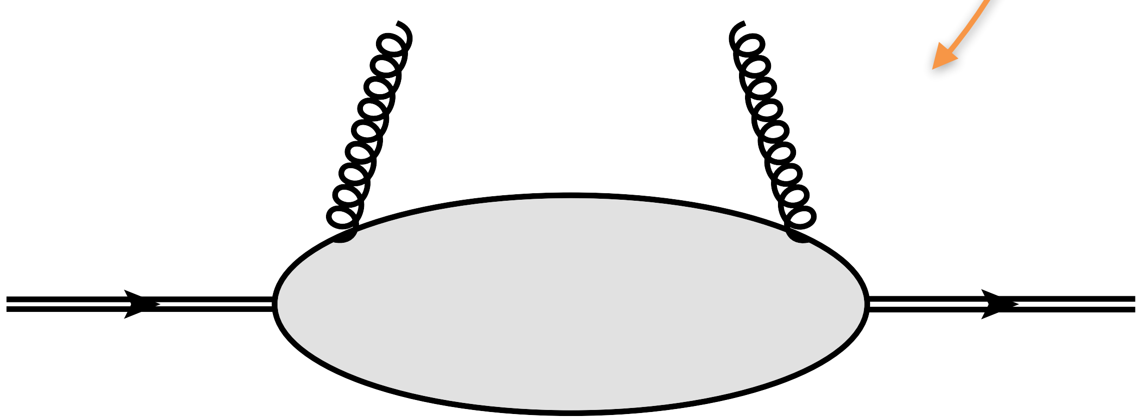
Quark TMD at moderate x:

J.C. Collins, D. E. Soper and G. Sterman, Nucl. Phys. B250, 199 (1985)
X. Ji, Jian-Ping Ma, and F. Yuan, Phys. Rev. D71, 034005 (2005)
J. C. Collins, Foundations of Perturbative QCD (CUP, 2011)





Gluon TMD:

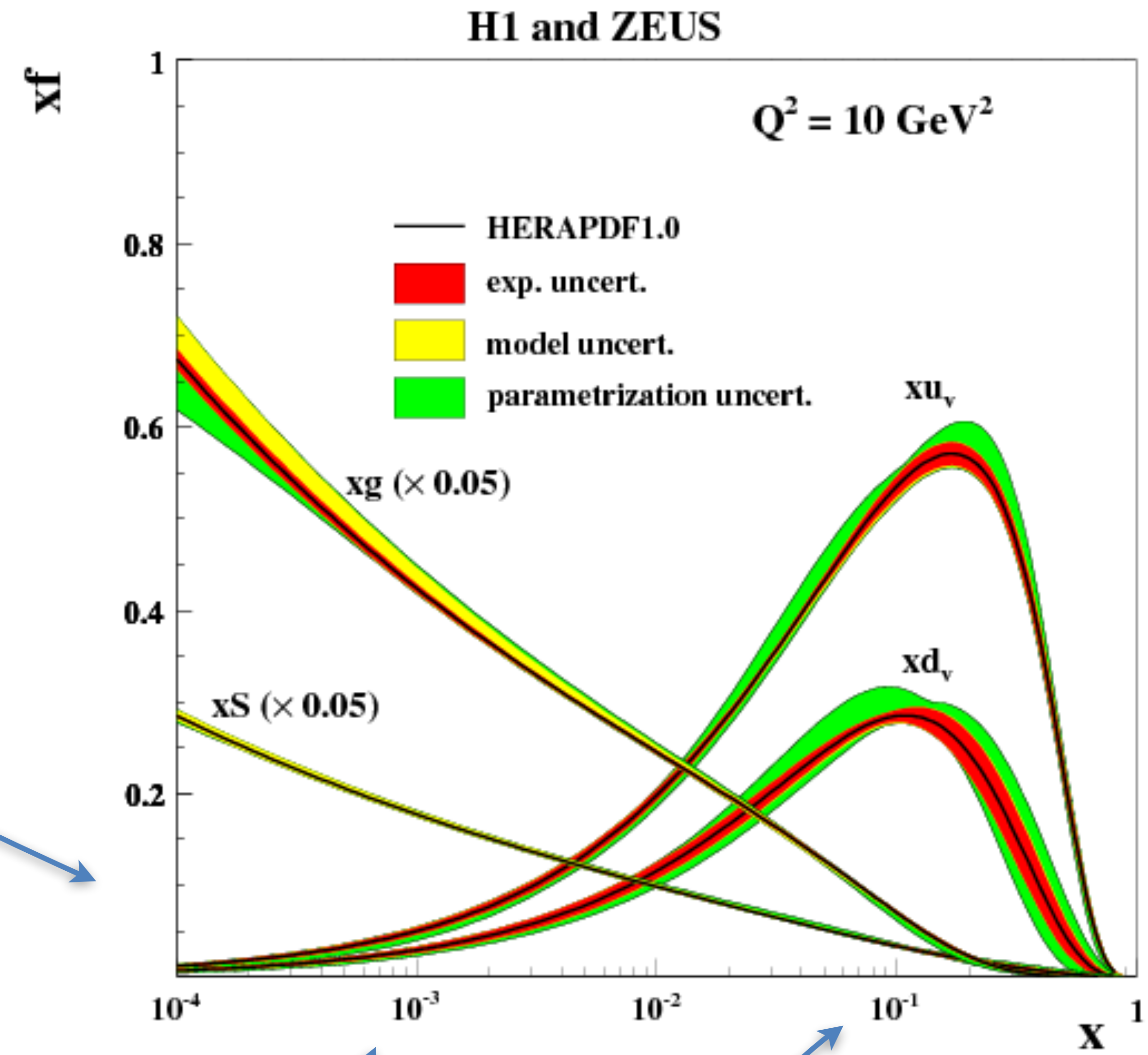


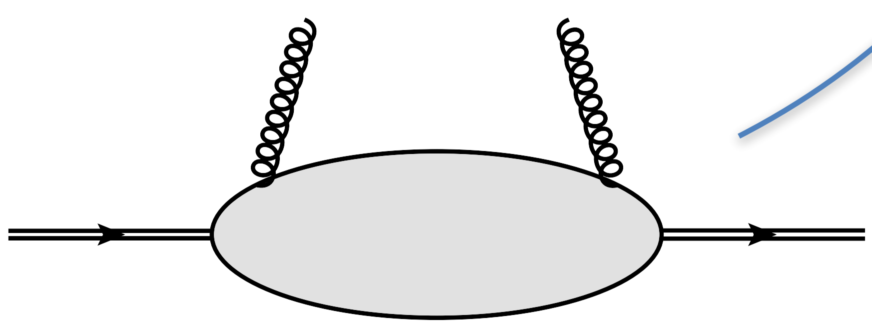
P. J. Mulders and J. Rodrigues, Phys. Rev. D63, 094021 (2001)
A. H. Mueller, Bo-Wen Xiao, and F. Yuan, Phys. Rev. D88, 114010 (2013)
M. G. Echevarria, T. Kasemets, P. J. Mulders, and C. Pisano, hep-ph/1502.0535

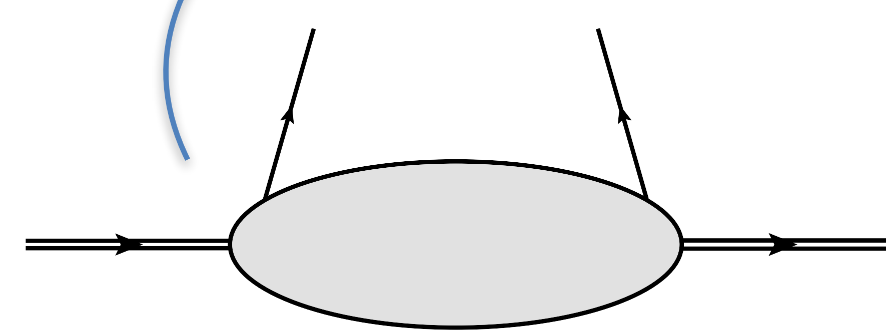
From moderate to small-x

I. Balitsky, Nucl. Phys. B463, 99 (1996)
Yu.V. Kovchegov, Phys. Rev. D60, 034008 (1999)

Non-linear effects at small x







Two definitions of gluon TMD

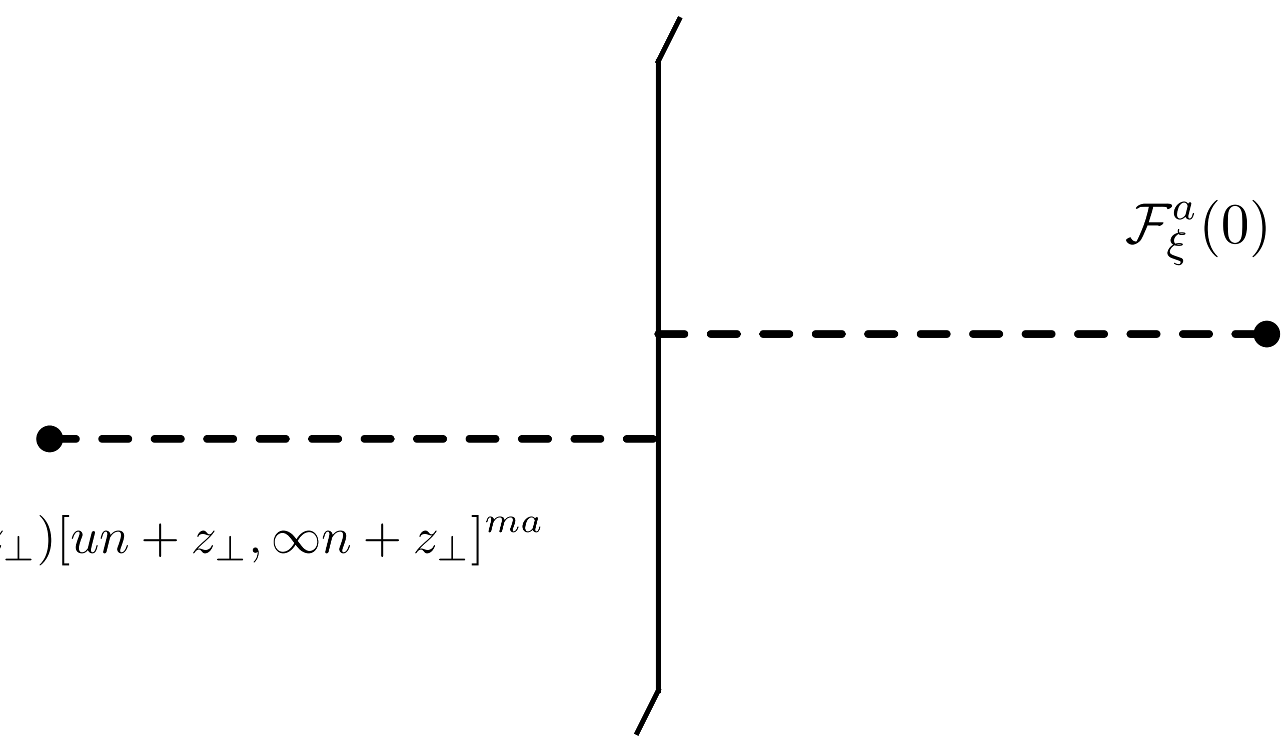
Moderate x:

$$\alpha_s \mathcal{D}(x, z_\perp) \equiv \frac{2\pi^2 \alpha_s}{N_c} G(x, z_\perp) = -\frac{1}{8\pi^2 (P\cdot n) x} \int du e^{-ixu(P\cdot n)} \langle P | \tilde{\mathcal{F}}^a_\xi (z_\perp + un) \mathcal{F}^{a\xi}(0) | P \rangle$$

there is no exponential factor

Small x (Weiszacker-Williams):

$$\alpha_s \mathcal{D}(x, z_\perp) \equiv \frac{2\pi^2 \alpha_s}{N_c} G(x, z_\perp) = -\frac{1}{8\pi^2 (P\cdot n) x} \int du \langle P | \tilde{\mathcal{F}}^a_\xi (z_\perp + un) \mathcal{F}^{a\xi}(0) | P \rangle$$



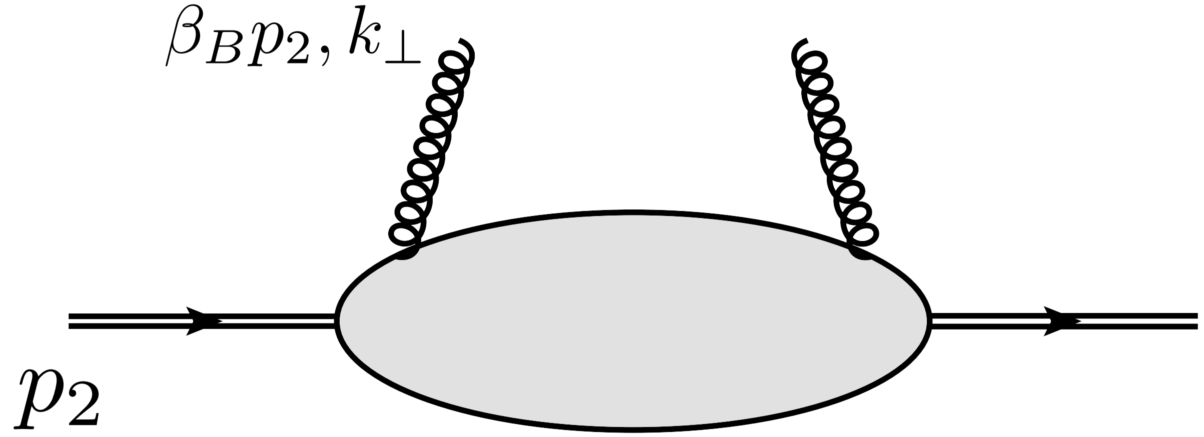
$$\mathcal{F}^a_\xi(0) \equiv [\infty n, 0]^{am} n^\mu g F^m_{\mu\xi}(0)$$

$$\tilde{\mathcal{F}}^a_\xi(z_\perp + un) \equiv n^\mu g \tilde{F}^m_{\mu\xi}(un + z_\perp)[un + z_\perp, \infty n + z_\perp]^{ma}$$

Definitions of gluon TMD

$$\beta_B \mathcal{D}(\beta_B, k_\perp) = -\frac{4}{\alpha_s \langle P|P\rangle} \int d^2x_\perp d^2y_\perp e^{ik_\perp(x_\perp - y_\perp)} \langle P|\tilde{\mathcal{F}}_i^a(\beta_B, x_\perp)\mathcal{F}^{ai}(\beta_B, y_\perp)|P\rangle$$

We consider evolution of this operator



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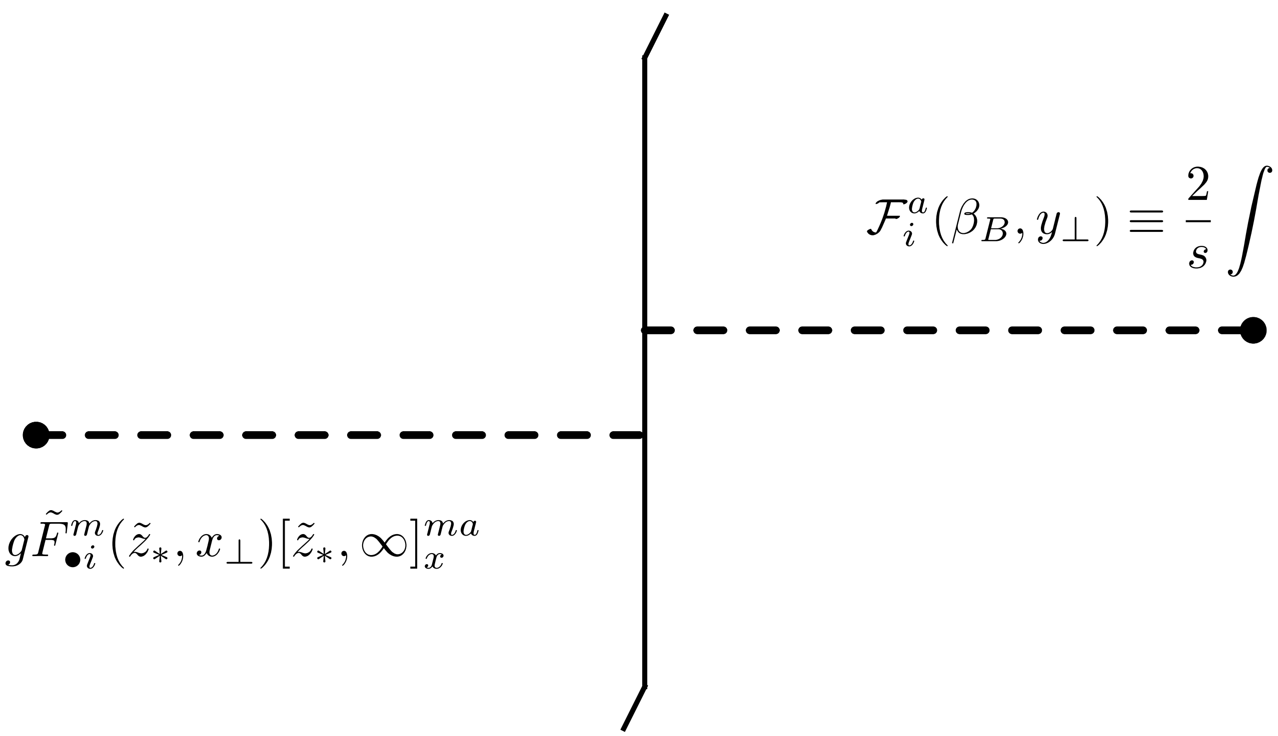


$$\mathcal{F}_i^a(\beta_B, y_\perp) \equiv \frac{2}{s}\int dz_* e^{i\beta_B z_*}[\infty, z_*]_y^{am} g F_{\bullet i}^m(z_*, y_\perp)$$

$$\tilde{\mathcal{F}}_i^a(\beta_B, x_\perp) \equiv \frac{2}{s}\int d\tilde{z}_* e^{-i\beta_B \tilde{z}_*} g\tilde{F}_{\bullet i}^m(\tilde{z}_*, x_\perp)[\tilde{z}_*, \infty]_x^{ma}$$

$$p = \alpha p_1 + \beta p_2 + p_\perp$$

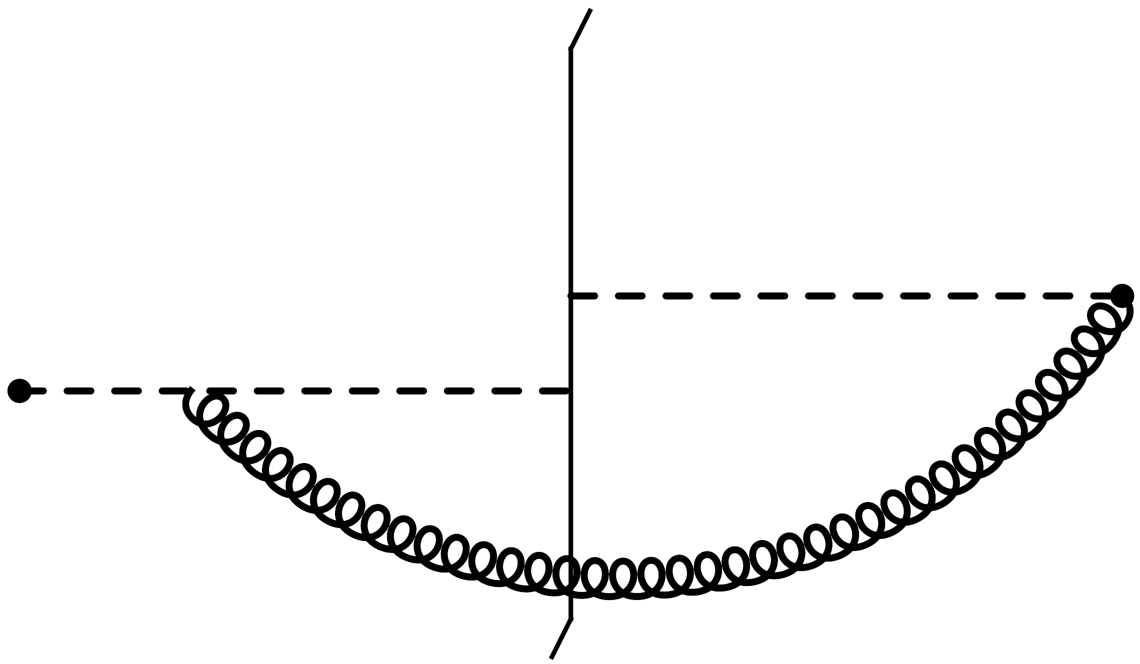
$$z_* = z_\mu p_2^\mu = \sqrt{\frac{s}{2}} z^-$$

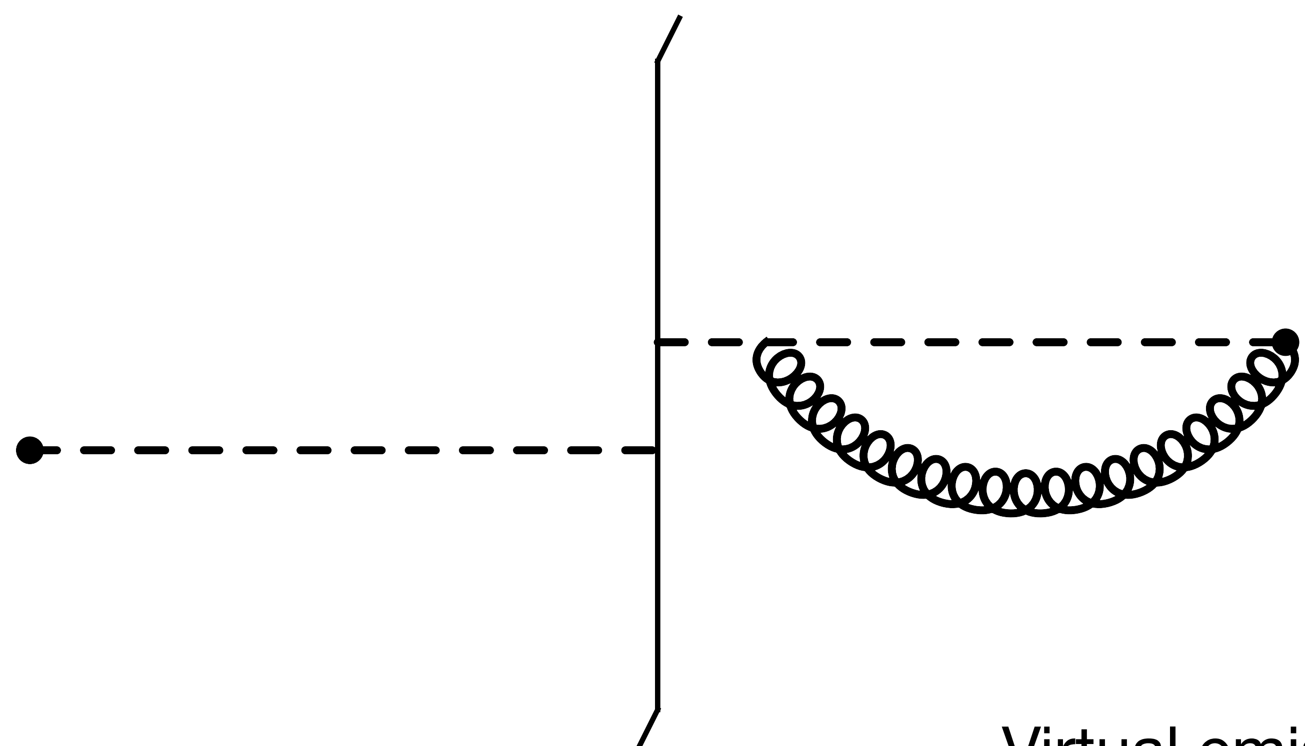
$$z_\bullet = z_\mu p_1^\mu = \sqrt{\frac{s}{2}} z^+$$

Evolution of gluon TMD

Note: there are several diagrams of this kind

Real emission

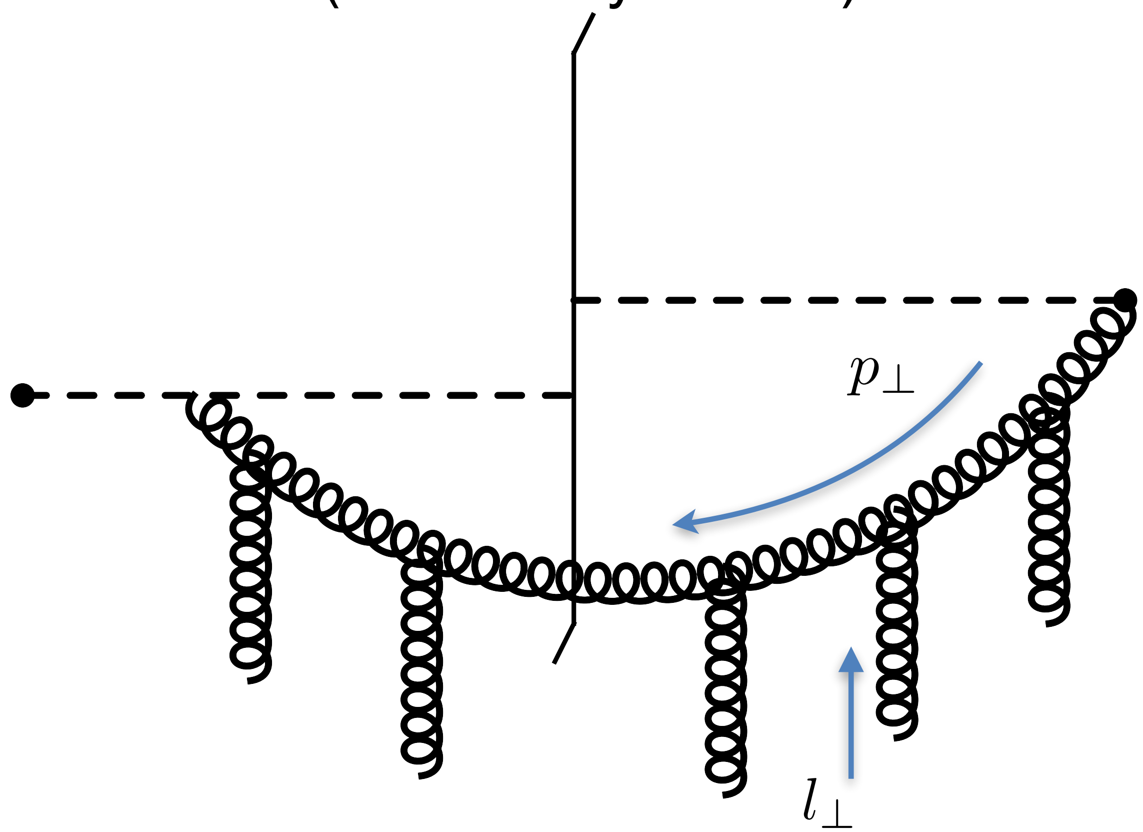




Virtual emission

Evolution at small and moderate x

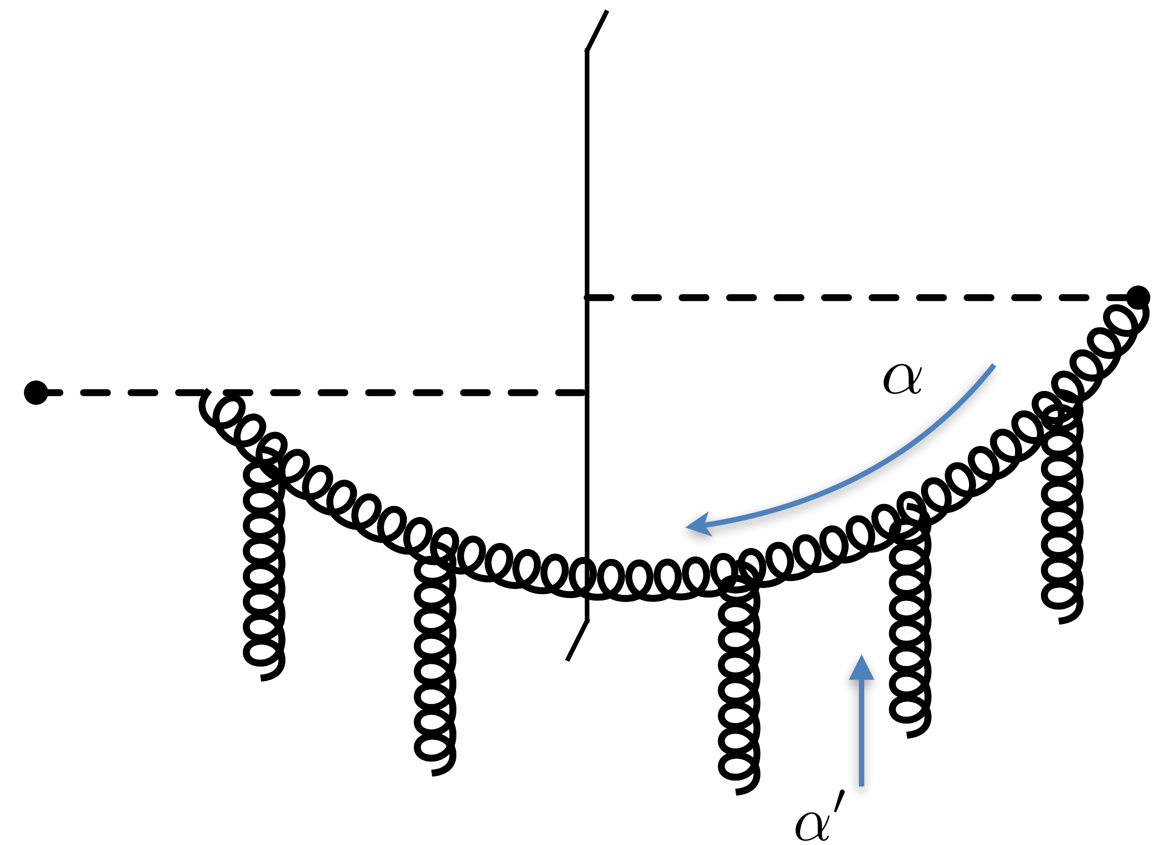
Moderate x (DGLAP dynamics)



- Ordering in transverse momentum
  $p_\perp > l_\perp$
- No ordering in rapidity
  $\alpha \sim \alpha'$

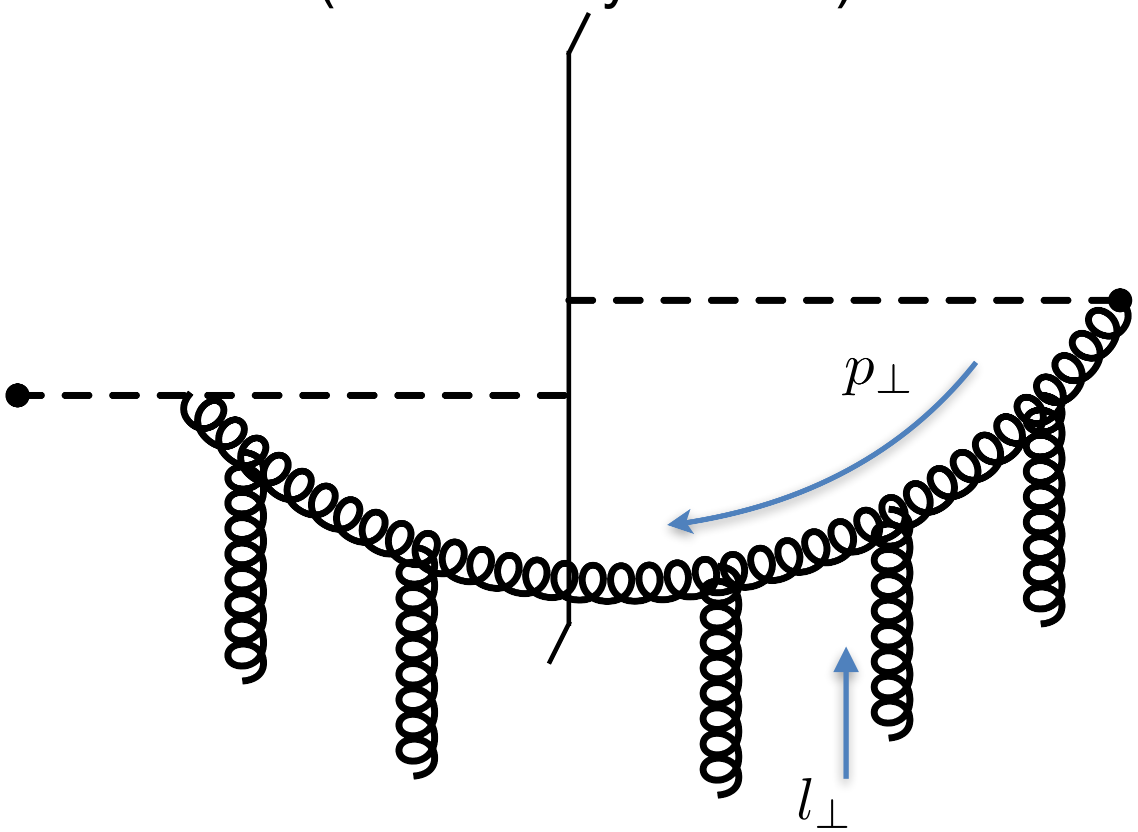
Small x

- Ordering in rapidity
  $\alpha > \alpha'$
- No ordering in transverse momentum
  $p_\perp \sim l_\perp$



Evolution at small and moderate x

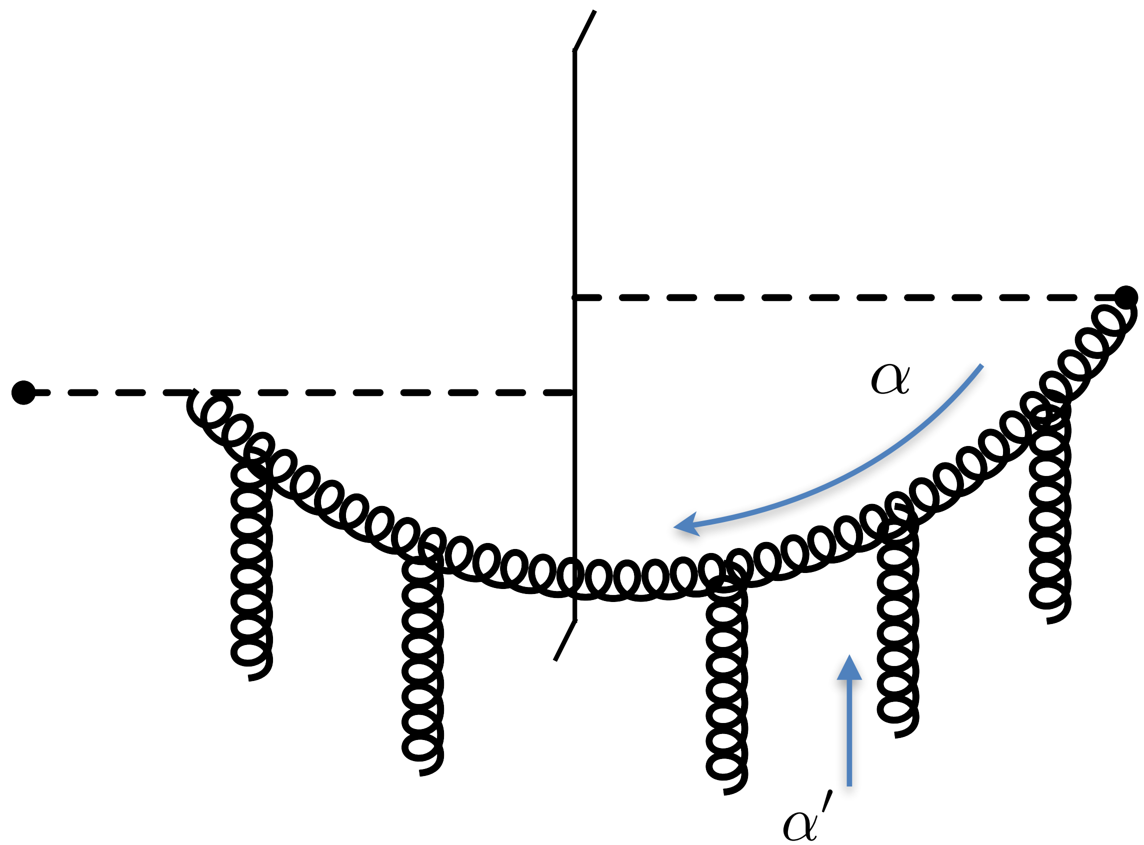
Moderate x (DGLAP dynamics)



light-cone expansion

Small x

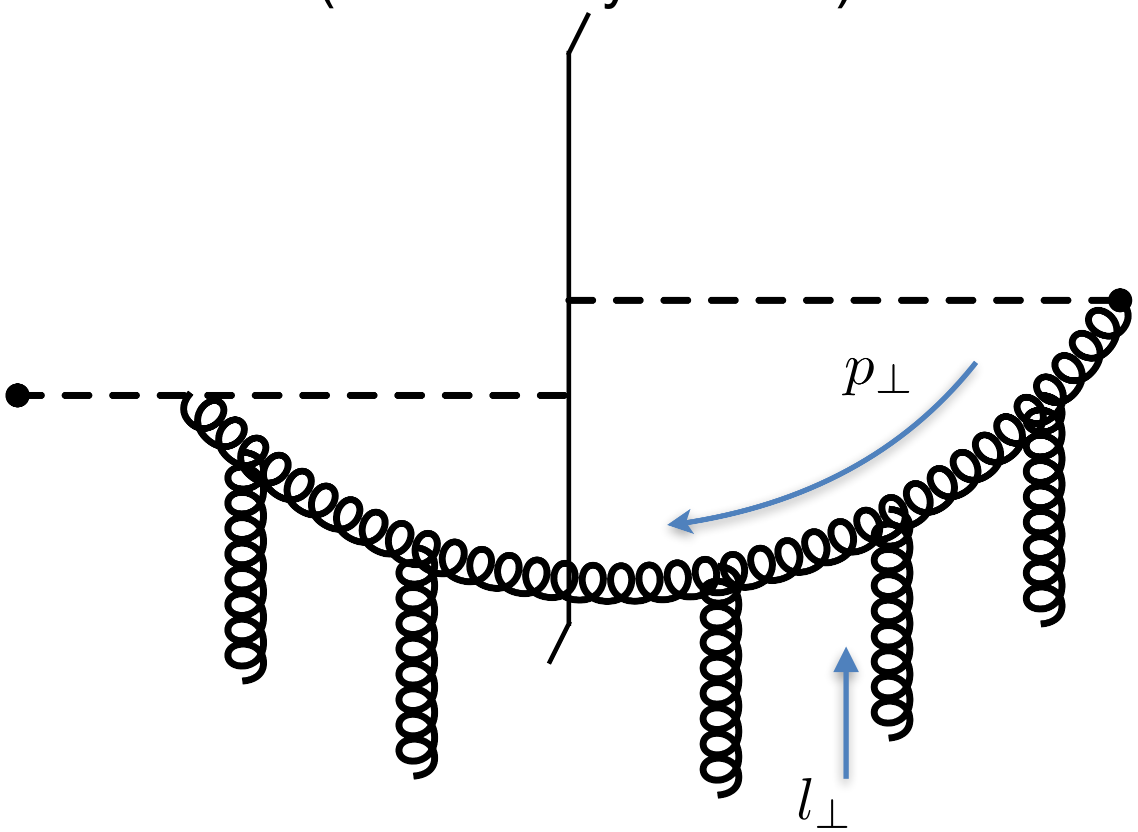
rapidity factorization approach



We consider emission in both limits and combine results

Light-cone expansion (moderate x)

Moderate x (DGLAP dynamics)



- Parameter of expansion $l_\perp / p_\perp \ll 1$
- In coordinate representation is an expansion in powers of deviation from the light-cone
- Gluon propagator is linear in F

$$\lim_{k^2\to 0} k^2 \langle A^a_\mu(k) A^b_\nu(y)\rangle = -ie^{i\frac{k_\perp^2}{\alpha s}y_*} e^{-ik_\perp y_\perp} \mathcal{G}^{ab}_{\mu\nu}(\infty, y_*; k_\perp)|_{y_\perp}$$

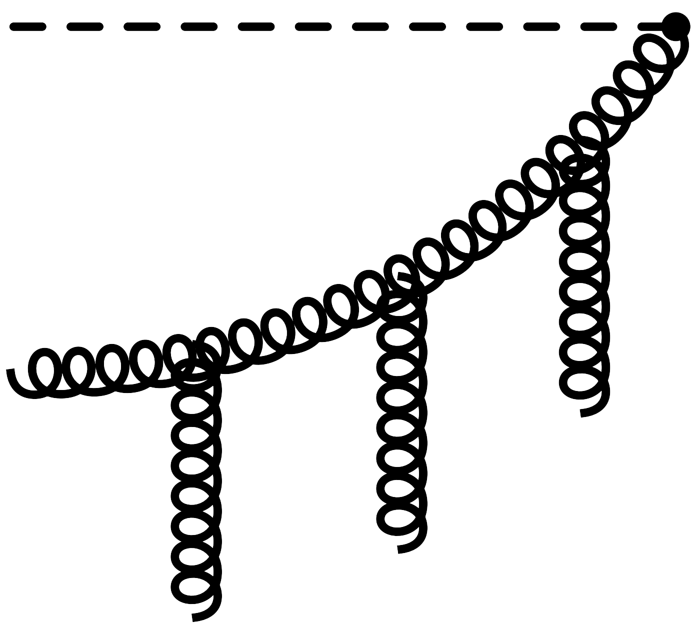
$$\mathcal{G}_{\mu\nu}(\infty, y_*, k_\perp)|_{y_\perp} = g_{\mu\nu}[\infty, y_*]_{y_\perp} + \int_{y_*}^{\infty} dz_* \Big(-\frac{4i}{\alpha s^2} k^j (z-y)_* g_{\mu\nu}[\infty, z_*]_{y_\perp} F_{\bullet j}[z_*, y_*]_{y_\perp} + \frac{4}{\alpha s^2}(p_{2\nu}\delta^j_\mu - p_{2\mu}\delta^j_\nu)[\infty, z_*]_{y_\perp} F_{\bullet j}[z_*, y_*]_{y_\perp}\Big)$$

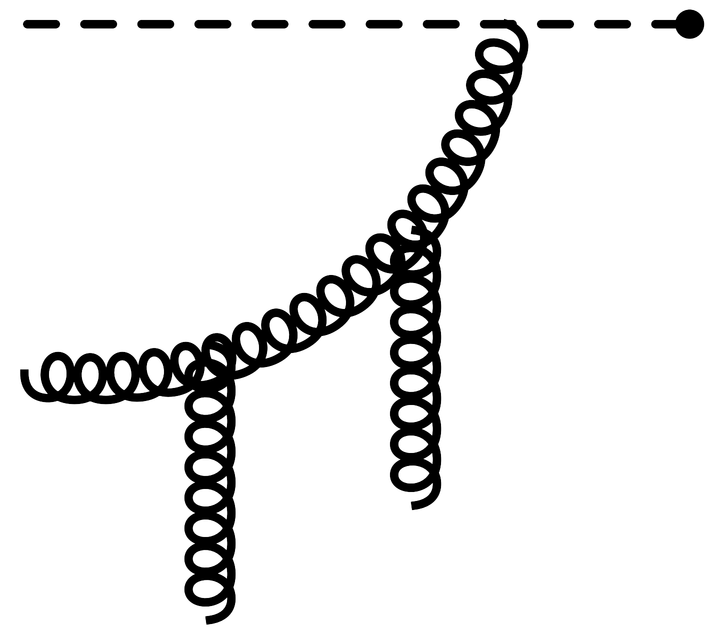
The structure at small x (rapidity factorization) is different

Light-cone expansion. Real emission

Lipatov vertex

$$L^{ab}_{\mu i}(k, y_\perp, \beta_B) = i \lim_{k^2 \to 0} k^2 \langle A^a_\mu(k) \mathcal{F}^b_i(\beta_B, y_\perp) \rangle$$

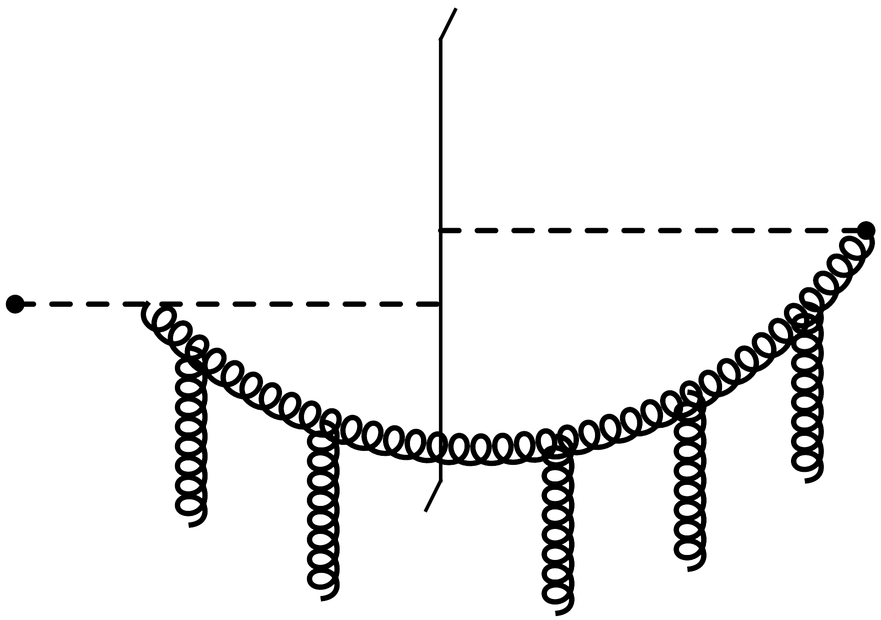


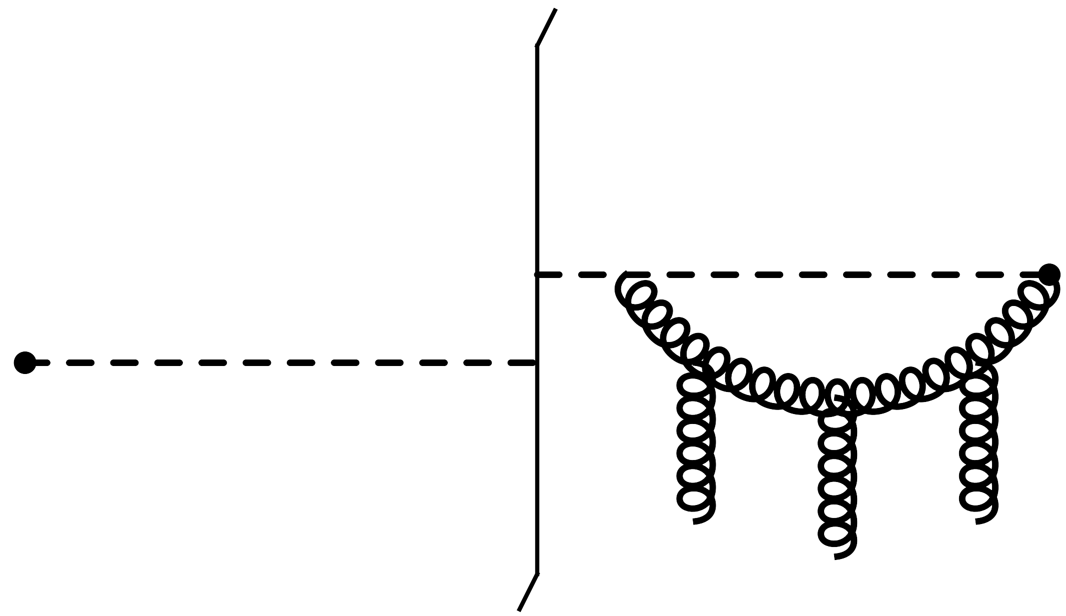


$$\langle \tilde{\mathcal{F}}^{ai}(\beta_B, x_\perp) \mathcal{F}^a_i(\beta_B, y_\perp) \rangle = -\int \frac{d\alpha}{2\alpha} d^2k_\perp \tilde{L}^{i\mu}(k, x_\perp, \beta_B) L_{\mu i}(k, y_\perp, \beta_B)$$

Light-cone expansion. Real and virtual emission

$$\langle\tilde{\mathcal{F}}^{ni}(\beta_B,x_\perp)\mathcal{F}^n_i(\beta_B,y_\perp)\rangle = 2N_c\int_0^\infty \frac{d\alpha}{\alpha}d^2k_\perp\Big\{e^{ik_\perp(x_\perp-y_\perp)}\Big[\frac{g^{jk}}{k_\perp^2}-\frac{2g^{jk}}{\alpha\beta_Bs+k_\perp^2}+\frac{2k_\perp^2g^{jk}}{(\alpha\beta_Bs+k_\perp^2)^2}-\frac{2k^jk^k}{(\alpha\beta_Bs+k_\perp^2)^2}$$
$$+\frac{4k_\perp^2k^jk^k}{(\alpha\beta_Bs+k_\perp^2)^3}-\frac{2k_\perp^4k^jk^k}{(\alpha\beta_Bs+k_\perp^2)^4}\Big]\tilde{\mathcal{F}}^n_k(\beta_B+\frac{k_\perp^2}{\alpha s},x_\perp)\mathcal{F}^n_j(\beta_B+\frac{k_\perp^2}{\alpha s},y_\perp)-\frac{\alpha\beta_Bs}{k_\perp^2(\alpha\beta_Bs+k_\perp^2)}\tilde{\mathcal{F}}^{ni}(\beta_B,x_\perp)\mathcal{F}^n_i(\beta_B,y_\perp)\Big\}$$





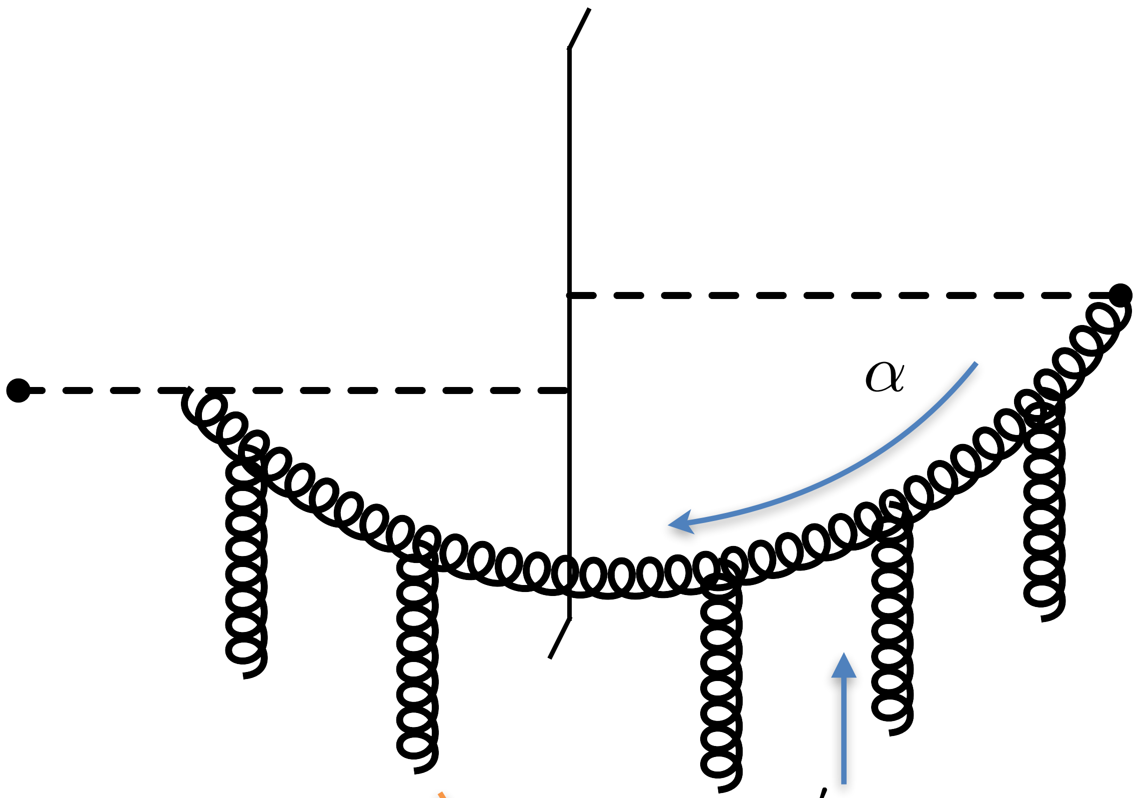
Collinear limit $x_\perp = y_\perp$

$$\mu^2\frac{d}{d\mu^2}\langle\tilde{\mathcal{F}}^{ni}(\beta_B,x_\perp)\mathcal{F}^n_i(\beta_B,x_\perp)\rangle = \frac{\alpha_s(\mu)}{\pi}N_c\int_{\beta_B}^1 dz\Big[\frac{1}{z(1-z)_+}-2+z(1-z)\Big]\tilde{\mathcal{F}}^{ni}(\frac{\beta_B}{z},x_\perp)\mathcal{F}^n_i(\frac{\beta_B}{z},x_\perp)$$

DGLAP equation

$(k_\perp^2 < \mu^2)$

Rapidity factorization approach (small x)

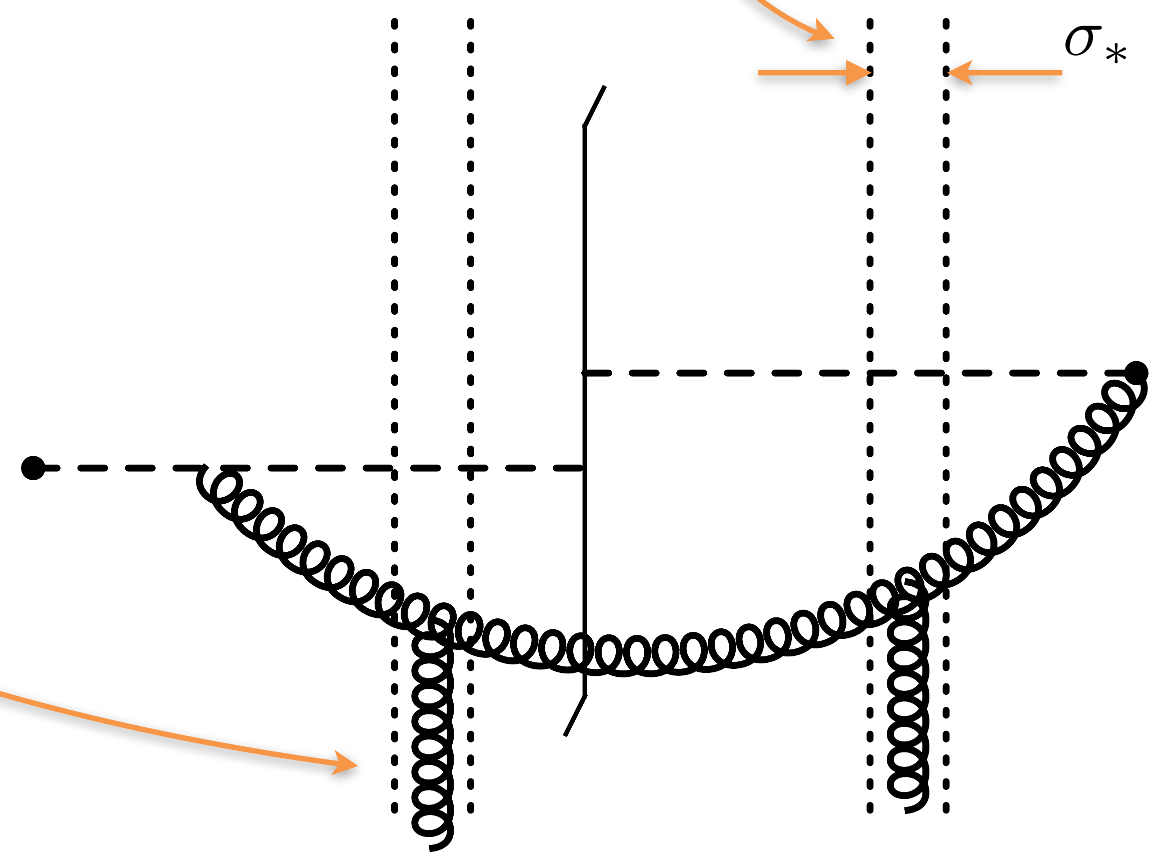


Separate slow $\alpha > \sigma'$

and fast fields $\alpha' < \sigma'$

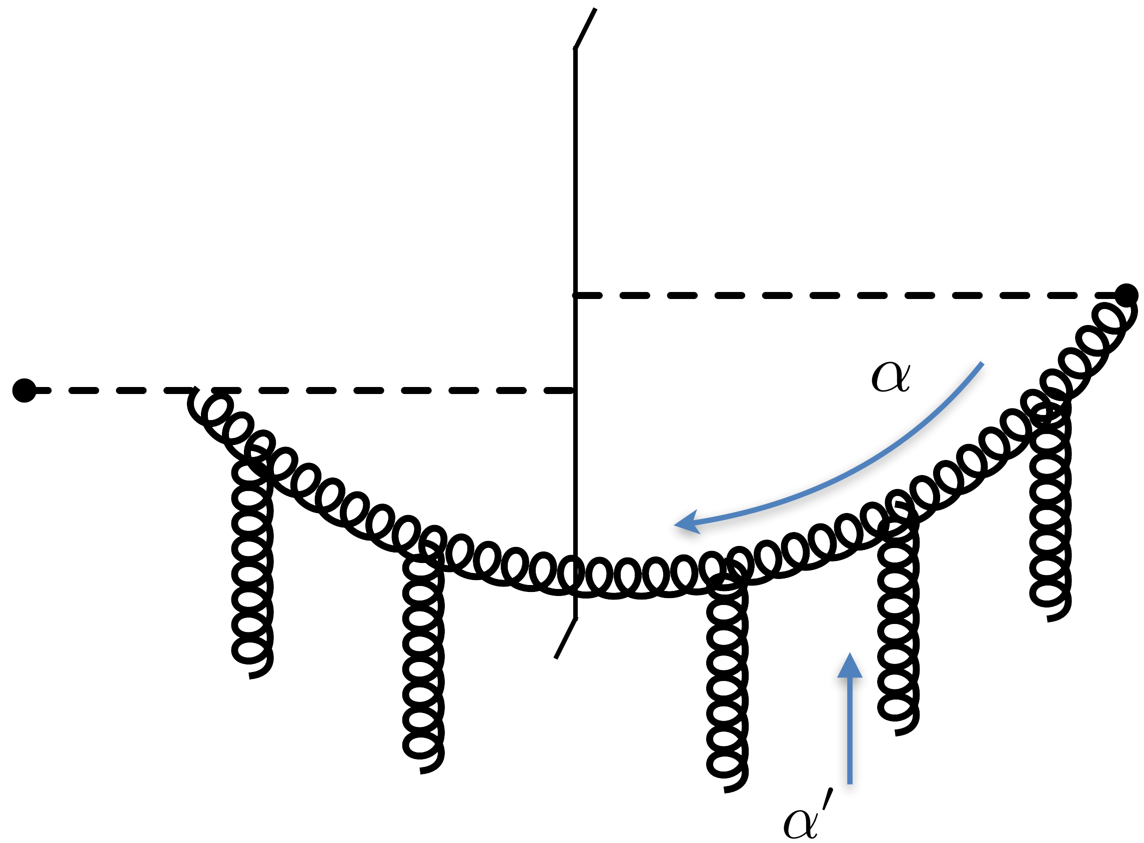
Fast fields form the shock-wave

To construct transition to moderate x we keep a finite width of the shock-wave



Interaction with external field is inside very thin shock-wave

Rapidity factorization approach. Gluon propagator



Parameter of expansion $\frac{p_\perp^2}{\alpha s}\sigma_* \ll 1$

No ordering in transverse momentum $p_\perp \sim l_\perp$

The gluon propagator has tortuous structure: nonlinear in F terms and quark contribution

Quark contribution

$$\lim_{k^2\to 0} k^2 \langle A_\mu^a(k) A_\nu^b(y)\rangle = -ie^{i\frac{k_\perp^2}{\alpha s}y_*} e^{-ik_\perp y_\perp} \mathcal{O}_{\mu\nu}^{ab}(\infty, y_*; k_\perp)|_{y_\perp}$$

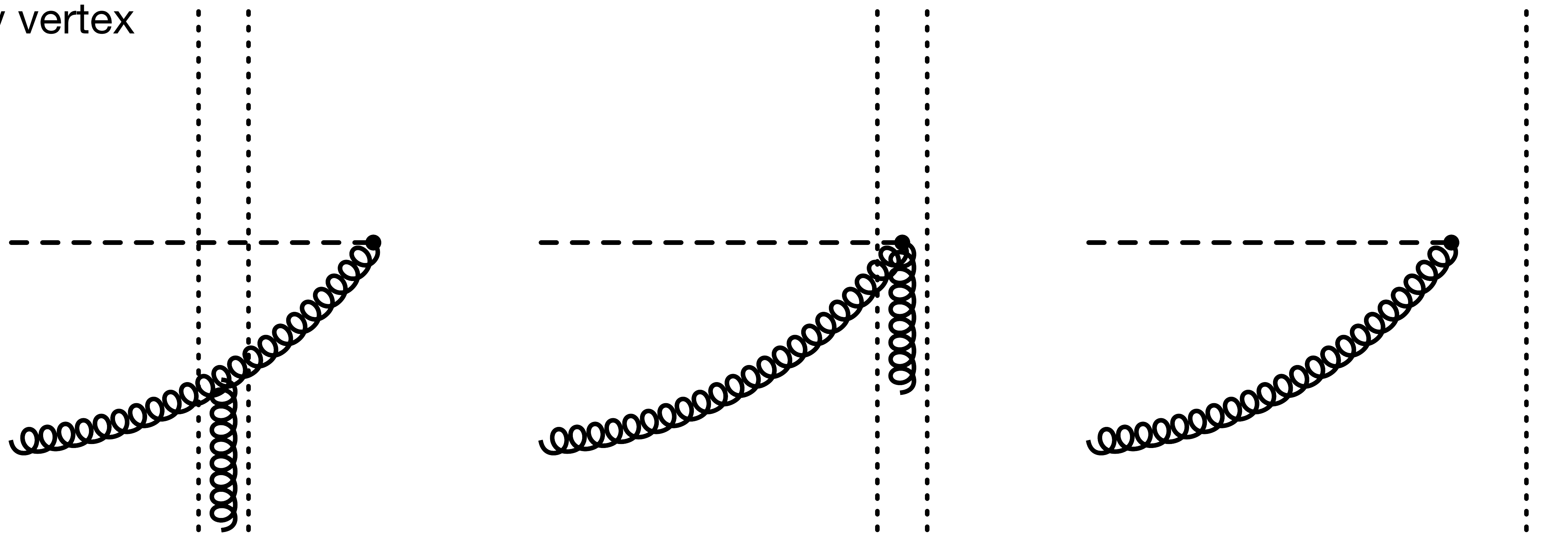
$$\mathcal{O}_{\mu\nu}^{ab}(\infty, y_*; k_\perp) = \mathcal{G}_{\mu\nu}^{ab}(\infty, y_*; k_\perp) + \mathcal{J}_{\mu\nu}^{ab}(\infty, y_*; k_\perp) + \bar{\mathcal{J}}_{\mu\nu}^{ab}(\infty, y_*; -k_\perp)$$

$$\mathcal{G}_{\mu\nu}(\infty, y_*, p_\perp) = g_{\mu\nu}[\infty, y_*] + \int_{y_*}^{\infty} dz_* \Big(-\frac{2i}{\alpha s^2}(z-y)_* g_{\mu\nu}\{2p^j[\infty, z_*]F_{\bullet j} - i[\infty, z_*]D^j F_{\bullet j}\} + \frac{4}{\alpha s^2}(p_{2\nu}\delta_\mu^j - p_{2\mu}\delta_\nu^j)[\infty, z_*]F_{\bullet j}\Big)[z_*, y_*]$$
$$+\frac{8}{\alpha s^3}\int_{y_*}^{\infty} dz_* \int_{y_*}^{z_*} dz'_*[ig_{\mu\nu}(z'-y)_* - \frac{2}{\alpha s}p_{2\mu}p_{2\nu}][\infty, z_*]F_{\bullet j}[z_*, z'_*]F_\bullet{}^j[z'_*, y_*]$$

Nonlinear terms

Rapidity factorization approach. Real emission

Lipatov vertex



We take into account position of the shock-wave

$U \equiv [\infty, -\infty]$

$$
\begin{aligned}
L^{ab}_{\mu i}(k, y_\perp, \beta_B) &= i \lim_{k^2\to 0} k^2 \langle A^a_\mu(k) \mathcal{F}^b_i(\beta_B, y_\perp)\rangle = 2e^{-ik_\perp y_\perp}\Big(\frac{p_{2\mu}}{\alpha s} - \frac{p_{1\mu}\alpha}{k_\perp^2}\Big)(\mathcal{F}_i(\beta_B, y_\perp) - i\partial_i U_{y_\perp} U^\dagger_{y_\perp})^{ab} \\
&+ (k_\perp| g_{\mu i}\Big\{\frac{\alpha\beta_B s}{\alpha\beta_B s + p_\perp^2} - U\frac{\alpha\beta_B s}{\alpha\beta_B s + p_\perp^2}U^\dagger\Big\} + 2\alpha p_{1\mu}\Big\{\frac{p_i}{\alpha\beta_B s + p_\perp^2} - U\frac{p_i}{\alpha\beta_B s + p_\perp^2}U^\dagger\Big\} - \frac{2ip_{1\mu}\alpha}{p_\perp^2}\partial_i U_{y_\perp} U^\dagger_{y_\perp} \\
&+ \Big\{\frac{2i}{\alpha s}p_{2\mu}\partial_i U - \frac{2i}{\alpha\beta_B s}\partial^\perp_\mu U p_i + \frac{2p_{2\mu}}{\beta_B\alpha^2 s^2}\partial_\perp^2 U p_i\Big\}\frac{\alpha\beta_B s}{\alpha\beta_B s + p_\perp^2}U^\dagger|y_\perp)^{ab}
\end{aligned}
$$

Takes into account dependence on $\beta_B \neq 0$

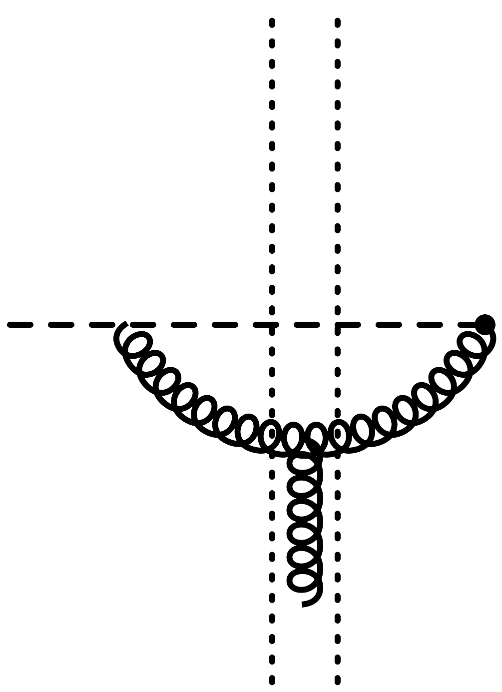
We apply the same strategy to calculation of the virtual correction

One loop correction for the whole region of transverse momentum

One can unify result obtained for $p_\perp \gg l_\perp$ (light-cone expansion) and $p_\perp \sim l_\perp$ (rapidity factorization approach)

$$\begin{aligned}
&L^{ab}_{\mu i}(k, y_\perp, \beta_B) = i \lim_{k^2\to 0} k^2 \langle A^a_\mu(k)\mathcal{F}^b_i(\beta_B, y_\perp)\rangle = \\
&\frac{2e^{-ik_\perp y_\perp}}{\alpha\beta_B s + k_\perp^2}\Big\{\frac{\alpha\beta_B s}{k_\perp^2}\Big(\frac{k_\perp^2}{\alpha s}p_{2\mu} - \alpha p_{1\mu}\Big)\delta_i^j - k_i\delta_\mu^j + \frac{\alpha\beta_B s g_{\mu i}k^j}{\alpha\beta_B s + k_\perp^2} + \frac{2\alpha k_i k^j}{\alpha\beta_B s + k_\perp^2}p_{1\mu}\Big\}\Big(\mathcal{F}^{ab}_j(\beta_B + \frac{k_\perp^2}{\alpha s}, y_\perp) - i\partial_j UU^{\dagger ab}\Big) \\
&+(k_\perp|\frac{\alpha\beta_B s g_{\mu i} + 2\alpha p_{1\mu}k_i}{\alpha\beta_B s + k_\perp^2}(2ik^j\partial_j U - \partial_\perp^2 U)\frac{1}{\alpha\beta_B s + p_\perp^2}U^\dagger + 2i\alpha p_{1\mu}\partial_i U\frac{1}{\alpha\beta_B s + p_\perp^2}U^\dagger - \frac{2ip_{1\mu}\alpha}{p_\perp^2}\partial_i U_{y_\perp}U^\dagger_{y_\perp} \\
&+\Big\{\frac{2i}{\alpha s}p_{2\mu}\partial_i U - \frac{2i}{\alpha\beta_B s}\partial^\perp_\mu U p_i + \frac{2p_{2\mu}}{\beta_B\alpha^2 s^2}\partial_\perp^2 U p_i\Big\}\frac{\alpha\beta_B s}{\alpha\beta_B s + p_\perp^2}U^\dagger|y_\perp)^{ab}
\end{aligned}$$

Virtual emission:



$$\begin{aligned}
\mathcal{F}^n_i(\beta_B, y_\perp) &= \frac{2}{s}\int_{-\infty}^{\infty} dy_* e^{i\beta_B y_*}\langle[\infty, y_*]^{nm}_y gF^m_{\bullet i}(y_*, y_\perp)\rangle \\
&= -if^{nkl}\int_{\sigma'}^{\sigma}\frac{d\alpha}{\alpha}\Big\{(y_\perp|\frac{1}{p_\perp^2}\Big(\alpha\beta_B s i\partial_i U + \partial_\perp^2 U p_i\Big)\frac{1}{\alpha\beta_B s + p_\perp^2}U^\dagger|y_\perp)^{kl} \\
&+(y_\perp|\frac{\alpha\beta_B s}{p_\perp^2(\alpha\beta_B s + p_\perp^2)}|y_\perp)\Big(\mathcal{F}_i(\beta_B, y_\perp) - i\partial_i U_{y_\perp}U^\dagger_{y_\perp}\Big)^{kl}\Big\}
\end{aligned}$$

- Valid in a full range of $p_\perp$
- Takes into account dependence on $\beta_B \neq 0$

Evolution equation of gluon TMD

Nonlinear equation

$$
\begin{aligned}
&\frac{d}{d\ln\sigma}\langle\tilde{\mathcal{F}}^{ai}(\beta_B,x_\perp)\mathcal{F}^a_i(\beta_B,y_\perp)\rangle = 2\alpha_s Tr\Big\{\\
&(x_\perp| -\frac{1}{2}\Big[\tilde{U}\frac{p_\perp^2 g^{ik}+2p^ip^k}{\sigma\beta_B s+p_\perp^2}\tilde{U}^\dagger-(\tilde{U}\to 1)\Big]\Big[U\frac{p_\perp^2 g_{ik}+2p_ip_k}{\sigma\beta_B s+p_\perp^2}U^\dagger-(U\to 1)\Big]\\
&+\breve{\tilde{\mathcal{F}}}_j(\beta_B+\frac{p_\perp^2}{\sigma s})\Big[\frac{p^ig^{jk}+p^kg^{ij}}{\sigma\beta_B s+p_\perp^2}+\frac{2p^ip^jp^k}{(\sigma\beta_B s+p_\perp^2)^2}\Big]\Big[U\frac{p_\perp^2 g_{ik}+2p_ip_k}{\sigma\beta_B s+p_\perp^2}U^\dagger-(U\to 1)\Big]\\
&+\Big[\tilde{U}\frac{p_\perp^2 g^{ik}+2p^ip^k}{\sigma\beta_B s+p_\perp^2}\tilde{U}^\dagger-(\tilde{U}\to 1)\Big]\Big[\frac{p_ig_{jk}+p_kg_{ij}}{\sigma\beta_B s+p_\perp^2}+\frac{2p_ip_jp_k}{(\sigma\beta_B s+p_\perp^2)^2}\Big]\breve{\mathcal{F}}^j(\beta_B+\frac{p_\perp^2}{\sigma s})\\
&+2\breve{\tilde{\mathcal{F}}}_j(\beta_B+\frac{p_\perp^2}{\sigma s})\Big[\frac{2p_\perp^2g^{jk}}{(\sigma\beta_B s+p_\perp^2)^2}+\frac{4p^jp^kp_\perp^2}{(\sigma\beta_B s+p_\perp^2)^3}-\frac{2p_\perp^4p^jp^k}{(\sigma\beta_B s+p_\perp^2)^4}\Big]\breve{\mathcal{F}}_k(\beta_B+\frac{p_\perp^2}{\sigma s})\\
&-\tilde{\mathcal{F}}^i(\beta_B+\frac{p_\perp^2}{\sigma s})\frac{p^j}{p_\perp^2}\Big[U\frac{p_\perp^2 g_{ij}+2p_ip_j}{\sigma\beta_B s+p_\perp^2}U^\dagger-(U\to 1)\Big]-\Big[\tilde{U}\frac{p_\perp^2 g^{ij}+2p^ip^j}{\sigma\beta_B s+p_\perp^2}\tilde{U}^\dagger-(\tilde{U}\to 1)\Big]\frac{p_j}{p_\perp^2}\mathcal{F}_i(\beta_B+\frac{p_\perp^2}{\sigma s})\\
&-2\tilde{\mathcal{F}}_i(\beta_B+\frac{p_\perp^2}{\sigma s})\Big[\frac{g^{ik}}{\sigma\beta_B s+p_\perp^2}+\frac{p^ip^k}{(\sigma\beta_B s+p_\perp^2)^2}\Big]\breve{\mathcal{F}}_k(\beta_B+\frac{p_\perp^2}{\sigma s})\\
&-2\breve{\tilde{\mathcal{F}}}_j(\beta_B+\frac{p_\perp^2}{\sigma s})\Big[\frac{g^{ij}}{\sigma\beta_B s+p_\perp^2}+\frac{p^ip^j}{(\sigma\beta_B s+p_\perp^2)^2}\Big]\mathcal{F}_i(\beta_B+\frac{p_\perp^2}{\sigma s})|y_\perp)\\
&+\tilde{\mathcal{F}}^i(\beta_B,x_\perp)\Big[(y_\perp|\frac{p^j}{p_\perp^2}U\frac{p_\perp^2 g_{ij}+2p_ip_j}{\sigma\beta_B s+p_\perp^2}U^\dagger|y_\perp)-(y_\perp|\frac{1}{\sigma\beta_B s+p_\perp^2}|y_\perp)U_i(y_\perp)\Big]\\
&+\Big[(x_\perp|\tilde{U}\frac{2p_ip_j+p_\perp^2 g_{ij}}{\sigma\beta_B s+p_\perp^2}\tilde{U}^\dagger\frac{p^j}{p_\perp^2}|x_\perp)-(x_\perp|\frac{1}{\sigma\beta_B s+p_\perp^2}|x_\perp)\tilde{U}_i(x_\perp)\Big]\mathcal{F}^i(\beta_B,y_\perp)\\
&+2\int\frac{d^2p_\perp}{p_\perp^2}e^{ip_\perp(x-y)_\perp}\tilde{\mathcal{F}}^i(\beta_B+\frac{p_\perp^2}{\sigma s},x_\perp)\mathcal{F}_i(\beta_B+\frac{p_\perp^2}{\sigma s},y_\perp)-2\tilde{\mathcal{F}}^i(\beta_B,x_\perp)\mathcal{F}_i(\beta_B,y_\perp)\int\frac{d^2p_\perp}{p_\perp^2}\frac{\sigma\beta_B s}{\sigma\beta_B s+p_\perp^2}
\end{aligned}
$$

- Valid for the whole range of σ and β_B
- No infrared divergency
- Real emission part contains kinematical constraint $p_\perp^2 < \sigma(1-\beta_B)s$

DGLAP limit

$$\beta_B = x_B \sim 1$$

$$k_\perp^2 \sim (x-y)_\perp^{-2} \sim s$$

$$\sigma \lesssim 1$$

$$\frac{d}{d\ln\sigma}\langle\tilde{\mathcal{F}}^{ni}(\beta_B,x_\perp)\mathcal{F}_i^n(\beta_B,y_\perp)\rangle = 2N_c\int d^2k_\perp\Big\{e^{ik_\perp(x_\perp-y_\perp)}\Big[\frac{g^{jk}}{k_\perp^2}-\frac{2g^{jk}}{\sigma\beta_B s+k_\perp^2}+\frac{2k_\perp^2 g^{jk}}{(\sigma\beta_B s+k_\perp^2)^2}-\frac{2k^j k^k}{(\sigma\beta_B s+k_\perp^2)^2}$$
$$+\frac{4k_\perp^2 k^j k^k}{(\sigma\beta_B s+k_\perp^2)^3}-\frac{2k_\perp^4 k^j k^k}{(\sigma\beta_B s+k_\perp^2)^4}\Big]\tilde{\mathcal{F}}_k^n\Big(\beta_B+\frac{k_\perp^2}{\sigma s},x_\perp\Big)\mathcal{F}_j^n\Big(\beta_B+\frac{k_\perp^2}{\sigma s},y_\perp\Big)-\frac{\sigma\beta_B s}{k_\perp^2(\sigma\beta_B s+k_\perp^2)}\tilde{\mathcal{F}}^{ni}(\beta_B,x_\perp)\mathcal{F}_i^n(\beta_B,y_\perp)\Big\}$$

- Evolution equation is linear
- In the collinear case reproduce DGLAP

Small x limit

$$\beta_B = x_B \sim \frac{(x-y)_\perp^{-2}}{s}$$

$$k_\perp^2 \sim (x-y)_\perp^{-2} \ll s$$

$$\frac{(x-y)_\perp^{-2}}{s} \ll \sigma \ll 1$$

$$\frac{d}{d\ln\sigma}\langle \tilde{U}^{ai}(x_\perp)U_i^a(y_\perp)\rangle = -4\alpha_s Tr\Big\{(x_\perp|\tilde{U}p_i\tilde{U}^\dagger\Big(\tilde{U}\frac{p_j}{p_\perp^2}\tilde{U}^\dagger - \frac{p_j}{p_\perp^2}\Big)\Big(U\frac{p^j}{p_\perp^2}U^\dagger - \frac{p^j}{p_\perp^2}\Big)Up^iU^\dagger|y_\perp)$$
$$-\tilde{U}^i(x_\perp)\Big[(y_\perp|\frac{p^j}{p_\perp^2}U\frac{p_ip_j}{p_\perp^2}U^\dagger|y_\perp) - \frac{1}{2}(y_\perp|\frac{1}{p_\perp^2}|y_\perp)U_i(y_\perp)\Big] - \Big[(x_\perp|\tilde{U}\frac{p_ip_j}{p_\perp^2}\tilde{U}^\dagger\frac{p^j}{p_\perp^2}|x_\perp) - \frac{1}{2}(x_\perp|\frac{1}{p_\perp^2}|x_\perp)\tilde{U}_i(x_\perp)\Big]U^i(y_\perp)\Big\}$$

- Nonlinear equation

$$\frac{d}{d\ln\sigma}\langle \tilde{U}_i^a(x_\perp)U_i^a(y_\perp)\rangle = -\frac{g^2}{8\pi^3}\int d^2z_\perp Tr\Big\{(-i\overrightarrow{\partial}_i^x + \tilde{U}_i^x)\Big(\tilde{U}_x\tilde{U}_z^\dagger - 1\Big)\frac{(x-y)_\perp^2}{(x-z)_\perp^2(z-y)_\perp^2}\Big(U_zU_y^\dagger - 1\Big)(i\overleftarrow{\partial}_i^y + U_i^y)\Big\}$$

I.Balitsky, A.T. (2014)

$$U_i \equiv i\partial_i U U^\dagger$$

Sudakov limit

$$\beta_B = x_B \sim 1$$

$$k_\perp^2 \sim (x-y)_\perp^{-2} \sim few\ GeV^2$$

$$\sigma \lesssim 1$$

No restriction for the virtual emission

$$\frac{d}{d\ln\sigma}\tilde{\mathcal{F}}^{ai}(x_\perp,\beta_B)\mathcal{F}_i^a(y_\perp,\beta_B) = -\frac{g^2N_c}{\pi}\int\frac{d^2p_\perp}{p_\perp^2}\left[1-e^{i(p,x-y)_\perp}\right]\tilde{\mathcal{F}}^{ai}(x_\perp,\beta_B)\mathcal{F}_i^a(y_\perp,\beta_B)$$

Real emission is restricted to the small $p_\perp$ region

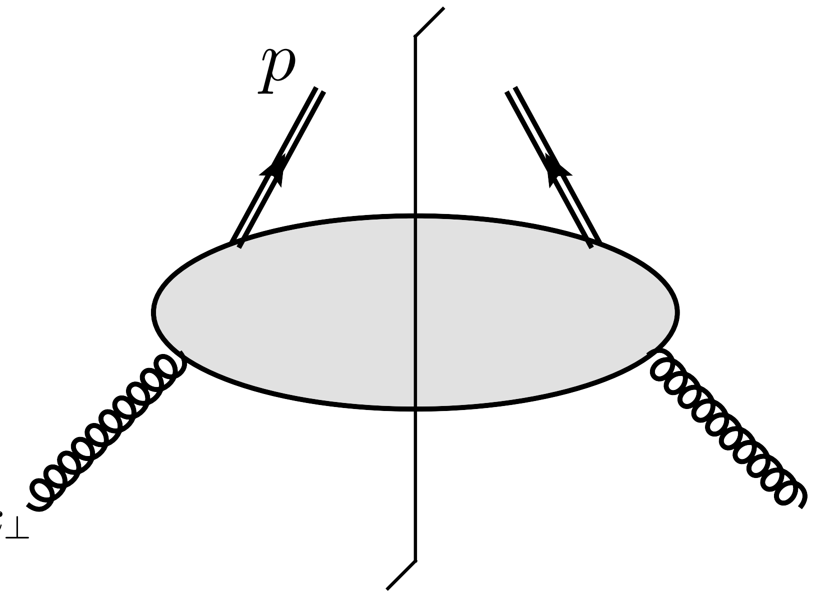
$$\mathcal{D}(\beta_B,k_\perp,\ln\sigma) \sim \exp\left\{-\frac{\alpha_sN_c}{2\pi}\ln^2\frac{\sigma s}{k_\perp^2}\right\}\mathcal{D}(\beta_B,k_\perp,\ln\frac{k_\perp^2}{s})$$

Fragmentation function

$$\alpha_s \mathcal{D}_f(x_F, z_\perp) = -\frac{z_F}{8\pi^2(p\cdot n)}\int du\, e^{i\frac{1}{z_F}(p\cdot n)u}\sum_X \langle 0|\tilde{\mathcal{F}}^a_i(z_\perp+un)|X+p\rangle\langle X+p|\mathcal{F}^{ai}(0)|0\rangle$$

$$\langle 0|U|p+X\rangle = 0$$

For fragmentation function non-linearity is lost



$k = p/z_F, k_\perp$

$$\frac{d}{d\ln\sigma}\langle\tilde{\mathcal{F}}^{ai}(-\beta_F,x_\perp)\mathcal{F}^a_i(-\beta_F,y_\perp)\rangle = 4\alpha_s N_c\int d^2p_\perp\Big\{\theta(\beta_F - \frac{p_\perp^2}{\sigma s} - 1)e^{ip_\perp(x-y)_\perp}$$
$$\Big[\frac{g^{ij}}{p_\perp^2} + \frac{2g^{ij}}{\sigma\beta_F s - p_\perp^2} - \frac{2p^ip^j - 2p_\perp^2 g^{ij}}{(\sigma\beta_F s-p_\perp^2)^2} - \frac{4p_\perp^2 p^ip^j}{(\sigma\beta_F s-p_\perp^2)^3} - \frac{2p^ip^jp_\perp^4}{(\sigma\beta_F s-p_\perp^2)^4}\Big]\tilde{\mathcal{F}}^a_i(-\beta_F+\frac{p_\perp^2}{\sigma s},x_\perp)\mathcal{F}^a_j(-\beta_F+\frac{p_\perp^2}{\sigma s},y_\perp)$$
$$-\frac{\theta(\sigma\beta_F s - p_\perp^2)}{p_\perp^2}\tilde{\mathcal{F}}^a_i(-\beta_F,x_\perp)\mathcal{F}^{ia}(-\beta_F,y_\perp)\Big\}$$

collinear case

$$\frac{d}{d\ln\sigma}\langle\tilde{\mathcal{F}}^{ai}(-\frac{1}{z_F},x_\perp)\mathcal{F}^a_i(-\frac{1}{z_F},x_\perp)\rangle = \frac{\alpha_s N_c}{\pi}\int_{z_F}^1\frac{d\rho}{\rho^3}\Big[\frac{1}{\rho(1-\rho)_+} - 2+\rho(1-\rho)\Big]\tilde{\mathcal{F}}^a_i(-\frac{\rho}{z_F},x_\perp)\mathcal{F}^{ia}(-\frac{\rho}{z_F},x_\perp)$$

Conclusion

- There is a difference between small and moderate x even at the level of definitions
- We’ve use light-cone expansion at moderate x and rapidity factorization at small x
- We’ve constructed evolution equation which is valid in both regions
- The equation reproduces different limits (BK, DGLAP and Sudakov)
- The equation is linear at moderate and non-linear at small x
- However evolution of the gluon fragmentation function is linear