# $\Lambda(1405)$ from Lattice QCD

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Graduate Research Fellowship Program

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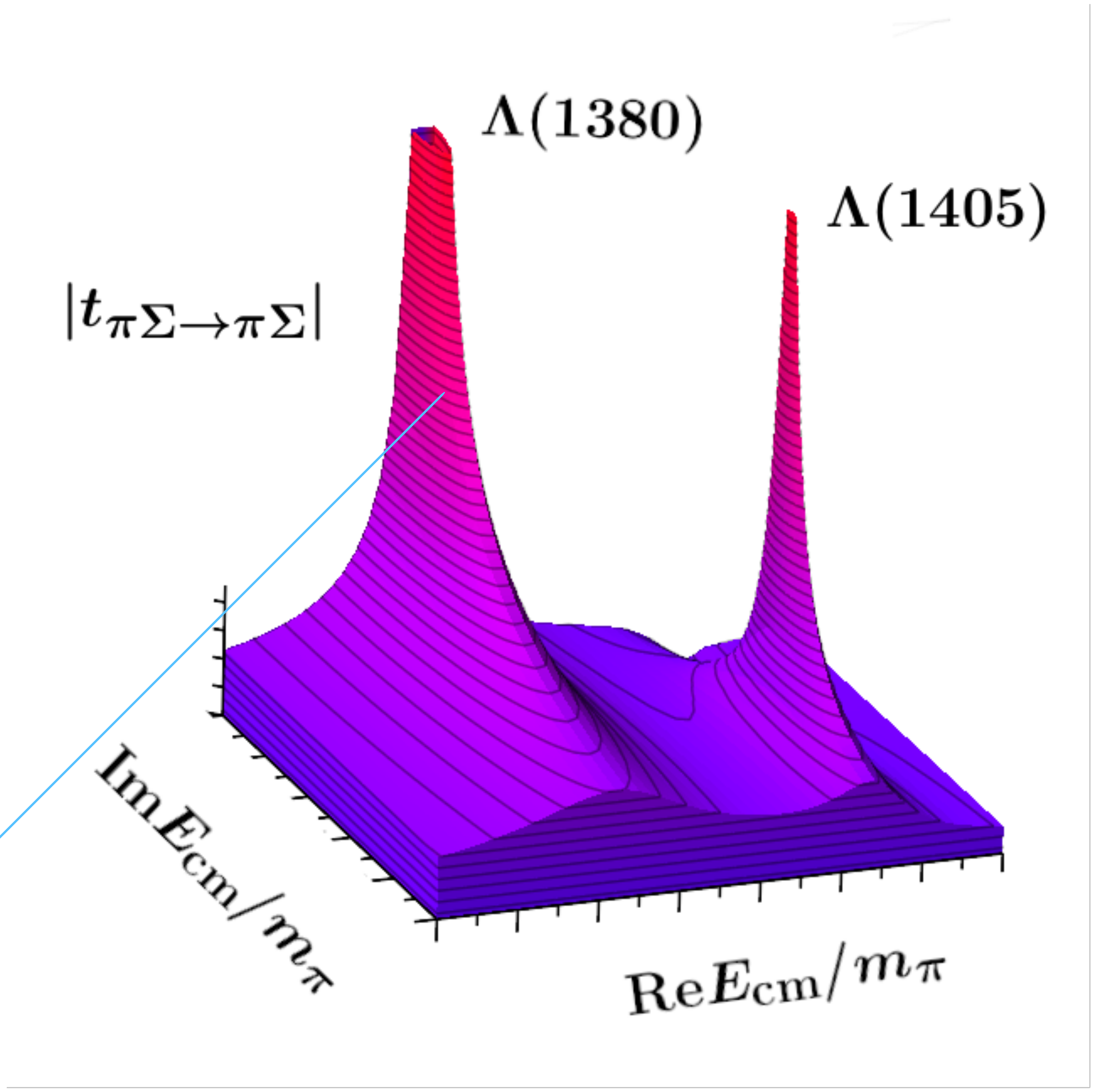
# $\Lambda(1405)$ from Lattice QCD

Outline

- **Nature of the Lambda (1405)**
- Lattice QCD
- Resonance Analysis
- Conclusions and Outlook




Phys. Rev. Lett. **132**, 051901 [arXiv:2307.10413]
Phys. Rev. D **109**, 014511 [arXiv:2307.13471]

# Nature of the $\Lambda(1405)$

Theoretical Prediction in 1959 by Dalitz and Tuan

Resonance Study in $K^{-}p \rightarrow \pi\Sigma$ Amplitude

[Dalitz & Tuan, PRL **2** (1959) 425]

+

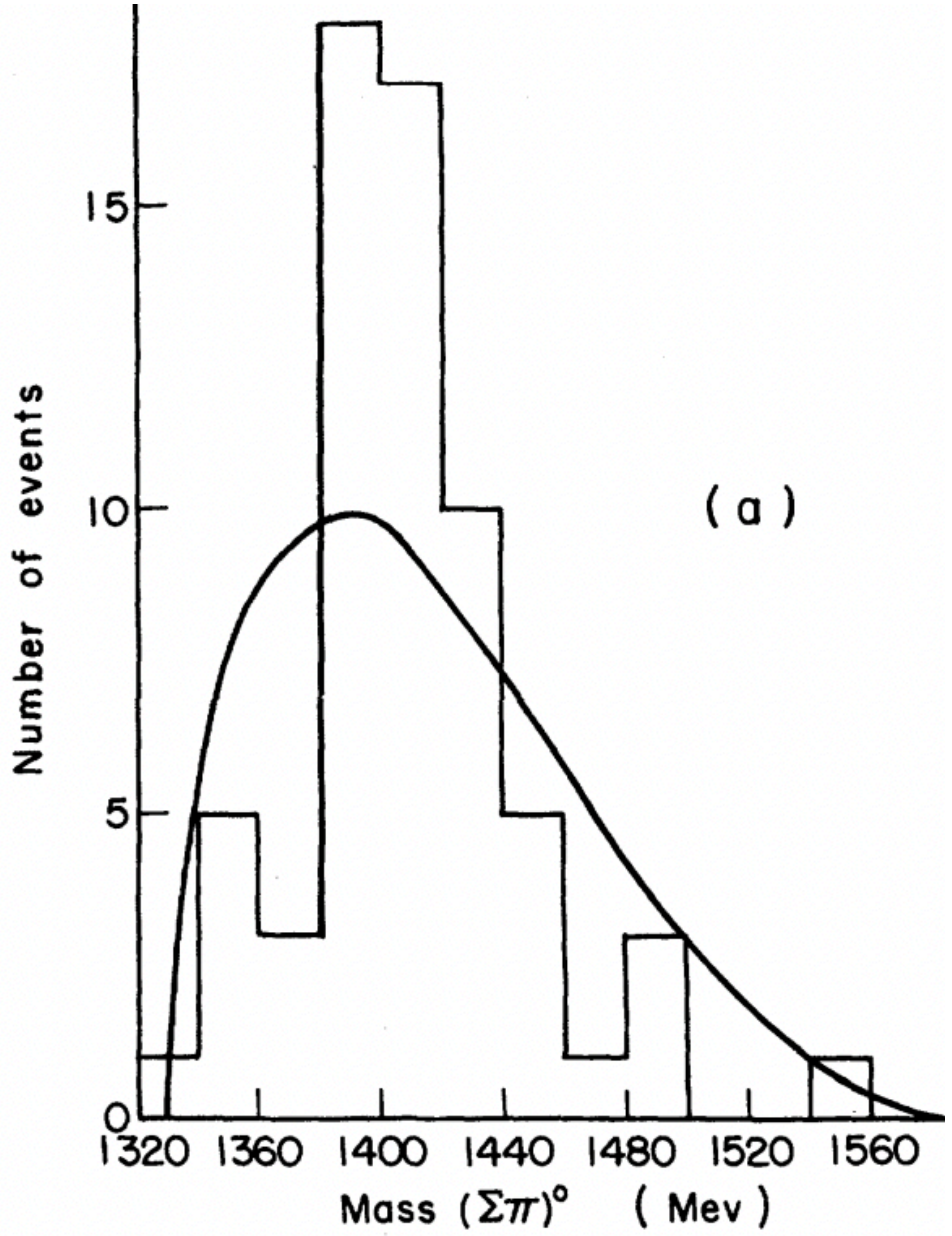
Experimental Evidence of Resonance

Enhancement in $\pi\Sigma$ mass spectrum in bubble chambers

[Alston et al., PRL **6** (1961) 698]

$$\Lambda(1405), I = 0, J^P = \frac{1}{2}^-$$

[Alston et al., PRL **6** (1961) 698]



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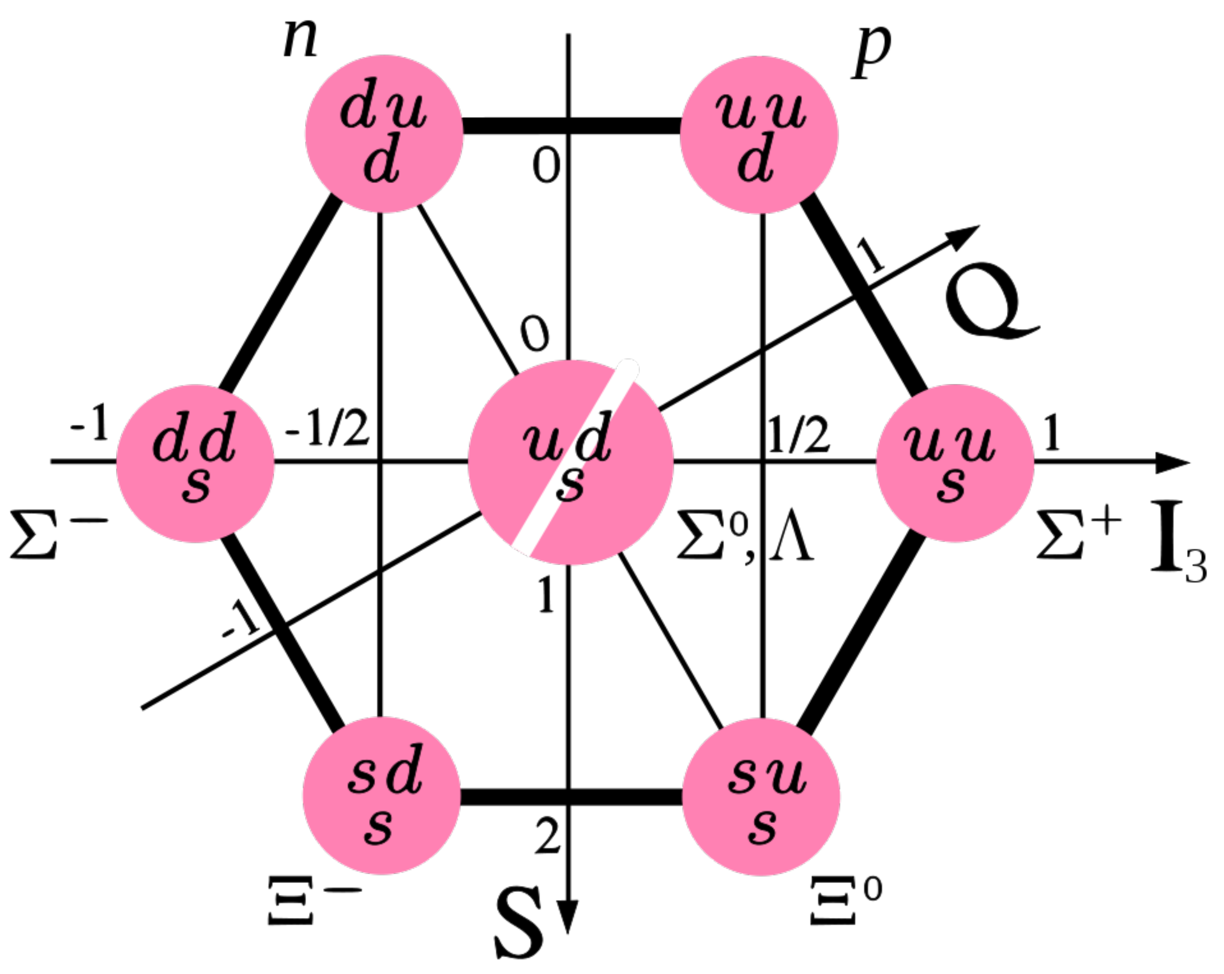
# Nature of the Λ(1405)

**Prediction** [Dalitz & Tuan, PRL **2** (1959) 425]

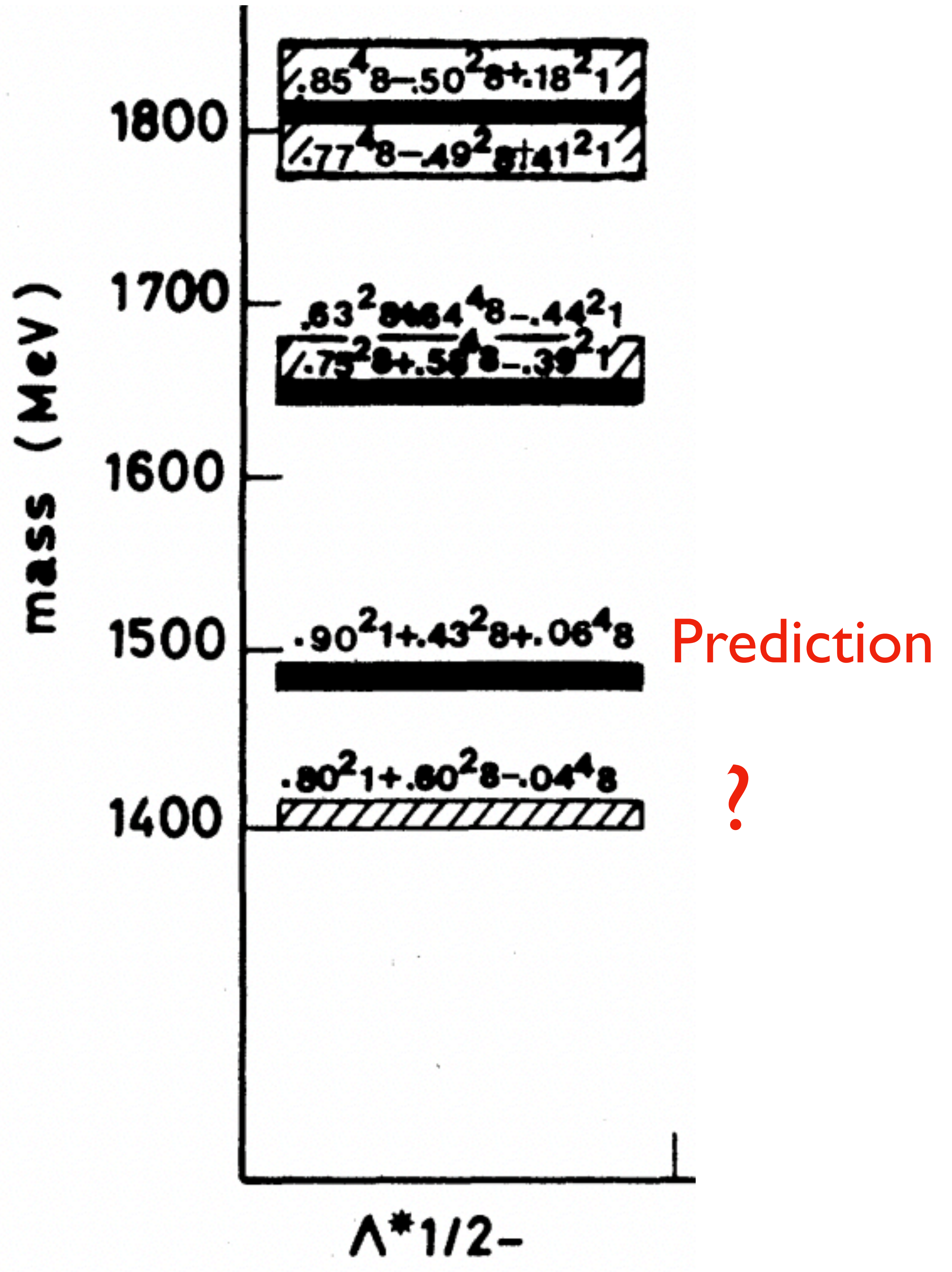
**Evidence** [Alston et al., PRL **6** (1961) 698]

Negative Parity Baryons in  Quark Model
Isgur & Karl

Λ mass low compared to prediction from QCD




[Isgur & Karl, PRD **18** (1978) 4187]



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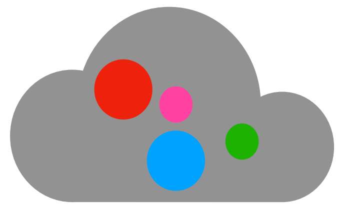
# Nature of the $\Lambda(1405)$

Prediction [Dalitz & Tuan, PRL **2** (1959) 425]

Evidence [Alston et al., PRL **6** (1961) 698]

Quark Model
[Isgur & Karl, PRD **18** (1978) 4187]

Cloudy-Bag Chiral Model
[Fink et al., PRC **41** (1990) 2720]

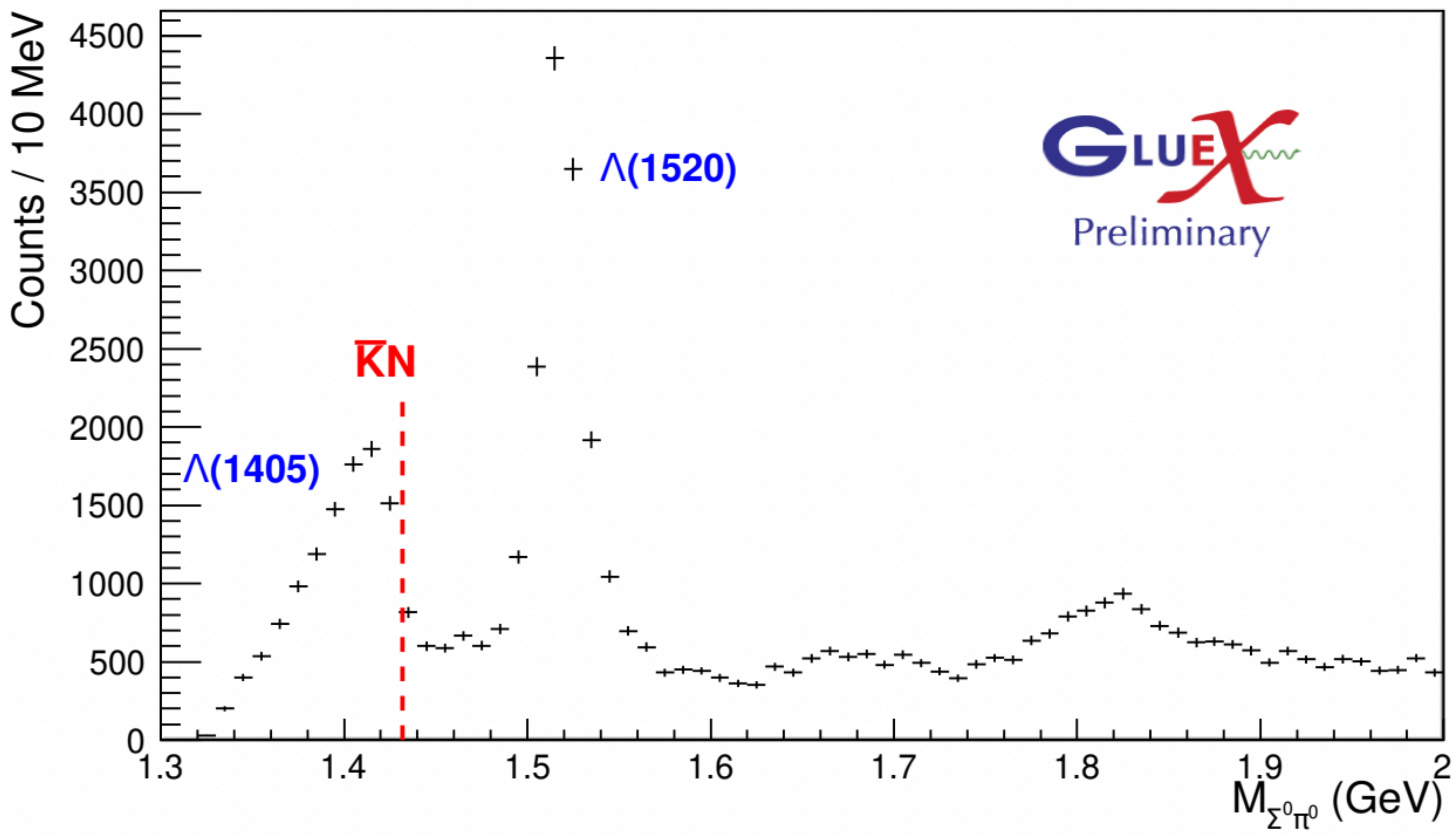


Chiral Coupled-Channel
[Oset & Ramos, NPA **635** (1997) 99]

SIDDHARTHA at DAΦNE: $K^{-}p$ Scattering Length
[Bazzi et al., PLB **704** (2011) 113]

Spin & Parity Measured @ CLAS. $J^P = \frac{1}{2}^-$
[Moriya et al., PRC **87** (2013) 035206]

[Wickramaarachchi et al., 2209.06230]



# Nature of the $\Lambda(1405)$

## One or Two Resonances?

PDG 2020

| | | |
|---|---|---|
| $\Lambda$ | $1/2^+$ | **** |
| $\Lambda(1380)$ | $1/2^-$ | ** |
| $\Lambda(1405)$ | $1/2^-$ | **** |
| $\Lambda(1520)$ | $3/2^-$ | **** |
| $\Lambda(1600)$ | $1/2^+$ | **** |
| $\Lambda(1670)$ | $1/2^-$ | **** |
| $\Lambda(1690)$ | $3/2^-$ | **** |

| | |
|---|---|
| * * ** | Existence is certain. |
| * * * | Existence is very likely. |
| ** | Evidence of existence is fair. |
| * | Evidence of existence is poor. |

### Experiment

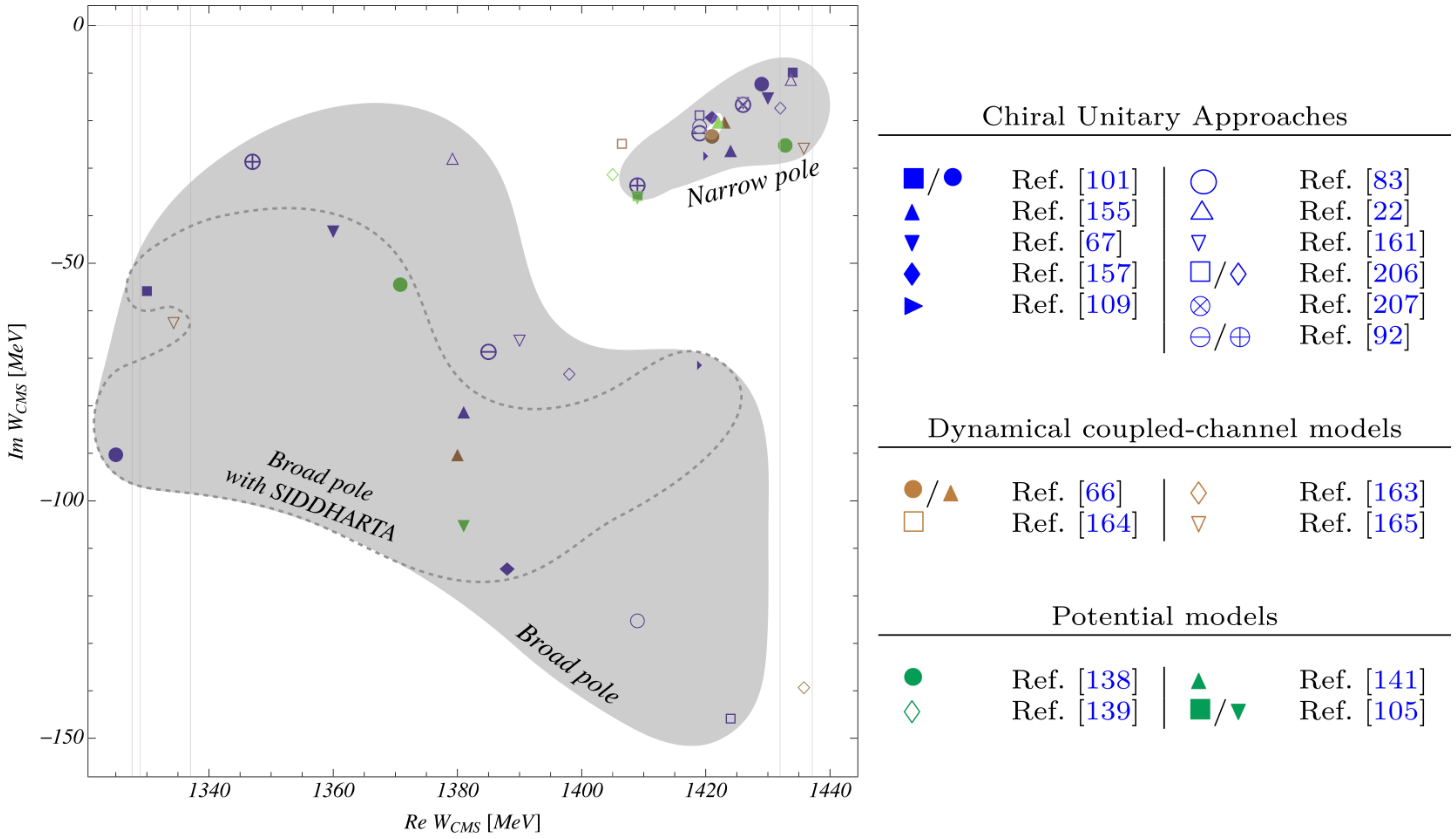
- J-PARC consistent w/ one pole
  [Aikawa et al., PLB, **837** (2023)137637]
- Multi-experiment analysis w/ one pole
  [Anisovich et al., EPJA **56** (2020)56:139]
- BGOOD & ALICE w/ two poles
  [Scheluchin et al., PLB **833** (2022)137375]
  [Acharya et al., EPJC **83** (2023)340]
- Different CLAS analysis w/ two poles
  [Mai & Meißner, EPJA **51**(2015)30]
  [Roca & Oset, PRC **87**(2013)055201]
- GlueX analysis w/ two poles
  [Wickramaarachchi et al., 2209.06230]

### Theory

- Simple SU(3) quark model w/ one pole
  [Isgur & Karl, PRD **18** (1978) 4187]
- Bag Model with Chirality w/ two poles
  [Fink et al., PRC **41** (1990) 2720]
- Chiral Unitarity approach w/ two poles
  [Mai, Eur. Phys. J. **230** (2021)10.1140]

# Nature of the $\Lambda(1405)$

[Mai, Eur. Phys. J. **230** (2021)10.1140]



# Nature of the $\Lambda(1405)$

## Lattice QCD?

- Lattice QCD Studies, none coupled-channel

Single-baryon three-quark fields

[Gubler et al., PRD **94** (2016)114518]

[Menadue et al., PRL **108** (2012)112001]

[Engel et al., PRD **87** (2013) 034502]

[Hall et al., *PRL* **114** (2015) 132002]

[Nakajima et al., *AIP* **594** (2001) 349]

[Nemoto et al., *NPA* **721** (2003) 879]

→

Insufficient to determine the Finite-Volume Spectrum!

[Lang & Verduci, PRD **87** (2013) 054502]
[Mohler et al., PRD **87** (2013) 034501]
[Wilson et al., PRD **92** (2015) 094502]

$N\pi$

[Bulava et al., BaSC, Nuc.Phys.B, 251402328]

$KN - \Sigma\pi$

[Bulava et al., BaSc, PRL **132** (2024) 5]
[Bulava et al., BaSc, PRD **109** (2024) 1]

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# Λ(1405) from Lattice QCD

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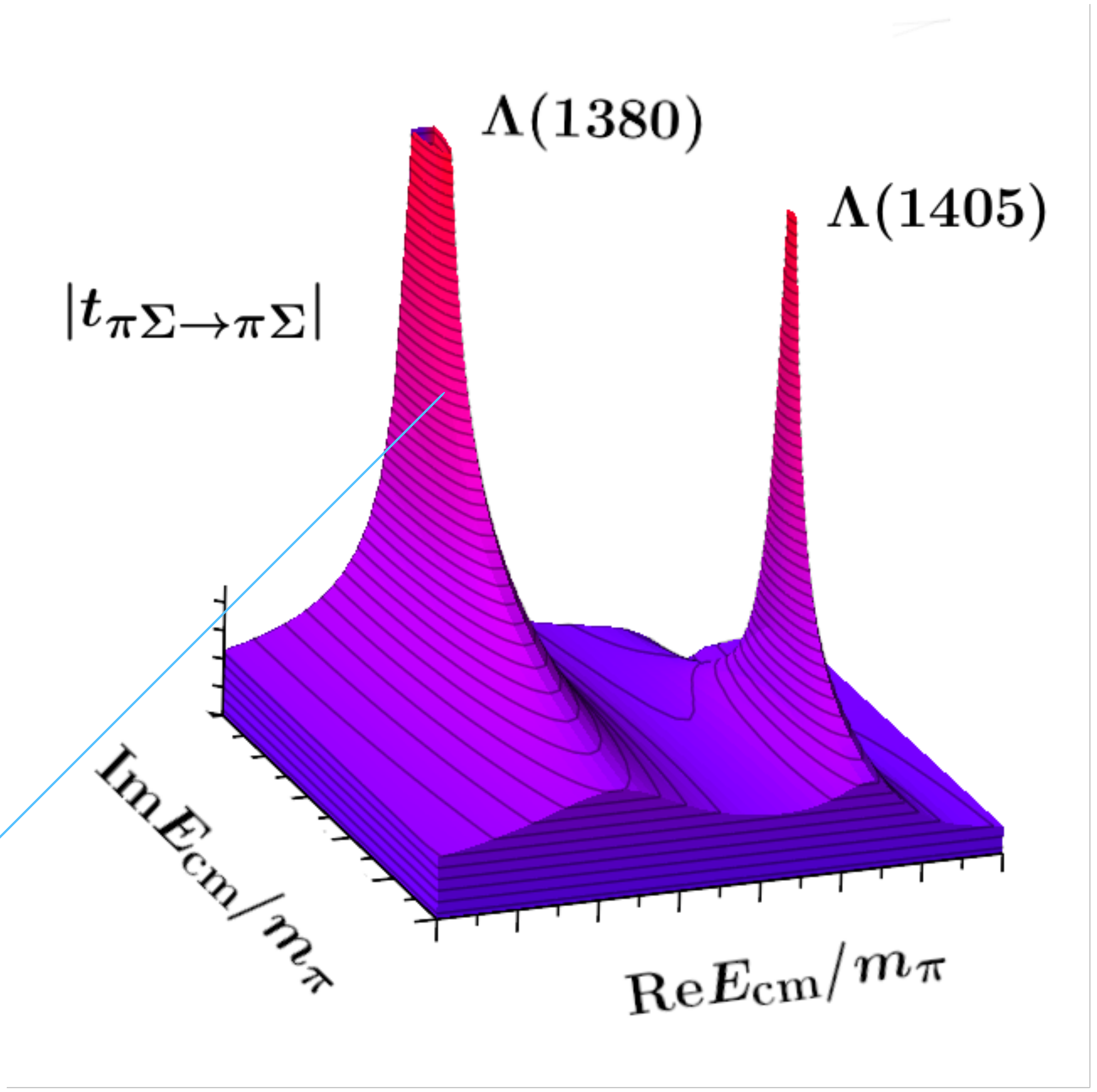
- Nature of the Lambda (1405)
- **Lattice QCD**
- Resonance Analysis
- Conclusions and Outlook

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# Lattice QCD

- Discrete, Euclidean spacetime lattice:

$$L,\ m_\pi\ ,\ a$$

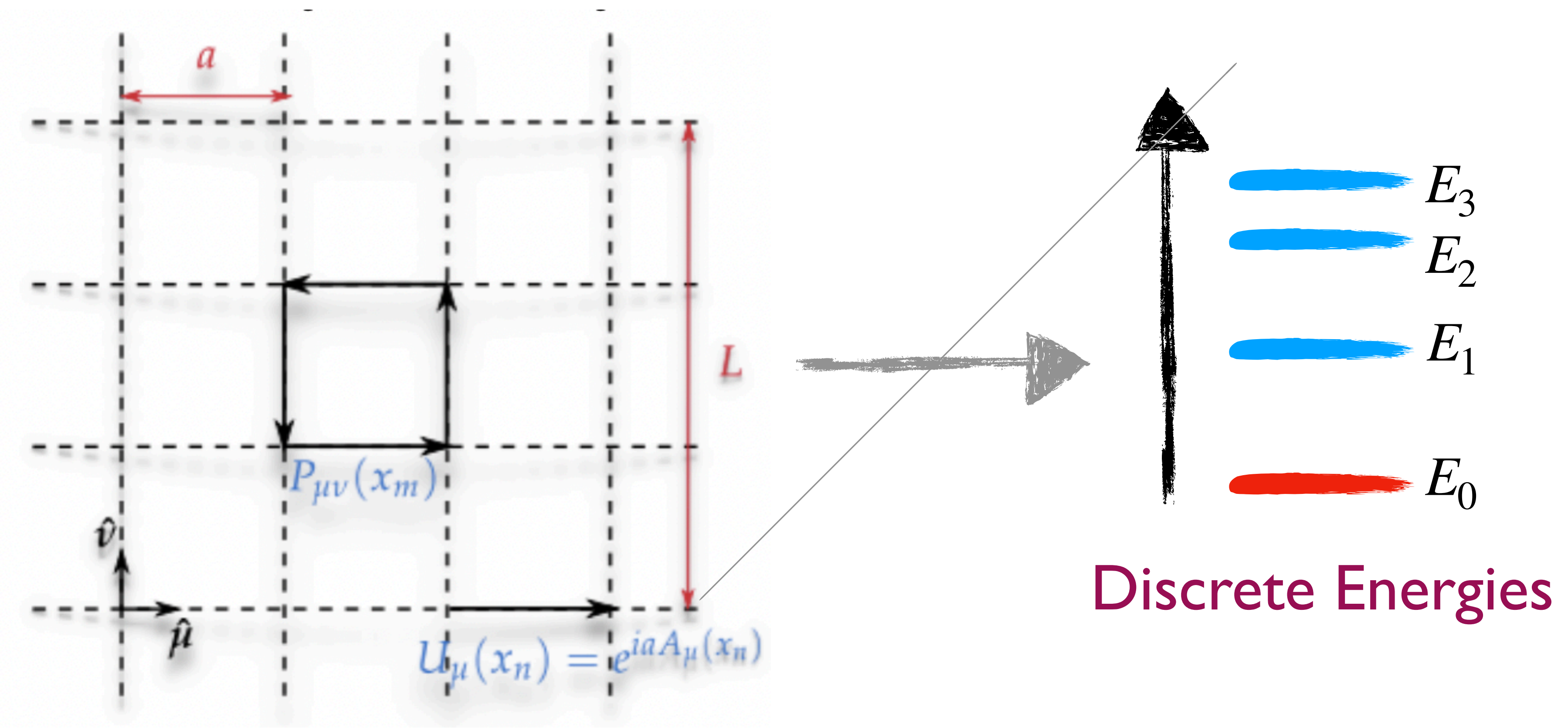
- Calculate correlation functions using Monte Carlo

$$C_L(t) = \langle O(t) O^\dagger(0) \rangle \longrightarrow \int dU\, e^{-S_g} \det K$$

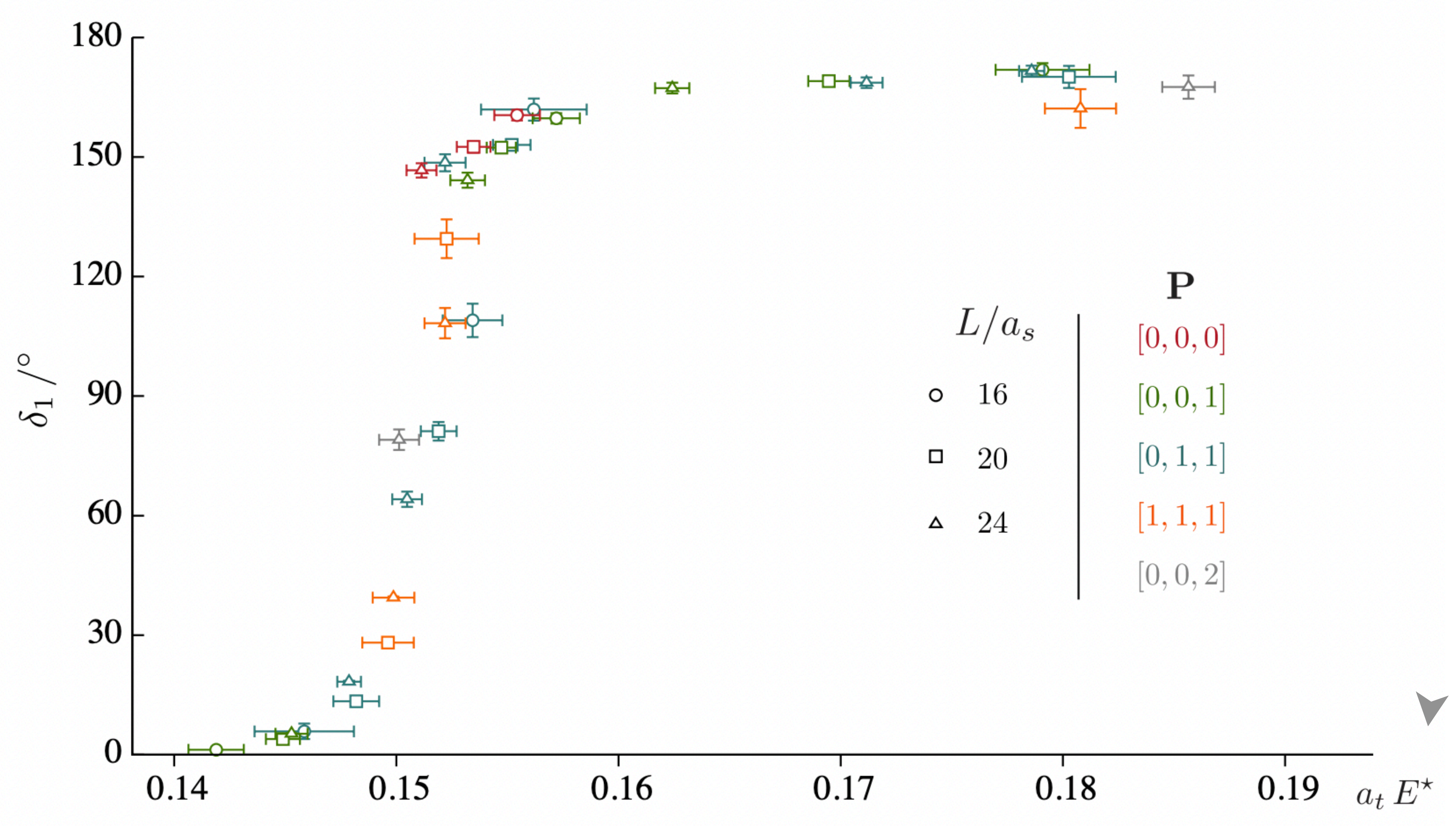
$$\sum_n |n\rangle\langle n|$$

$$C_L(t) = \sum_{\mathbf{n}} Z_n Z_n^\dagger e^{-E_n t} \xrightarrow[t \to \infty]{} e^{-E_0 t}$$

- Extract finite-volume spectrum and map to physical observables
  Hadron Masses
  Matrix Elements
  Scattering Amplitudes




[Dudek et al., PRD **87** (2014)034505]



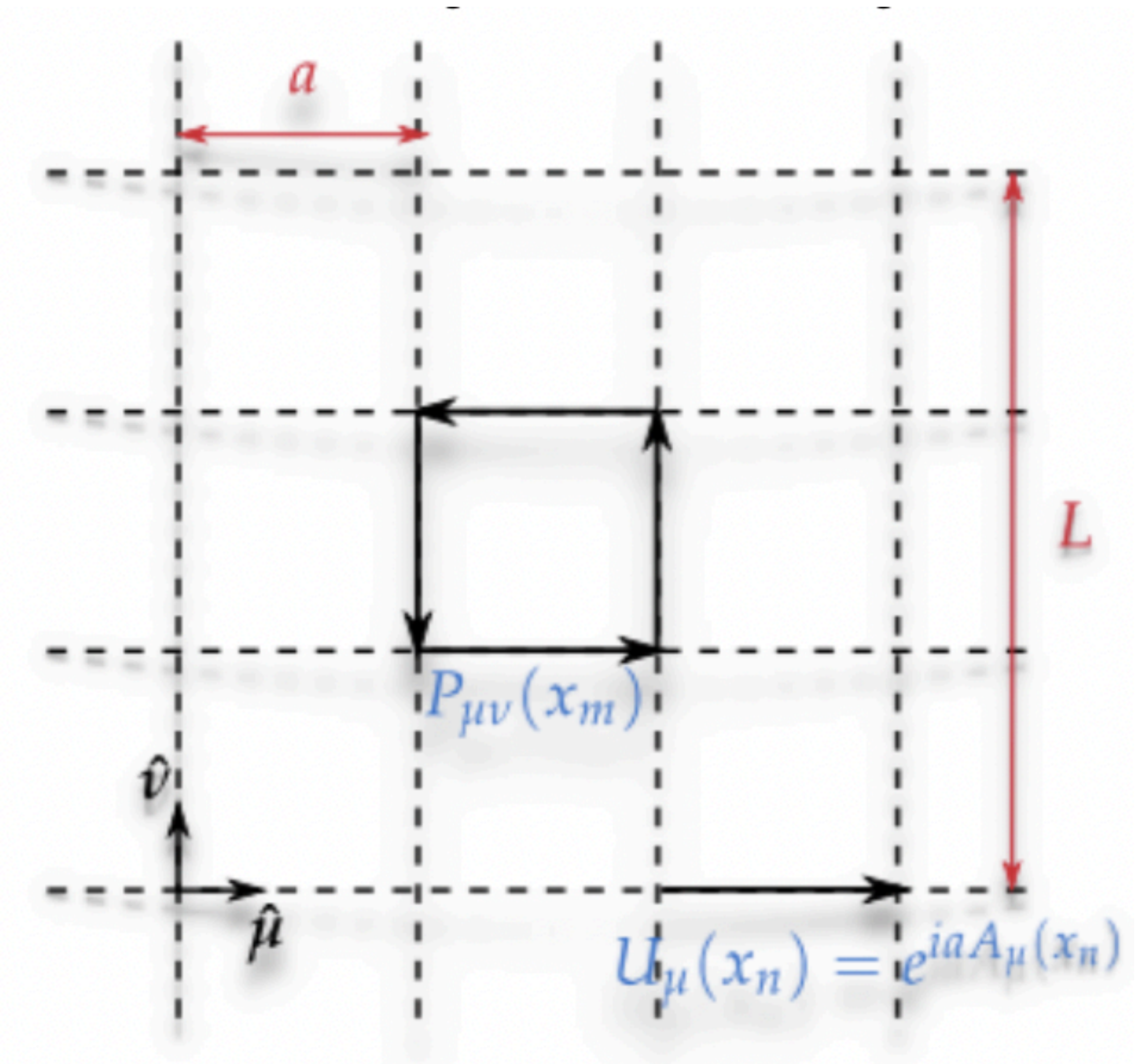
# Lattice QCD



Ensemble used called **D200** generated by Coordinated Lattice Simulations (CLS)
[Bruno et al., JHEP **02** (2015) 044]
[Straßberger et al.,arXiv:2112.06696]

| $a[fm]$ | $(L/a)^3 \times (T/a)$ | $m_\pi$ | $m_K$ |
|---|---|---|---|
| $0.0633(4)(6)$ | $64^3 \times 128$ | $\approx 200$ MeV | $\approx 487$ MeV |

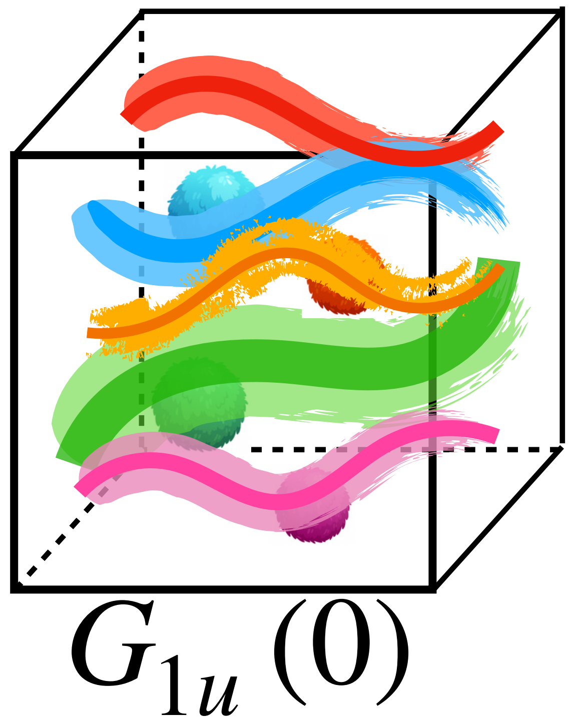
- Heavier-than-physical degenerate u- and d-quarks,
  Lighter-than-physical s-quarks
  $N_f = 2 + 1$
- Tree-level improved Lüscher-Weisz gauge action
- Non-perturbatively $\mathcal{O}(a)$-improved Wilson fermion action
- 2000 gauge configurations
- Open temporal BCs



# Lattice QCD

## Finite Volume Spectrum

- Time-to-time slice correlators
  Stochastic Laplacian-Heaviside Method
  [Morningstar et al., PRD **83** (2011)114505]
  - Quark smearing method to maximize overlap to onto finite-volume energy states
  - Use of single- and multi-hadron operators in each Irrep symmetry channel.
  - Construct large Hermitian correlation matrix




$\Lambda\,[G_{1u}(0)]$

$\pi\,[A_{1u}^{-}(0)]\ \Sigma\,[G_{1g}(0)]$

$\bar{K}\,[A_{1u}(0)]\ N\,[G_{1g}(0)]$

$\pi\,[A_{1u}^{-}(0)]\ \Sigma\,[G_{1g}(0)]$

$\bar{K}\,[A_2(1)]\ N\,[G_{1g}(0)]$

| Symmetry Channel | |
|---|---|
| Total momentum | $\mathbf{P}$ |
| Irreducible Rep of Cubic Group | $\Lambda$ |
| Strangeness | $S$ |
| Isospin | $I$ |

# Lattice QCD

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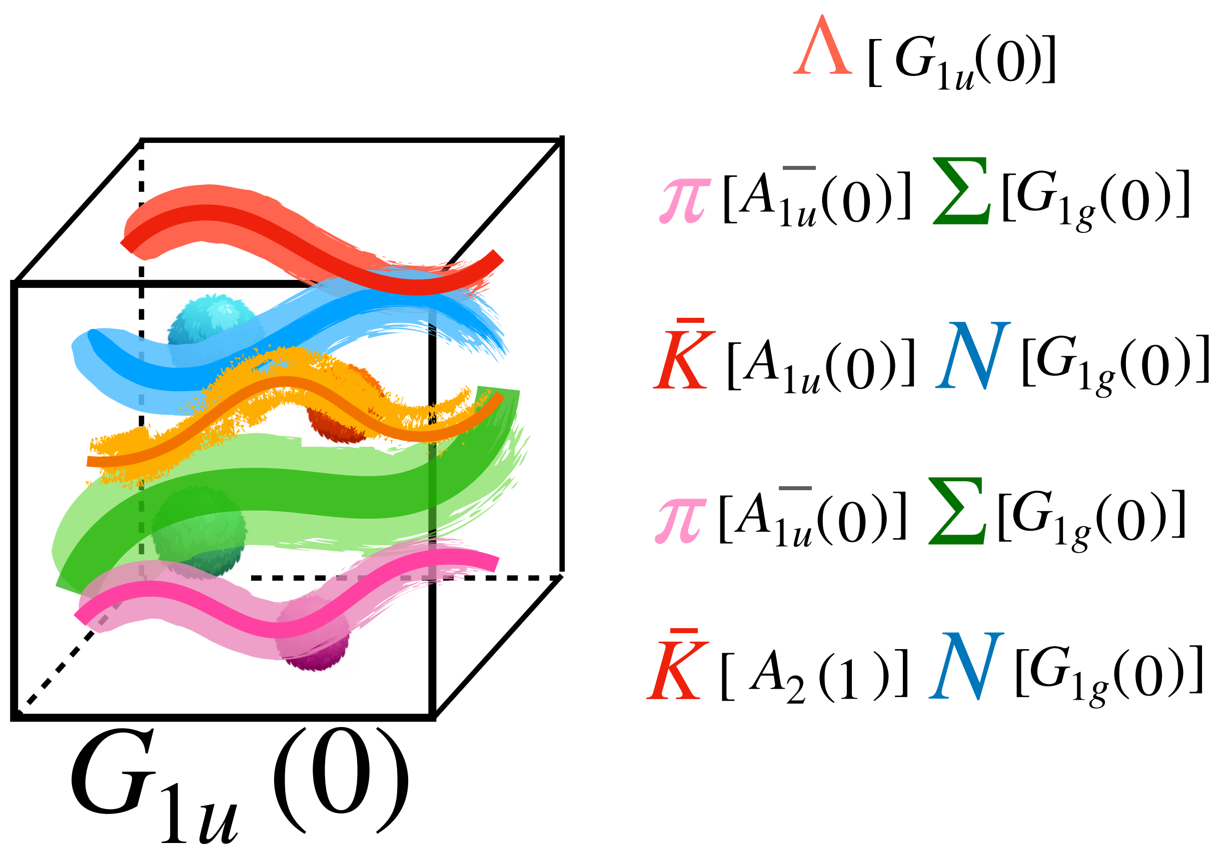


| Symmetry Channel | |
|---|---|
| Total momentum | $\mathbf{P}$ |
| Irreducible Rep of Cubic Group | $\Lambda$ |
| Strangeness | $S$ |
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$$C_{ij}(t) = \begin{array}{c} \\ \pi \\ N \\ \vdots \end{array}\!\!\begin{pmatrix} \pi & N & \cdots \\ \pi\pi & \pi N & \pi\cdots \\ N\pi & NN & \cdots\cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

- Extraction of Energy Spectra
  Generalized Eigenvalue Problem (GEVP)
  [Blossier et al., JHEP **04** (2009) 094]

$$C_{ij}(t) = \langle O_i(t) O_j^{\dagger}(0) \rangle$$

$$C(t)\, v_n = \lambda_n(t, t_0)\, C(t_0)\, v_n$$

$$\lambda_n(t, t_0) \sim e^{-E_n(t-t_0)} \xrightarrow{t\to\infty} E_n + O(e^{-\Delta E_n t})$$

# Lattice QCD

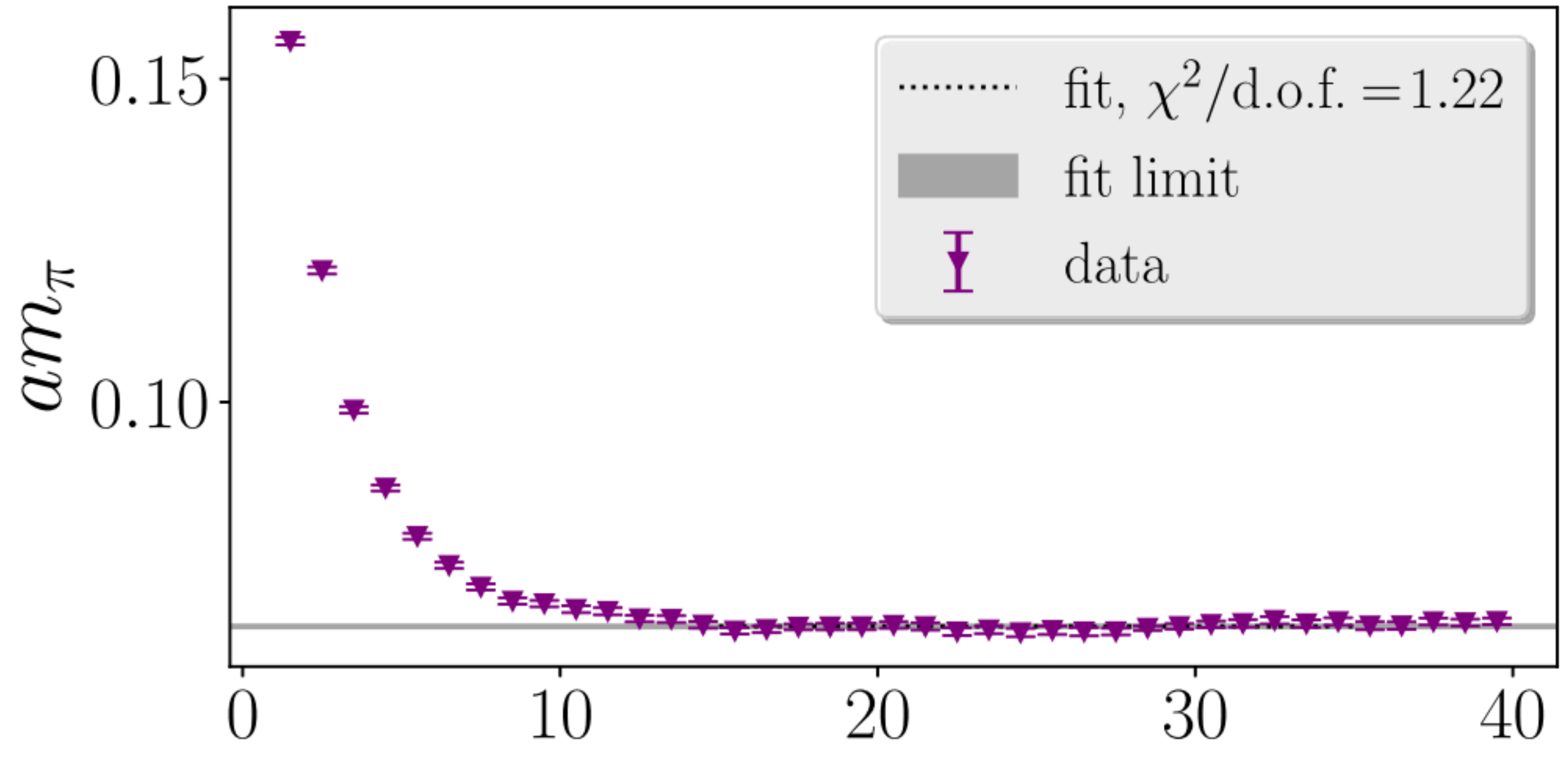
## Energy Determinations

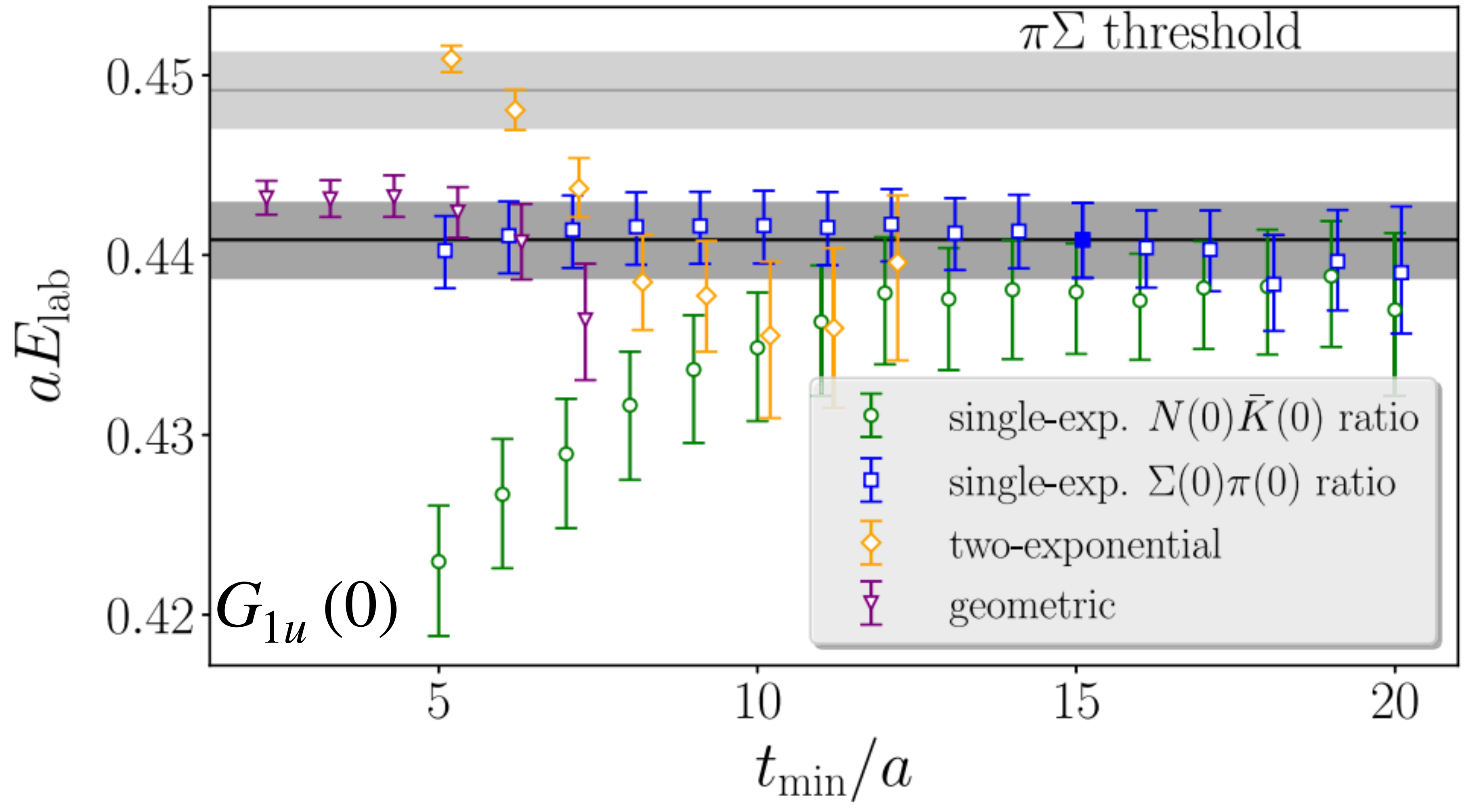
- GEVP + Energy Shift from Ratio Fits

$$R_n(t) = \frac{C_{\text{meson-baryon}}(t)}{C_{\text{meson}}(t)C_{\text{baryon}}(t)} \longrightarrow \Delta E_n \longrightarrow E_{\text{lab}}$$

  - Reduced uncertainties and excited state contamination
  - Can lead to false plateaus!
  - Compare against multi-exponential ansatz

- Final fit criteria
  - $\chi^2$/dof $< 1.5$
  - Agreement of fit results with nearby $t_{min}$
  - Consistent with various fit forms and plateau region.

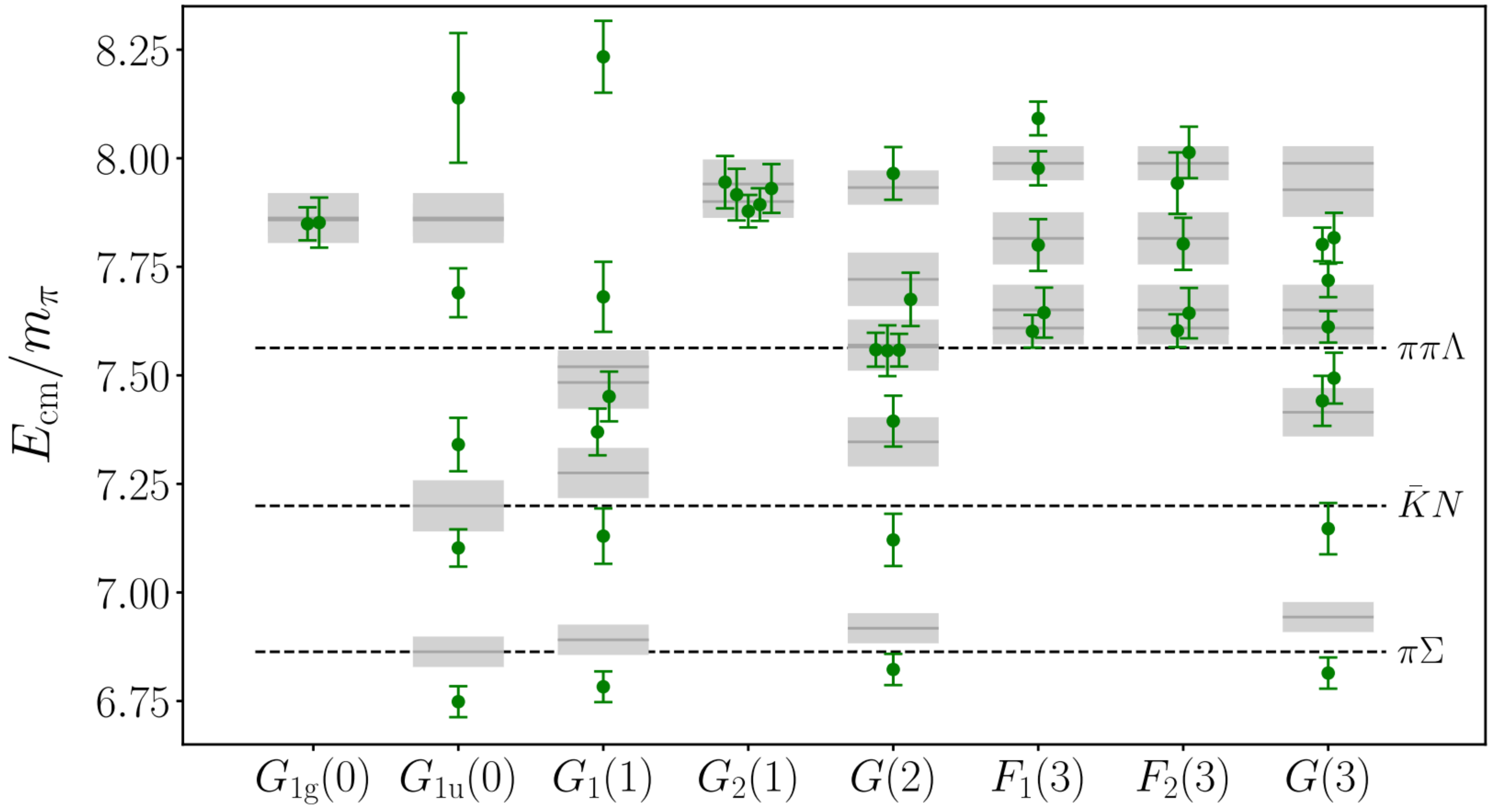


# Lattice QCD

## Finite-Volume Spectrum

$I = 0, \quad S = -1 \quad J^P = 1/2^-$




### Baryon FV Irreps

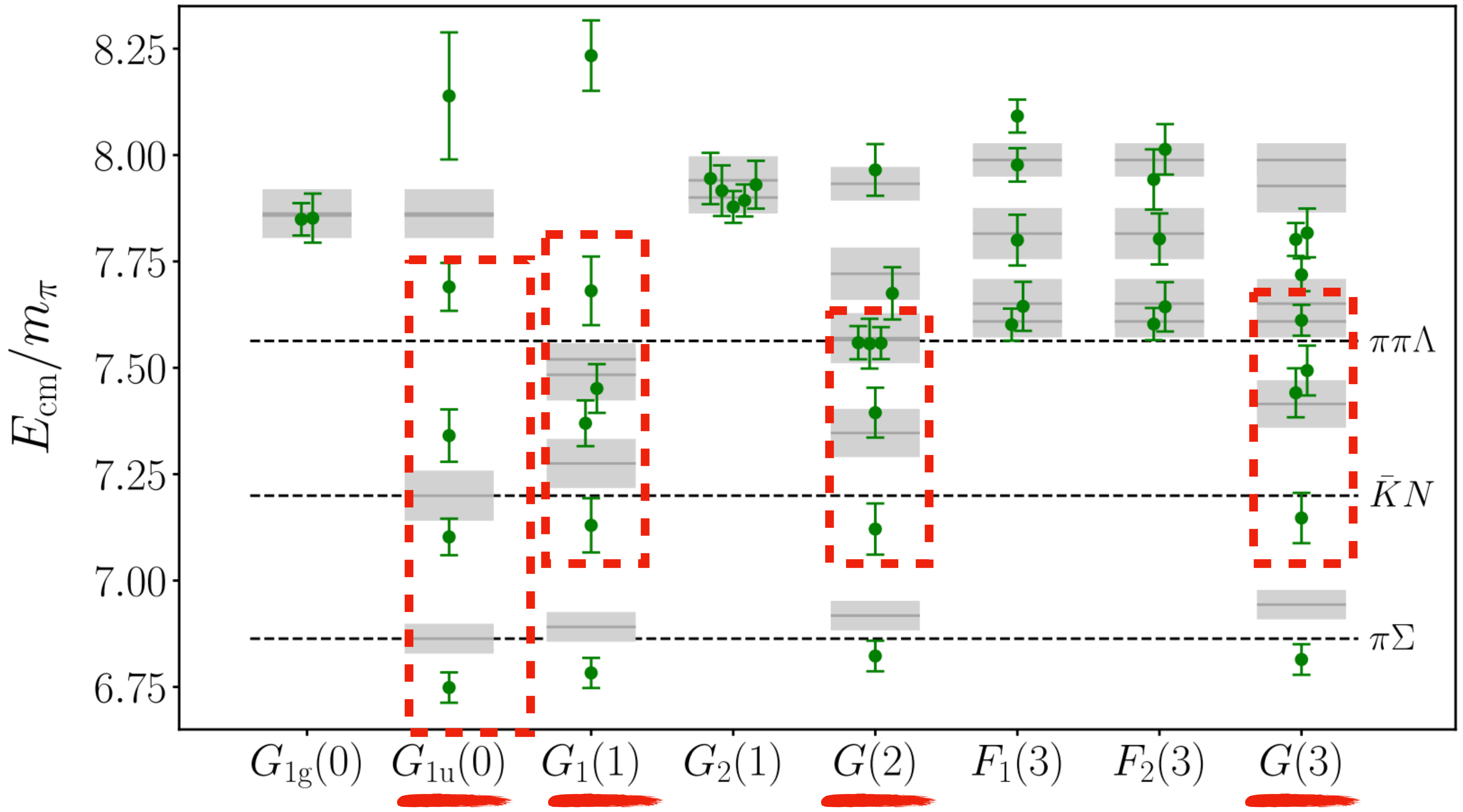
| $\Lambda(\boldsymbol{d}^2)$ | $(2J, L)$ content for $L \leq 2$ |
|---|---|
| $H_g(0)$ | $(3,1)$ |
| $H_u(0)$ | $(3,2),\ (5,2)$ |
| $G_{1g}(0)$ | $(1,1)$ |
| $G_{1u}(0)$ | $(1,0)$ |
| $G_{2g}(0)$ | |
| $G_{2u}(0)$ | $(5,2)$ |
| $G_1(1),\ G_1(4)$ | $(1,0),\ (1,1),\ (3,1),\ (3,2),\ (5,2)$ |
| $G_2(1),\ G_2(4)$ | $(3,1),\ (3,2),\ (5,2)$ |
| $G(2)$ | $(1,0),\ (1,1),\ (3,1),\ (3,2),\ (5,2)$ |
| $F_1(3)$ | $(3,1),\ (3,2),\ (5,2)$ |
| $F_2(3)$ | $(3,1),\ (3,2),\ (5,2)$ |
| $G(3)$ | $(1,0),\ (1,1),\ (3,1),\ (3,2),\ (5,2)$ |

- 47 Energy Levels

# Lattice QCD

## Finite-Volume Spectrum

$I = 0, \quad S = -1 \quad J^P = 1/2^-$

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- 47 Energy Levels
- S-wave $J^P = \frac{1}{2}^-$ analysis ➤ 15 energy levels

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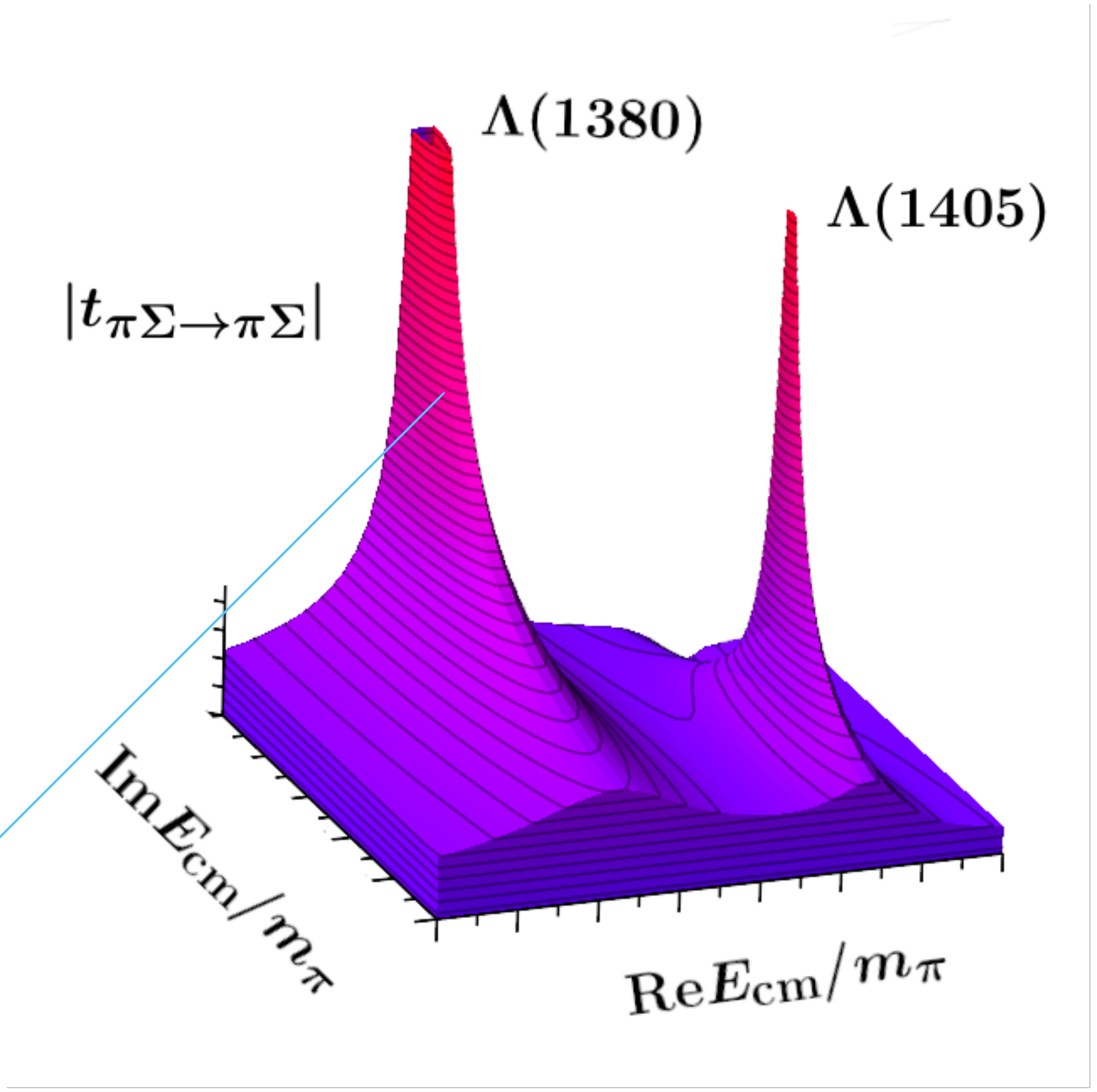


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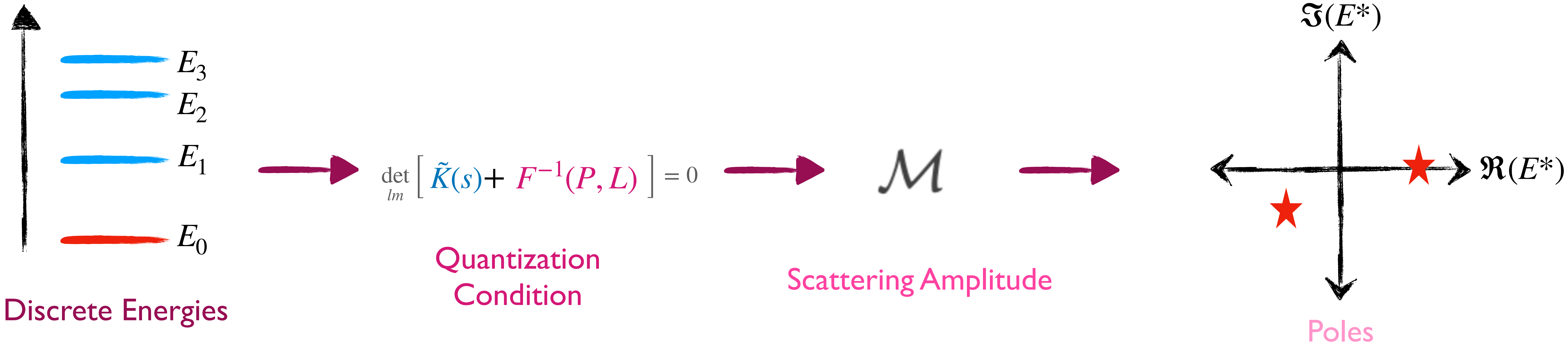
# Resonance Analysis

- Multi-channel Two-particle Scattering Amplitude
  [M. Lüscher, NPB **354** (1991) 53]
  [R. Briceño, PRD **89** (2014) 074507]

$$\begin{pmatrix} \pi\Sigma \to \pi\Sigma & \pi\Sigma \to Kp \\ Kp \to \pi\Sigma & Kp \to Kp \end{pmatrix}$$



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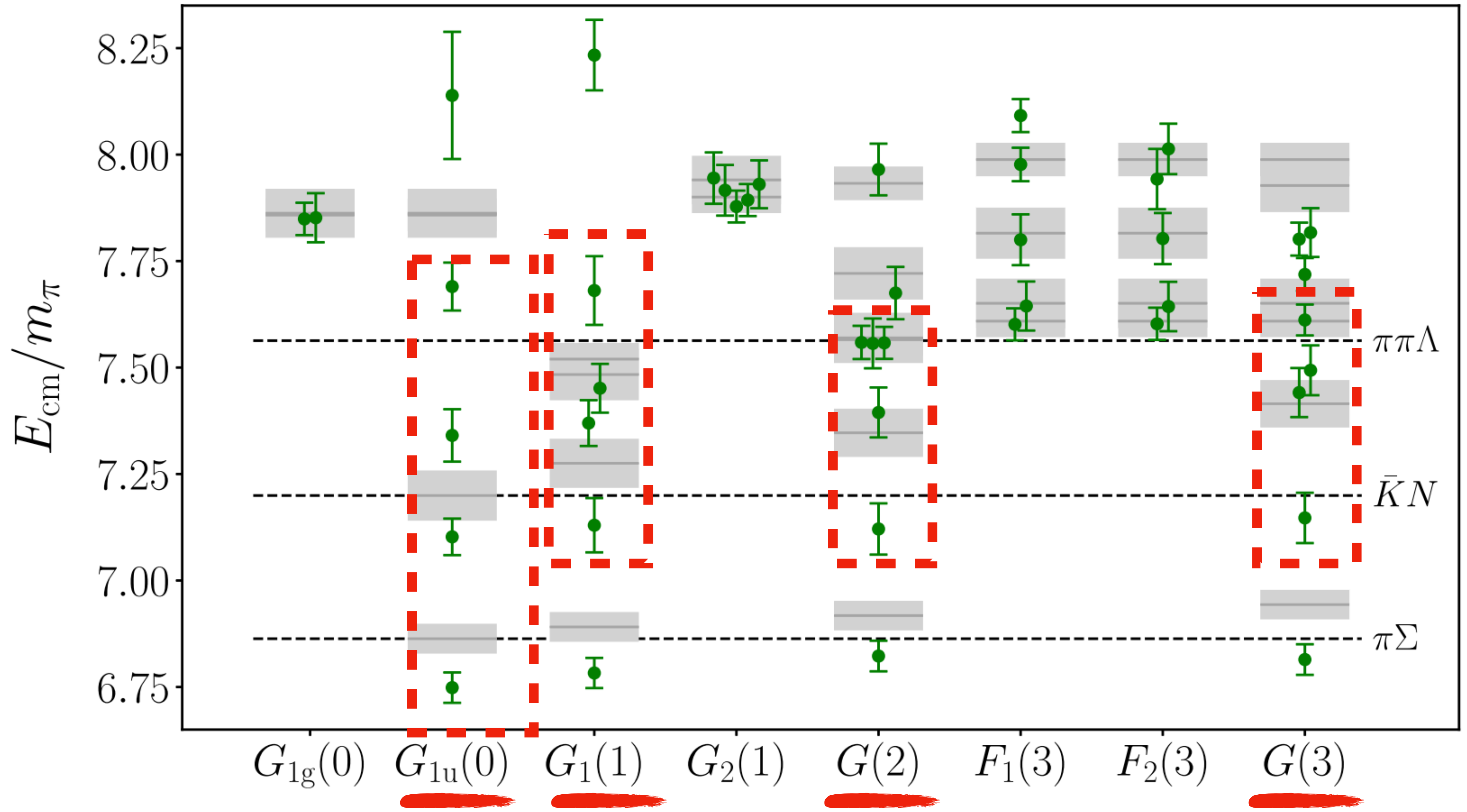



$$\det_{lm}\left[\tilde{K}(s) + F^{-1}(P,L)\right] = 0$$

- Determinant over matrix $J, m_J, l, s, a$ (particle species)
- Valid below inelastic/three-particle threshold
- Truncate $l_{max}$ for analysis
  Keep s-wave
  Checked impact of higher partial waves

# Resonance Analysis

$$\det\left[\underbrace{\begin{pmatrix}\widetilde{K}_{\pi\Sigma\to\pi\Sigma} & \widetilde{K}_{\pi\Sigma\to\bar{K}N}\\ \widetilde{K}_{\bar{K}N\to\pi\Sigma} & \widetilde{K}_{\bar{K}N\to\bar{K}N}\end{pmatrix}}_{\text{Multi-channel matrix}} + \underbrace{\begin{pmatrix}F^{-1}_{\pi\Sigma}(E_n,\vec{P},L) & 0\\ 0 & F^{-1}_{\bar{K}N}(E_n,\vec{P},L)\end{pmatrix}}_{\text{Zeta Function}}\right]=0$$

- K-matrix is real, symmetric, diagonal in total angular momentum $J$
- Test Parametrizations for K-matrix and its inverse (6)
- Parameterizations are flexible enough to allow 0,1,2 poles

$$\begin{cases}\widetilde{K}_{ij}=\frac{m_\pi}{E_{\mathrm{cm}}}\left(A_{ij}+B_{ij}\Delta_{\pi\Sigma}(E_{\mathrm{cm}})\right)\\ \widetilde{K}_{ij}=\widehat{A}_{ij}+\widehat{B}_{ij}\Delta_{\pi\Sigma}(E_{\mathrm{cm}})\\ \widetilde{K}^{-1}_{ij}=\frac{E_{\mathrm{cm}}}{m_\pi}\left(\widetilde{A}_{ij}+\widetilde{B}_{ij}\Delta_{\pi\Sigma}(E_{\mathrm{cm}})\right)\\ \widetilde{K}_{ij}=\frac{\widehat{C}_{ij}}{m_\pi}\left(2E_{\mathrm{cm}}-M_i-M_j\right)\\ \vdots\end{cases}$$

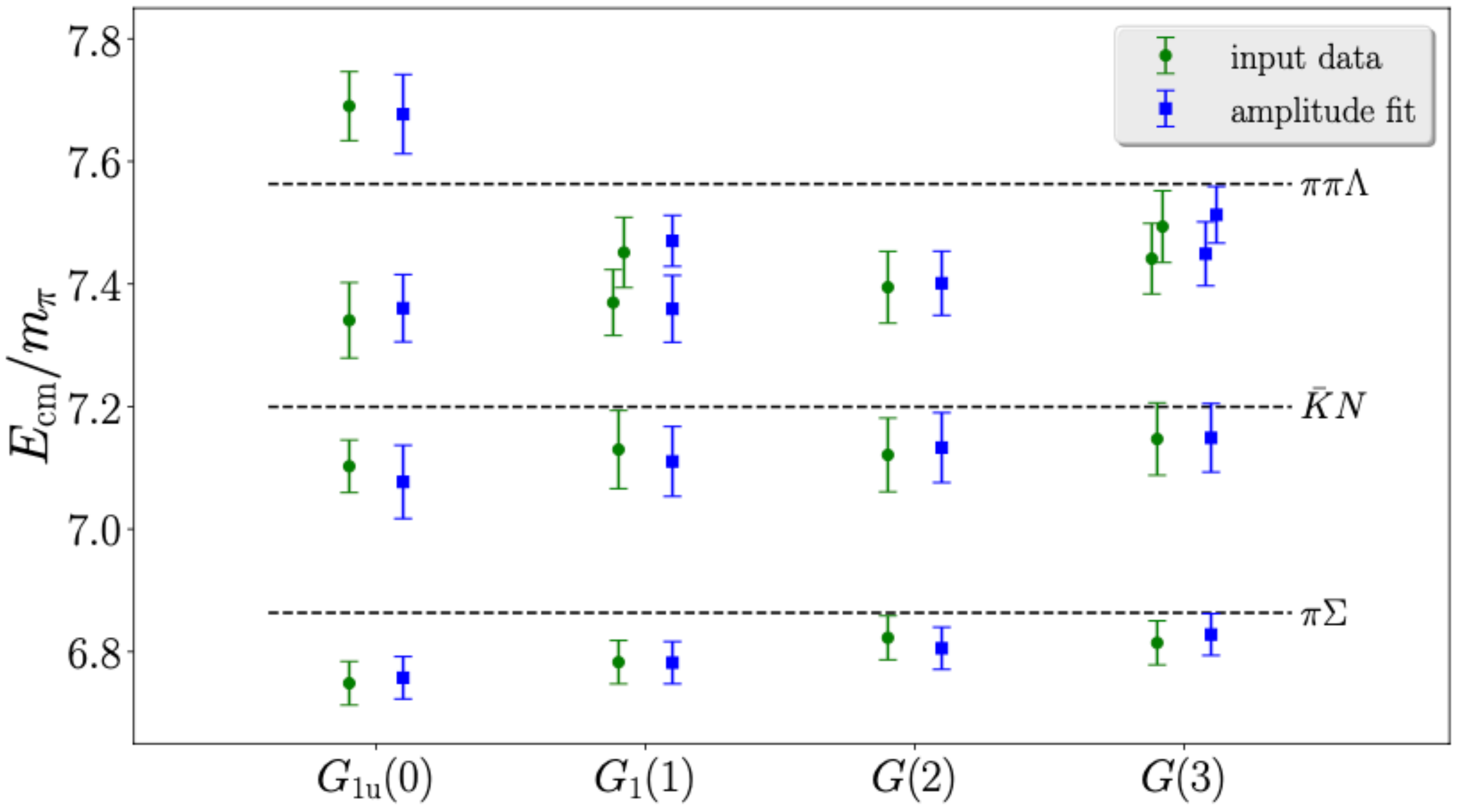
- Use best-fit to find pole positions as vanishing eigenvalues of inverse amplitude

$$\mathcal{T}^{-1}(E_{\mathrm{pole}})=0$$

$$\Delta_{\pi\Sigma}(E_{\mathrm{cm}})=\frac{E^2_{\mathrm{cm}}-(m_\pi+m_\Sigma)^2}{(m_\pi+m_\Sigma)^2},$$

# Resonance Analysis

## Fitting the spectrum




- Fit shifts w.r.t. non-interacting energy levels

$$\Delta E_i = E_{\rm cm}^{\rm latt} - E_{\rm cm}^{\rm free}$$

- Minimize correlated $\chi^2$ with residues

$$\delta_i = \Delta E_{{\rm cm},i} - \Delta E_{{\rm cm},i}^{\rm QC}$$

- Preferred fit based on lowest AIC $\chi^2 - 2\,{\rm dof}$

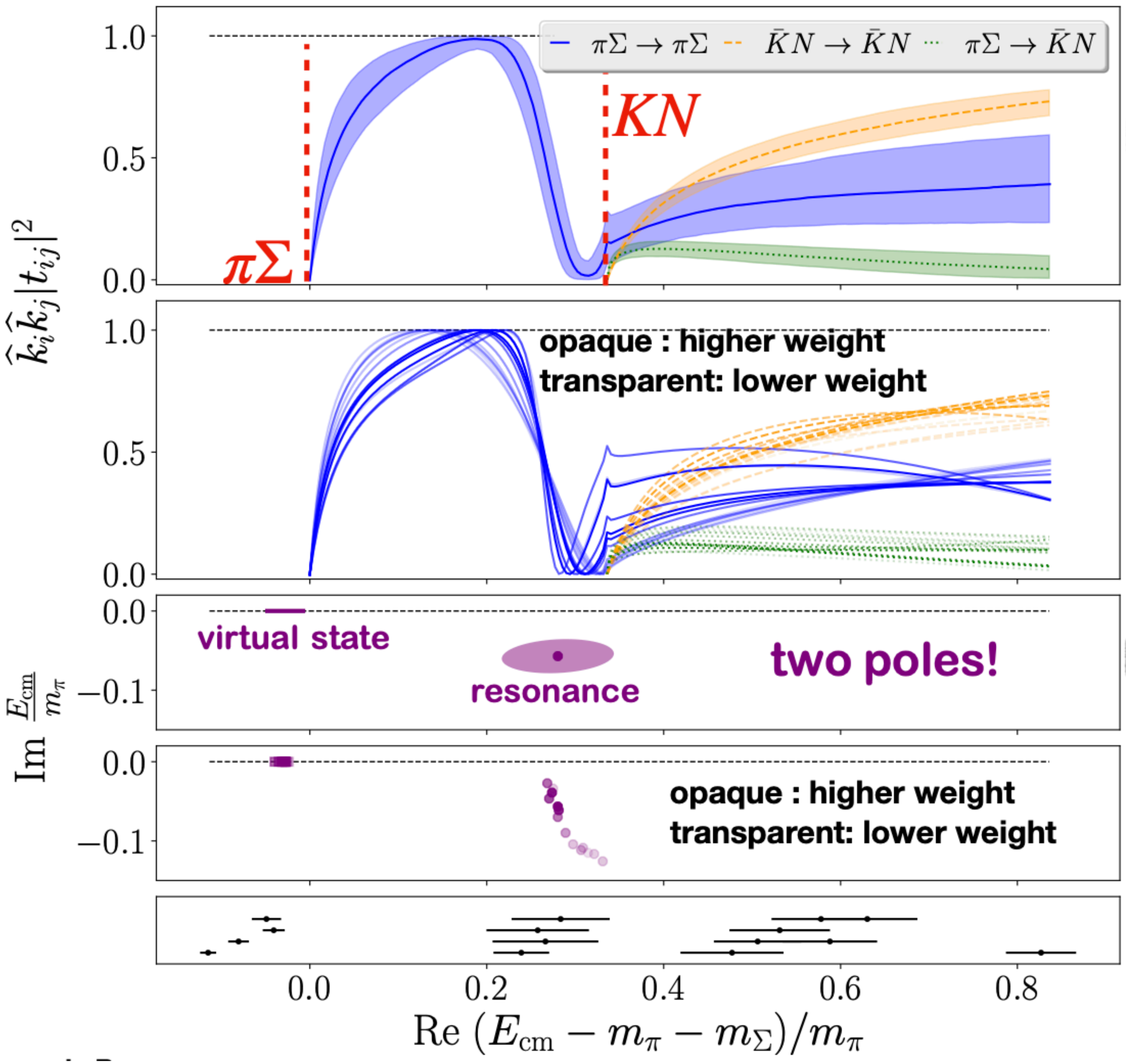
  Akaike Information Criterion

$$\tilde{K}_{ij} = \frac{m_\pi}{E_{\rm cm}}\left(A_{ij} + B_{ij}\Delta_{\pi\Sigma}(E_{\rm cm})\right)$$

4 parameters $B_{11} = B_{00} = 0$ (fixed)

15 energies $\chi^2/{\rm dof} = 0.96$

# Resonance Analysis




## Scattering Transition Amplitude

$$t^{-1} = \widetilde{K}^{-1} - i\hat{k}$$

or

$$t = \frac{m_\pi}{E_{cm} - E_{pole}} \begin{pmatrix} c^2_{\pi\Sigma} & c_{\pi\Sigma}c_{\bar{K}N} \\ c_{\pi\Sigma}c_{\bar{K}N} & c^2_{\bar{K}N} \end{pmatrix}$$

- Scattering Amplitude for all parametrization
  All find two poles!
- one resonance and one virtual bound state
- Four different Riemann sheets

# Resonance Analysis

Two poles with (sign Im $k_{\pi\Sigma}$, sign Im $k_{KN}$) = ( − , + )

## Virtual Bound State

$$E_1 = 1392(9)_{stat}(2)_{model}(16)_a \; MeV$$

$$\left|\frac{c^{(1)}_{\pi\Sigma}}{c^{(1)}_{\bar{K}N}}\right| = 1.9(4)_{\mathrm{stat}}(6)_{\mathrm{model}}$$

Stronger coupling to $\Sigma\pi$

## Resonance

$$E_2 = 1455(13)_{stat}(2)_{model}(17)_a - i11.5(4.4)_{stat}(4.0)_{model}(0.1)_a MeV$$

$$\left|\frac{c^{(2)}_{\pi\Sigma}}{c^{(2)}_{\bar{K}N}}\right| = 0.53(9)_{\mathrm{stat}}(10)_{\mathrm{model}}$$

Stronger coupling to $KN$

- Qualitative agreement with chiral approaches [ PDG, Section 83]

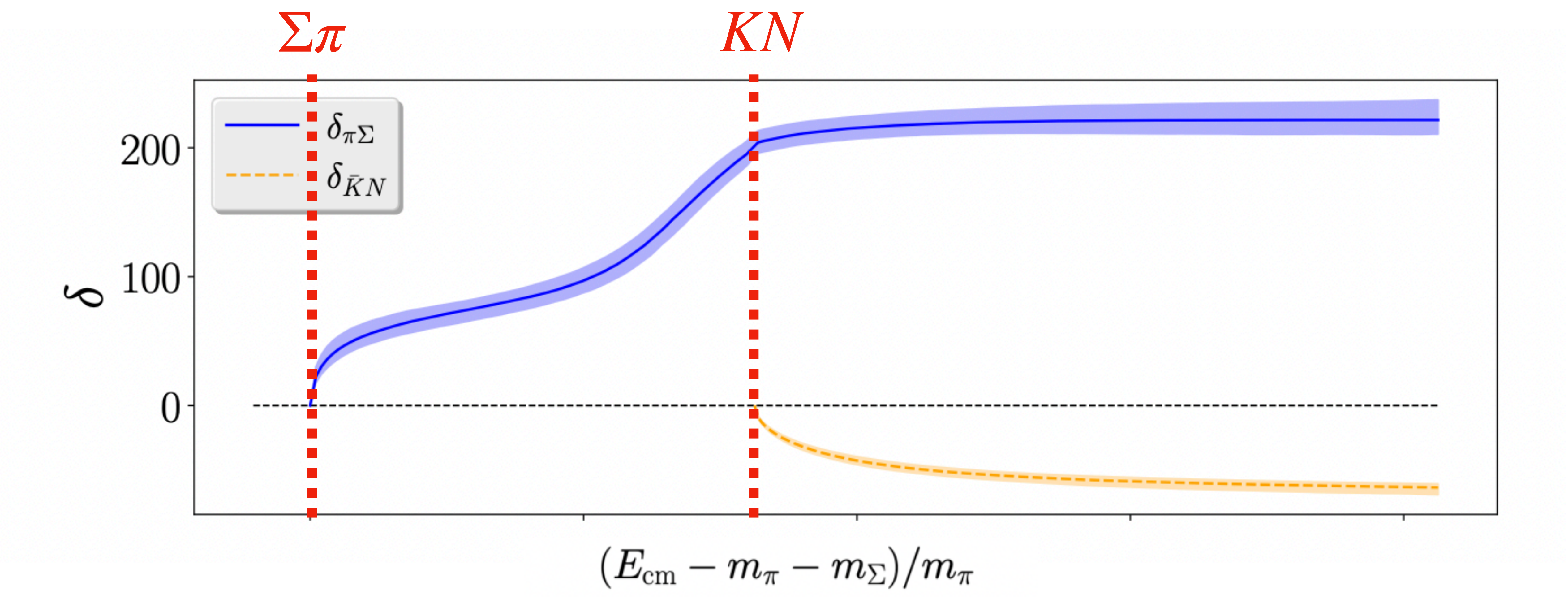
$$\mathrm{Re}\, E_1 = 1325 - 1380\,\mathrm{MeV}$$

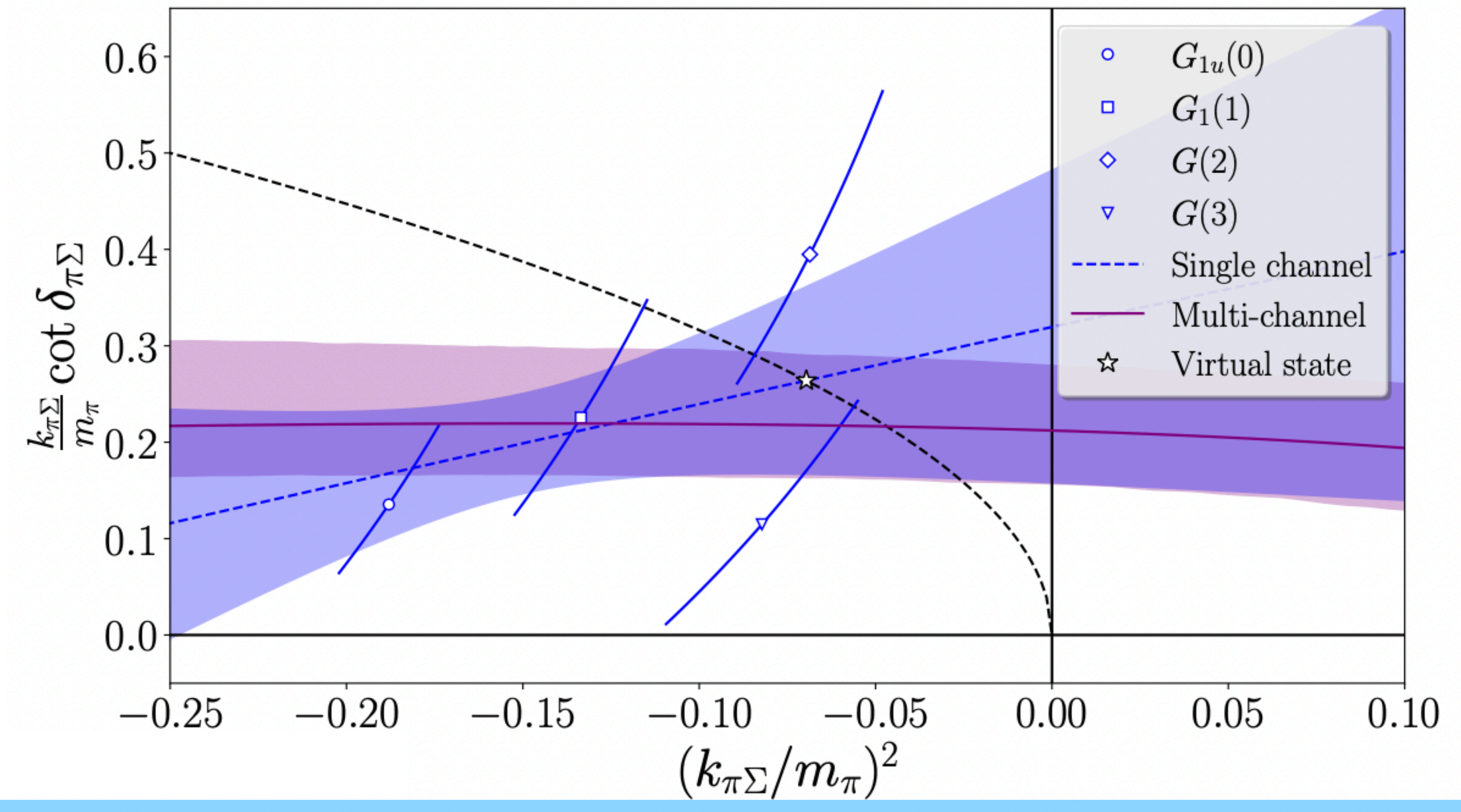
$$\mathrm{Re}\, E_2 = 1421 - 1434\,\mathrm{MeV}$$

- Lower pole on the real axis
  - Unphysical pion mass effect

# Resonance Analysis

## Phase & Single Channel




- Rapid Increase in phase shift after $\Sigma\pi$
  **Virtual State**
- Crosses 90 degrees
  **Resonance**
- Single Channel Lüscher Analysis
  Evidence of the lower pole as a virtual bound state
- Valid below the $KN$ threshold
  Agreement with multi-channel analysis

$$E_1 = 1389(8)_{stat}(16)_a\ MeV$$

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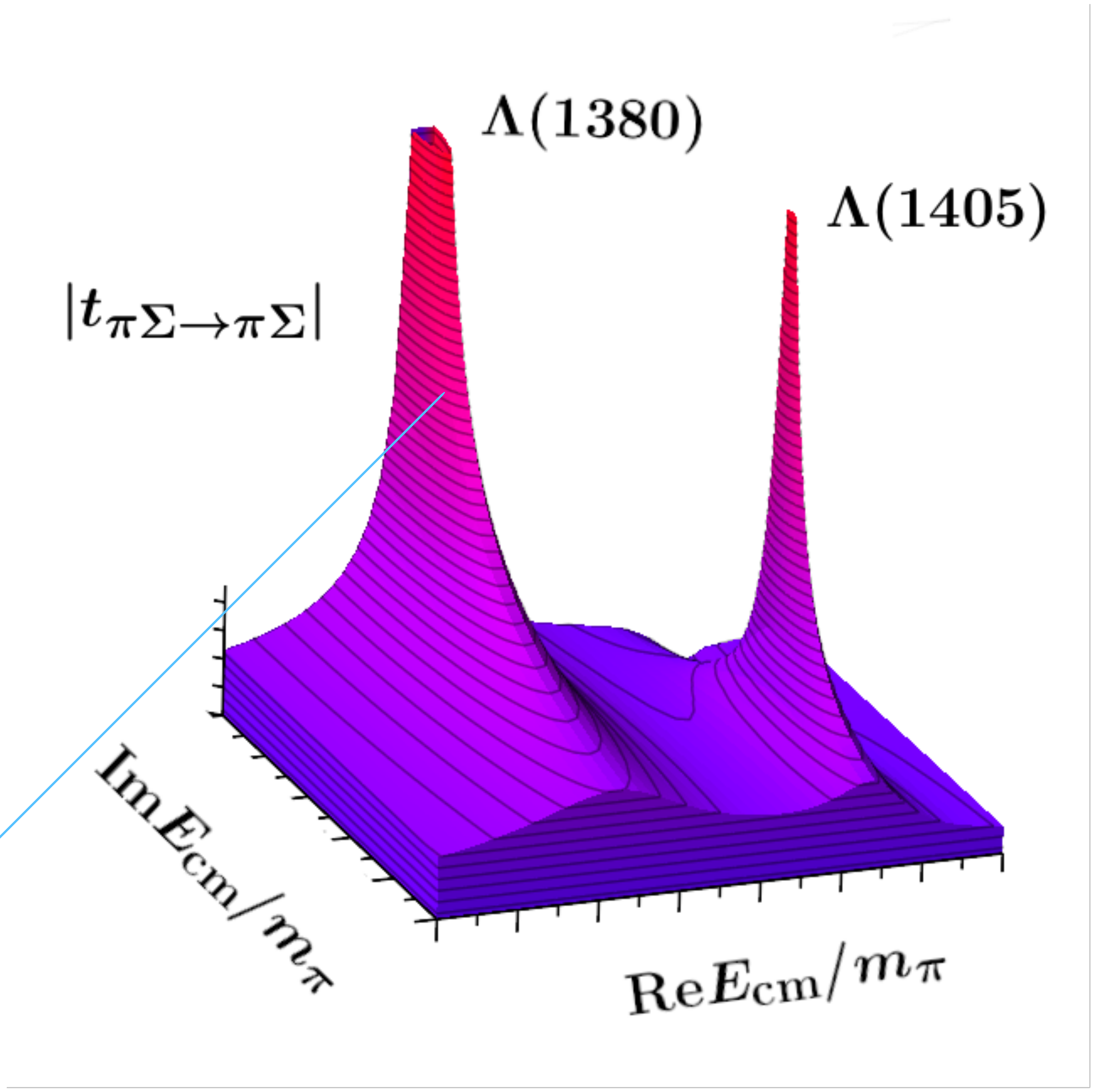
- **Conclusions and Outlook**



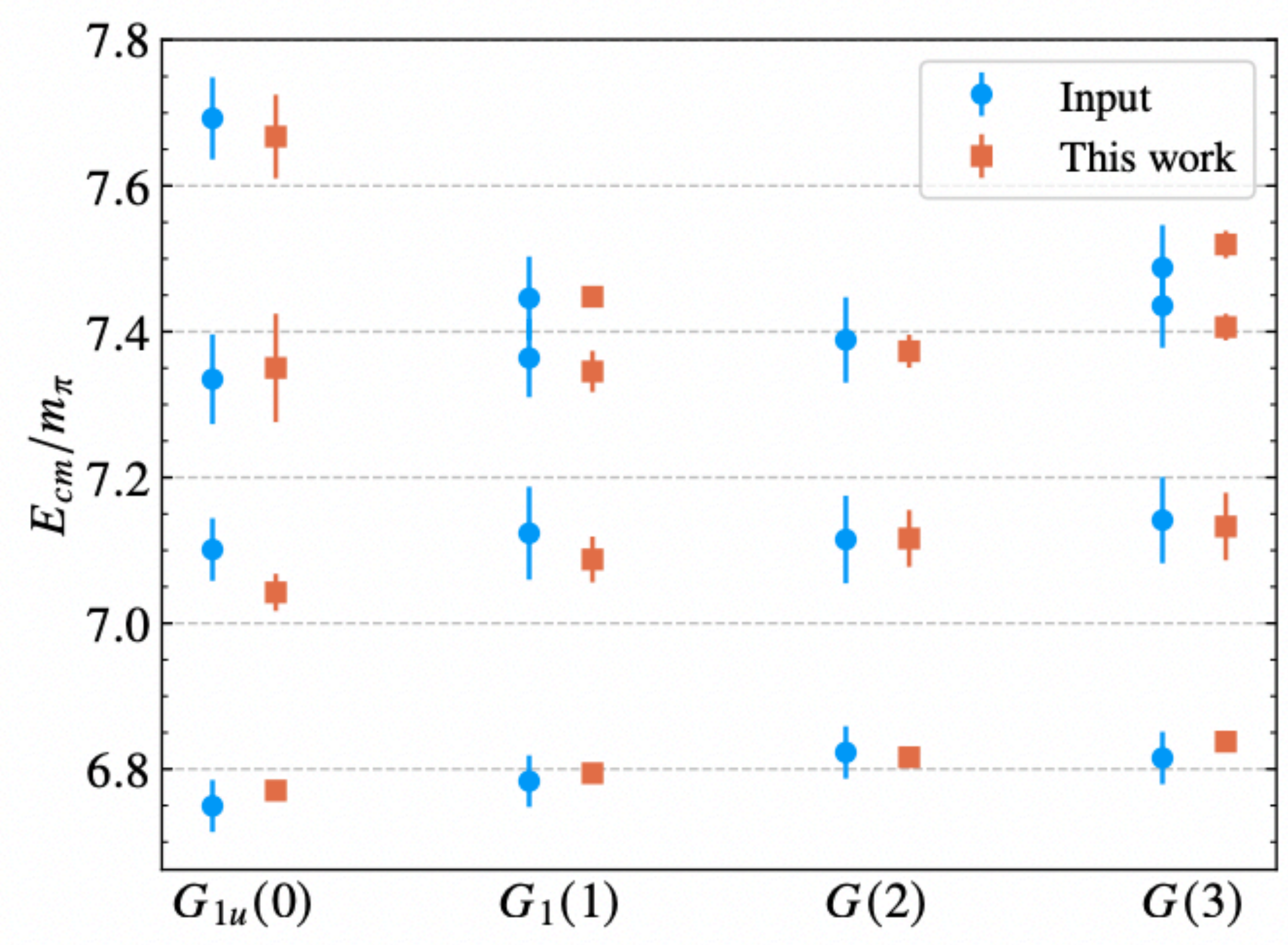

Phys. Rev. Lett. **132**, 051901 [arXiv:2307.10413]

Phys. Rev. D **109**, 014511 [arXiv:2307.13471]

# Conclusion

## Pole Trajectories

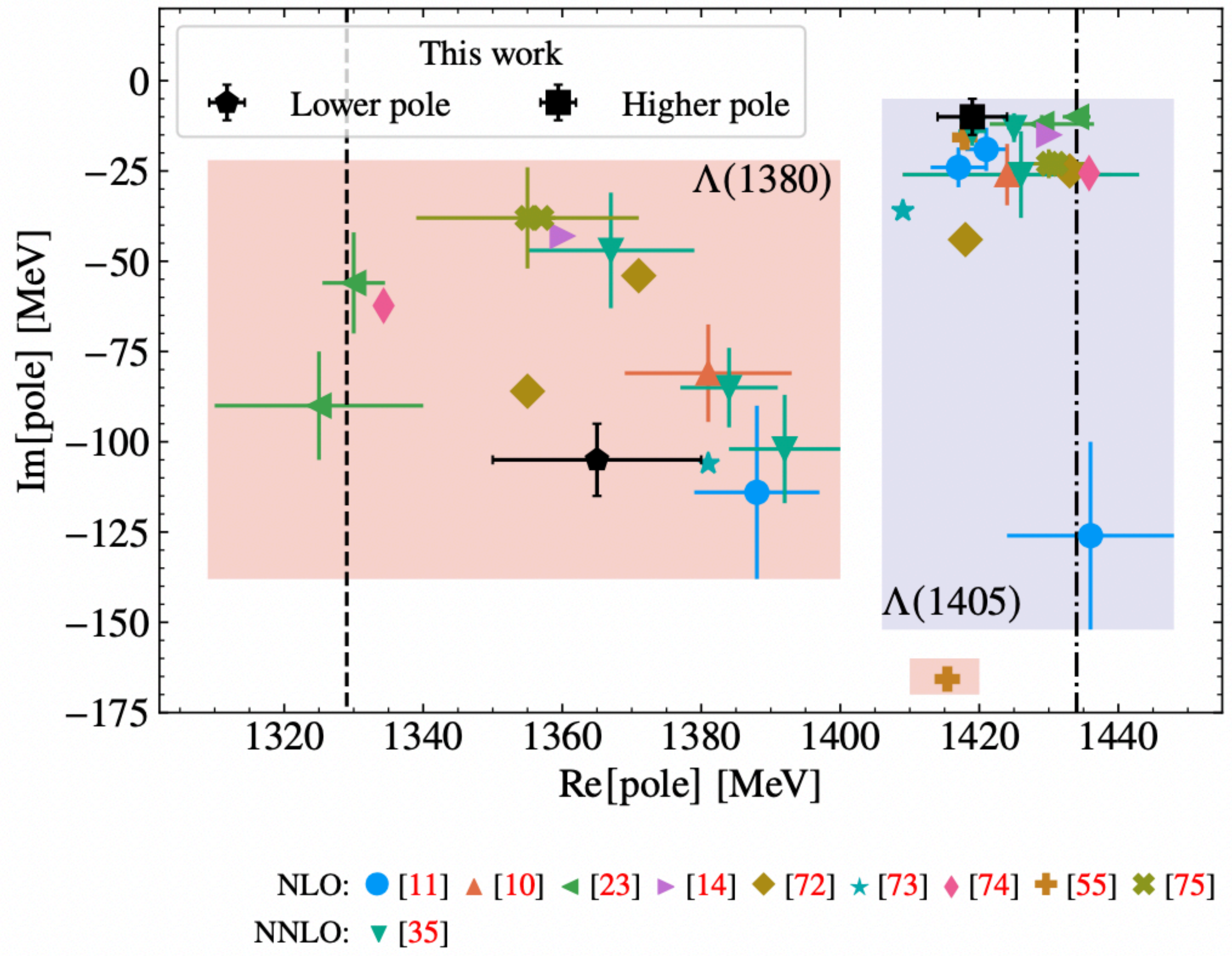
Use results of our study w/
Chiral Unitary Approach at NLO

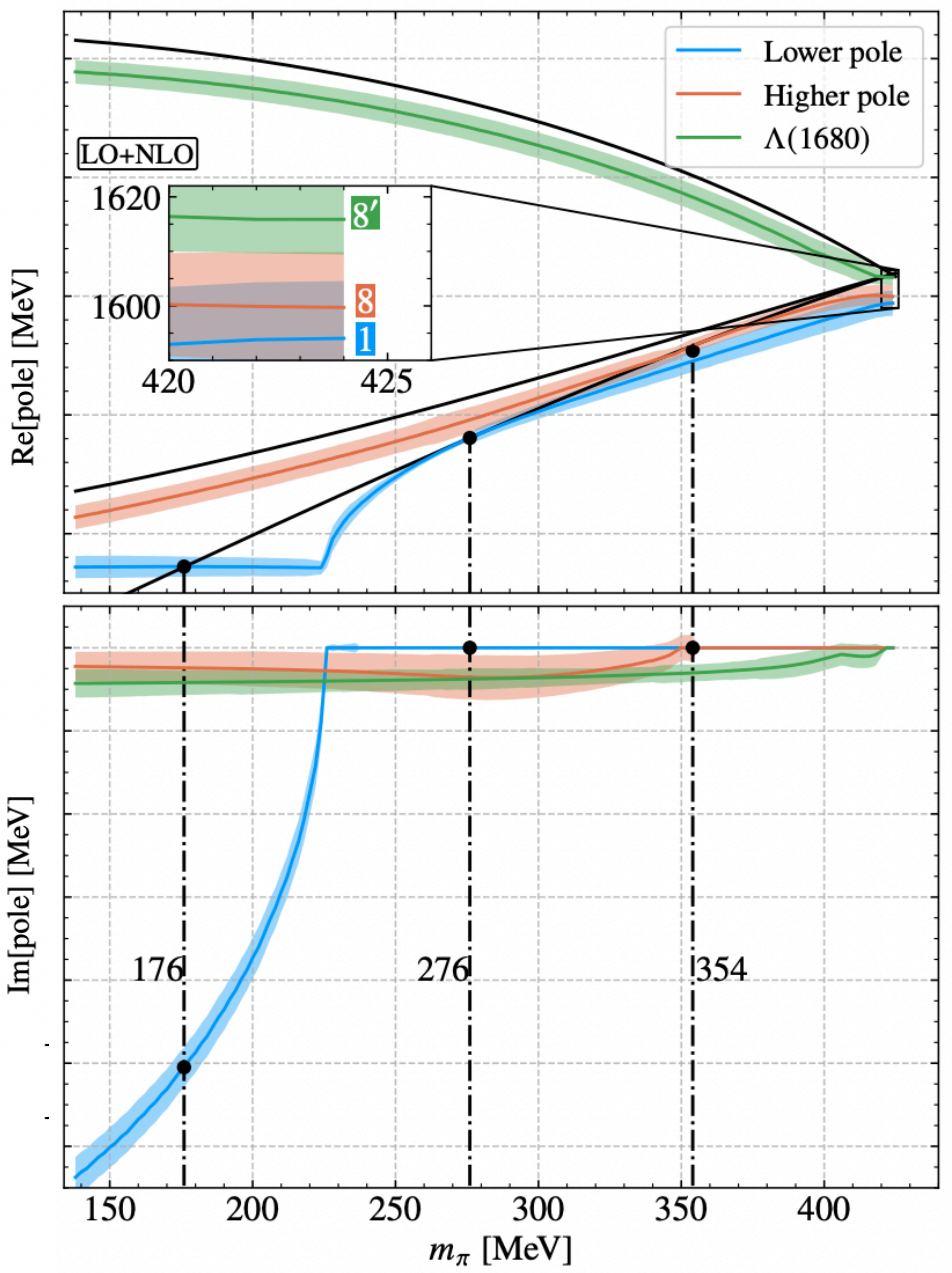



**Pole trajectories of the Λ(1405) helps establish its dynamical nature**

Zejian Zhuang,[1] R. Molina,[1,*] Jun-Xu Lu,[2] and Li-Sheng Geng[2,3,4,5]

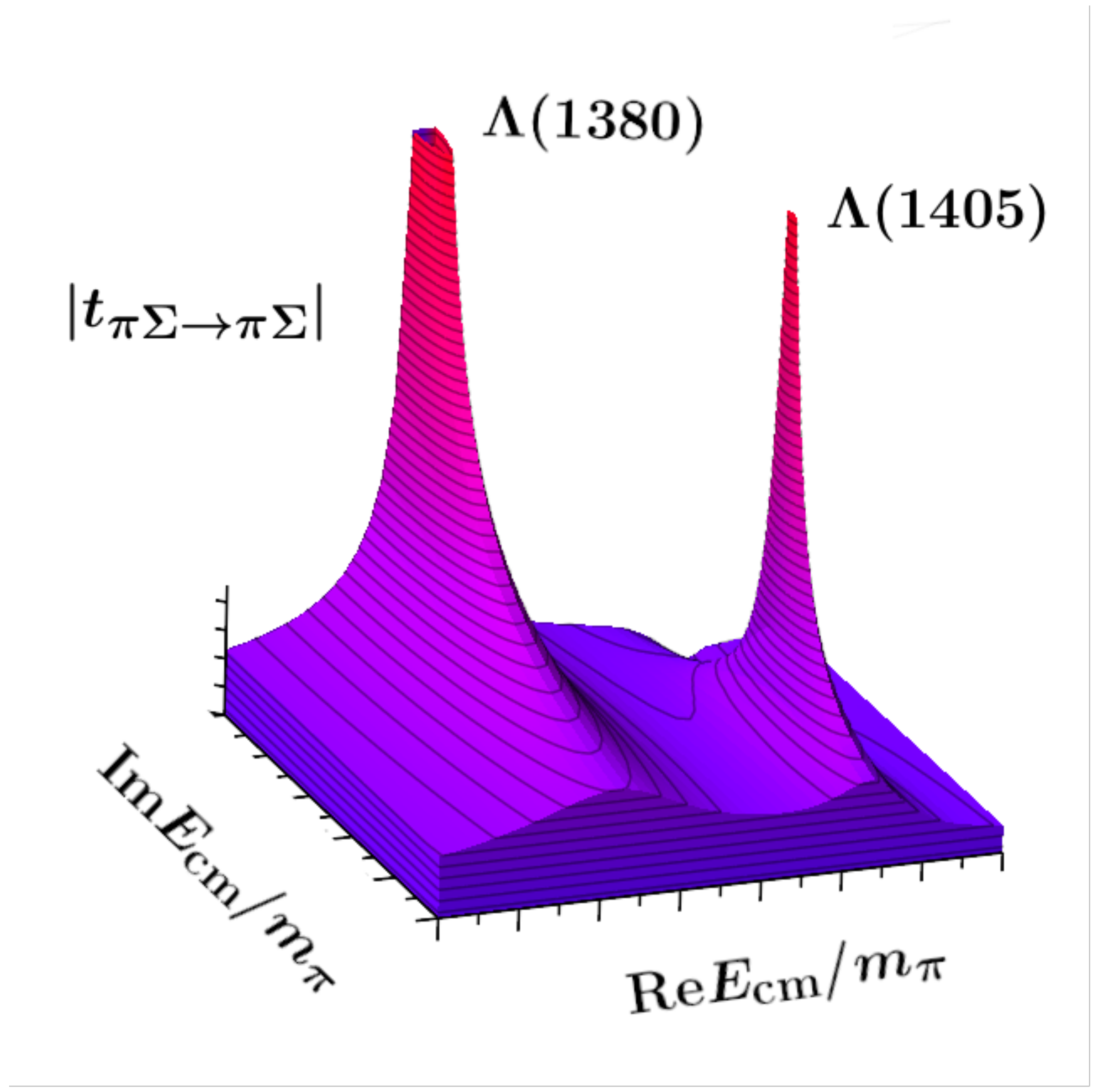
Comparison of results at $m_\pi = 200\ MeV$



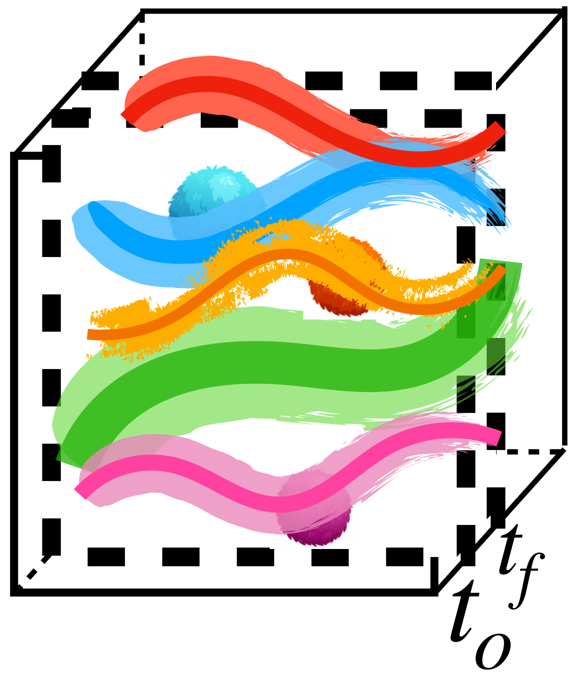
# Conclusion

- First LQCD study of coupled-channel $KN - \Sigma\pi$ scattering in $\Lambda(1405)$ region
- We find for $m_\pi \sim 200\ MeV$, a virtual bound state and a resonance
  - $E_1 = 1392(9)_{stat}(2)_{model}(16)_a\ MeV$
  - $E_2 = [1455(13)_{stat}(2)_{model}(17)_a - i11.5(4.4)_{stat}(4.0)_{model}(0.1)_a]\ MeV$
- Each parametrization of amplitudes supports **two-pole picture**
- Results agree with phenomenological extractions from chiral unitary models




- <u>Outlook</u>
  - Explore quark mass dependence on poles: strange mass dependence
  - Explore impact of three-hadron operators
  - Calculations at the physical point (or near)

# Finite-Volume Spectrum




Once the correlators are generated and diagonalized,

**Fit forms are used to extract the finite-volume spectra**

( All are fit forms on the diagonal elements of the rotated correlators)

- Single Exponential $$C(t) = A_n e^{-tE_n}$$
- Two-Exponential $$C(t) = A_n e^{-tE_n} + A_1 e^{-tD^2}$$
- Geometric $$C(t) = \frac{A_n e^{-tE_n}}{1 - Be^{-Mt}}$$
- Ratio of Corrs $$C(t) = \frac{D_n(t)}{C_A(\mathbf{d}_A^2, t)C_B(\mathbf{d}_b^2, t)} = A_n e^{-t\Delta E_n}$$

**Effective Mass**

$$\ln(C(t)/C(t+a))$$

**Hadron fits**

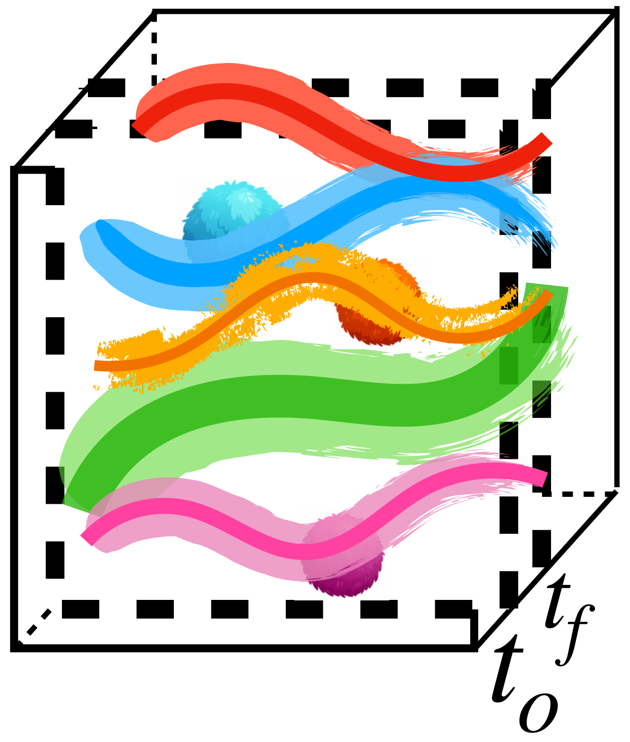
$$[t_{min}, t_{max}]$$

$$t_{max} = 35a_{pion,kaon}$$

$$t_{max} = 25a_{hadrons}$$

# Finite-Volume Spectrum

## Multi-Hadron Energies




- Ratio of correlators $$C(t) = \frac{D_n(t)}{C_A(\mathbf{d}_A^2, t)C_B(\mathbf{d}_b^2, t)}$$
- Channels $(A, B) = (\pi, \Sigma)$ or $(\bar{K}, N)$
- Non-interacting energies $$E_n^{\text{non. int.}} = \sqrt{m_A^2 + \left(\frac{2\pi \boldsymbol{d}_A}{L}\right)^2} + \sqrt{m_B^2 + \left(\frac{2\pi \boldsymbol{d}_B}{L}\right)^2}$$
- Single-exp ansatz for interaction shift $a\Delta E_n$
- Lab-frame energy $aE_n^{lab} = a\Delta E_n + aE_n^{non-int}$

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# Resonance Analysis

- Effective Range Expansion (ERE)

$$\tilde{K}_{ij} = \frac{m_\pi}{E_{\rm cm}}\left(A_{ij} + B_{ij}\Delta_{\pi\Sigma}\right)$$

- ERE w/o energy

$$\tilde{K}_{ij} = A_{ij} + B_{ij}\Delta_{\pi\Sigma}$$

- ERE inverse

$$\tilde{K}^{-1}_{ij} = \frac{E_{cm}}{m_\pi}\left(A_{ij} + B_{ij}\Delta_{\pi\Sigma}\right)$$

- Blatt-Biederharn

$$\tilde{K}^{-1}_{ij} = R_\theta F\, R^{-1}_\theta \quad F = \begin{pmatrix} f_0(E_{\rm cm}) & 0 \\ 0 & f_1(E_{\rm cm}) \end{pmatrix}$$

$$R \in SU(2)$$

- LO Weinberg-Tomozawa

$$\tilde{K}_{ij} = A_{ij}\left(2E_{\rm cm} - M_i - M_j\right)$$

$$M_0 = m_\Sigma \text{ and } M_1 = m_N$$

- Linear expansion

$$\tilde{K}_{ij} = A_{ij} + \frac{B_{ij}}{m_\pi}\left(E_{\rm cm} - M_\Sigma - M_N\right)$$

**Strategy**

$\tilde{K}_{ij}$

↓

$$\det_{lm}\left[\tilde{K}(s) + F^{-1}(P, L)\right] = 0$$

↓

Minimize $\chi^2$

↓

$\tilde{K}^*_{ij}$

PWA13/ATHOS8

# Other fits

3. An ERE of $\widetilde{K}^{-1}$ of the form

$$\widetilde{K}_{ij}^{-1} = \frac{E_{\rm cm}}{m_\pi}\left(\widetilde{A}_{ij} + \widetilde{B}_{ij}\Delta_{\pi\Sigma}(E_{\rm cm})\right). \qquad (15)$$

4. A Blatt-Biederharn [84] parametrization:

$$\widetilde{K} = C\ F\ C^{-1}, \qquad (16)$$

where

$$C = \begin{pmatrix} \cos\epsilon & \sin\epsilon \\ -\sin\epsilon & \cos\epsilon \end{pmatrix}, \qquad (17)$$

$$F = \begin{pmatrix} f_0(E_{\rm cm}) & 0 \\ 0 & f_1(E_{\rm cm}) \end{pmatrix}, \qquad (18)$$

and

$$f_i(E_{\rm cm}) = \frac{m_\pi}{E_{\rm cm}}\frac{a_i + b_i\Delta_{\pi\Sigma}(E_{\rm cm})}{1 + c_i\Delta_{\pi\Sigma}(E_{\rm cm})}. \qquad (19)$$

TABLE X. Fit results for $\widetilde{K}$ parametrization class 3 shown in Eq. (15). Errors are propagated through the derivative method. Empty entries indicate parameters set to zero in a fit. AIC refers to Akaike Information Criterion.

| Fit | $\widetilde{A}_{00}$ | $\widetilde{A}_{11}$ | $\widetilde{A}_{01}$ | $\widetilde{B}_{00}$ | $\widetilde{B}_{11}$ | $\widetilde{B}_{01}$ | $\chi^2$/dof | AIC |
|---|---|---|---|---|---|---|---|---|
| a | 0.092(21) | −0.036(15) | 0.082(20) | 0.28(15) | | | 11.73/(15 − 4) | −10.27 |
| b | 0.114(25) | −0.041(24) | 0.096(19) | | 0.19(16) | | 14.57/(15 − 4) | −7.43 |
| c | 0.137(33) | −0.019(14) | 0.119(21) | | | −0.142(85) | 13.10/(15 − 4) | −8.90 |

TABLE XI. Fit results for $\widetilde{K}$ parametrization class 4 shown in Eq. (16). Errors are propagated through the derivative method. Empty entries indicate parameters set to zero in a fit. AIC refers to Akaike Information Criterion.

| Fit | $a_0$ | $a_1$ | $b_0$ | $b_1$ | $c_0$ | $c_1$ | $\epsilon$ | $\chi^2$/dof | AIC |
|---|---|---|---|---|---|---|---|---|---|
| a | 5.7(1.2) | −11.4(1.2) | | −27(15) | | | 0.451(56) | 13.27/(15 − 4) | −8.73 |
| b | 13.7(4.1) | −14.06(86) | −37(17) | | | | 0.349(75) | 10.63/(15 − 4) | −11.37 |
| c | 5.8(1.2) | −11.8(1.1) | | | | −1.62(95) | 0.468(48) | 13.54/(15 − 4) | −8.46 |
| d | 12.2(3.4) | −14.06(87) | | | 5.8(3.2) | | 0.360(82) | 11.13/(15 − 4) | −10.87 |

# Higher Partial Waves

**Check for effect of higher partial waves using leves in nontrivial irreps**

**Parametrize p-wave K-matrix with simple form**

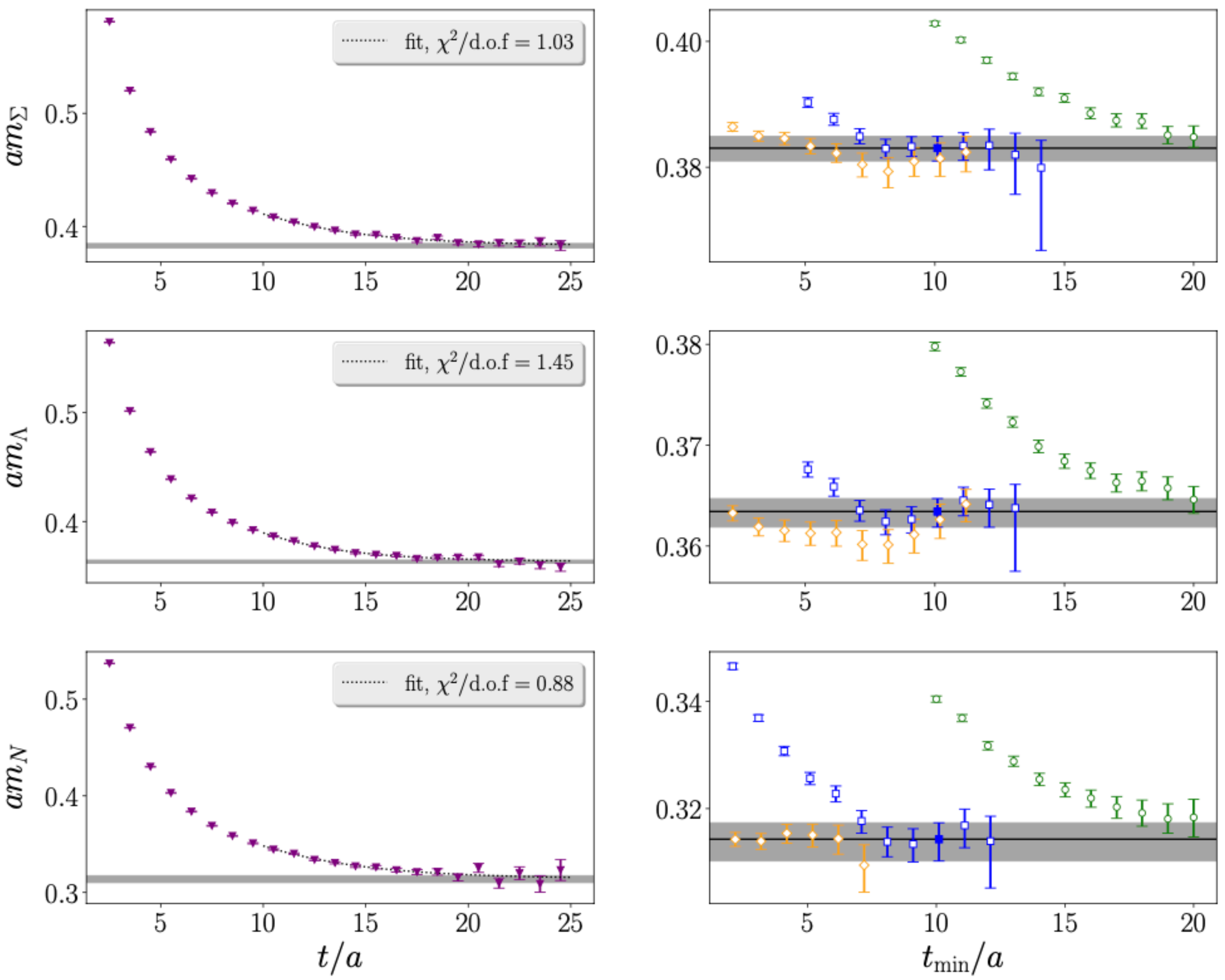
$$\tilde{K}^{J^P} = \mathrm{diag}\left(A_{00}^{J^P}, A_{11}^{J^P}\right).$$

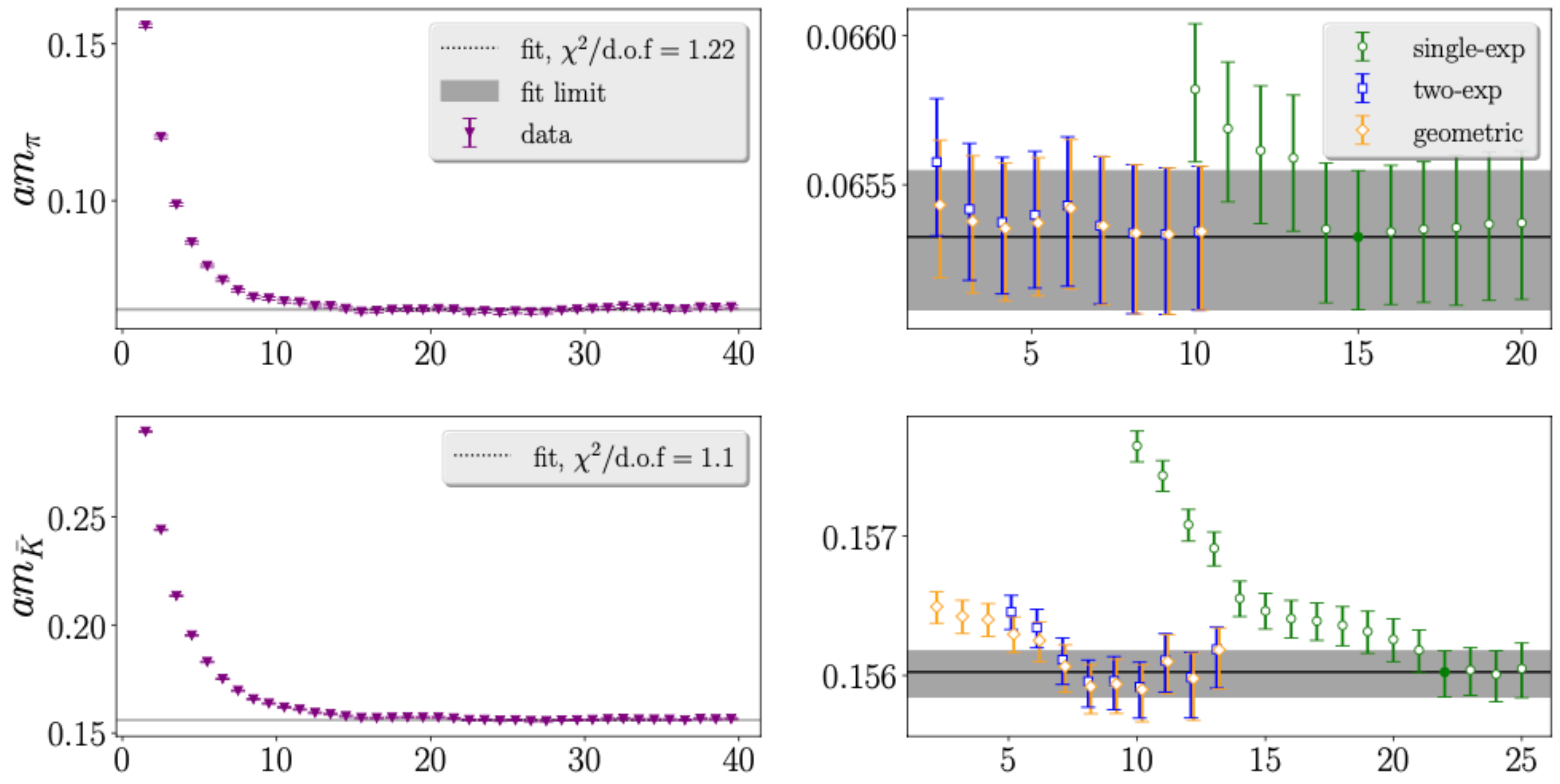
**Impact on s-wave parameters is negligible**

TABLE XII. Fit results for $\widetilde{K}$ parametrization class 1 shown in Eq. (13) for the $J^P = 1/2^-$ wave, and Eq. (32) for the $J^P = 1/2^+, 3/2^+$ waves using $\ell_{\max} = 1$. Errors are propagated through the derivative method. Empty entries indicate parameters set to zero in a fit. AIC refers to Akaike Information Criterion.

| $J^P$ partial waves | $A_{00}$ | $A_{11}$ | $A_{01}$ | $B_{01}$ | $A_{00}^{1/2^+}$ | $A_{11}^{1/2^+}$ | $A_{00}^{3/2^+}$ | $A_{11}^{3/2^+}$ | $\chi^2$/dof | AIC |
|---|---|---|---|---|---|---|---|---|---|---|
| $1/2^-$ | 4.1(1.2) | −10.5(1.1) | 10.3(1.3) | −29(15) | | | | | 10.52/(15-4) | −11.48 |
| $1/2^-$ and $1/2^+$ | 4.1(1.2) | −10.5(1.1) | 10.3(1.3) | −30(15) | 0.0088(39) | 0.031(15) | | | 10.52/(17-6) | −11.48 |
| $1/2^-$ and $3/2^+$ | 4.1(1.1) | −10.9(1.1) | 10.4(1.3) | −32(15) | | | 0.0172(48) | 0.0218(48) | 14.10/(21-6) | −15.90 |

# Single Hadron Masses



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# Resonance Analysis

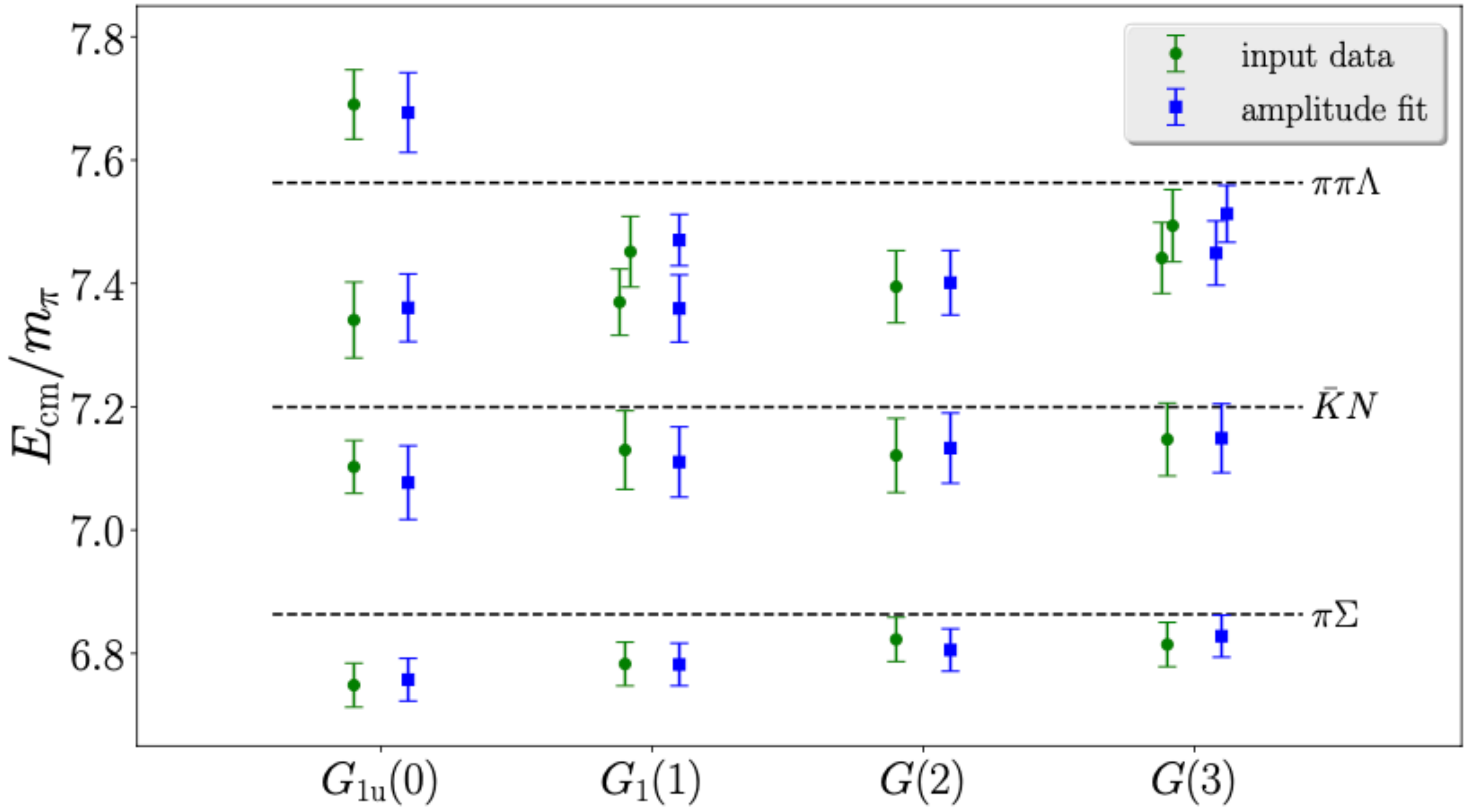
$$\mathrm{AIC} = \chi^2 - 2n_{d.o.f.}$$

$\tilde{K}^*_{ij}$ Effective Range Expansion (ERE)

$$\tilde{K}_{ij} = \frac{m_\pi}{E_{\mathrm{cm}}}\left(A_{ij} + B_{ij}\Delta_{\pi\Sigma}(E_{\mathrm{cm}})\right)$$

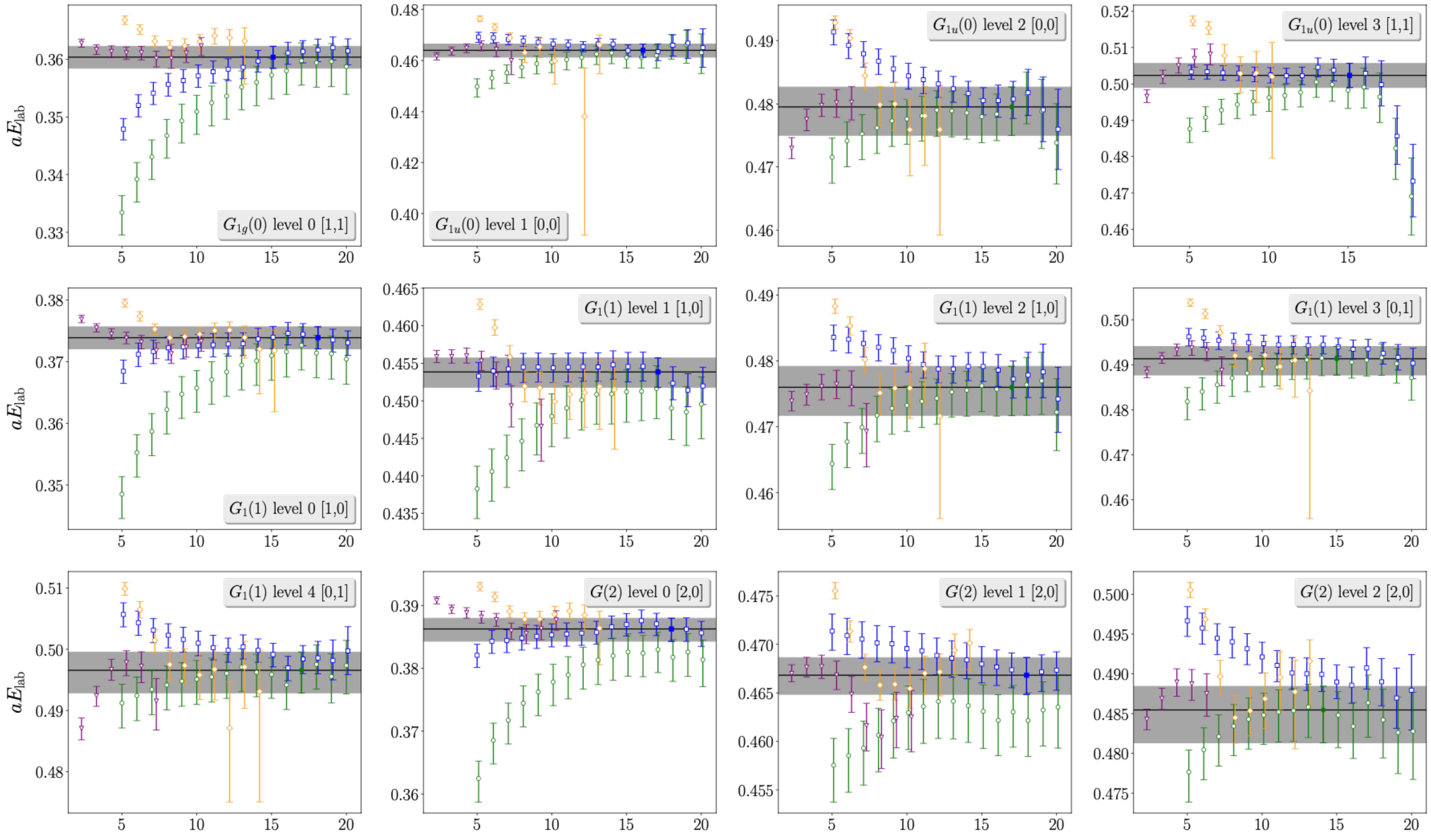
| Fit | $A_{00}$ | $A_{11}$ | $A_{01}$ | $B_{00}$ | $B_{11}$ | $B_{01}$ | $\chi^2/d.o.f.$ | AIC |
|---|---|---|---|---|---|---|---|---|
| a | 1.5(1.4) | −8.78(72) | 8.30(65) | | | | 15.68/(15 − 3) | −8.32 |
| b | 4.1(1.2) | −10.5(1.1) | 10.3(1.3) | | | −29(15) | 10.52/(15 − 4) | −11.48 |
| c | 2.3(1.3) | −8.62(58) | 7.60(80) | | −18(11) | | 12.29/(15 − 4) | −9.71 |
| d | 15.1(5.3) | −11.8(1.3) | 7.6(1.3) | −56(19) | | | 11.48/(15 − 4) | −10.52 |
| e | 9.6(6.2) | −12.7(3.4) | 11.1(2.8) | −23(26) | 18(31) | −37(29) | 9.70/(15 − 6) | −8.30 |

$$A_{00} = 4.1(1.8), \qquad A_{11} = -10.5(1.1),$$
$$A_{01} = 10.3(1.5), \qquad B_{01} = -29(18),$$




$$\det_{lm}\left[\tilde{K}(s) + F^{-1}(P,L)\right] = 0$$

# T-min plots



$aE_{\rm lab}$
$G_{1g}(0)$ level 0 [1,1]
$G_{1u}(0)$ level 1 [0,0]
$G_{1u}(0)$ level 2 [0,0]
$G_{1u}(0)$ level 3 [1,1]
$G_1(1)$ level 0 [1,0]
$G_1(1)$ level 1 [1,0]
$G_1(1)$ level 2 [1,0]
$G_1(1)$ level 3 [0,1]
$G_1(1)$ level 4 [0,1]
$G(2)$ level 0 [2,0]
$G(2)$ level 1 [2,0]
$G(2)$ level 2 [2,0]

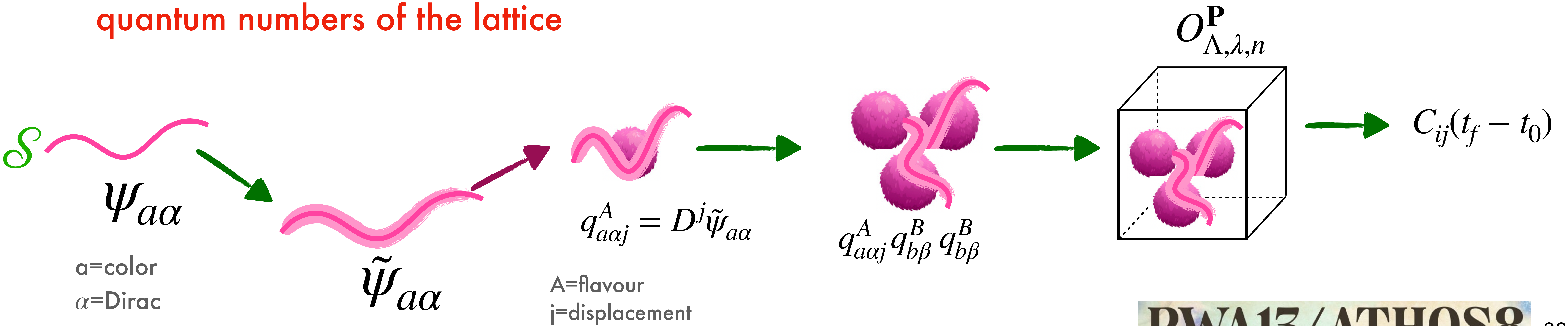
# Lattice QCD

## Correlation Matrices

- Single-hadrons built w/ Laplacian-Heaviside smeared quark fields
  LapH Method [Peardon et al., PRD **80** (2009) 054506]
- Time-to-time slice correlators of multi-hadron operators in larger volumes w/ Stochastic Laplacian-Heaviside
  [Morningstar et al., PRD **83** (2011)114505]
- Obtain $N \times N$ Hermitian correlation matrices projected to quantum numbers of the lattice

**Quantum numbers of Operators**

| | |
|---|---|
| Lattice Momentum | $\mathbf{P}$ |
| Irreducible Rep of Cubic Group | $\Lambda$ |
| Row of irrep | $\lambda$ |
| Spin, isospin,... | $S, I, ..$ |



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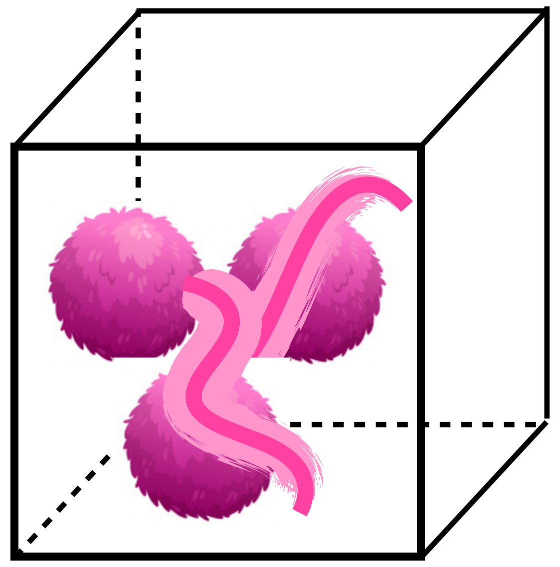
# Lattice QCD

## Correlation Matrices

### Define time-slice to time-slice operators

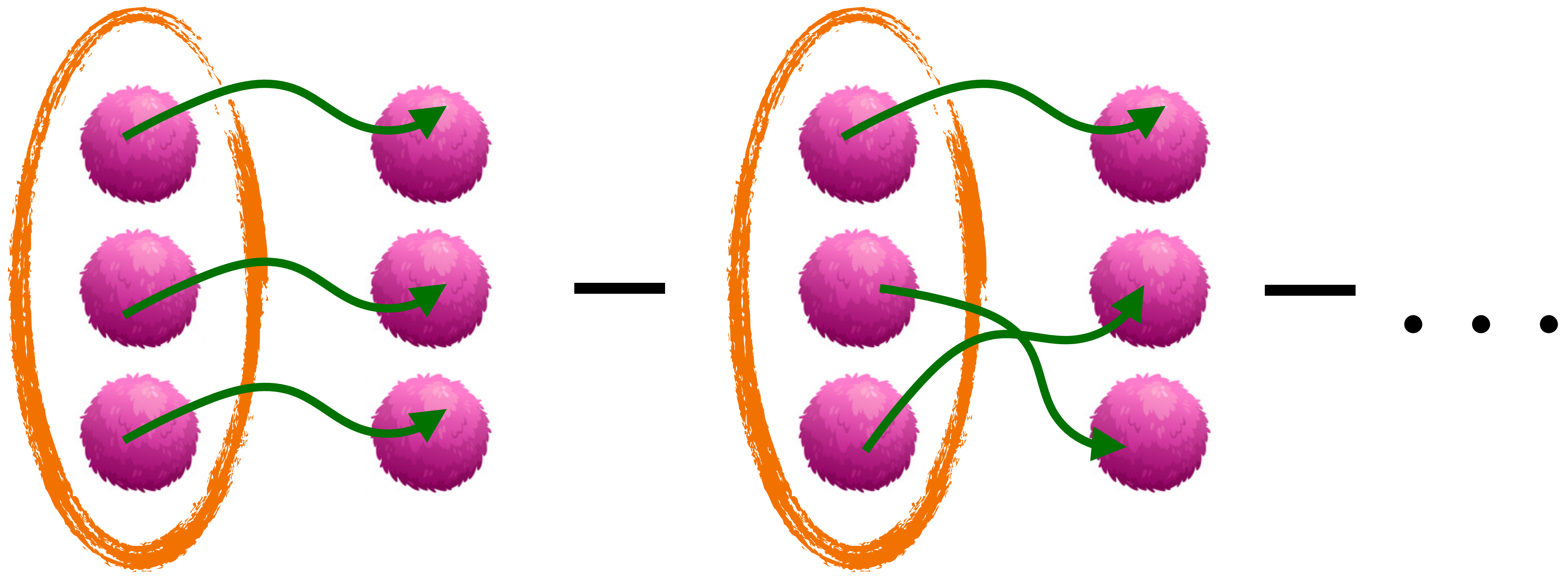
$O^{\mathbf{P}}_{\Lambda,\lambda,n}$



$$C_{ij}(t_f - t_0) = \sum \left\langle O^{\mathbf{P}}_{\Lambda,\lambda,n}(t_f)\, O^{\mathbf{P}}_{\Lambda,\lambda,n}(t_i) \right\rangle$$

$C_{ij}$

SLapH Estimated



## Finite Volume Irreps

Baryon groups

| $J$ | irreps, $\Lambda(\text{dim})$ |
|---|---|
| $\frac{1}{2}$ | $G_1(2)$ |
| $\frac{3}{2}$ | $H(4)$ |
| $\frac{5}{2}$ | $H(4) \oplus G_2(2)$ |
| $\frac{7}{2}$ | $G_1(2) \oplus H(4) \oplus G_2(2)$ |
| $\frac{9}{2}$ | $G_1(2) \oplus {}^1H(4) \oplus {}^2H(4)$ |

Meson groups

| Group | $\lvert\mathbf{p}L/2\pi\rvert^2$ | $\Gamma$ | $\ell$ |
|---|---|---|---|
| $O_h$ | 0 | $A_1^+$ | 0, 4 |
| | | $A_2^+$ | >4 |
| | | $E^+$ | 2, 4 |
| | | $T_1^+$ | 4 |
| | | $T_2^+$ | 2, 4 |
| $C_{4v}$ | 1 | $A_1$ | 0, 2,4 |
| | | $A_2$ | >4 |
| | | $B_1$ | 2, 4 |
| | | $B_2$ | 2, 4 |
| | | $E$ | 2, 4 |
| $C_{2v}$ | 2 | $A_1$ | 0, 2, 4 |
| | | $A_2$ | 2, 4 |
| | | $B_1$ | 2, 4 |
| | | $B_2$ | 2, 4 |
| $C_{3v}$ | 3 | $A_1$ | 0, 2, 4 |
| | | $A_2$ | >4 |
| | | $E$ | 2, 4 |

[Morningstar et al., PRD **88** (2013) 014511]