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Radiative decay of the resonant K^* and the $\gamma K \to K\pi$ amplitude from lattice QCD

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(for the Hadron Spectrum Collaboration)

$\gamma K \to \pi K$ and the K^* resonance from lattice QCD

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current induced transitions to hadron-hadron resonances

for example

low energy pion photoproduction, $\gamma N \to \pi N$ in which the Δ resonance appears

meson resonance production in semileptonic heavy-flavor decays, e.g. $B \to \ell\ell\, K^* \to \ell\ell\, K\pi$

or things not easily measurable but of theoretical interest, $\gamma\{\omega,\phi\} \to \{\pi\pi, K\bar{K}\}$

$f_0(980)$ flavor content & spatial size ?

can compute with lattice QCD – **finite-volume** matrix elements from three-point functions

"large" finite-volume corrections controlled by the hadron-hadron scattering amplitude

complication of presence of multiple J^P owing to cubic boundary

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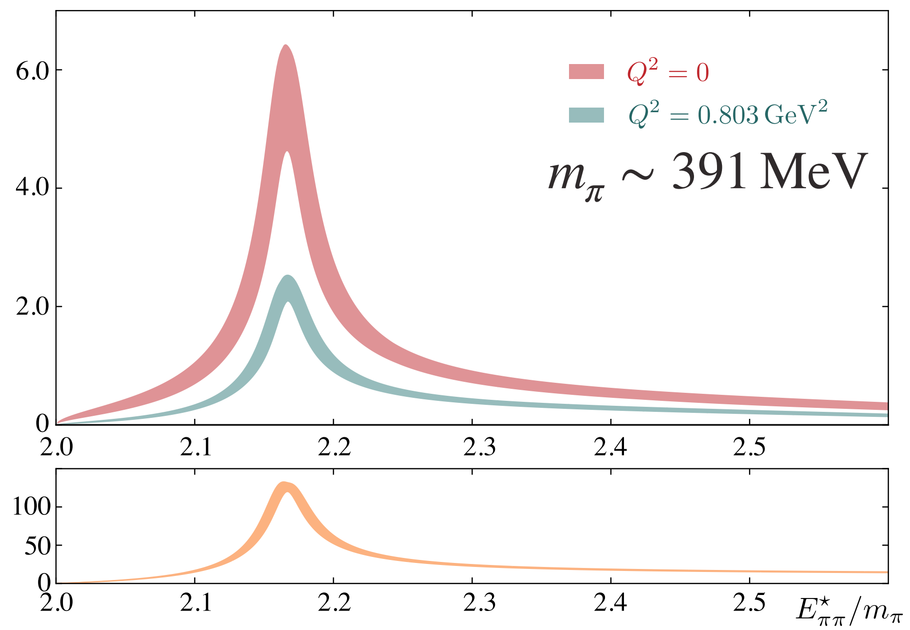
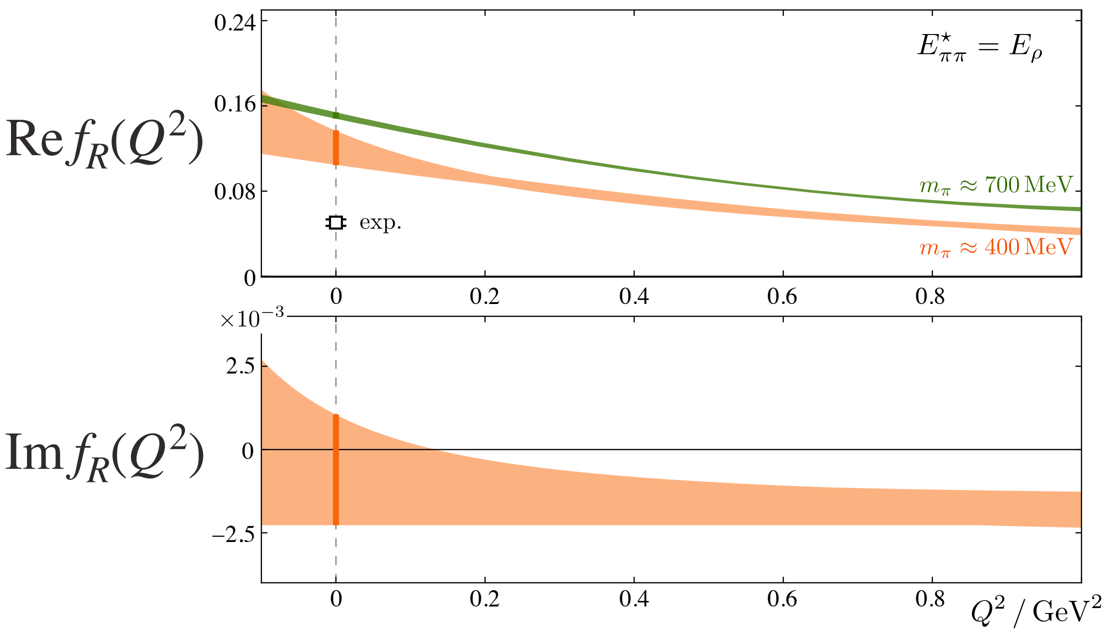
complication of presence of multiple J^P owing to cubic boundary

to date, only concrete application to $\gamma\pi \to \pi\pi$



analytic continuation to the ρ pole



but $\pi\pi$ is "special", no $J^P = 0^+$ with isospin=1, so $J^P = 1^-$ is always lowest partial wave can compute with lattice QCD – **finite-volume** matrix elements from three-point functions

"large" finite-volume corrections controlled by the hadron-hadron scattering amplitude

complication of presence of multiple J^P owing to cubic boundary

next simplest case $\gamma K \to \pi K$

πK with isospin=½ : 0^+ ("κ"), 1^- (K^*), …

no amplitude $\gamma K \to (\pi K)_{0^+}$ but still an effect from 0^+ in finite-volume …

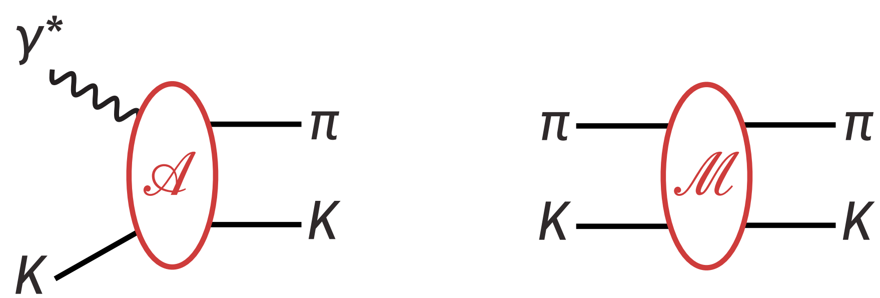
the process of interest is

current + stable hadron → resonance → hadron–hadron pair

actually don't really need there to be a resonance

e.g. $\gamma K \to \pi K$ in a *P*-wave

after the current produces $K\pi$ … … $K\pi$ strongly rescatters



$$\mathcal{H}(Q^2, E^\star_{K\pi}) \equiv \langle K \,|\, j \,|\, K\pi; E^\star_{K\pi}\rangle$$

suspressing kinematic variables, helicity and lorentz indices

$$= \mathcal{A}\big(Q^2, E^\star_{K\pi}\big) \cdot \frac{1}{k^\star_{K\pi}} \cdot \mathcal{M}^{\ell=1}(E^\star_{K\pi})$$

removing an 'excess' P-wave threshold factor

unitarity insists that production amplitude, $\mathcal{A}$, is **real** in the region of interest

(free of singularities, polynomial in $(E^\star_{K\pi})^2$)

Omnès function also an option here

★ means cm-frame

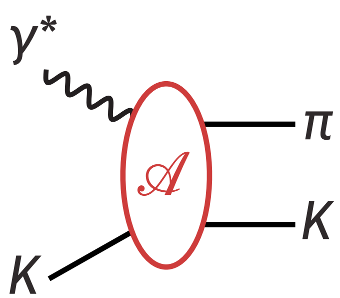
resonance transition form-factors (in infinite volume)

the process of interest is

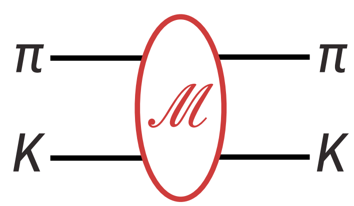
current + stable hadron → resonance → hadron–hadron pair

e.g. $\gamma K \to \pi K$ in a P-wave

after the current produces $K\pi$ …



… $K\pi$ strongly rescatters



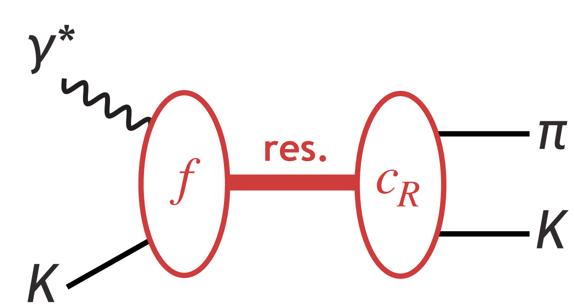
$$\mathcal{H}(Q^2, E^\star_{K\pi}) \equiv \langle K \,|\, j \,|\, K\pi; E^\star_{K\pi} \rangle$$

$$= \mathcal{A}\big(Q^2, E^\star_{K\pi}\big) \cdot \frac{1}{k^\star_{K\pi}} \cdot \mathcal{M}^{\ell=1}(E^\star_{K\pi})$$

strong scattering amplitude, $\mathcal{M}$, can have resonance poles $\mathcal{M}^{\ell=1}(s) \sim \dfrac{c_R^2}{s_0 - s}$ $\sqrt{s_0} = m_R - i\tfrac{1}{2}\Gamma_R$

hence $\mathcal{H}(Q^2, s) \sim \dfrac{c_R\, f(Q^2)}{s_0 - s}$

residue at the complex pole



infinite volume

continuum of scattering states

$\mathscr{M}(E^\star)$

finite volume

discrete spectrum of states

$E_n(L)$

$E_n(L)$ are solutions of

$$\det\left[F^{-1}(E^\star;L) + \mathscr{M}(E^\star)\right] = 0$$

kinematic finite-volume functions

spectra obtained from two-point correlation functions $C_{ij}(t) = \langle 0 \,|\, \mathcal{O}_i(t)\, \mathcal{O}_j^\dagger(0) \,|\, 0 \rangle$

evaluate with a large basis of operators to form a matrix

and diagonalize $\mathbf{C}(t)\, v_n = \lambda_n(t,t_0)\, \mathbf{C}(t_0)\, v_n$

eigenvalues given energies

$\lambda_n(t,t_0) \sim e^{-E_n(t-t_0)}$

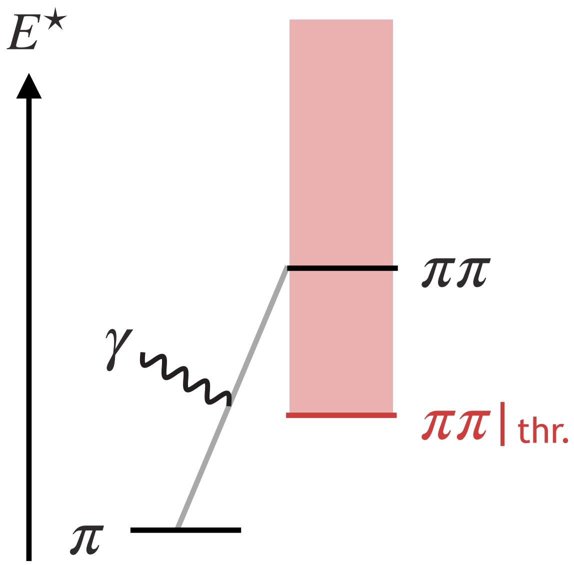
eigenvectors give **optimal operators**

$\Omega_n \sim \sum_i (v_n)_i\, \mathcal{O}_i$

produce just one state in the 'tower'

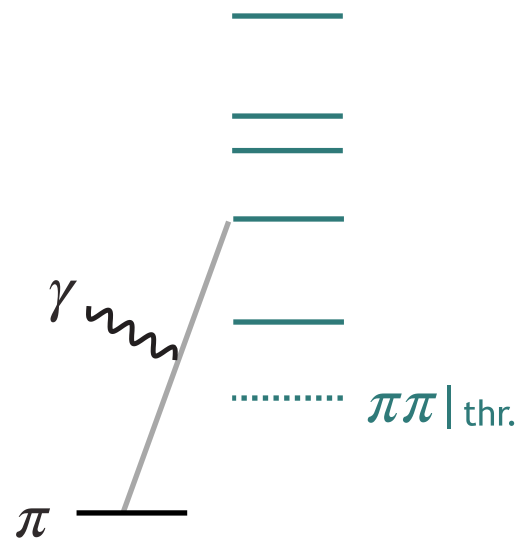
current matrix-elements in a finite-volume – cartoon of $\gamma\pi \to \pi\pi$

infinite volume



can transition
to any energy
in the $\pi\pi$ continuum

finite volume



can only transition
to one of the discrete
f.v. eigenstates

finite-volume matrix element

$${}_L\langle \pi | j | \pi\pi; E_n^\star \rangle_L$$

single hadron state

$$|\pi\rangle_L \sim |\pi\rangle_\infty + \mathcal{O}(e^{-m_\pi L})$$

hadron-hadron state

$$|\pi\pi; E_n^\star\rangle_L \sim \sqrt{\tilde{R}_n}\, |\pi\pi; E_{\pi\pi}^\star = E_n^\star\rangle_\infty$$

effective f.v.
normalization

c.f. "Lellouch-Lüscher" factor

$$\tilde{R}_n(L) \equiv 2E_n \cdot \lim_{E \to E_n} (E - E_n) \Big(F^{-1}(E^\star; L) + \mathcal{M}(E^\star) \Big)^{-1}$$

effective f.v. normalization
depends on the scattering
amplitude

what's different in $\gamma K \to \pi K$?

cubic nature of lattice puts spectra in **irreducible representations** of a **reduced group of rotations**

in $\pi\pi$ case, this has limited impact because even and odd ℓ are in different isospins

consequence of Bose symmetry

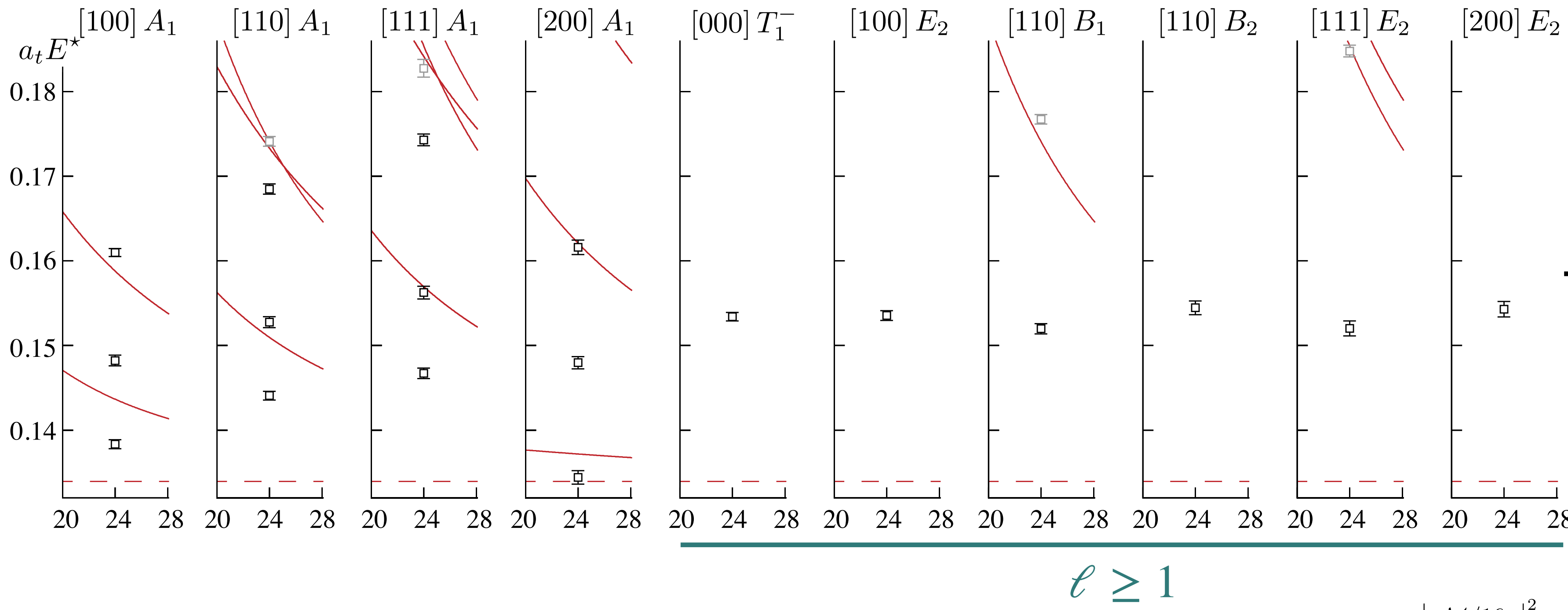
in πK case, there is no Bose symmetry

$\mathbf{p}_{K\pi}\ \Lambda$	$[000]A_1^+$	$[000]T_1^-$	$[100]A_1$	$[100]E_2$	$[110]A_1$	$[110]B_1$	$[110]B_1$	$[111]A_1$	$[111]E_2$	$[200]A_1$
$\ell \leq 2$	0	1	$0,1,2$	$1,2$	$0,1,2$	$1,2$	$1,2$	$0,1,2$	$1,2$	$0,1,2$

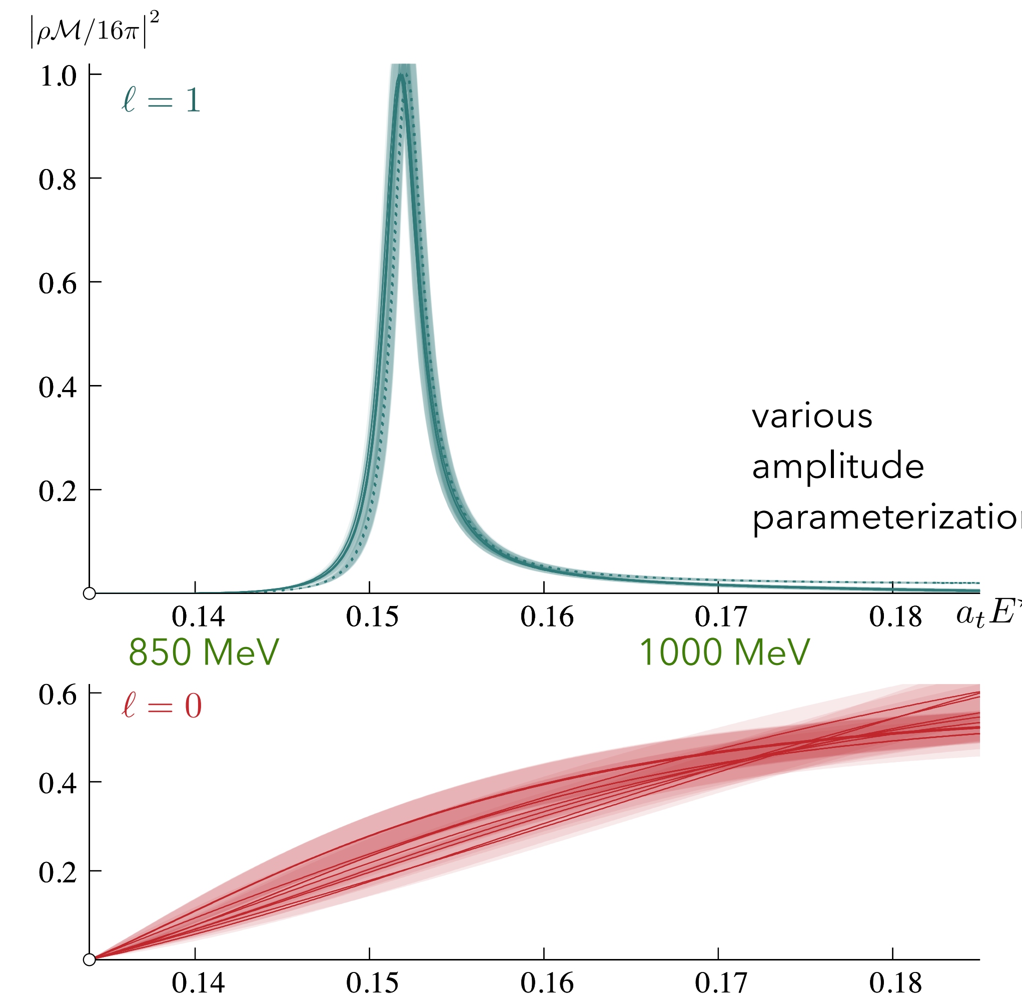
spectrum in some irreps sensitive to scattering in both $\ell = 0, \ell = 1$

finite-volume spectrum → *S*,*P*-wave amplitudes

$m_\pi \sim 284\,\mathrm{MeV}$



$$\det\left[F^{-1}(E^\star; L) + \mathcal{M}(E^\star)\right] = 0$$



various amplitude parameterizations

$\ell = 2$ found to be negligible in this energy region relation between finite-volume matrix element, and infinite-volume matrix element, $\mathscr{H}$

$$\left| {}_L\langle K \,|\, j \,|\, K\pi \rangle_L \right| \propto \left(\mathscr{H} \cdot \tilde{R}_n \cdot \mathscr{H} \right)^{1/2}$$

where the residue of the finite-volume hadron-hadron propagator appears

$$\tilde{R}_n(L) \equiv 2E_n \cdot \lim_{E \to E_n} \left(E - E_n\right) \Big(\underbrace{F^{-1}(E^\star; L)}_{\text{matrix in } \ell = 0,1} + \underbrace{\mathscr{M}(E^\star)}_{\text{diagonal matrix in } \ell = 0,1} \Big)^{-1}$$

$E_n(L)$ are solutions of

$$\det\left[F^{-1}(E^\star; L) + \mathscr{M}(E^\star) \right] = 0$$

what's different in $\gamma K \to \pi K$?

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using an eigen-decomposition $F + \mathscr{M}^{-1} = \sum_i \mu_i \, \mathbf{w}_i \, \mathbf{w}_i^\intercal$ $\qquad \mathbf{w}_i = \begin{pmatrix} \mathbf{w}_i^{\ell=0} \\ \mathbf{w}_i^{\ell=1} \end{pmatrix}$

the residue factorizes $\tilde{R}_n = \left(- \dfrac{2E_n^\star}{\underbrace{\mu_0^{\star\,\prime}}_{\text{slope of zero crossing eigenvalue}}} \right) \mathscr{M}^{-1} \mathbf{w}_0 \, \underbrace{\mathbf{w}_0^\intercal}_{\text{zero crossing eigenvector}} \mathscr{M}^{-1}$

only the zero-crossing eigenvalue is relevant

what's different in $\gamma K \to \pi K$?

relation between finite-volume matrix element, and infinite-volume matrix element, $\mathscr{H}$

$$\left| {}_L\langle K \,|\, j \,|\, K\pi \rangle_L \right| \propto \left(\mathscr{H} \cdot \tilde{R}_n \cdot \mathscr{H} \right)^{1/2}$$

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using an eigen-decomposition $F + \mathscr{M}^{-1} = \sum_i \mu_i \, \mathbf{w}_i \, \mathbf{w}_i^\mathsf{T}$

the residue factorizes $\tilde{R}_n = \left(-\frac{2E_n^\star}{\mu_0^{\star\prime}} \right) \mathscr{M}^{-1} \mathbf{w}_0 \, \mathbf{w}_0^\mathsf{T} \mathscr{M}^{-1}$

$\mu_0^{\star\prime}$: slope of zero crossing eigenvalue

$\mathbf{w}_0^\mathsf{T}$: zero crossing eigenvector

and the net finite-volume correction is $F(Q^2, E_{K\pi}^\star = E_n^\star) = \frac{1}{\tilde{r}_n(L)} F_L(Q^2, E_n^\star)$

remember, no $\gamma K \to (K\pi)_{\ell=0}$ amplitude

$$\mathscr{H} = \mathscr{A} \cdot \frac{1}{k_{K\pi}^\star} \cdot \mathscr{M}^{\ell=1}$$

$$\mathscr{A} = \underbrace{K}_{\text{kinematic factor}} \cdot \underbrace{F}_{\text{form-factor}}$$

where $\tilde{r}_n(L) = \sqrt{-\frac{2E_n^\star}{\mu_0^{\star\prime}}} \left| \mathbf{w}_0^{\ell=1} \right| \frac{1}{k_{K\pi}^\star}$

evaluating $\gamma K \to \pi K$?

$$F(Q^2, E^\star_{K\pi} = E^\star_n) = \frac{1}{\tilde{r}_n(L)} F_L(Q^2, E^\star_n)$$

extract finite-volume form-factor, $F_L(Q^2, E^\star_n)$, from lattice QCD computed three-point functions

compute the finite-volume corrections, $\tilde{r}_n(L)$, using lattice QCD obtained scattering amplitudes

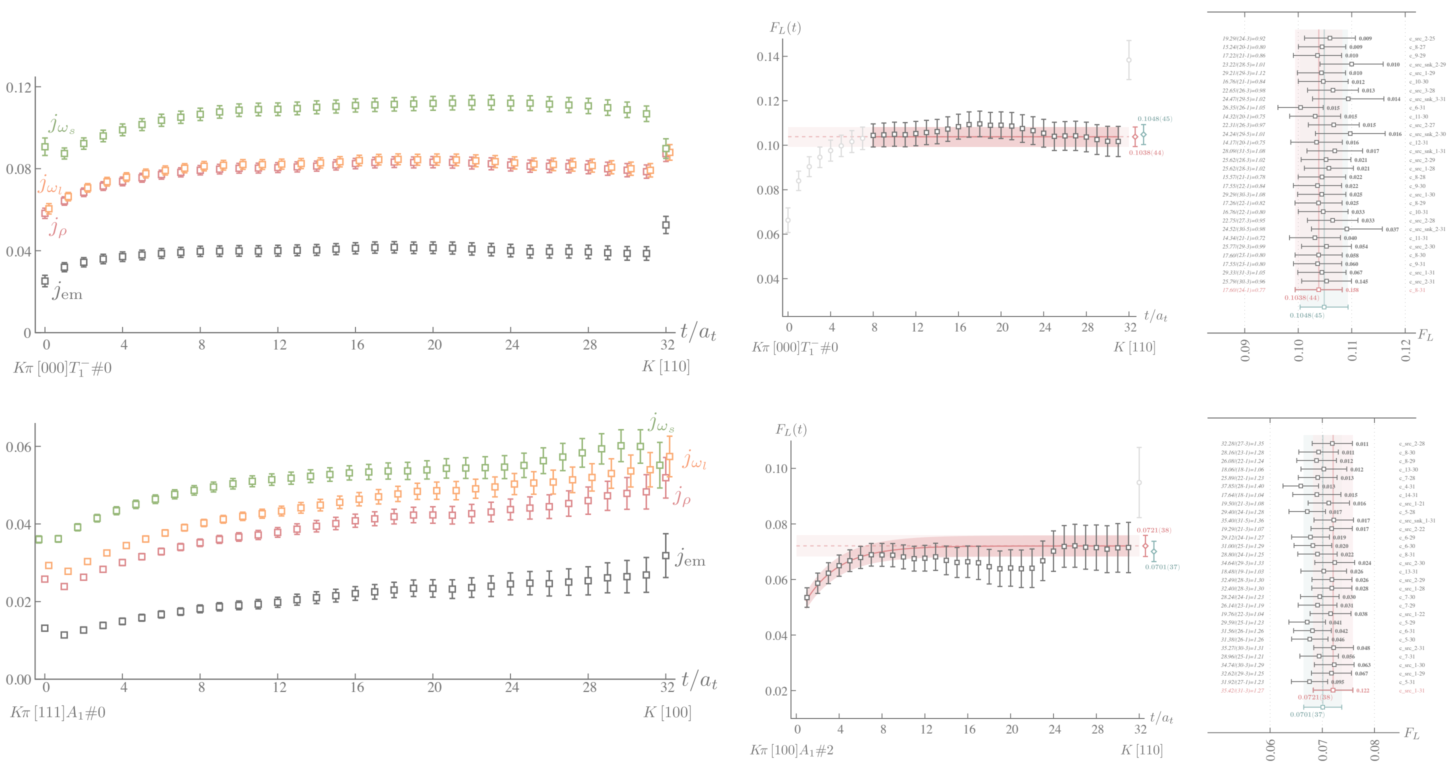
$$\tilde{r}_n(L) = \sqrt{-\frac{2E^\star_n}{\mu_0^{\star\,\prime}}} \left| \mathbf{w}_0^{\ell=1} \right| \frac{1}{k^\star_{K\pi}}$$

three-point functions

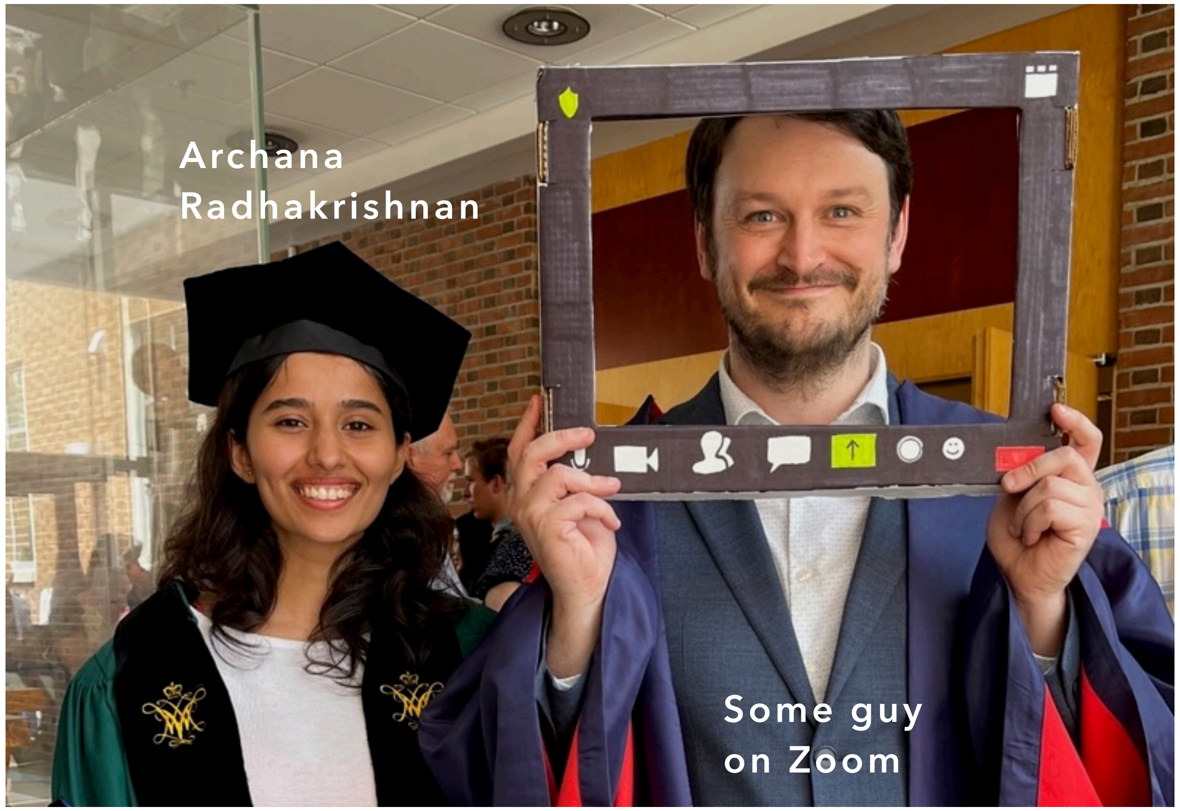
$$\langle 0 | \Omega_K(\mathbf{p}_K, \Delta t)\, j(\mathbf{q}, t)\, \Omega^\dagger_{K\pi}(\mathbf{p}_{K\pi}, 0) | 0 \rangle = e^{-E_K(\Delta t - t)}\, e^{-E_n t} \cdot K \cdot F_L(Q^2, E_n^\star) + \dots\,,$$

just a single $\Delta t = 32\, a_t$

a range of kaon and current three-momenta
for each kaon-pion discrete energy level



COVID-lockdown-era project



three-point functions — our kinematical coverage

$$\langle 0 | \Omega_K(\mathbf{p}_K, \Delta t)\, j(\mathbf{q}, t)\, \Omega^\dagger_{K\pi}(\mathbf{p}_{K\pi}, 0) | 0 \rangle = e^{-E_K(\Delta t - t)}\, e^{-E_n t} \cdot K \cdot F_L(Q^2, E^\star_n) + \dots ,$$

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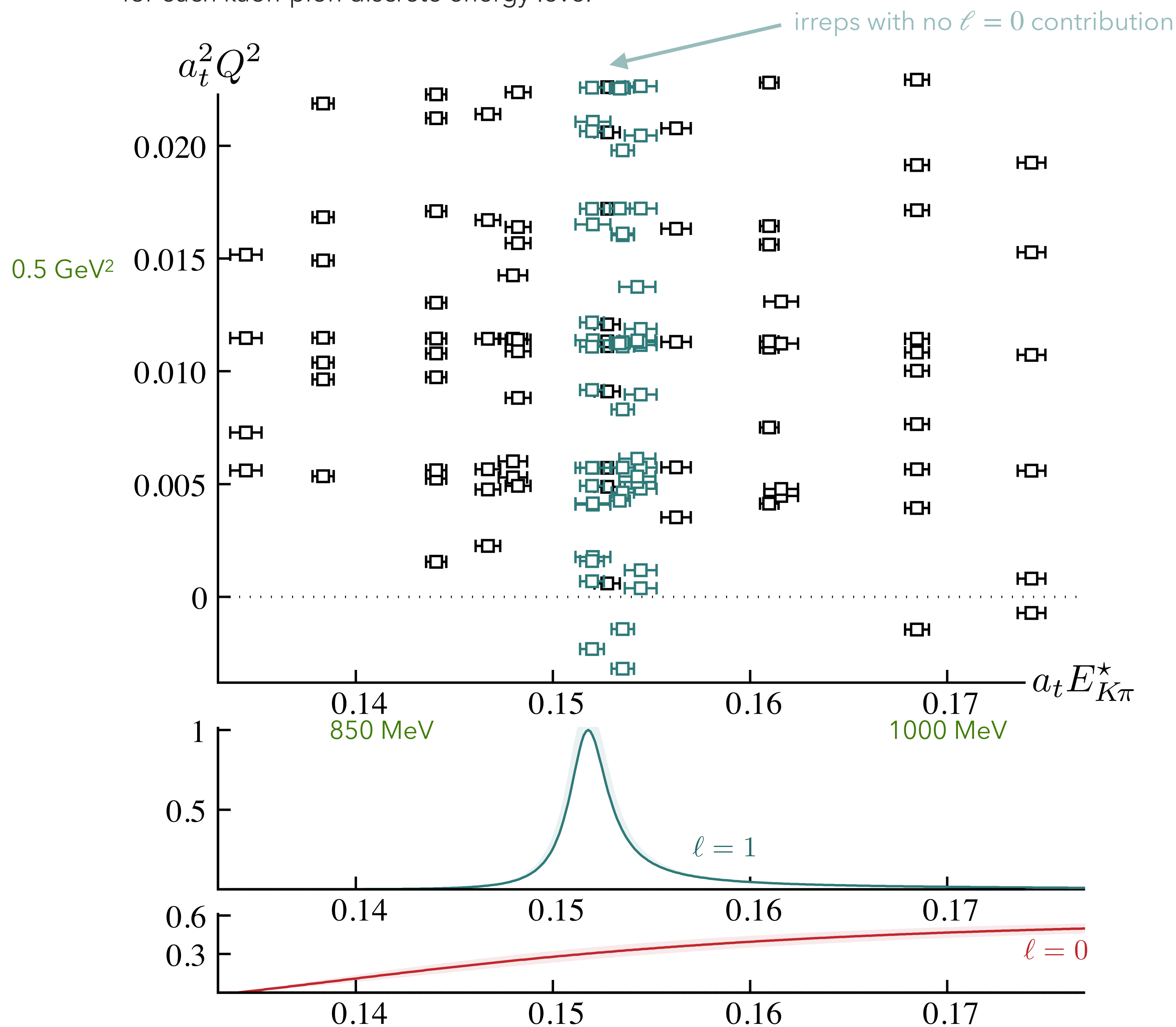
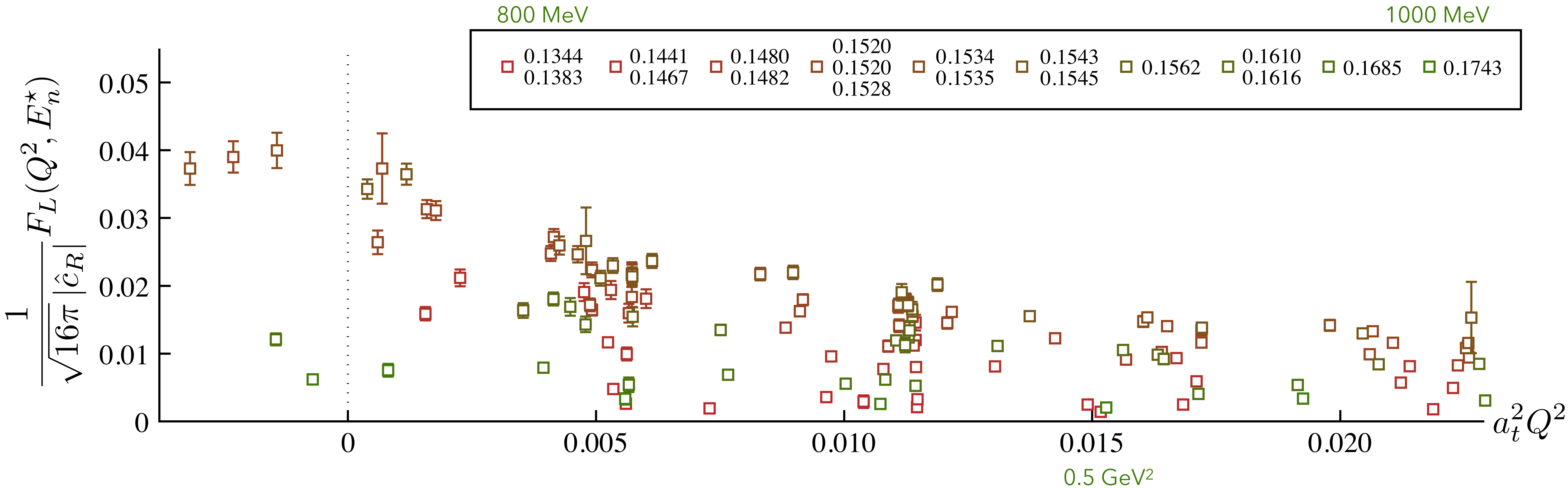


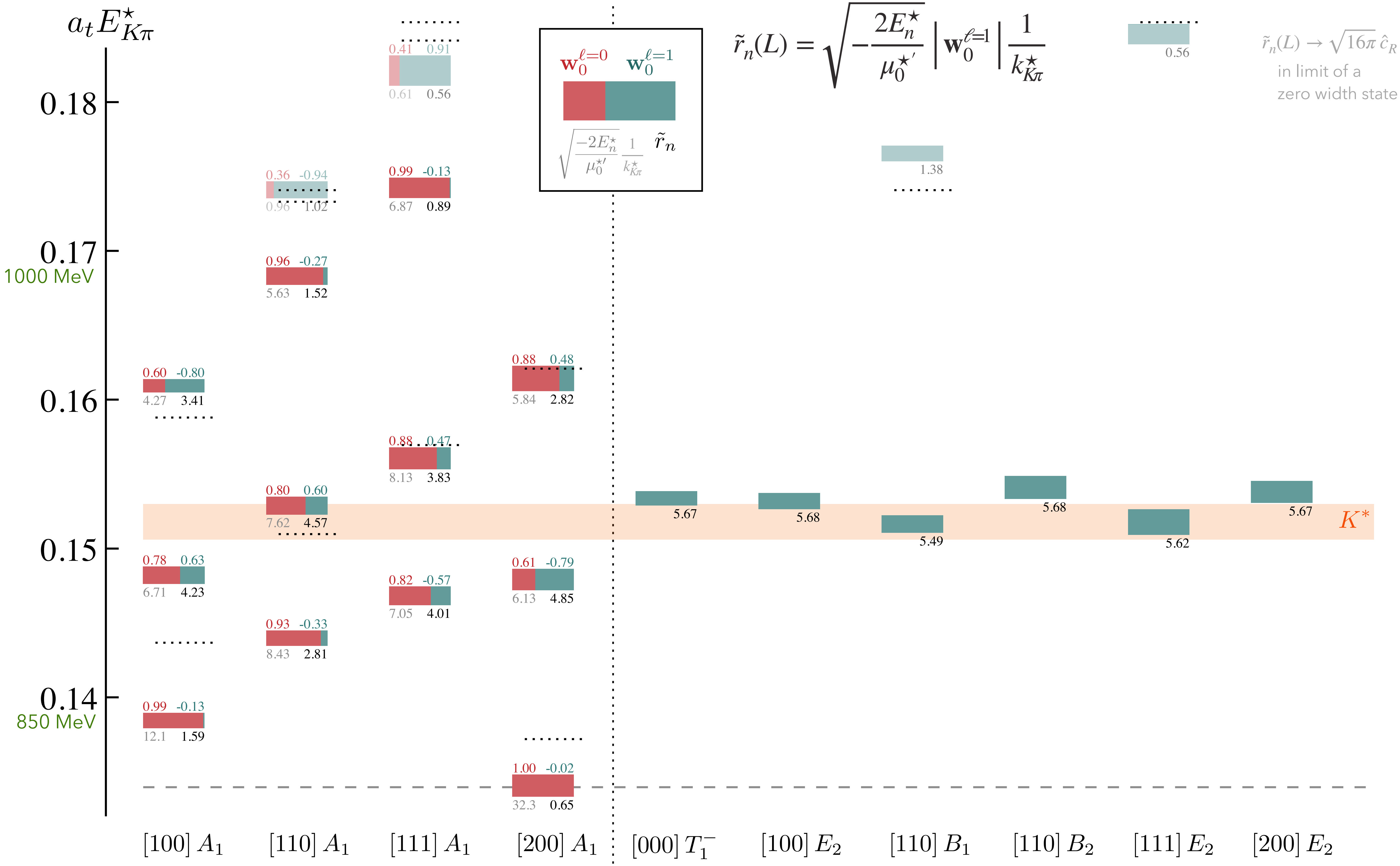
finite-volume form-factor



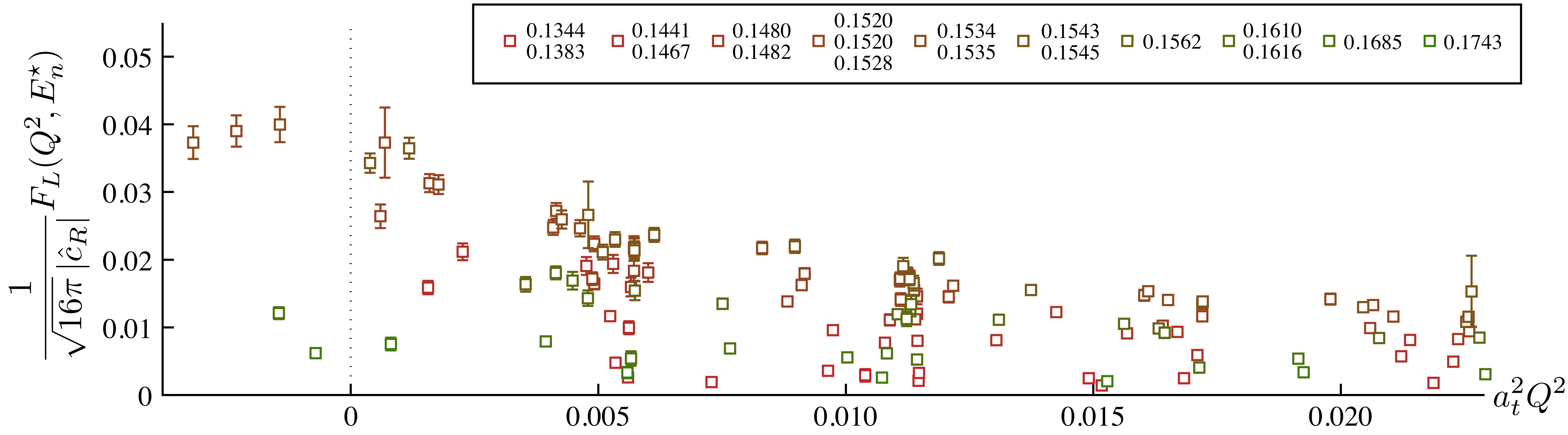
finite-volume correction factors

$$F(Q^2, E^\star_{K\pi} = E^\star_n) = \frac{1}{\tilde{r}_n(L)} F_L(Q^2, E^\star_n)$$

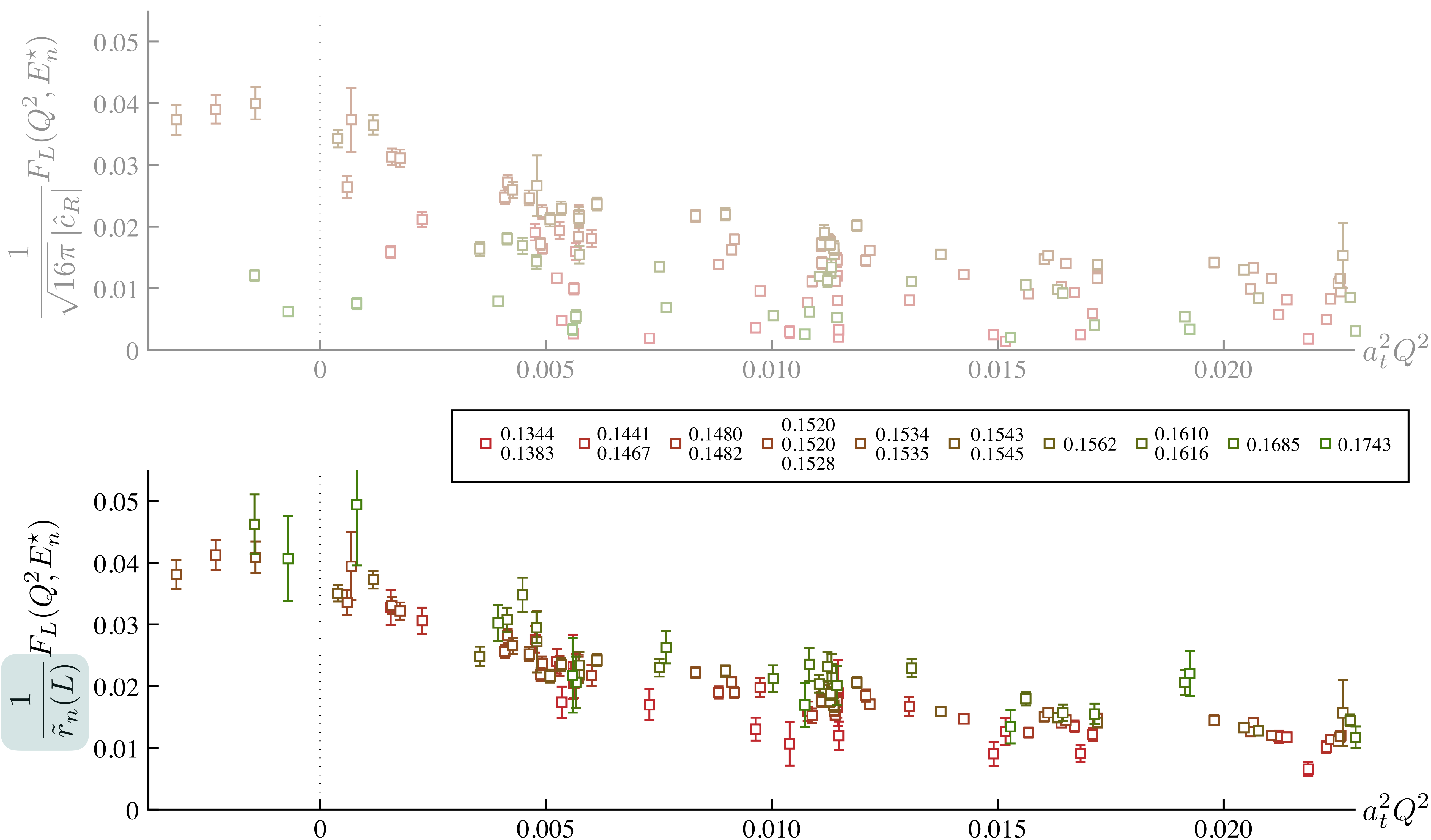
$$\tilde{r}_n(L) = \sqrt{-\frac{2E^\star_n}{\mu_0^{\star\prime}}}\left|\mathbf{w}_0^{\ell=1}\right|\frac{1}{k^\star_{K\pi}}$$



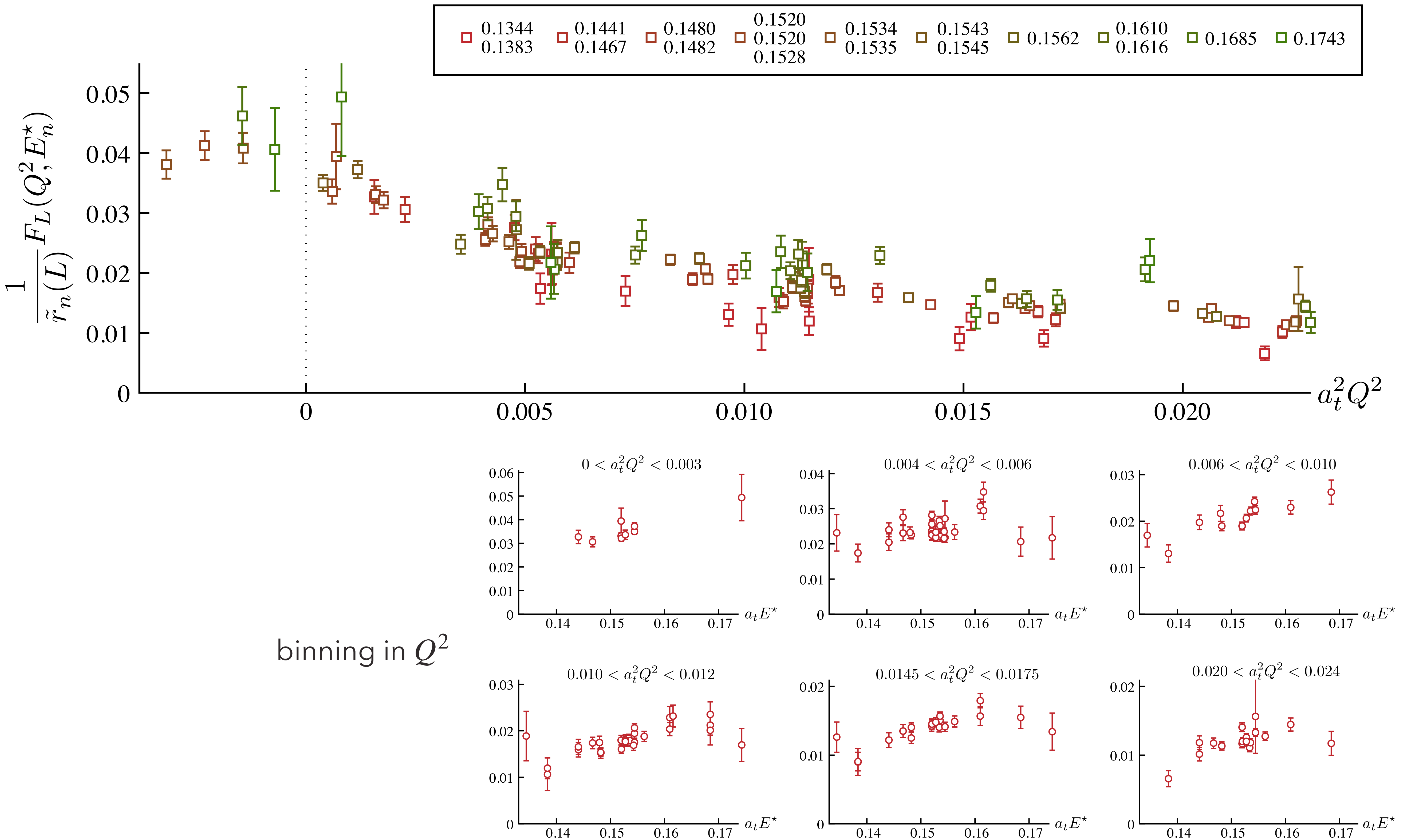
finite-volume form-factor



infinite-volume form-factor



infinite-volume form-factor



binning in Q^2

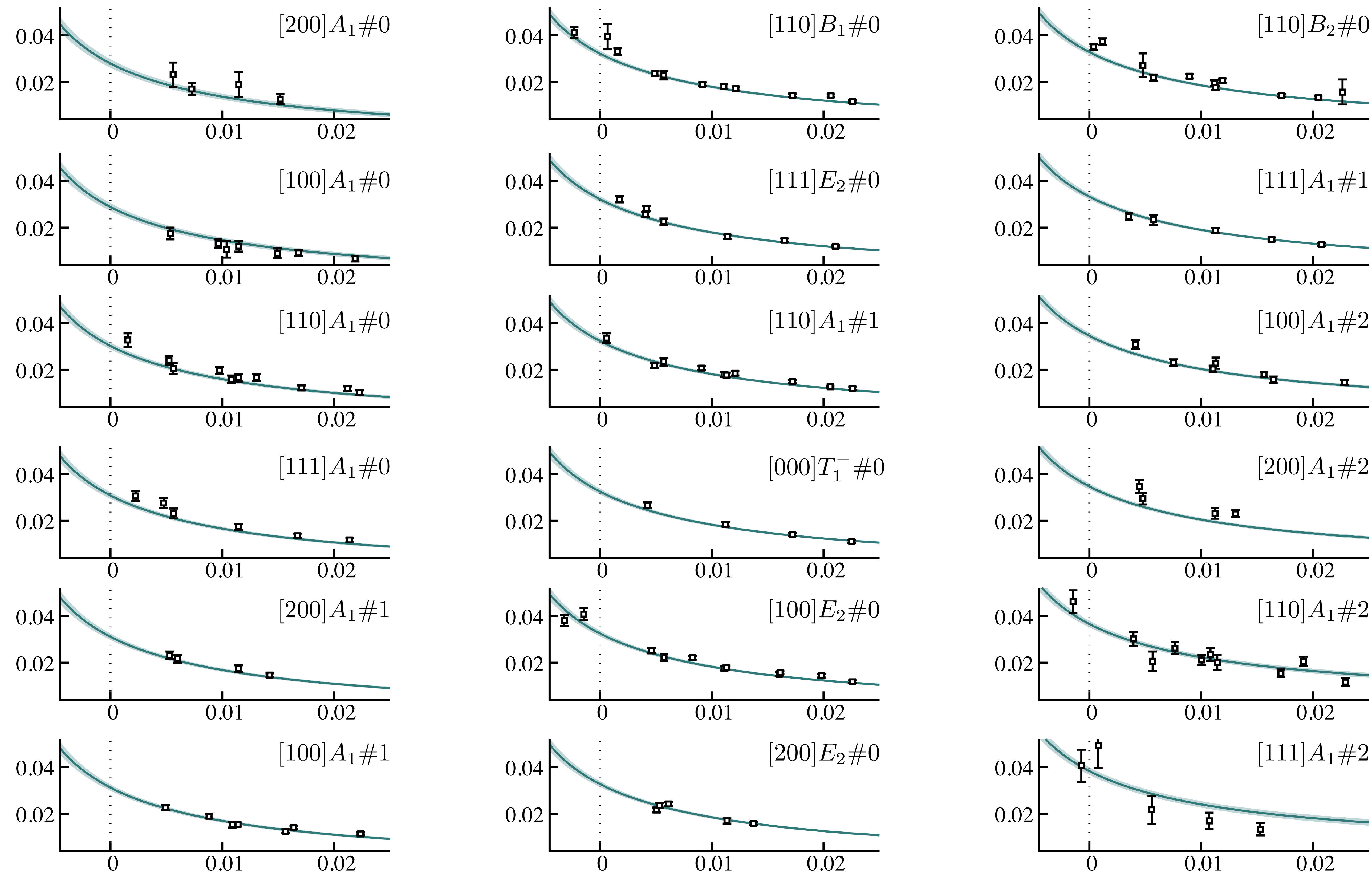
modest energy dependence as expected

global fitting of all the infinite-volume form-factor data

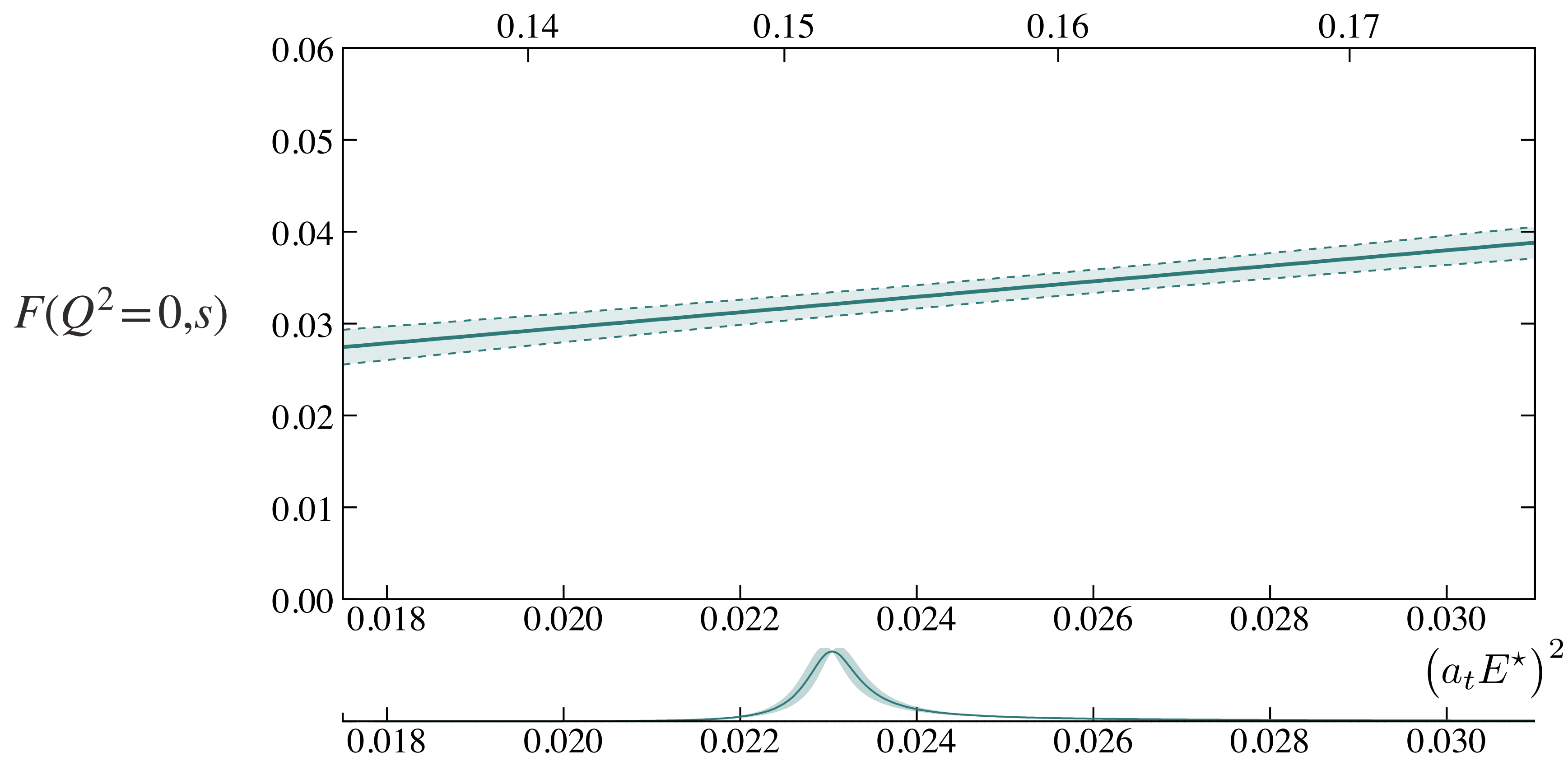
energy dependent conformal mapping fit here

$$F(Q^2, s) = \left(b_{0,0} + b_{0,1}\frac{s-s_0}{s_0}\right) + b_{1,0}\cdot\left(z(Q^2)-z(0)\right) + b_{2,0}\cdot\left(z(Q^2)-z(0)\right)^2$$

128 data points, 4 free params



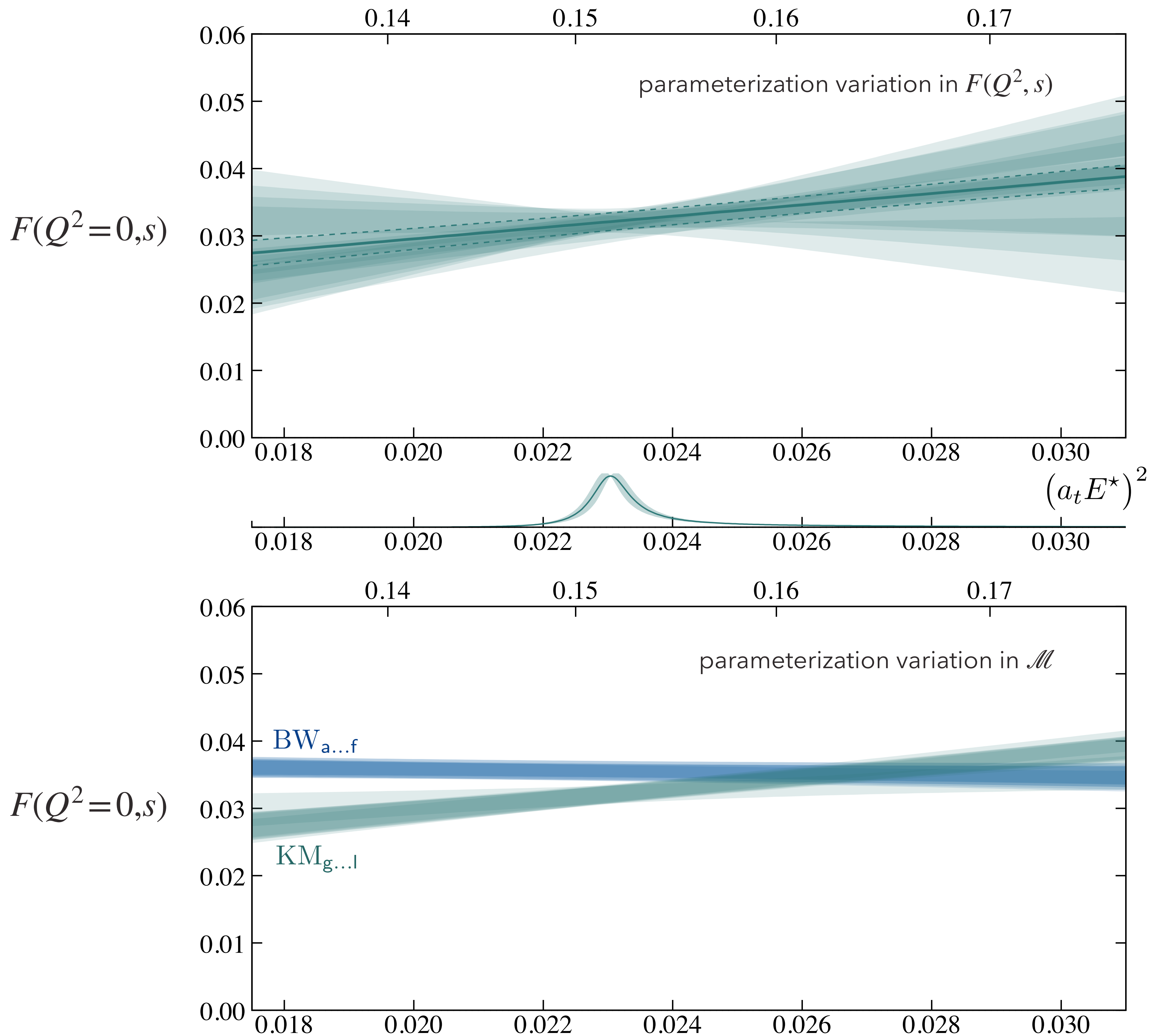
global fitting of all the infinite-volume form-factor data



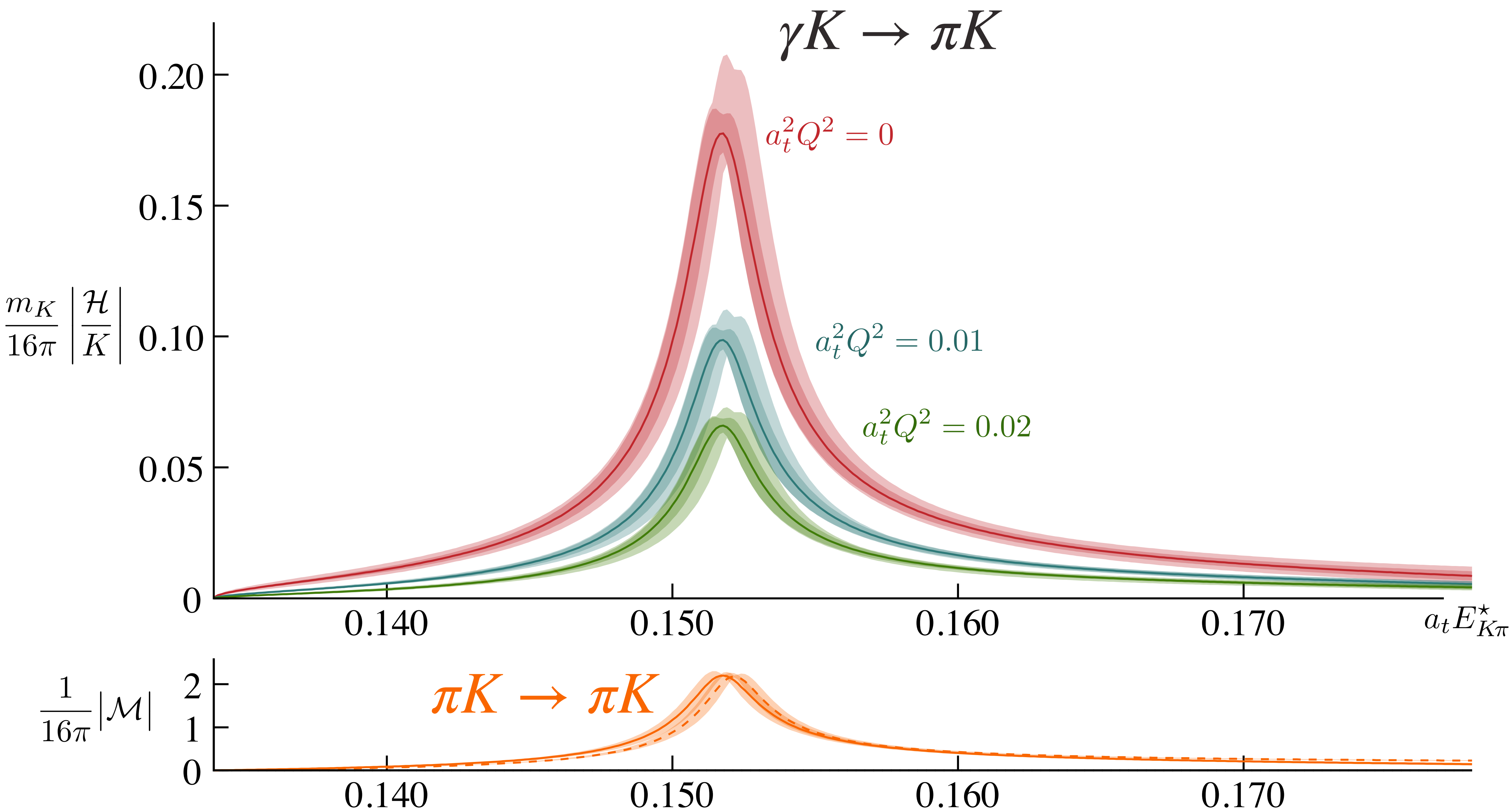
modest energy dependence as expected

energy dependent conformal mapping fit here $F(Q^2,s) = \left(b_{0,0} + b_{0,1}\frac{s-s_0}{s_0}\right) + b_{1,0}\cdot\left(z(Q^2)-z(0)\right) + b_{2,0}\cdot\left(z(Q^2)-z(0)\right)^2$

parameterization variation

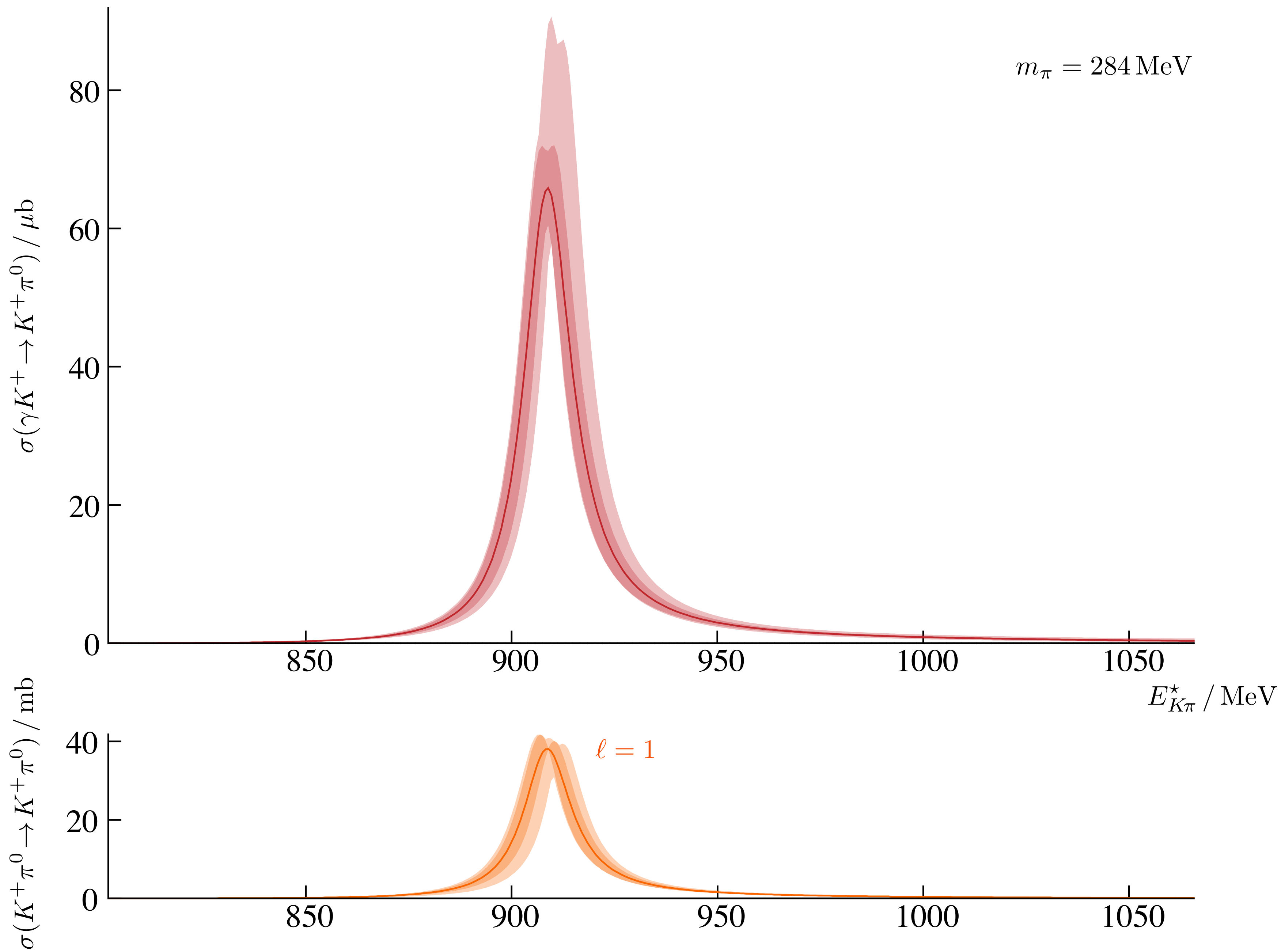


the transition amplitude

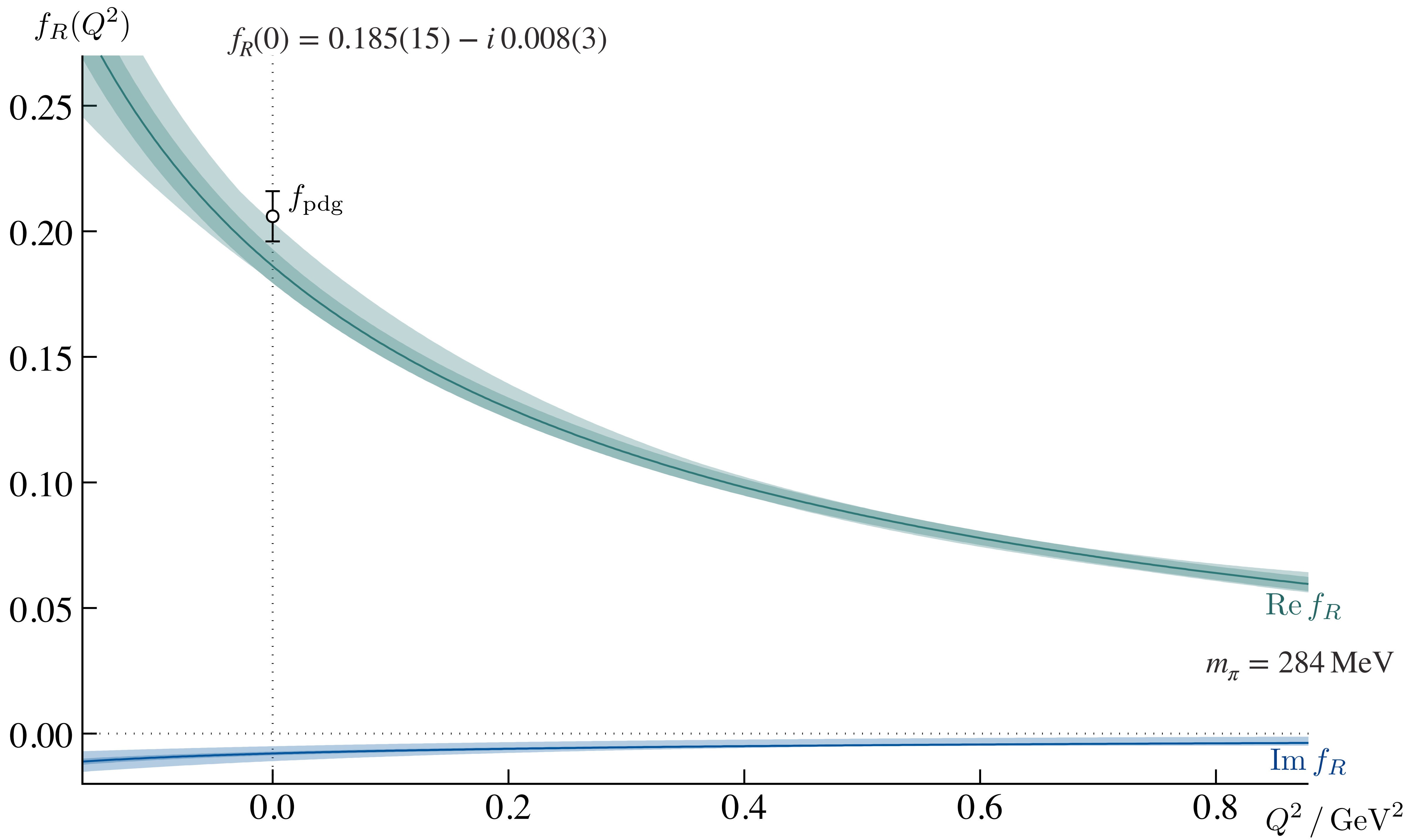


$$\left|\mathcal{H}\left(\gamma K^+ \to K^+\pi^0\right)\right| = \frac{1}{\sqrt{3}}\left|\mathcal{H}\left(\gamma K^+ \to (K\pi)_{1/2,+1/2}\right)\right|.$$

$$\sigma\left(\gamma K^+ \to K^+\pi^0\right) = \frac{1}{3}\alpha\frac{k^\star_{K\gamma}}{k^\star_{K\pi}}\frac{1}{m_K^2}\left|F\,\mathcal{M}\right|^2$$

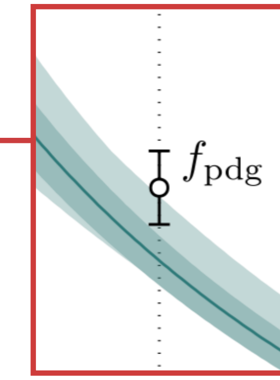


resonance transition form-factor

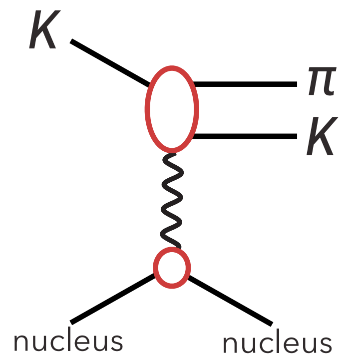


$$\mathscr{H}(Q^2, s) \sim \frac{c_R\, f(Q^2)}{s_0 - s}$$

experimental determination



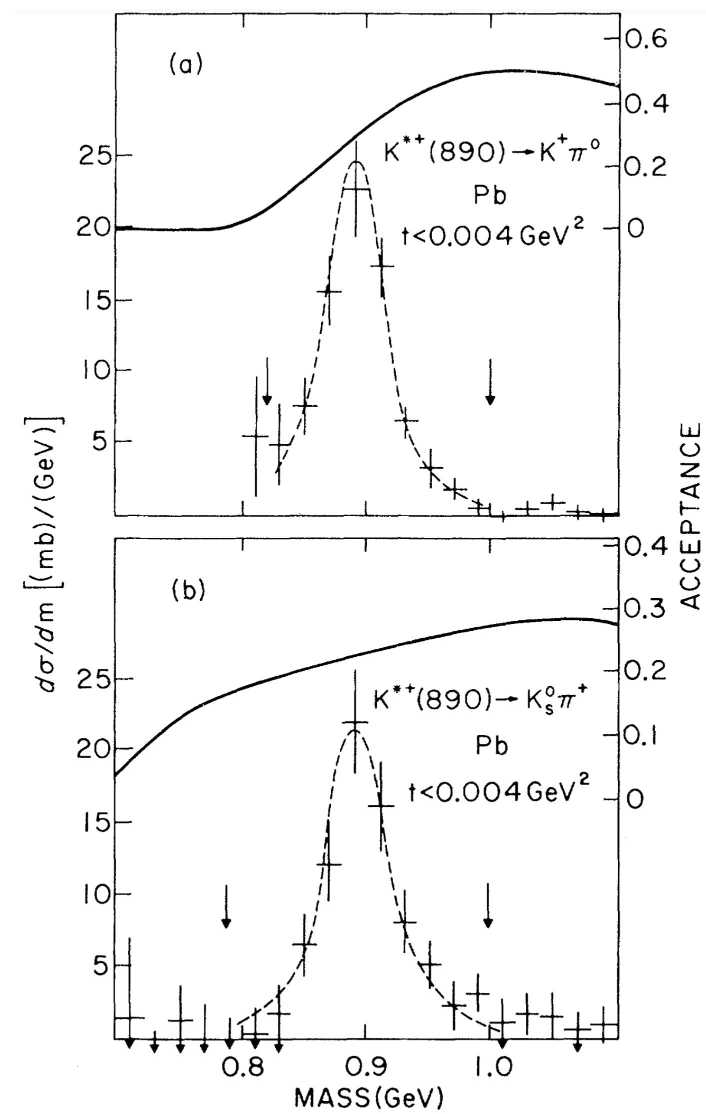
handful of Primakoff experiments in the 70s, 80s



(very forward production of πK using $K^\pm, K_L^0$ beams on nuclear targets)

pdg average of a couple of experiments $\Gamma(K^{*\pm} \to K^\pm \gamma) = 50(5)\,\mathrm{keV}$

$\Gamma(K^{*0} \to K^0 \gamma) = 116(10)\,\mathrm{keV}$



very simplistic analysis scheme

$$\frac{d\sigma}{dt\,dm} = 3\pi\alpha Z^2 \frac{\Gamma_o}{k_o^3} \frac{t - t_{min,o}}{t^2} \,|\, f_{C_o} |^2 BW(m);$$

$$BW(m) = \frac{1}{\pi} \frac{m^2 \Gamma^{tot}}{[m^2 - m_o^2] + [m_o \Gamma^{tot}]^2} \left| \frac{g(k)}{g(k_o)} \right|^2$$

$$\Gamma(K^{*+} \to K^+ \gamma) = \frac{4}{3} \alpha \frac{k^{\star 3}_{K\gamma}}{m_K^2} |f|^2$$

loss of rigor here
this is not the pole residue

$$|f_{\mathrm{pdg}}| = 0.206(10)$$

stress-tested the **1+J→2** finite-volume formalism in a case with an ‘unwanted’ lower partial wave

similar formalism describes **coupled-channels**
see **Felipe Ortega**’s talk (tomorrow) for an application

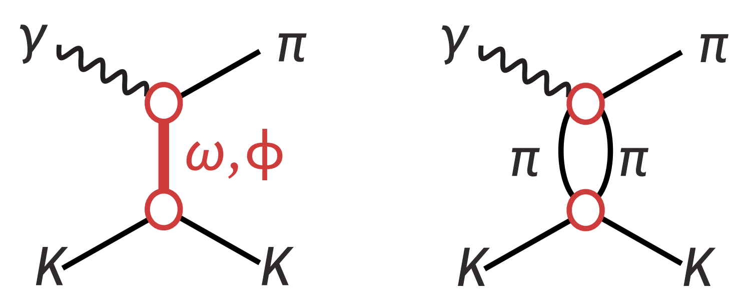
consistent **production amplitude** at 128 kinematic points, shows expected **mild energy dependence**

K^* **transition form-factor** extracted from scattering resonance pole,
reasonable ball-park agreement with experiment (considering computation at ‘wrong’ light quark mass)

dispersive approach (Dax, Stamen, Kubis)

inputs:

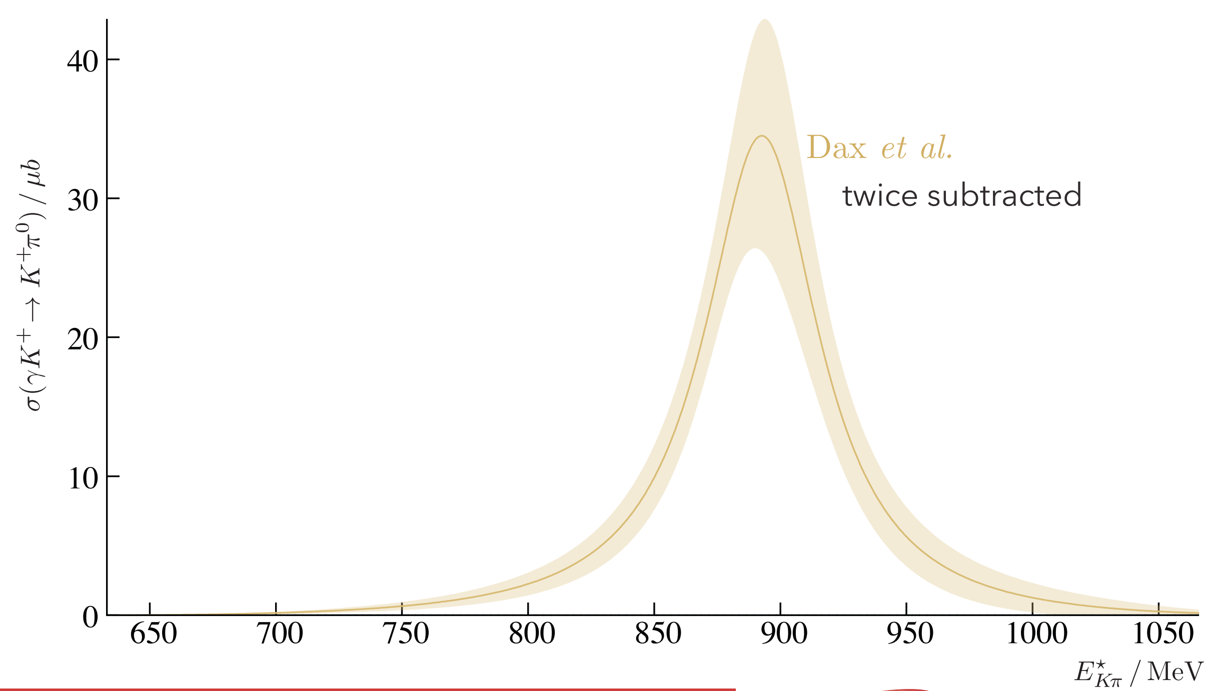
parameterized t-channel amplitudes



s-channel $K\pi$ scattering – Omnès from elastic phase-shift

"free" params: dispersion subtraction constants (one or two)

constrain with
(a) chiral anomaly
(b) experimental width $K^* \to K\gamma$



a crude extrapolation – assume constant couplings

pdg

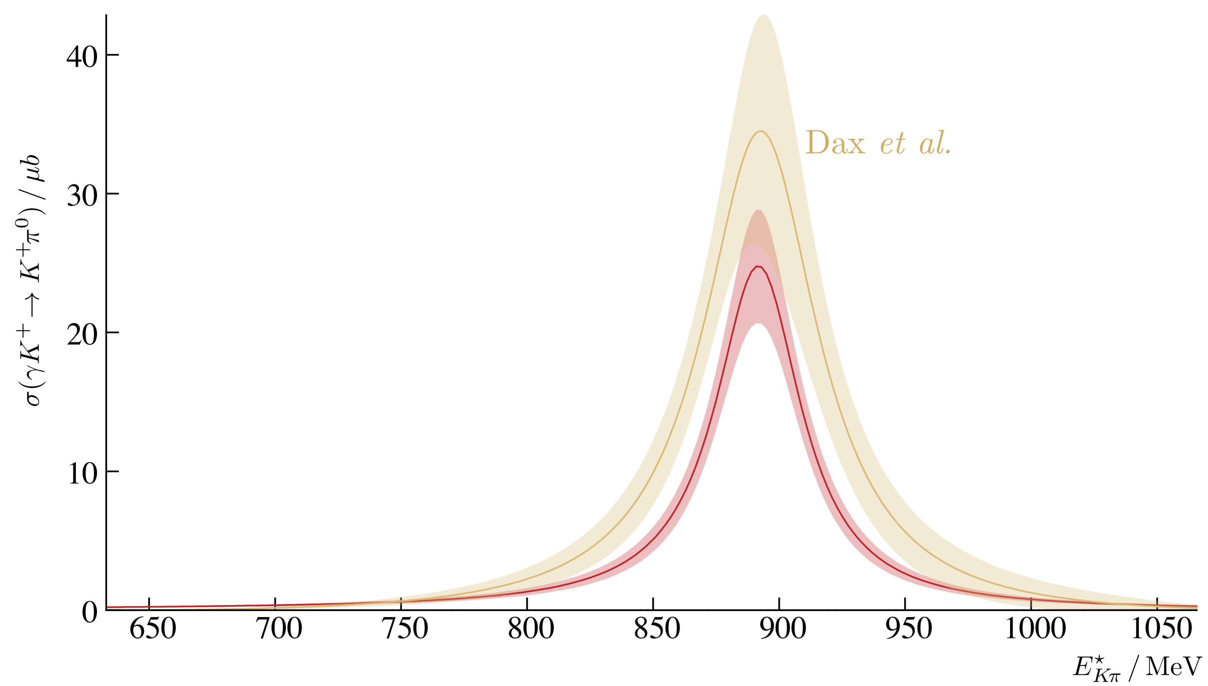
hadronic width $\Gamma_R = 3 \cdot \Gamma(K^+\pi^0) = 3 \cdot \frac{2}{3} \frac{k_{K\pi}^{\star 3}}{m_R^2} |\hat{c}_R|^2 = 42(3)\,\mathrm{MeV}$

$K^*(892)^\pm$ hadroproduced full width $\Gamma = 51.4 \pm 0.8$ MeV
$K^*(892)^\pm$ in τ decays full width $\Gamma = 46.2 \pm 1.3$ MeV

radiative width $\Gamma(K^{*+}\to K^+\gamma) = \frac{4}{3}\alpha \frac{k_{K\gamma}^{\star 3}}{m_K^2} |f_R(0)|^2 = 40(6)\,\mathrm{keV}$

$\Gamma(K^{*\pm} \to K^\pm\gamma) = 50(5)\,\mathrm{keV}$

cross-section $$\sigma(\gamma K^+ \to K^+\pi^0) = \frac{2\pi}{k_{K\gamma}^{\star 2}} \frac{m_R^2 \Gamma_R \Gamma(K^{*+}\to K^+\gamma)}{\left|(m_R - i\Gamma_R/2)^2 - s\right|^2}$$



describing the Q^2 dependence – finite-volume form-factors

simple, singularity-free, parameterizations

"exp poly"

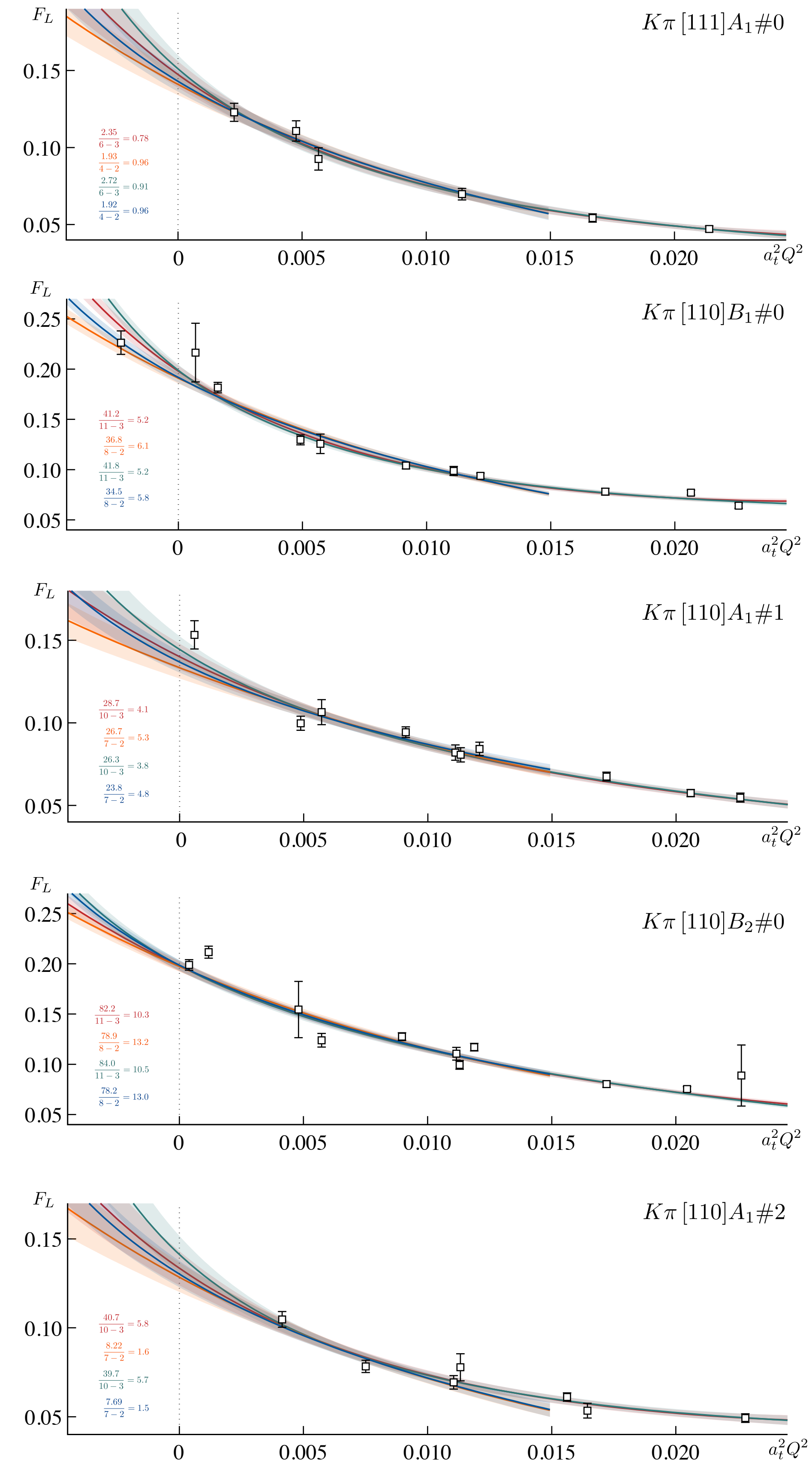
$$F_L(Q^2) = f_{0L} \cdot \exp\left[-\sum_{n=1}^{N} a_n \left(\frac{Q^2}{4m_\pi^2} \right)^n \right]$$

"conformal mapping"

$$F_L(Q^2) = \sum_{n=0}^{N} b_{nL} \left(z(Q^2) - z(0) \right)^n$$

$$z(Q^2) = \frac{\sqrt{Q^2 + t_{\text{cut}}} - \sqrt{Q_0^2 + t_{\text{cut}}}}{\sqrt{Q^2 + t_{\text{cut}}} + \sqrt{Q_0^2 + t_{\text{cut}}}}$$

$t_{\text{cut}} = (2m_\pi)^2$
$a_t^2\, Q_0^2 = 0.0035$



$f_{0L}\ \exp\left[-a_1 \frac{Q^2}{4m_\pi^2} - a_2 \left(\frac{Q^2}{4m_\pi^2} \right)^2 \right]$

$f_{0L}\ \exp\left[-a_1 \frac{Q^2}{4m_\pi^2} \right]$ $\quad a_t^2 Q^2 < 0.015$

$\sum_{n=0}^{2} b_{nL} \left(z(q^2) - z(0) \right)^n$

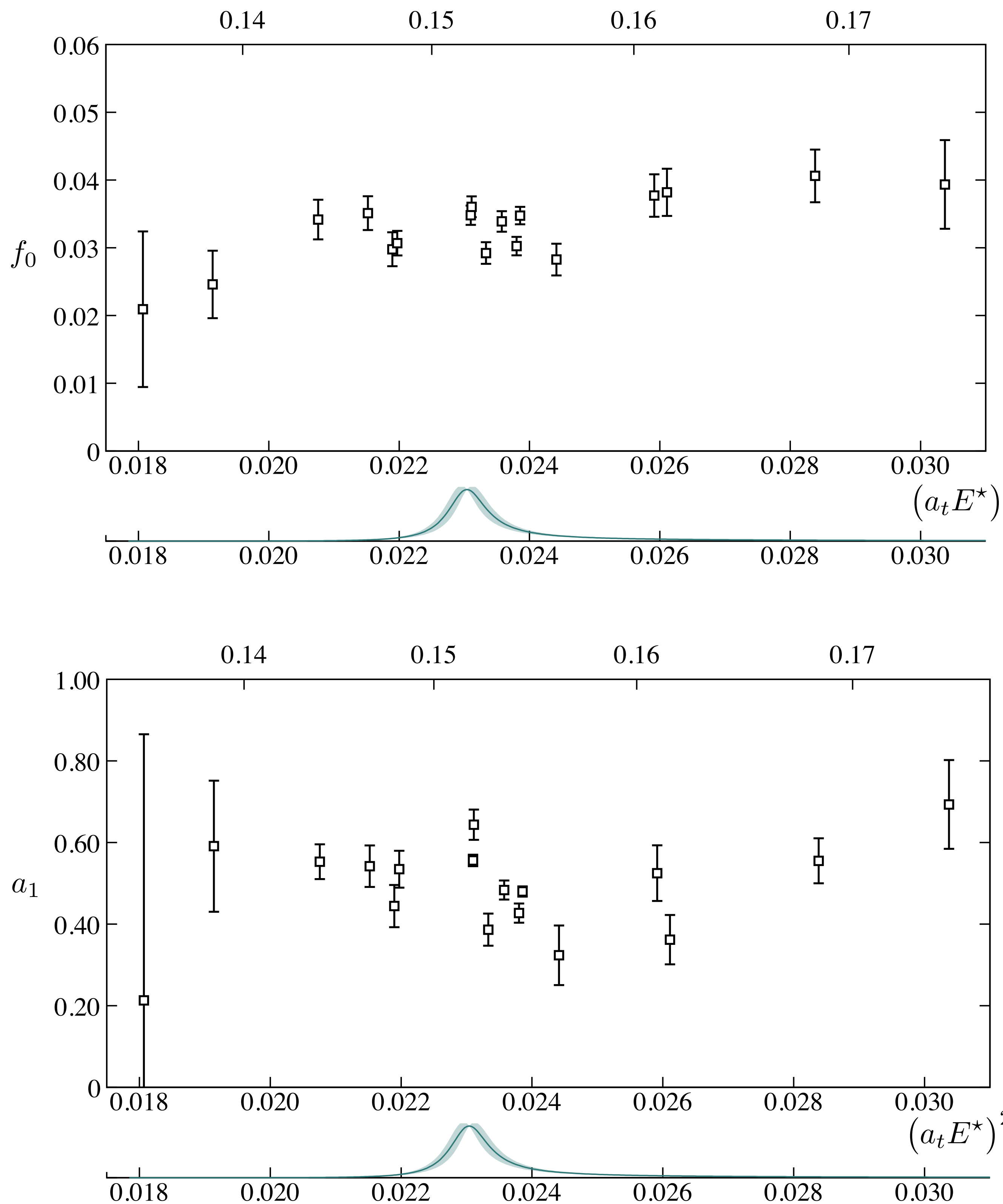
$\sum_{n=0}^{1} b_{nL} \left(z(q^2) - z(0) \right)^n$ $\quad a_t^2 Q^2 < 0.015$

energy dependence after finite volume correction

e.g. *"exp poly"* fit form

$$F_L(Q^2) = f_{0L} \cdot \exp\left[-a_1\left(\frac{Q^2}{4m_\pi^2}\right)\right]$$

$$f_0 = \frac{1}{\tilde{r}_n(L)} f_{0L}$$



modest energy dependence over a broad energy region

$$\left\langle r^2 \right\rangle_{K^{*+},K^+} \equiv \frac{1}{f_R(0)} \cdot \left(-6 \frac{d}{dQ^2} f_R(Q^2) \right) \Bigg|_{Q^2=0}$$

$$\mathrm{Re} \left\langle r^2 \right\rangle^{1/2}_{K^{*+},K^+} = 0.69(4)\,\mathrm{fm} \qquad \left\langle r^2 \right\rangle^{1/2}_{K^+} = 0.55(2)\,\mathrm{fm}$$

needs some thought on how to use this information …

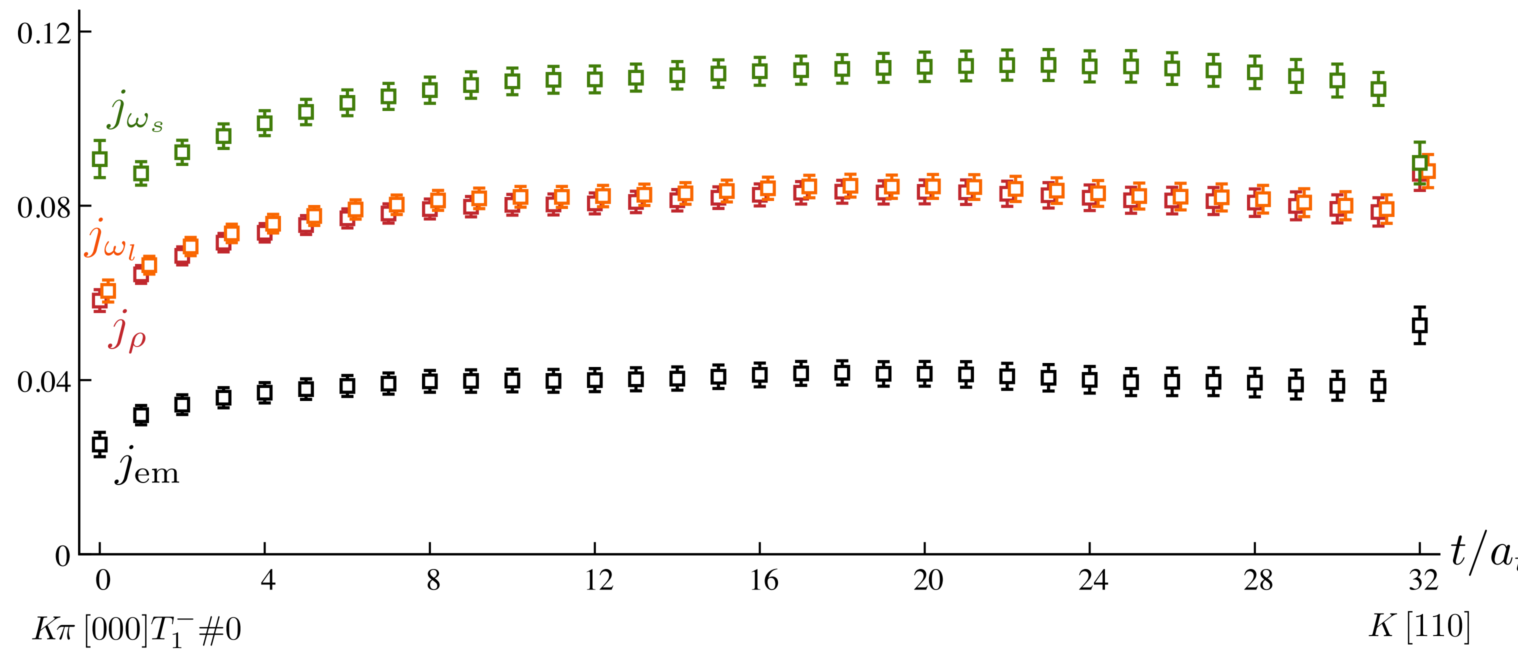
$$j_{\text{em,phys}} = Z_V^l\left(\frac{1}{\sqrt{2}}j_{\rho,\text{lat}}+\frac{1}{3\sqrt{2}}j_{\omega_l,\text{lat}}\right) + Z_V^s\left(-\frac{1}{3}j_{\omega_s,\text{lat}}\right)$$

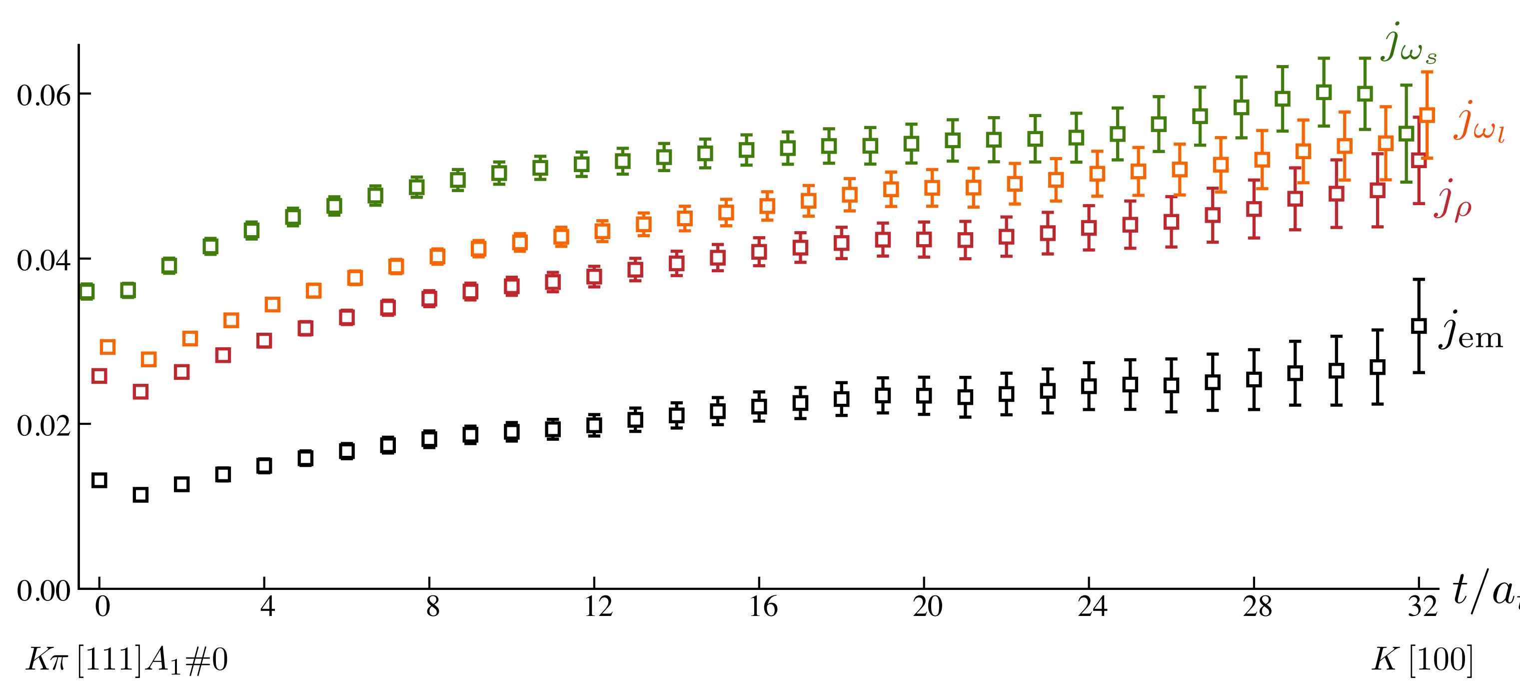
$$\bar{\psi}\Gamma\psi = \bar{\psi}\gamma_i\psi+\tfrac{1}{4}(1-\xi)a_t\partial_4\left(\bar{\psi}\sigma_{4i}\psi\right)$$

$$j_\rho \equiv \frac{1}{\sqrt{2}}\left(\bar{u}\Gamma u - \bar{d}\Gamma d\right), j_{\omega_l} \equiv \frac{1}{\sqrt{2}}\left(\bar{u}\Gamma u + \bar{d}\Gamma d\right), j_{\omega_s} \equiv \bar{s}\Gamma s$$

We compute a set of three-point functions based upon the following choices:

- at $t=0$, an optimized operator corresponding to each black point in Figure 1, having any allowed lattice rotation of the specified momentum. If the irrep is more than one-dimensional, all rows are considered;
- at all $0 \le t/a_t \le 32$ a spatial current insertion having momentum [000], [100], [110], [111] or [200] (and *not* rotations of these specific directions). Rather than three cartesian directions for the current, the subductions of a vector for the relevant momentum are used;
- at $\Delta t/a_t = 32$, an optimized operator for a kaon with a momentum $\le$ [211], with all allowed lattice rotations considered.

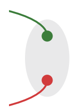




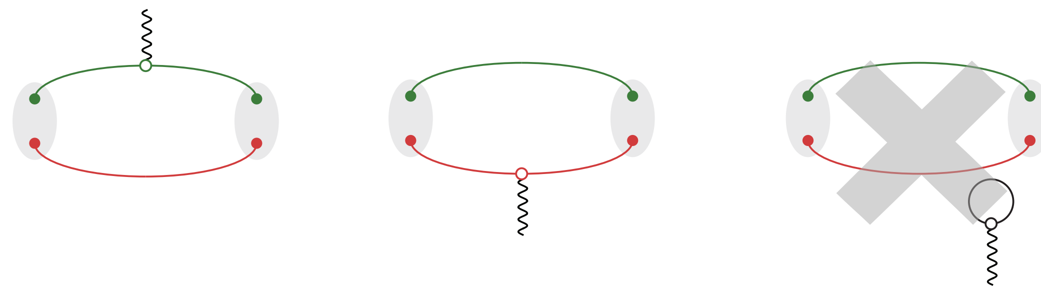
computing the three-point correlation functions

$\langle 0 \,|\, \Omega_K(\mathbf{p}_K, \Delta t)\, j(\mathbf{q}, t)\, \Omega^\dagger_{K\pi}(\mathbf{p}_{K\pi}, 0) \,|\, 0 \rangle$

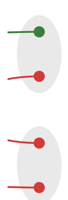
"$K\pi$" optimized operators feature both



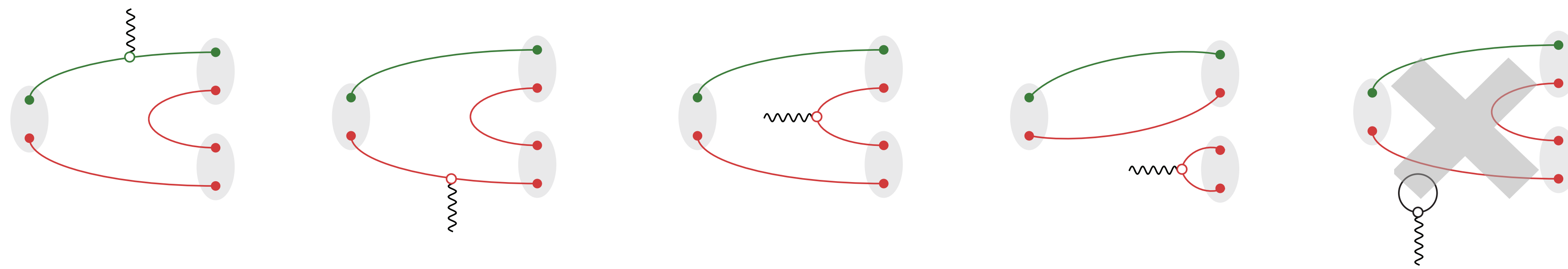
single-meson-like



and



meson-meson-like



current lands on strange quarks and light quarks

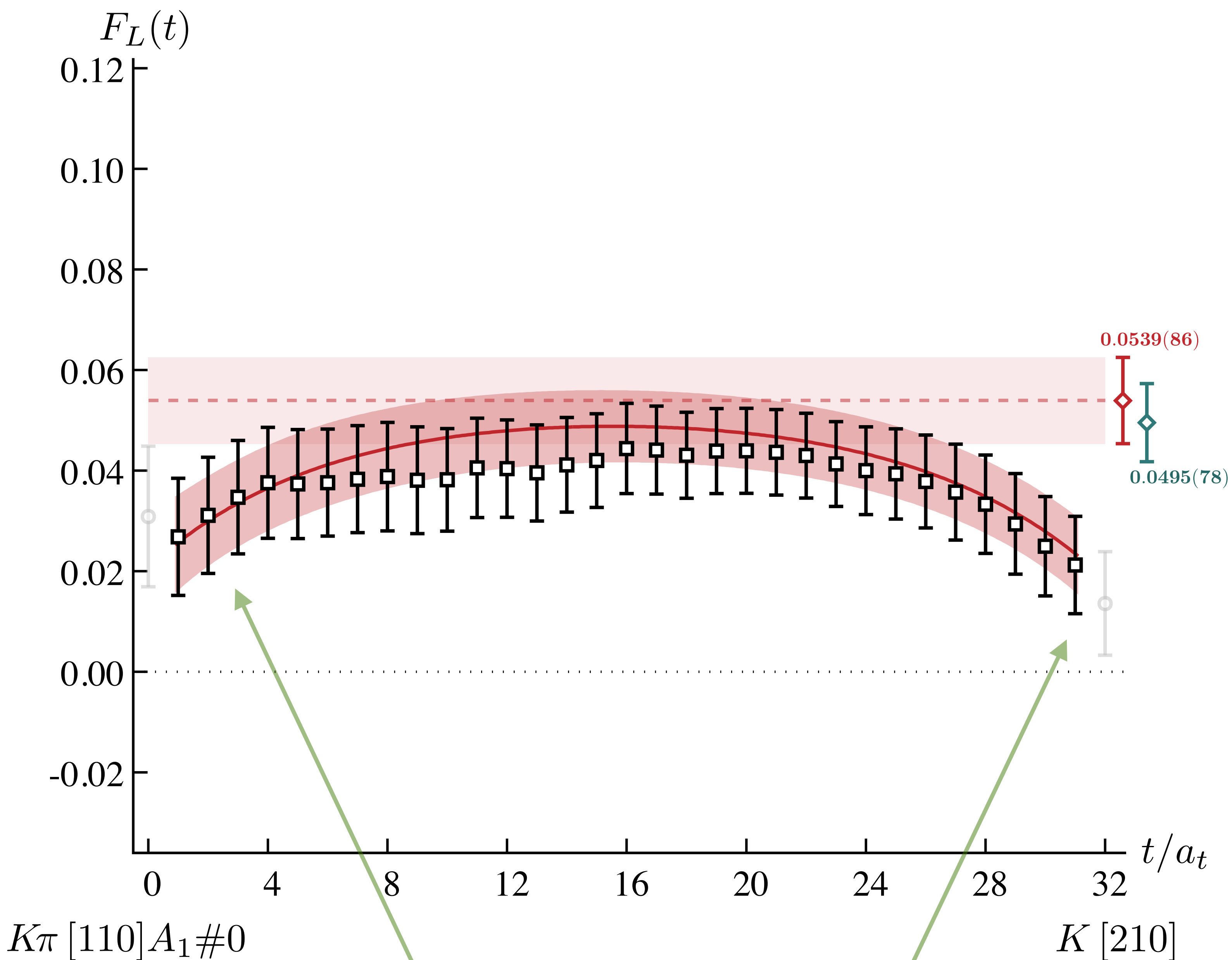
completely disconnected contributions neglected

zero in the SU(3) flavor limit,
also OZI arguments suggest small

vector current renormalizations determined non-perturbatively using pion and kaon form-factors at $Q^2 = 0$
tree-level improved current for anisotropic action used (typically modest effect)

timeslice fitting

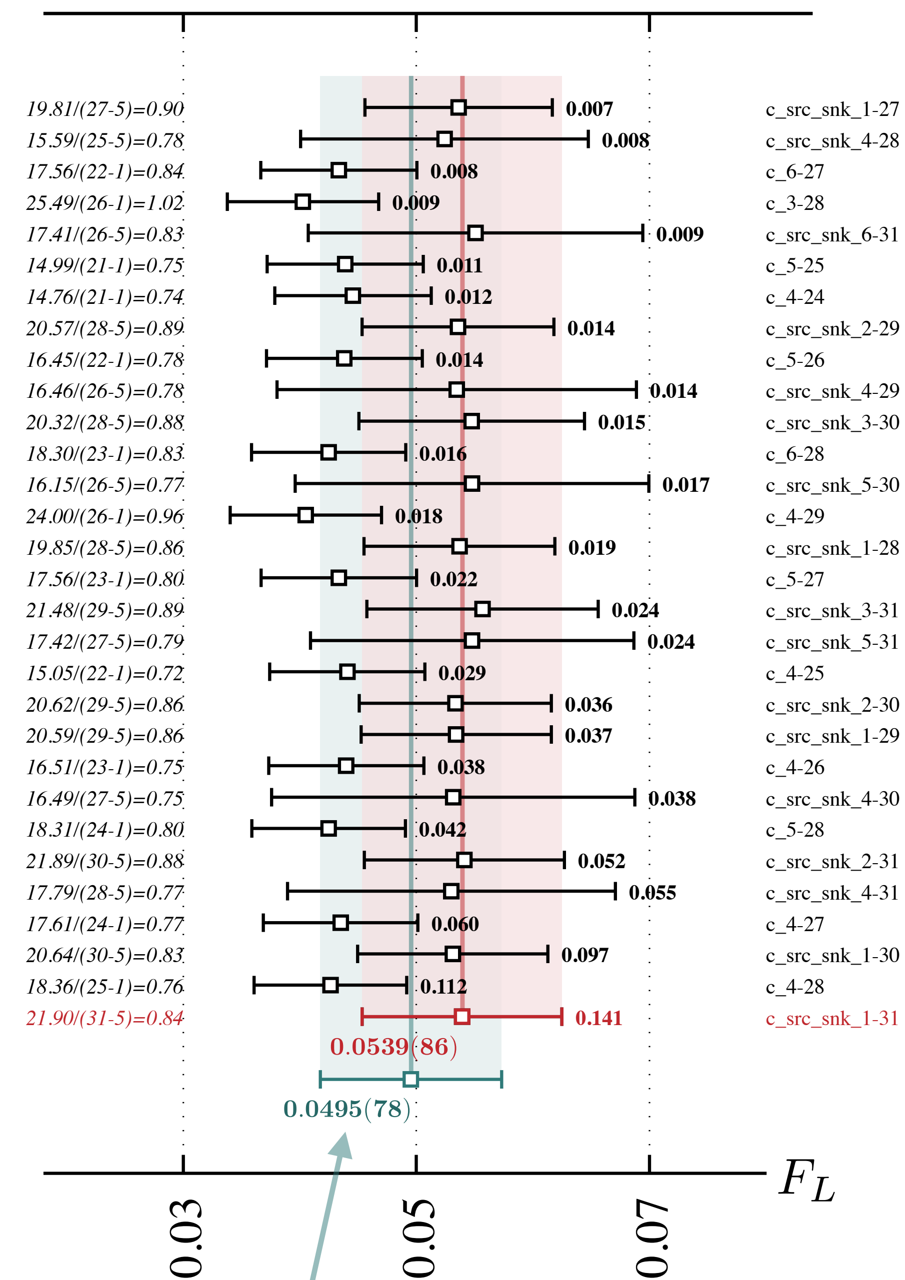
$$F_L(t) \equiv e^{E_K(\Delta t - t)} \cdot e^{E_n t} \cdot \frac{1}{K} \cdot \langle 0 \,|\, \Omega_K(\Delta t)\, j(t)\, \Omega^\dagger_{K\pi}(0) \,|\, 0 \rangle$$



modest excited state contribution
through use of optimized operators

$$F_L + a_{\rm src}\, e^{-\delta E_{\rm src} t} + a_{\rm snk}\, e^{-\delta E_{\rm snk}(\Delta t - t)}$$

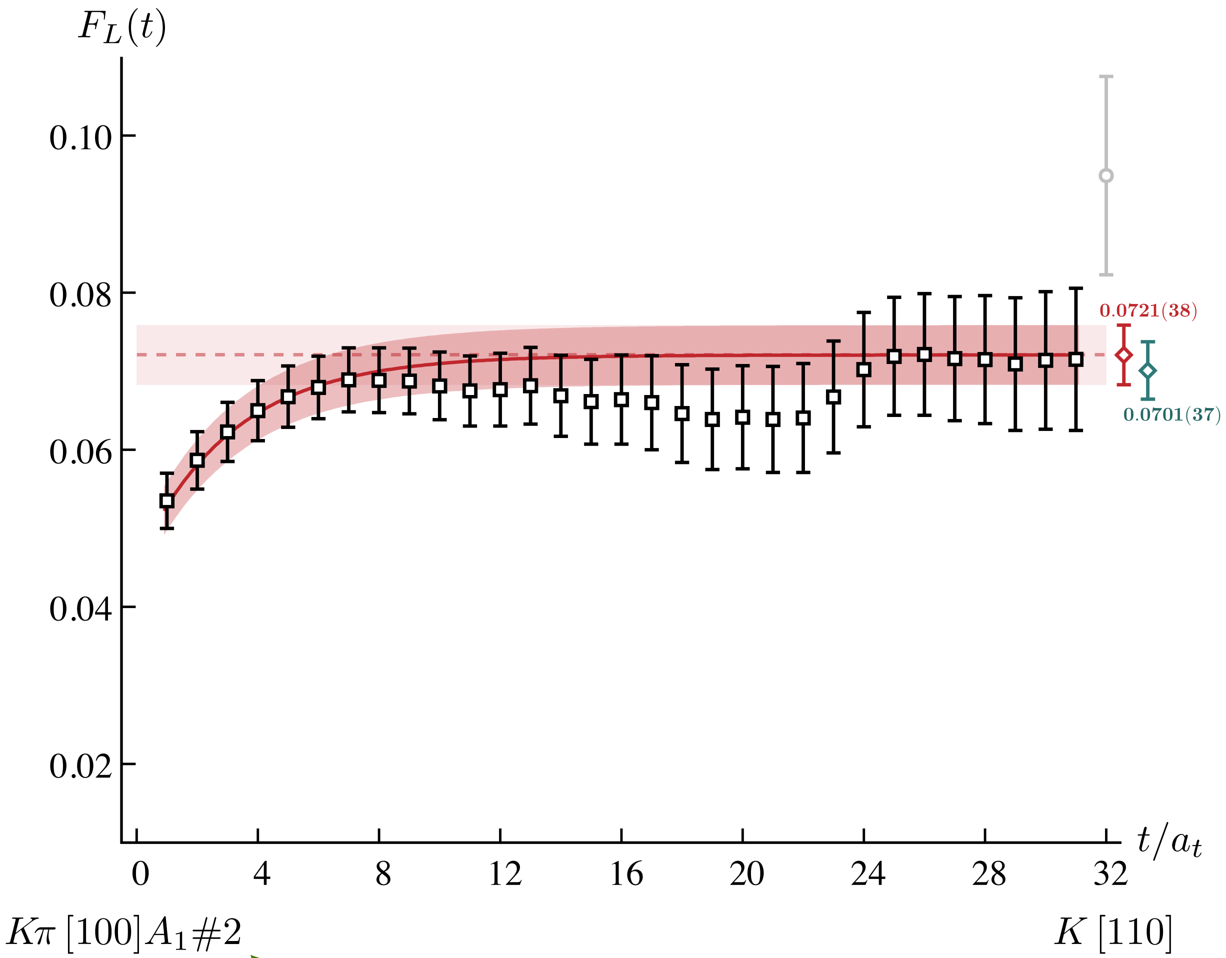
varying fit ranges, source & sink exponentials



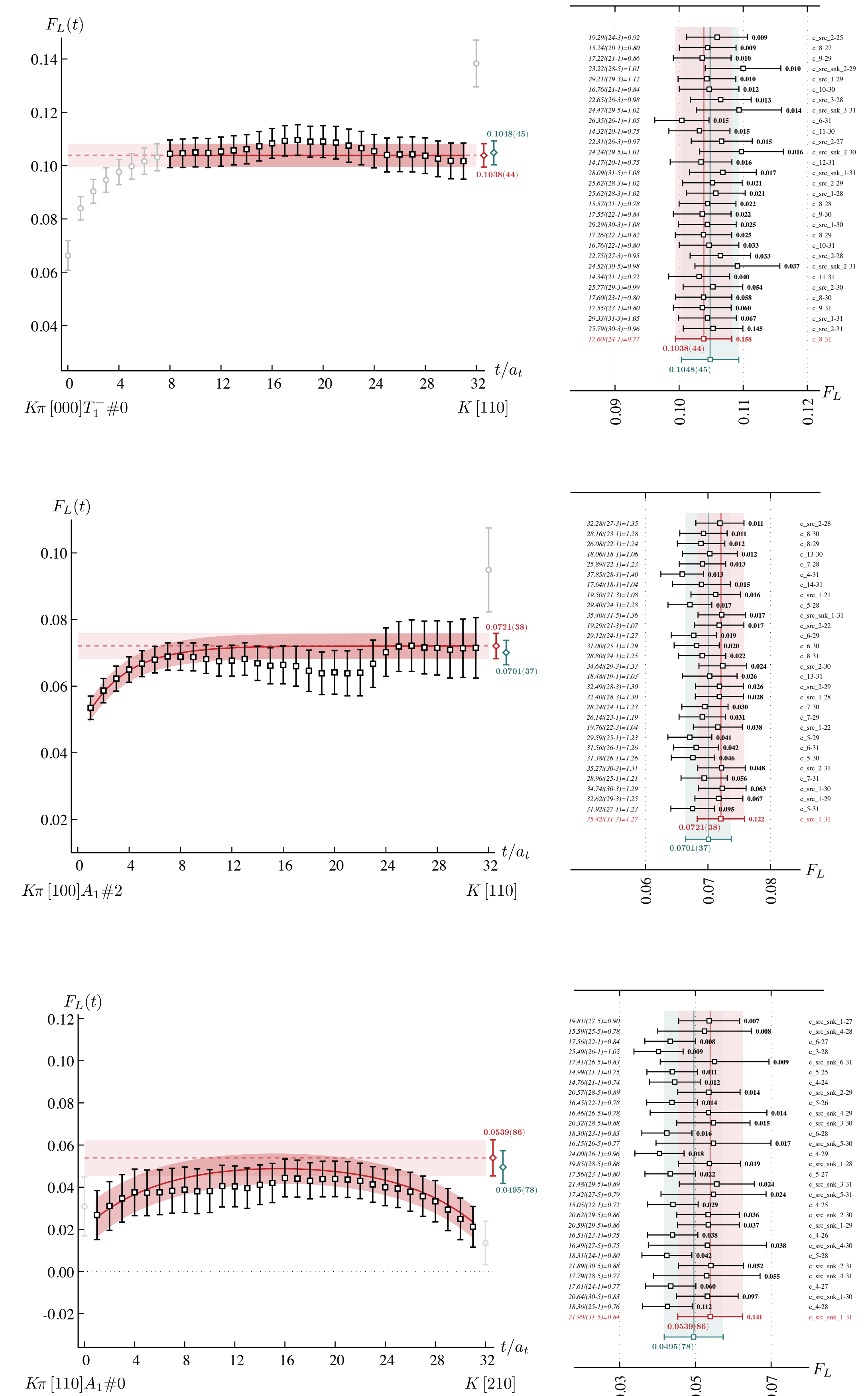
'model average' of
30 best descriptions
using AIC as probability $\exp\left[-\frac{1}{2}(\chi^2 - 2N_{\rm dof})\right]$

example timeslice fitting

well over a thousand correlation functions fit this way



second excited state accessed
through an optimized operator

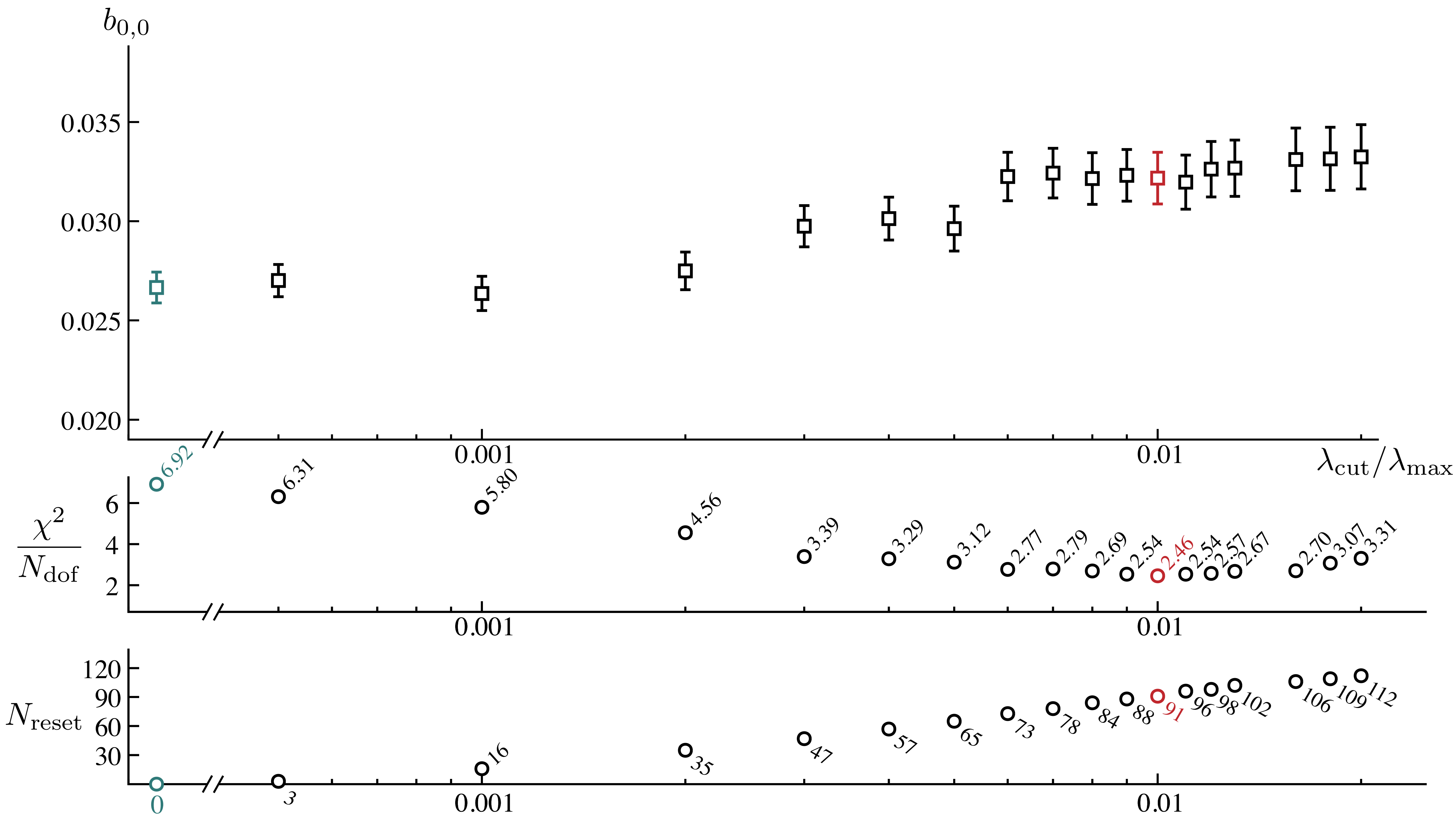


data correlation issues

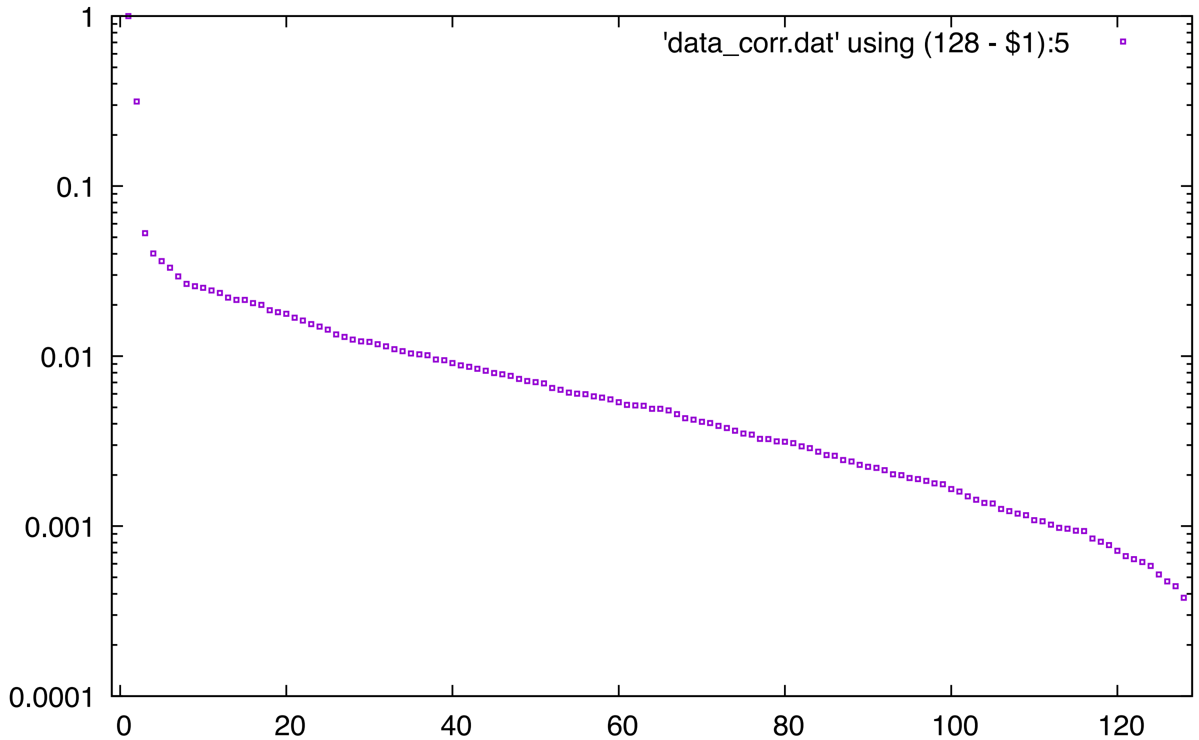
128 data points, 348 configurations – how well determined is the data correlation ?

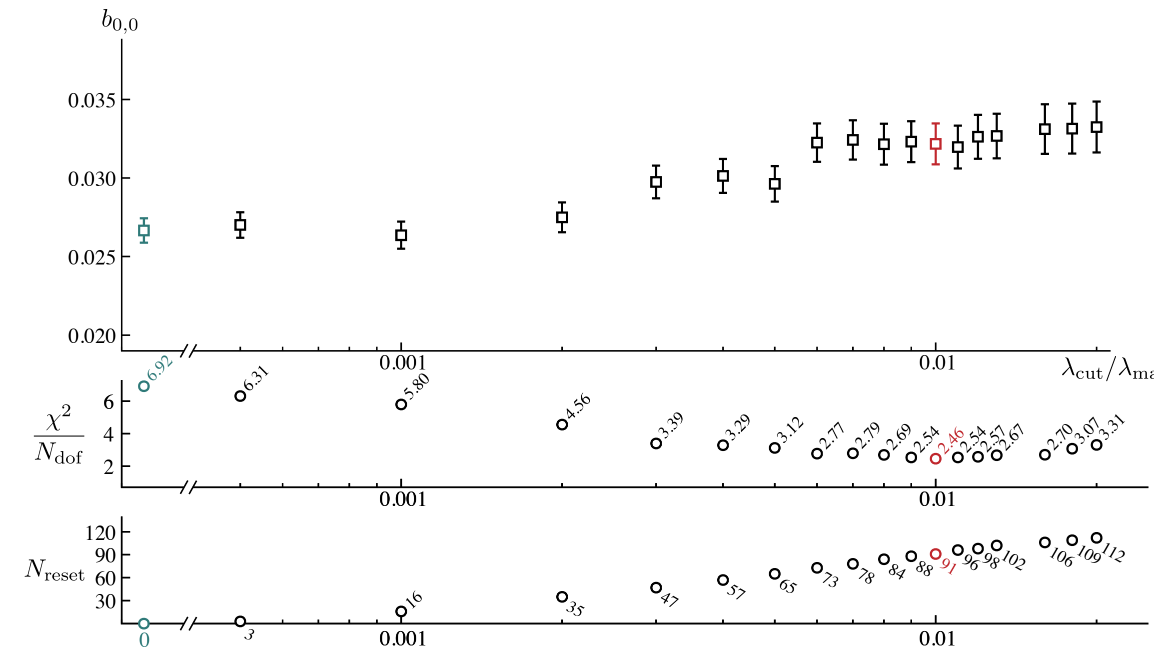
$$\mathbb{C}_{ij} = \frac{1}{N(N-1)} \sum_{a=1}^{N} \big(y_i(a) - \bar{y}_i\big)\big(y_j(a) - \bar{y}_j\big)$$

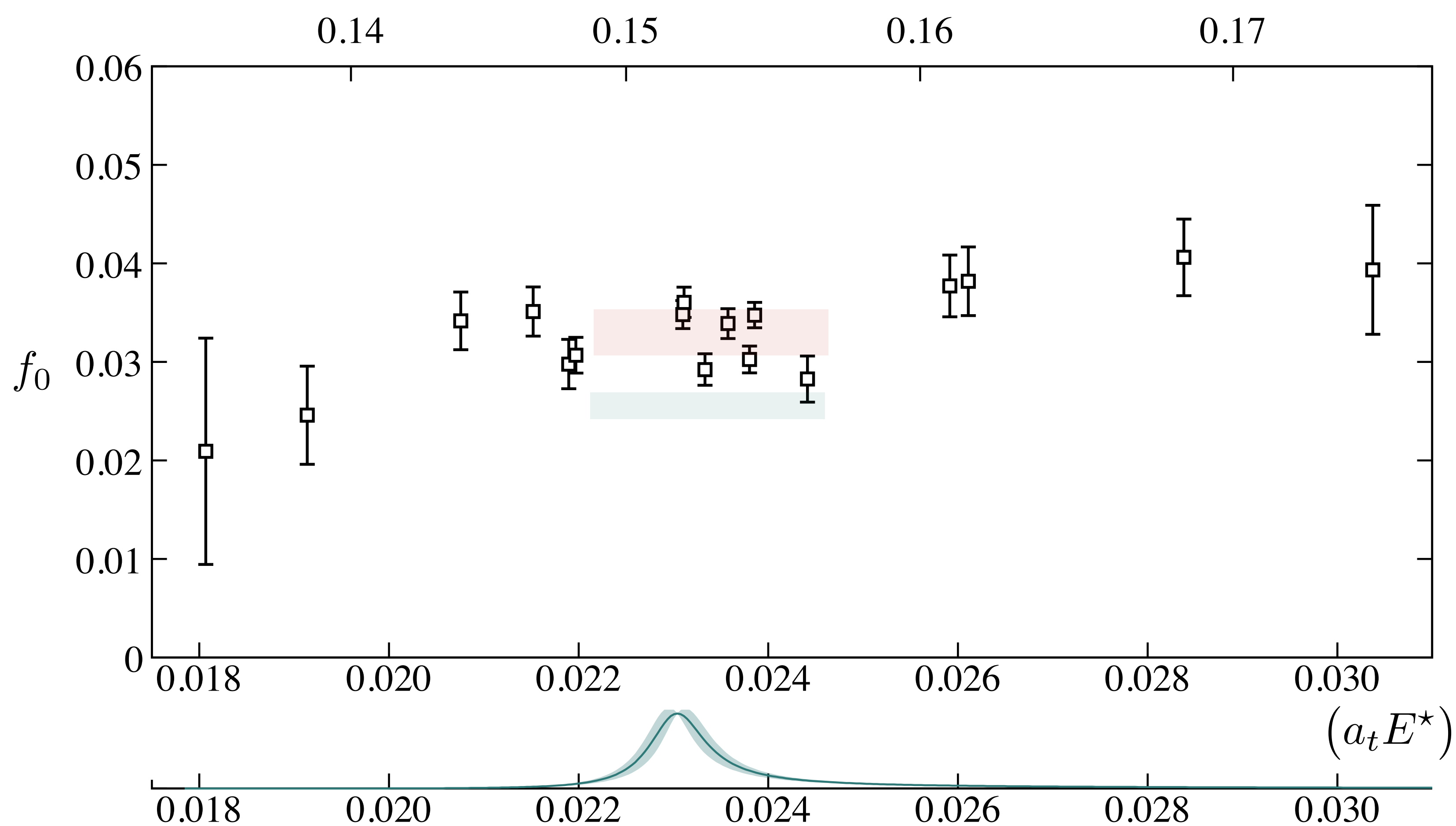
348 outer products



data correlation





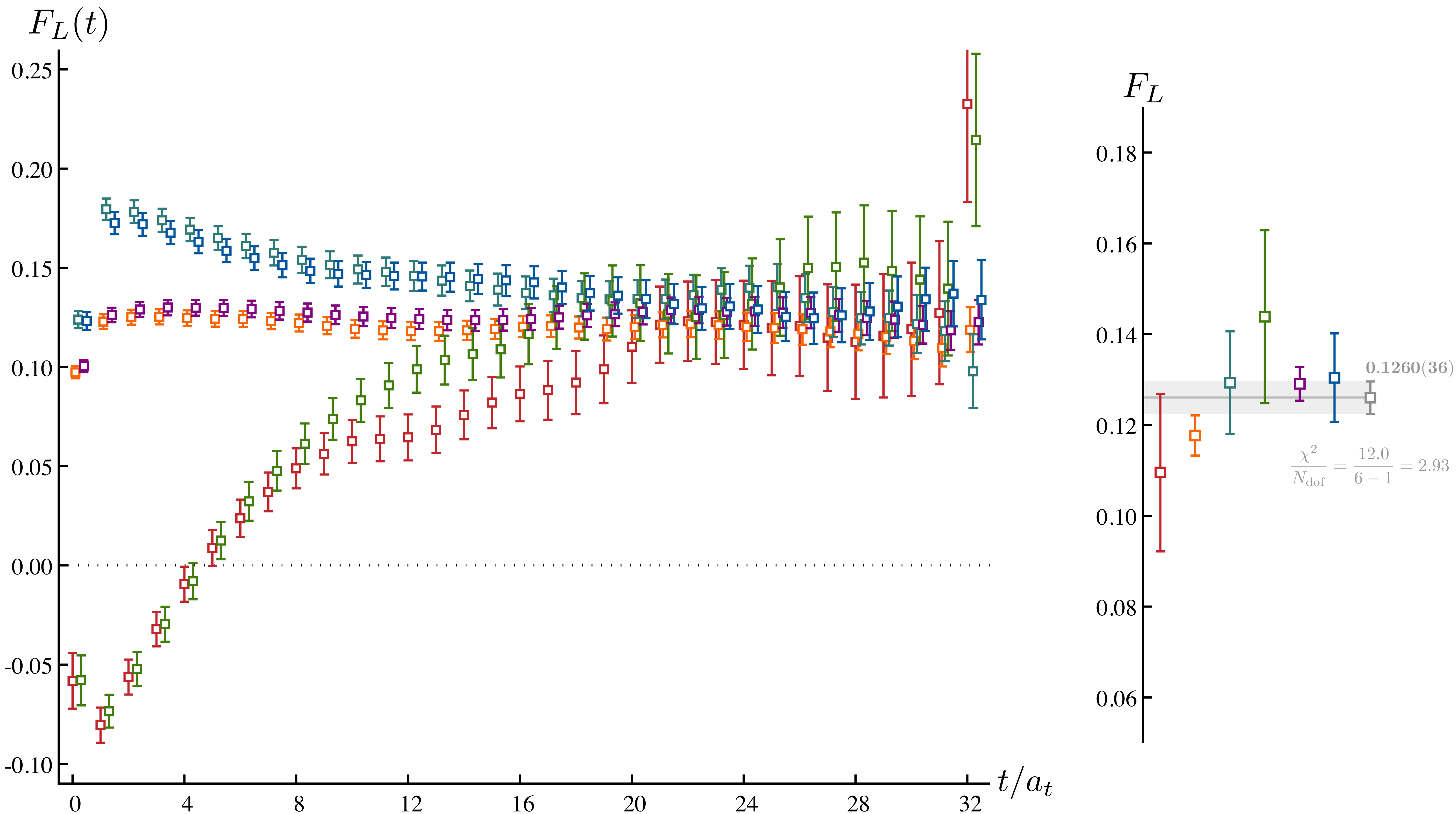


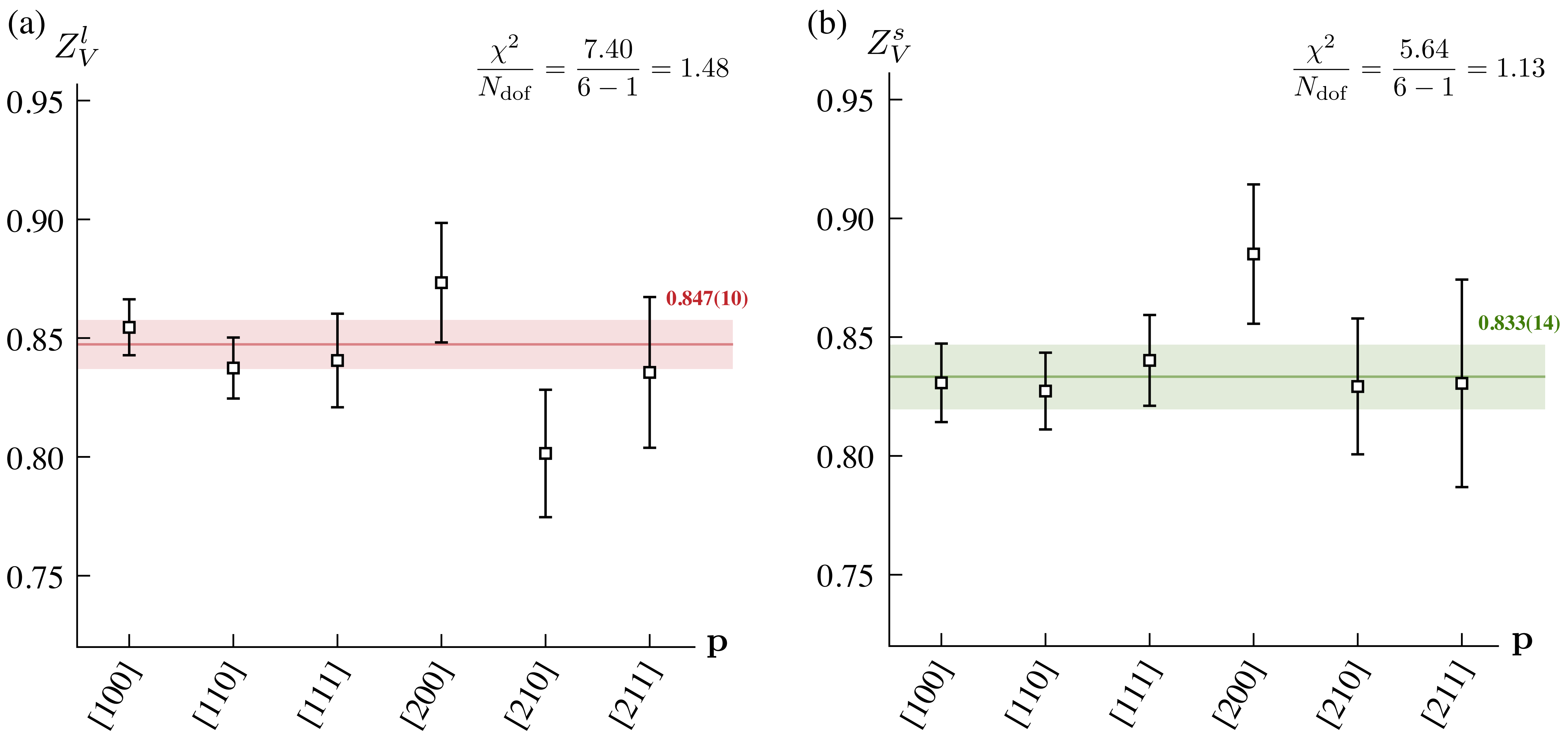
finite-volume correction and data correlation

	$[200]A_1\{0\}\,0.1344(8)$	$[100]A_1\{0\}\,0.1383(5)$	$[110]A_1\{0\}\,0.1441(5)$	$[111]A_1\{0\}\,0.1467(6)$	$[200]A_1\{1\}\,0.1480(7)$	$[100]A_1\{1\}\,0.1482(6)$	$[110]A_1\{1\}\,0.1485(6)$	$[000]T_1^-\{0\}\,0.1534(5)$	$[100]E_2\{0\}\,0.1535(6)$	$[110]B_1\{0\}\,0.1520(6)$	$[110]B_2\{0\}\,0.1545(8)$	$[111]E_2\{0\}\,0.1520(9)$	$[200]E_2\{0\}\,0.1543(9)$	$[111]A_1\{1\}\,0.1562(7)$	$[200]A_1\{2\}\,0.1616(9)$	$[100]A_1\{2\}\,0.1610(5)$	$[110]A_1\{2\}\,0.1685(6)$	$[111]A_1\{2\}\,0.1743(7)$	$[110]B_1\{1\}\,0.1767(5)$
$[200]A_1\{0\}\,0.1344(8)$	+1.00	+0.99	+0.96	+0.94	+0.96	+0.91	+0.59	+0.80	+0.79	+0.91	+0.77	+0.87	+0.79	+0.08	-0.32	-0.35	-0.57	-0.69	-0.70
$[100]A_1\{0\}\,0.1383(5)$		+1.00	+0.98	+0.97	+0.95	+0.89	+0.53	+0.80	+0.78	+0.89	+0.77	+0.86	+0.79	+0.01	-0.32	-0.33	-0.56	-0.67	-0.65
$[110]A_1\{0\}\,0.1441(5)$			+1.00	+1.00	+0.93	+0.86	+0.45	+0.77	+0.76	+0.85	+0.75	+0.82	+0.76	-0.09	-0.34	-0.32	-0.55	-0.63	-0.58
$[111]A_1\{0\}\,0.1467(6)$				+1.00	+0.94	+0.88	+0.45	+0.77	+0.76	+0.85	+0.75	+0.82	+0.77	-0.09	-0.34	-0.32	-0.54	-0.62	-0.57
$[200]A_1\{1\}\,0.1480(7)$					+1.00	+0.98	+0.71	+0.88	+0.87	+0.97	+0.86	+0.94	+0.87	+0.20	-0.24	-0.29	-0.48	-0.61	-0.65
$[100]A_1\{1\}\,0.1482(6)$						+1.00	+0.73	+0.82	+0.81	+0.93	+0.80	+0.90	+0.81	+0.22	-0.31	-0.37	-0.51	-0.63	-0.70
$[110]A_1\{1\}\,0.1485(6)$							+1.00	+0.86	+0.86	+0.85	+0.86	+0.87	+0.86	+0.82	+0.32	+0.21	+0.07	-0.13	-0.30
$[000]T_1^-\{0\}\,0.1534(5)$								+1.00	+1.00	+0.96	+1.00	+0.99	+1.00	+0.54	+0.25	+0.20	-0.02	-0.19	-0.26
$[100]E_2\{0\}\,0.1535(6)$									+1.00	+0.95	+1.00	+0.98	+1.00	+0.55	+0.27	+0.23	+0.01	-0.16	-0.23
$[110]B_1\{0\}\,0.1520(6)$										+1.00	+0.95	+0.99	+0.96	+0.43	-0.00	-0.06	-0.28	-0.44	-0.51
$[110]B_2\{0\}\,0.1545(8)$											+1.00	+0.98	+1.00	+0.55	+0.29	+0.25	+0.03	-0.14	-0.21
$[111]E_2\{0\}\,0.1520(9)$												+1.00	+0.98	+0.49	+0.10	+0.05	-0.17	-0.34	-0.42
$[200]E_2\{0\}\,0.1543(9)$													+1.00	+0.55	+0.27	+0.22	-0.00	-0.17	-0.24
$[111]A_1\{1\}\,0.1562(7)$														+1.00	+0.71	+0.59	+0.52	+0.35	+0.15
$[200]A_1\{2\}\,0.1616(9)$															+1.00	+0.99	+0.93	+0.85	+0.78
$[100]A_1\{2\}\,0.1610(5)$																+1.00	+0.94	+0.87	+0.85
$[110]A_1\{2\}\,0.1685(6)$																	+1.00	+0.98	+0.91
$[111]A_1\{2\}\,0.1743(7)$																		+1.00	+0.96
$[110]B_1\{1\}\,0.1767(5)$																			+1.00

fit, then average

averaging ‘equivalent’ correlation functions



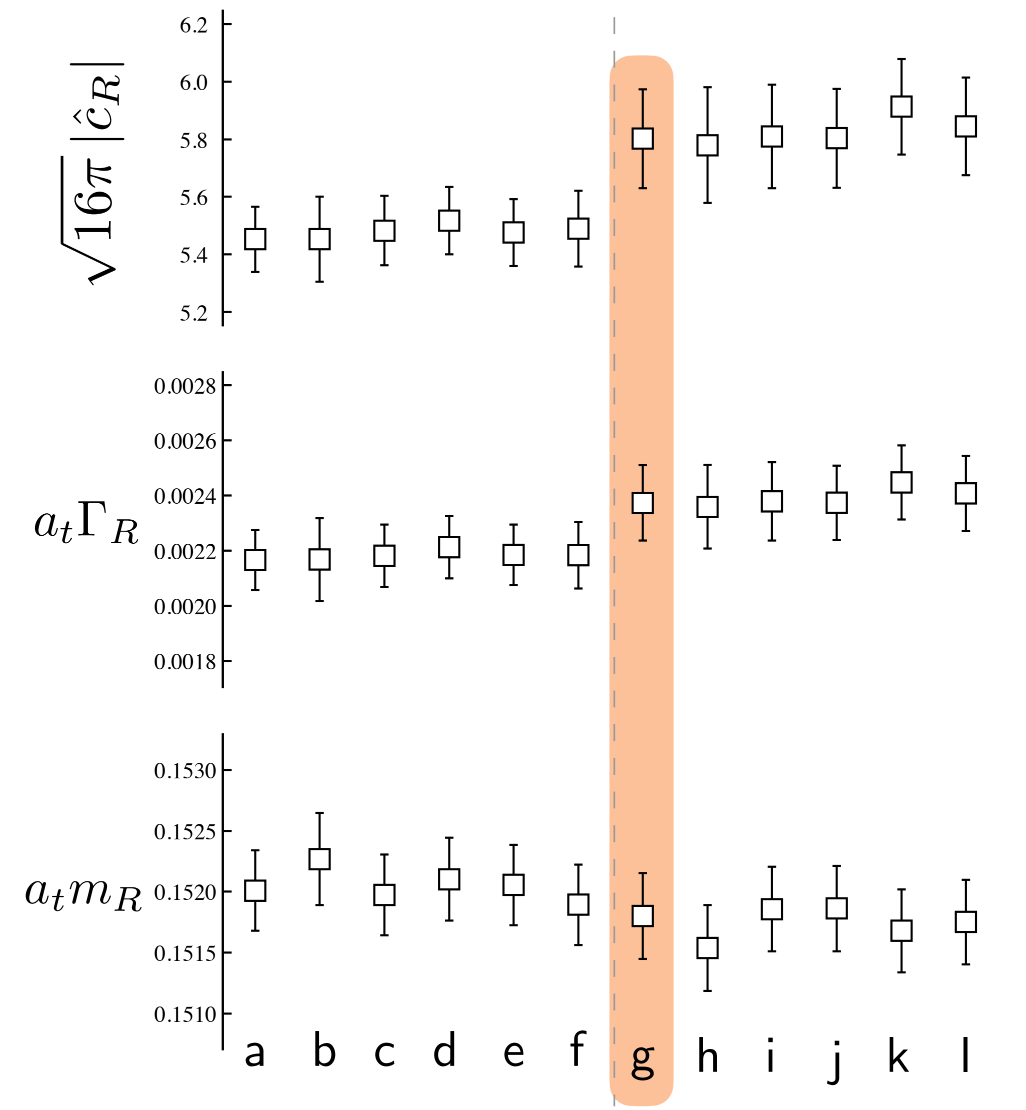

(a)
Z_V^l
$\frac{\chi^2}{N_{\rm dof}} = \frac{7.40}{6-1} = 1.48$
0.95
0.90
0.85
0.80
0.75
0.847(10)
p
[100]
[110]
[111]
[200]
[210]
[211]
(b)
Z_V^s
$\frac{\chi^2}{N_{\rm dof}} = \frac{5.64}{6-1} = 1.13$
0.95
0.90
0.85
0.80
0.75
0.833(14)
p
[100]
[110]
[111]
[200]
[210]
[211]

scattering amplitude parameterization variation

$$\mathscr{A}^{\mu}_{\lambda_{K\pi}}\left(\mathbf{p}_K, \mathbf{p}_{K\pi}; Q^2, E^\star_{K\pi}\right) = \frac{2}{m_K}\epsilon^{\mu\nu\rho\sigma}(\mathbf{p}_K)_\nu\,(\mathbf{p}_{K\pi})_\rho\,\epsilon_\sigma(\mathbf{p}_{K\pi}, \lambda_{K\pi})\;\cdot F(Q^2, E^\star_{K\pi})$$

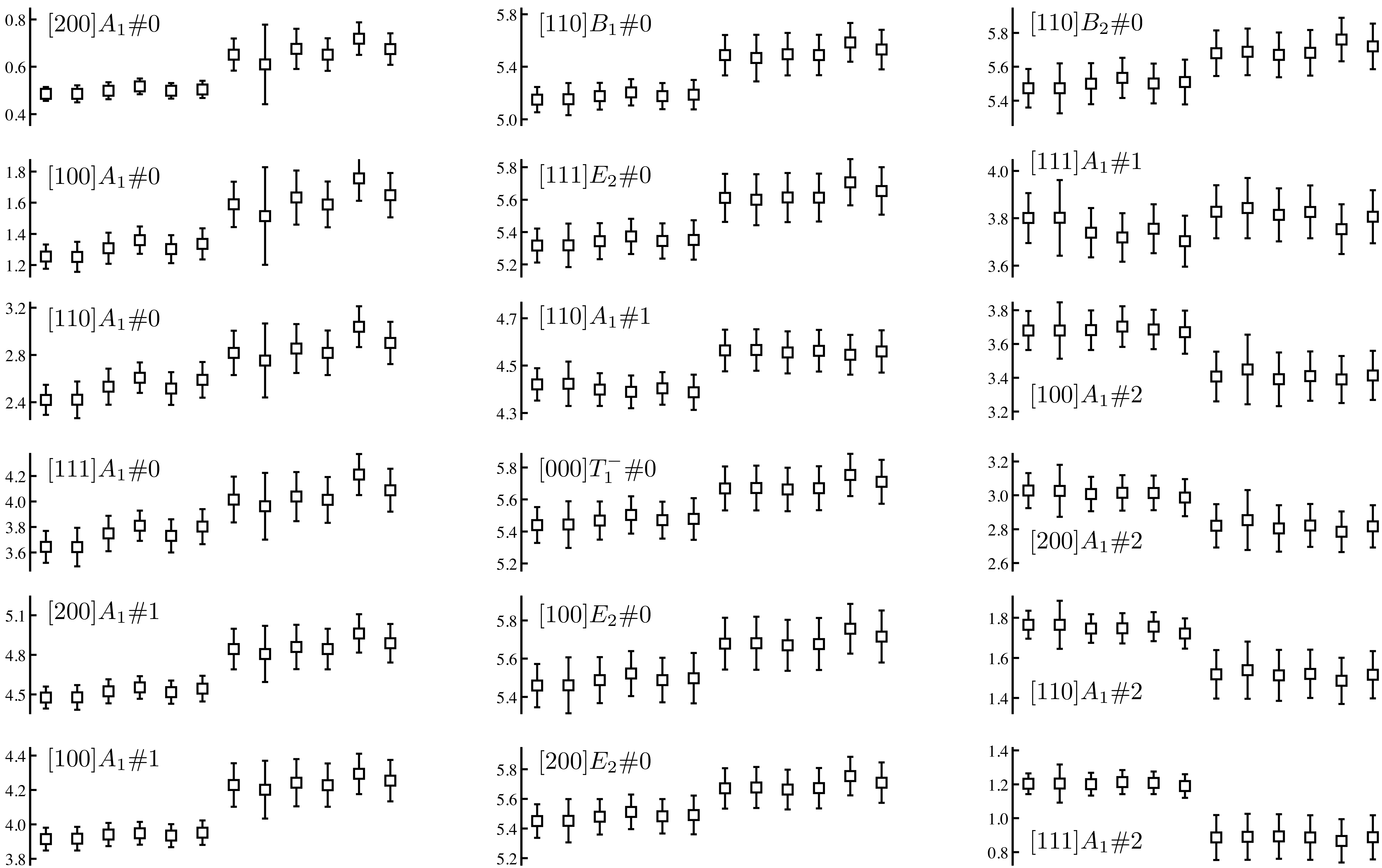
$$\left(K^\mu\, F\sqrt{16\pi}\,\hat{c}_R\right)\cdot\frac{1}{\left(m_R - i\Gamma_R/2\right)^2 - E^{\star 2}_{K\pi}}\cdot\left(\sqrt{16\pi}\,\hat{c}_R\, k^\star_{K\pi}\right)$$

$$f_R(Q^2) \equiv F\left(Q^2, m_R - i\tfrac{1}{2}\Gamma_R\right)\cdot\sqrt{16\pi}\,\hat{c}_R$$



$$a_t\, m_R = 0.1518(4)$$
$$a_t\, \Gamma_R = 0.0024(1)$$
$$\sqrt{16\pi}\,\hat{c}_R = 5.80(17) - i\,0.19(3)$$

finite-volume correction parameterization variation



For the choices $\mathrm{BW}_{\mathsf{a}\ldots\mathsf{f}}$ the P–wave amplitude is a Breit-Wigner,

$$\mathcal{M}^{\ell=1}(s) = \frac{16\pi}{\rho(s)} \frac{\sqrt{s}\,\Gamma(s)}{m_{\mathrm{BW}}^2 - s - i\sqrt{s}\,\Gamma(s)}\,, \quad \Gamma(s) = g_{\mathrm{BW}}^2 \frac{k^{\star 3}}{s}\,,$$

where m_{BW}, g_{BW} are free parameters. The S-wave amplitudes are

$$\mathcal{M}_{\mathsf{a}}^{\ell=0}(s) = \frac{16\pi}{\left(\gamma_0 + \gamma_1\left(\frac{s-s_{\mathrm{thr}}}{s_{\mathrm{thr}}}\right)\right)^{-1} + I_{\mathrm{thr}}(s)}\,,$$

$$\mathcal{M}_{\mathsf{b}}^{\ell=0}(s) = \frac{16\pi}{\left(\gamma_0 + \gamma_1\left(\frac{s-s_{\mathrm{thr}}}{s_{\mathrm{thr}}}\right) + \gamma_2\left(\frac{s-s_{\mathrm{thr}}}{s_{\mathrm{thr}}}\right)^2\right)^{-1} + I_{\mathrm{thr}}(s)}\,,$$

$$\mathcal{M}_{\mathsf{c}}^{\ell=0}(s) = \frac{16\pi\,(s - s_A)}{\left(\gamma_0 + \gamma_1\left(\frac{s-s_{\mathrm{thr}}}{s_{\mathrm{thr}}}\right)\right)^{-1} - i\rho(s)\,(s - s_A)}\,,$$

$$\mathcal{M}_{\mathsf{d}}^{\ell=0}(s) = \frac{16\pi}{\left(\gamma_0 + \gamma_1\left(\frac{s-s_{\mathrm{thr}}}{s_{\mathrm{thr}}}\right)\right)^{-1} - i\rho(s)}\,,$$

$$\mathcal{M}_{\mathsf{e}}^{\ell=0}(s) = \frac{16\pi\,(s - s_A)}{\gamma_0 + \gamma_1\left(\frac{s-s_{\mathrm{thr}}}{s_{\mathrm{thr}}}\right) + I_{\mathrm{thr}}(s)\,(s - s_A)}\,,$$

$$\mathcal{M}_{\mathsf{f}}^{\ell=0}(s) = \frac{16\pi}{\rho(s)} \frac{k^\star}{a^{-1} + \frac{1}{2} r k^{\star 2} - i k^\star}\,,$$

$$\mathcal{M}^{\ell=0}(s) = \frac{16\pi}{\left(\gamma_0 + \gamma_1\left(\frac{s-s_{\mathrm{thr}}}{s_{\mathrm{thr}}}\right)\right)^{-1} + I_{\mathrm{thr}}(s)}\,,$$

$$\mathcal{M}_{\mathsf{g}}^{\ell=1}(s) = \frac{16\pi}{\frac{1}{4k^{\star 2}}\left(\frac{g^2}{m^2-s} + \gamma_0\right)^{-1} + I_{\mathrm{pole}}(s)}$$

$$\mathcal{M}_{\mathsf{h}}^{\ell=1}(s) = \frac{16\pi}{\frac{1}{4k^{\star 2}}\left(\frac{g^2}{m^2-s} + \gamma_0 + \gamma_1\left(\frac{s-s_{\mathrm{thr}}}{s_{\mathrm{thr}}}\right)\right)^{-1} + I_{\mathrm{pole}}(s)}$$

$$\mathcal{M}_{\mathsf{i}}^{\ell=1}(s) = \frac{16\pi}{\frac{1}{4k^{\star 2}}\left(\frac{\left(g_0 + g_1\frac{s-s_{\mathrm{thr}}}{s_{\mathrm{thr}}}\right)^2}{m^2-s}\right)^{-1} + I_{\mathrm{pole}}(s)}$$

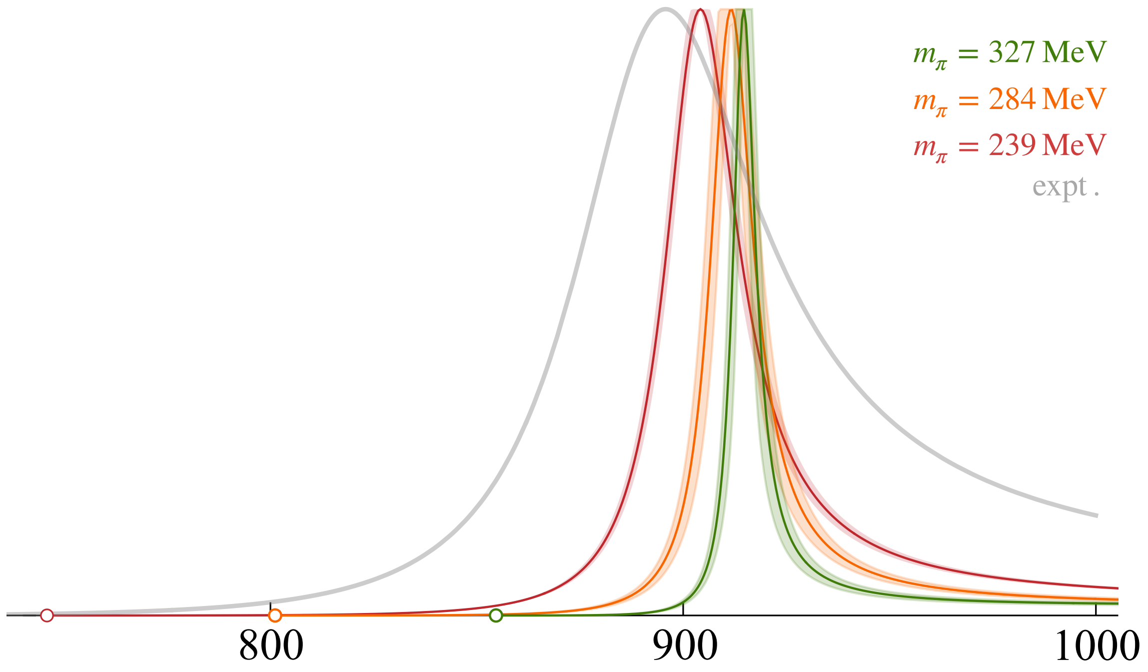
$$\mathcal{M}_{\mathsf{j}}^{\ell=1}(s) = \frac{16\pi}{\frac{1}{4k^{\star 2}}\left(\frac{\left(g_0 + g_1\frac{s-s_{\mathrm{thr}}}{s_{\mathrm{thr}}}\right)^2}{m^2-s} + \gamma_0\right)^{-1} + I_{\mathrm{pole}}(s)}$$

$$\mathcal{M}_{\mathsf{k}}^{\ell=1}(s) = \frac{16\pi}{\frac{1}{4k^{\star 2}}\left(\frac{g^2}{m^2-s} + \gamma_0\right)^{-1} - i\rho(s)}$$

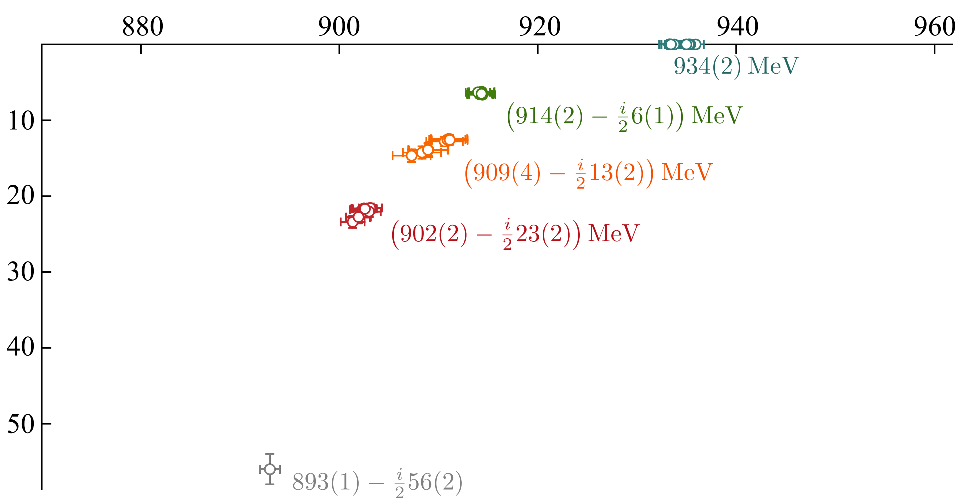
$$\mathcal{M}_{\mathsf{l}}^{\ell=1}(s) = \frac{16\pi}{\frac{1}{4k^{\star 2}}\left(\frac{g^2}{m^2-s} + \gamma_0\right)^{-1} + I_{\mathrm{pole}}(s)}$$

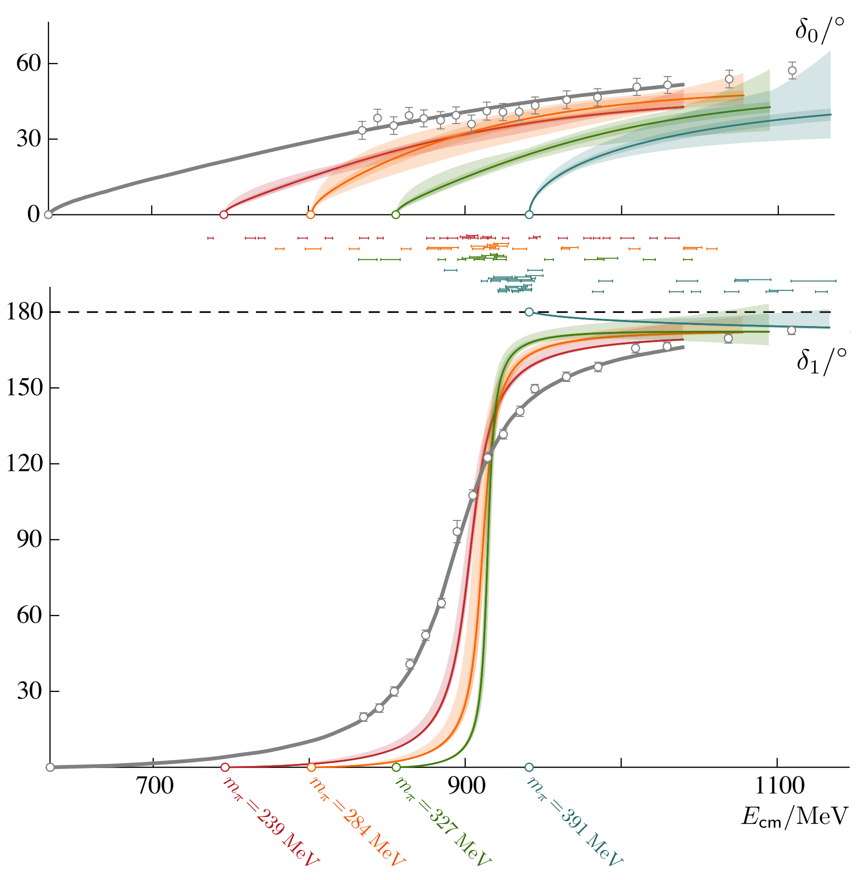
aside: variation with light quark mass

$\pi K \to \pi K \quad \ell = 1$ elastic scattering PRL 123 042002 (2019)



K^* resonance pole position

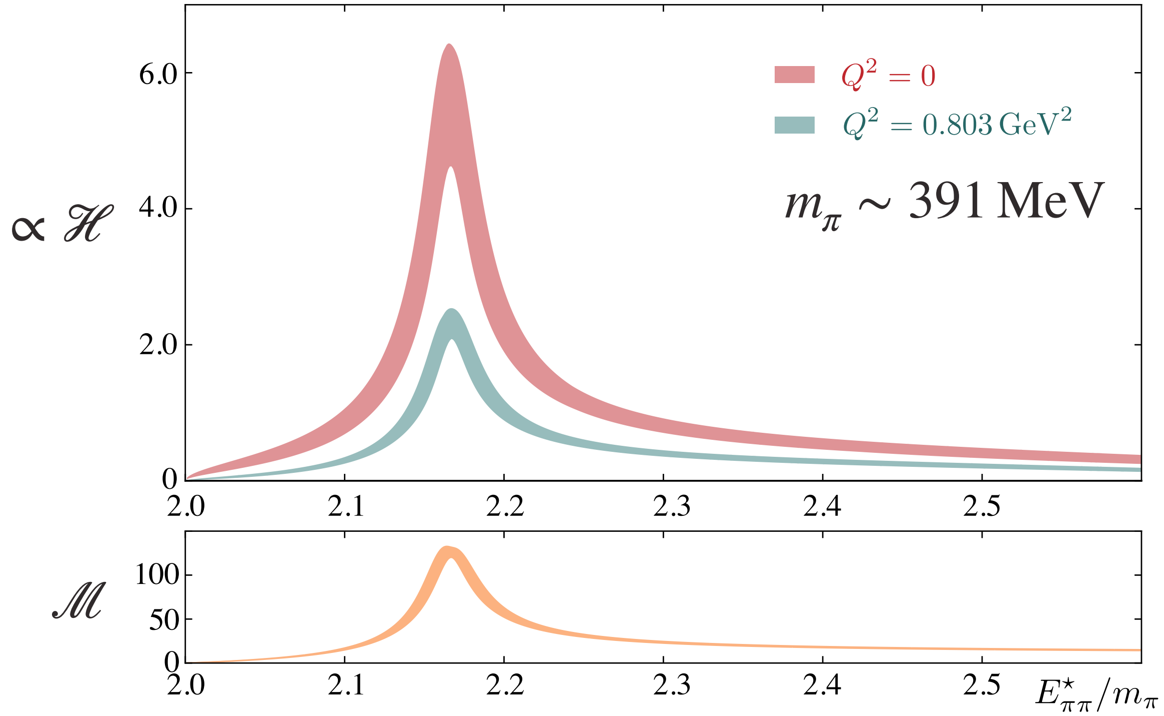




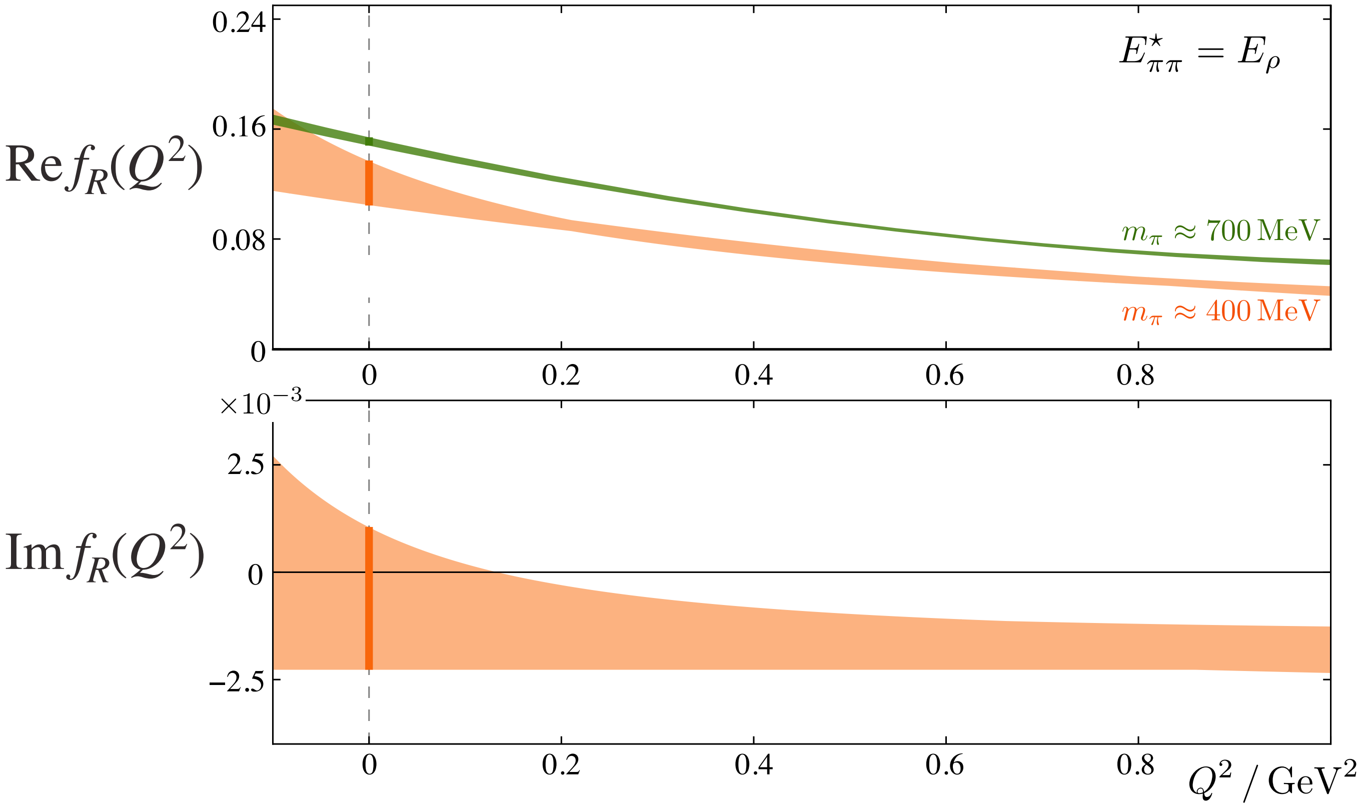
only previous example: $\gamma\pi \to \pi\pi$

just a single partial-wave, $\ell = 1$, needs to be considered here

PRL 115 242001 (2015)
PRD 93 114508 (2016)



analytic continuation to the ρ pole



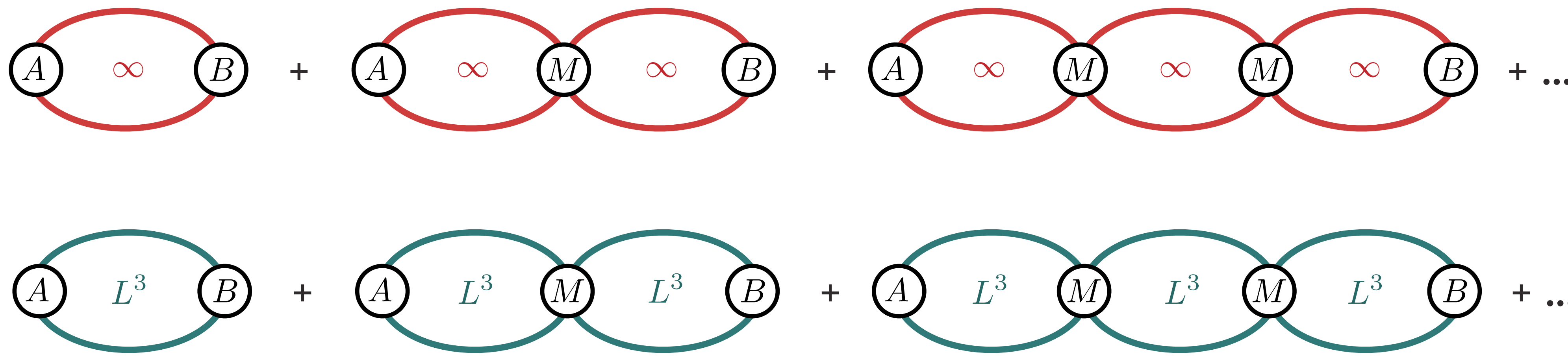
consider a two-point correlation function — operators with the quantum numbers of a two-hadron system

$$C_L(t, \mathbf{P}) = \int_L d^3\mathbf{x} \int_L d^3\mathbf{y}\; e^{-i\mathbf{P}\cdot(\mathbf{x}-\mathbf{y})} \langle 0 | A(\mathbf{x}, t)\, B^\dagger(\mathbf{y}, 0) | 0 \rangle$$

now consider the 'all-orders' skeleton perturbative expansion for this

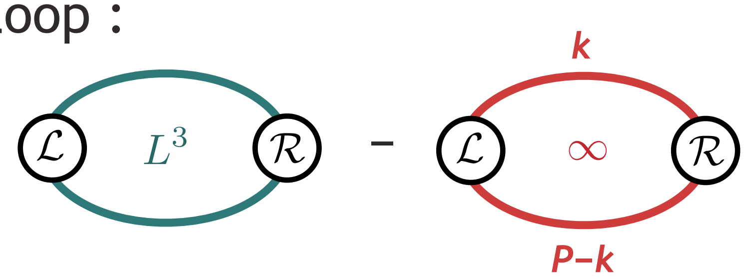
in infinite volume

in finite volume



where the colored lines are fully-dressed propagators,
and where we are below three-hadron thresholds, so diagrams with three lines can't go on-shell

basic loop :



$$= -\left[\frac{1}{L^3}\sum_{\mathbf{k}} - \int\frac{d^3\mathbf{k}}{(2\pi)^3}\right]\int\frac{dk_4}{2\pi}\,\mathcal{L}(P-k,k)\,\Delta(k)\Delta(P-k)\,\mathcal{R}^\dagger(P-k,k)$$

finite volume — infinite volume — dressed propagators [only the poles matter]

performing the k_4 integration

$$= -\left[\frac{1}{L^3}\sum_{\mathbf{k}} - \int\frac{d^3\mathbf{k}}{(2\pi)^3}\right]\frac{1}{2\omega_k}\,\frac{1}{2\omega_{P-k}}\,\mathcal{L}\,\frac{1}{E-\omega_k-\omega_{P-k}+i\epsilon}\,\mathcal{R}^\dagger\Big|_{k_4=i\omega_k}$$

for smooth functions of $\boldsymbol{k}$,
the difference between Σ and ∫
is exponentially suppressed

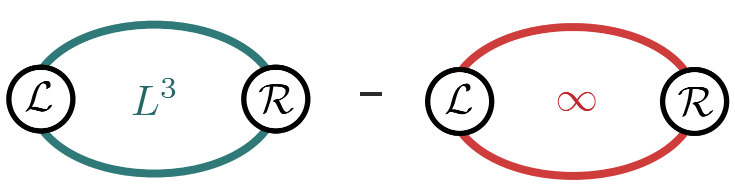
[Poisson summation formula]

but there is a **pole** at

$$E = \omega_k + \omega_{P-k}$$

this ensures **on-shell** dominance in $\mathcal{L}, \mathcal{R}^\dagger$

expanding in partial-waves



$$= -\mathcal{L}_{\ell m}(P)\,F_{\ell m,\ell' m'}(P,L)\,\mathcal{R}^\dagger_{\ell' m'}(P)$$

with $F_{\ell m,\ell' m'}(P,L) = -\left[\frac{1}{L^3}\sum_{\mathbf{k}} - \int\frac{d^3\mathbf{k}}{(2\pi)^3}\right]\frac{4\pi\,Y_{\ell m}(\hat{\mathbf{k}}^\star)\,Y^*_{\ell m}(\hat{\mathbf{k}}^\star)}{2\omega_k\,2\omega_{P-k}\,(E-\omega_k-\omega_{P-k}+i\epsilon)}\left(\frac{k^\star}{q^\star}\right)^{\ell+\ell'}$

a 3+1 field theory derivation

consider a two-point correlation function – operators with the quantum numbers of a two-hadron system

$$C_L(t,\mathbf{P}) = \int_L d^3\mathbf{x} \int_L d^3\mathbf{y}\; e^{-i\mathbf{P}\cdot(\mathbf{x}-\mathbf{y})} \langle 0 | A(\mathbf{x},t)\, B^\dagger(\mathbf{y},0) | 0 \rangle$$

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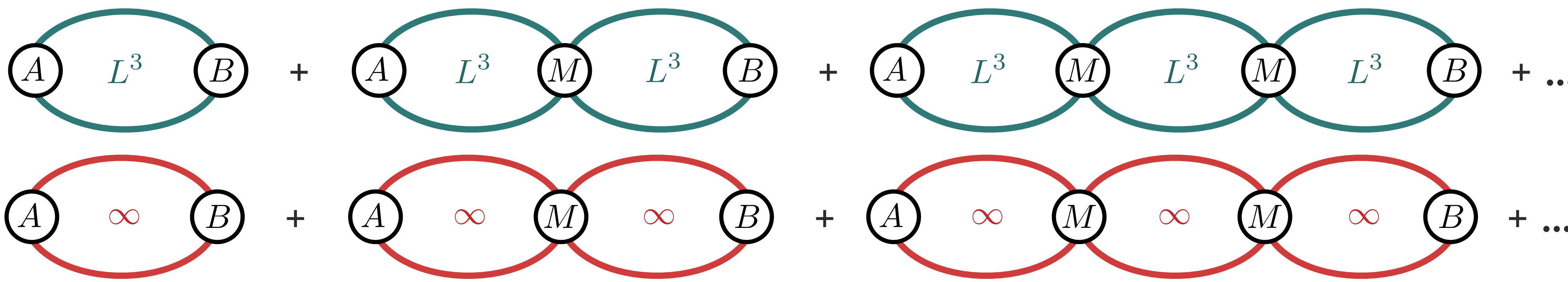


$$C_L - C_\infty = \tilde{A}\,(-F)\,\tilde{B} + \tilde{A}\,(-F)M(-F)\,\tilde{B} + \tilde{A}\,(-F)M(-F)M(-F)\,\tilde{B} + \ldots$$

a geometric series can be summed: $\tilde{A}\left[F^{-1} + M\right]^{-1}\tilde{B}$

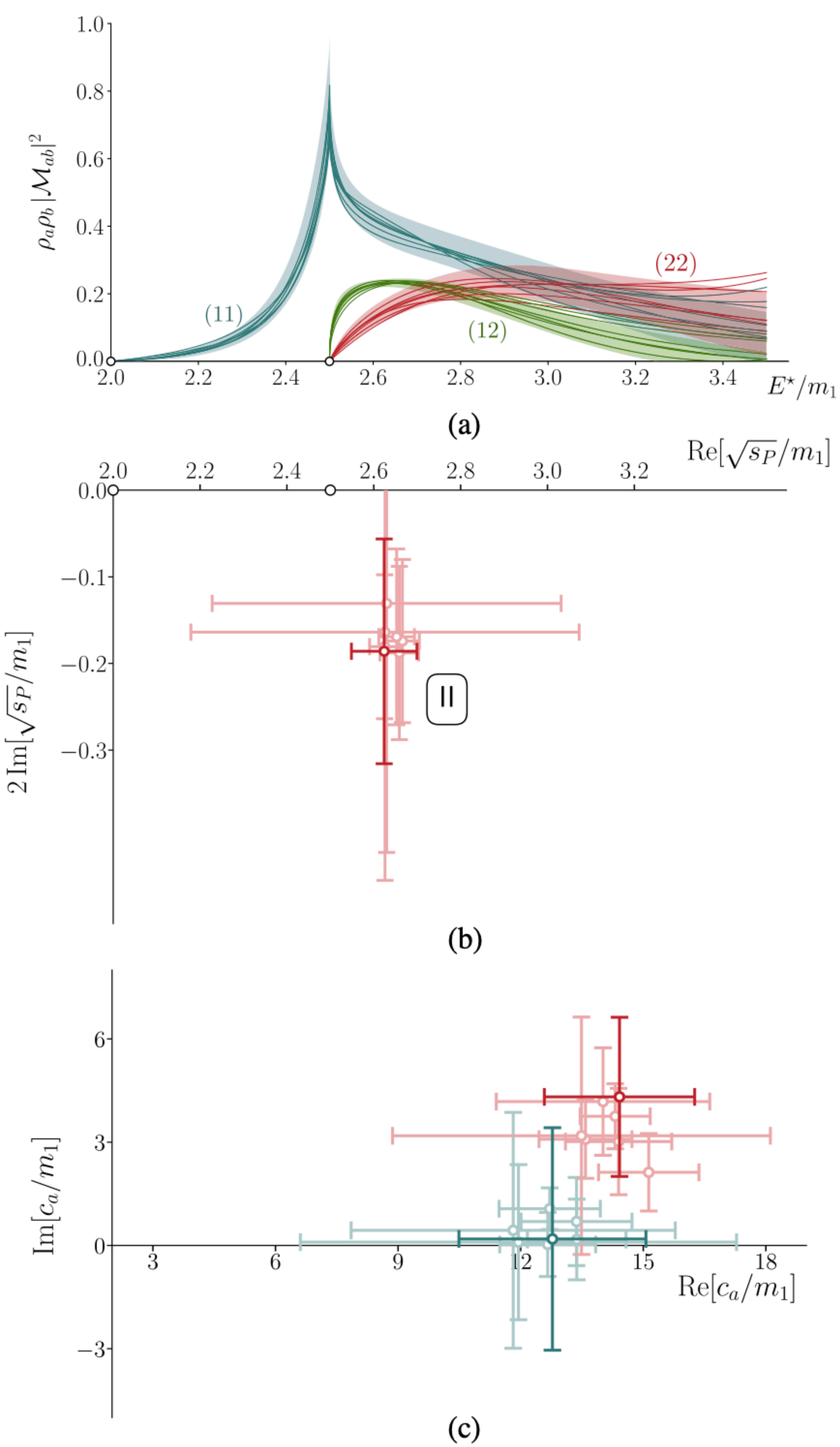
giving $C_L(t,\mathbf{P}) = L^3 \int \frac{dE}{2\pi}\, e^{iEt} \Big[C_\infty(E,\mathbf{P}) - \tilde{A}\left[F^{-1}(E,\mathbf{P},L) + M(E,\mathbf{P})\right]^{-1}\tilde{B} \Big]$

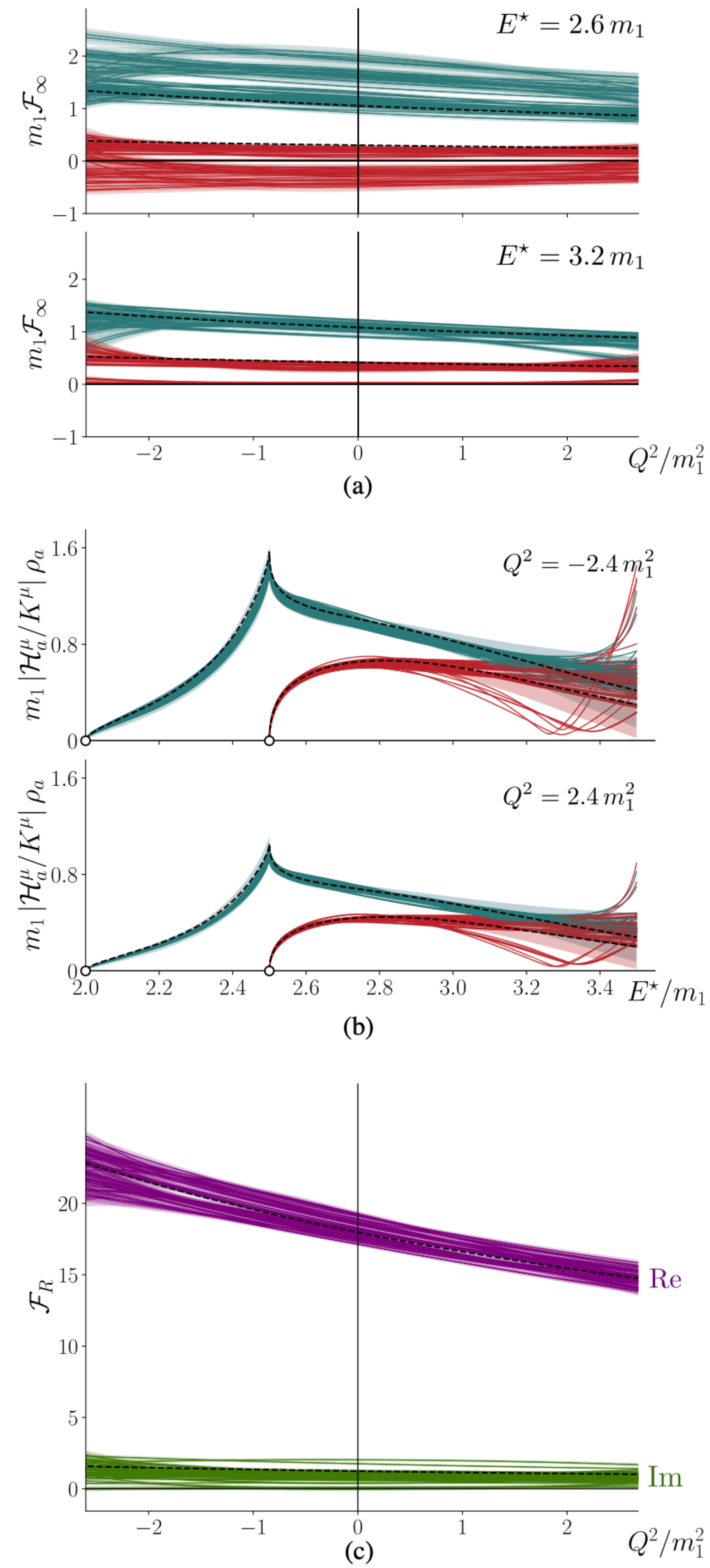
discrete spectral decomposition for finite-volume requires poles in *E*

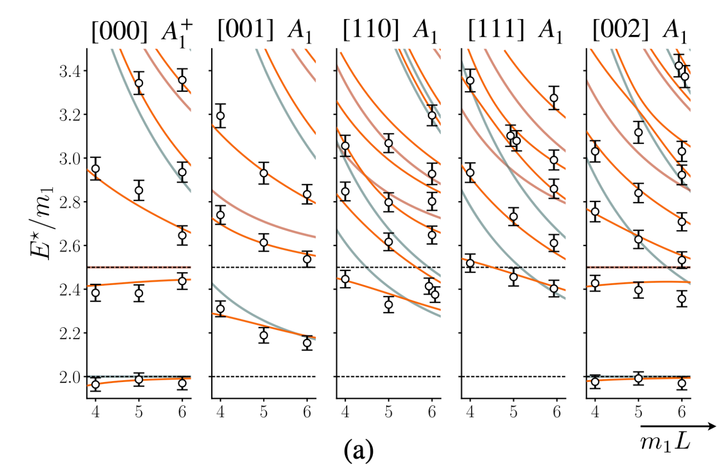
⇒ divergence of $\left[F^{-1}(E,\mathbf{P},L) + M(E,\mathbf{P})\right]^{-1}$

$$\Rightarrow \det\left[F^{-1}(E,\mathbf{P},L) + M(E,\mathbf{P})\right] = 0$$

coupled-channel toy model







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Constraining $1 + \mathcal{J} \to 2$ coupled-channel amplitudes in a finite volume

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