Ambiguities in the Partial-Wave Analysis of Meson Photoproduction

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1 FLORIDA STATE UNIVERSITY 1851

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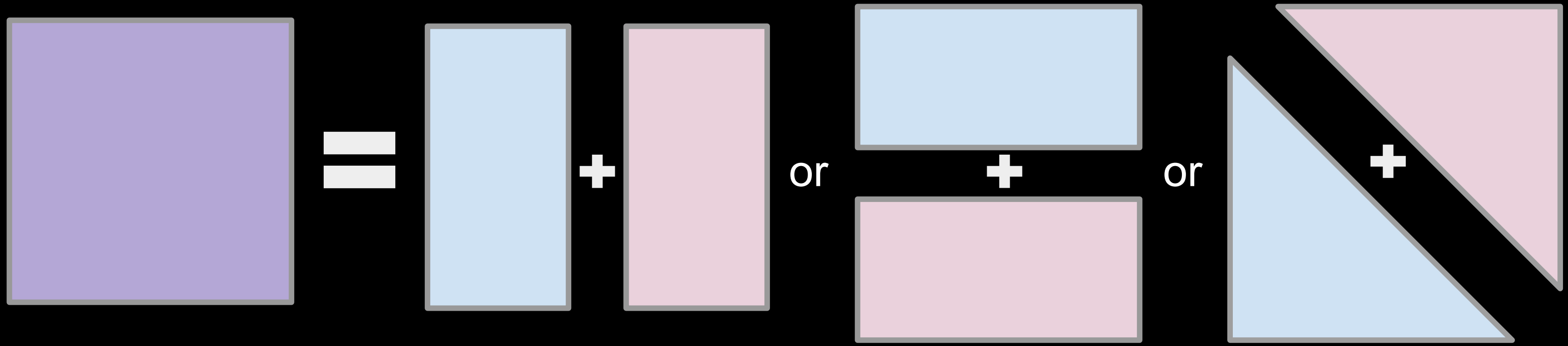
3 WILLIAM & MARY CHARTERED 1693

An Old Problem in a New Light

- Ambiguities in PWA is a known problem studied for years
- Solutions on how to address ambiguities exist See Julien B.'s talk: Tuesday 12:20 am
- JPAC recently claimed "there are no mathematical ambiguities in partial-wave analysis of two mesons produced with a linearly polarized photon beam"
  - This is true…for a limited waveset or for Barrelet zeroes' ambiguities
- Linearly polarized photon amplitudes have more story to share…

Let's be Clear on Ambiguities

- We focus on continuous ambiguities from Under-Constrained Solutions (UCS), e.g. many shapes that can make a square
- Barrelet zeros are discrete ambiguities, e.g. colors on the shapes combine to make purple



Amplitudes for Linearly Polarized Photon

$$I(\Phi,\Omega) \approx (1-P_\gamma)\left[\left|\sum_{\ell,m}[J_\ell]^{(-)}_m \mathrm{Im}(Z^m_\ell)\right|^2 + \left|\sum_{\ell,m}[J_\ell]^{(+)}_m \mathrm{Re}(Z^m_\ell)\right|^2\right]$$
$$+(1+P_\gamma)\left[\left|\sum_{\ell,m}[J_\ell]^{(+)}_m \mathrm{Im}(Z^m_\ell)\right|^2 + \left|\sum_{\ell,m}[J_\ell]^{(-)}_m \mathrm{Re}(Z^m_\ell)\right|^2\right]$$

$$Z^m_l(\Omega,\Phi) = Y^m_l(\Omega)e^{-i\Phi}$$

- Remember: positive (negative) reflectivity = natural (unnatural) parity exchange
- Are there enough equations to obtain unique free parameters?

Describing Moments with Amplitudes

- Moment expand the intensity with orthogonal angular functions
- Angular moments can be expressed in terms of amplitudes
- For 2 pseudoscalars the formula is:

$$H(L,M) = \sum \rho^{\ell\ell'}_{mm'} \left[\sqrt{\frac{2L+1}{4\pi}} \sqrt{\frac{2\ell'+1}{2\ell+1}} \langle \ell' 0, L0, | \ell 0 \rangle \langle \ell' m', LM | \ell m \rangle \right]$$

- ρ is the spin density matrix containing the conjugate of the amplitudes from which the magnitudes and phases are extracted
- Allows to count equations and number of parameters

From Moment Equations to Matrices

- The moment expansion can be generalized as:

$$\sum_{k}^{N} a_{ik} f_k = H_i$$

  - a_{mk} = all constants (Clebsch-Gordan, normalization, symmetry, etc.)
  - f_k = every combination of free complex parameters $\mathrm{Re}(J_m^{\varepsilon} J_m^{\varepsilon *})$
- The equation $Af = H$ can be expressed as:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ a_{M1} & a_{M2} & \cdots & a_{MN} \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_N \end{bmatrix} = \begin{bmatrix} H_1 \\ H_2 \\ \vdots \\ H_M \end{bmatrix}$$

Test if Model is Properly Constrained

- The *rank* of A gives the number of independent equations
- For N amplitudes, there are 2N-2 free parameters (these are magnitudes and phases)
- Therefore, UCS ambiguities are present when

$$\mathit{rank}(A) < 2N-2$$

- Means infinitely many values, within some boundaries, will yield the same value of moments/describe the same angular distribution
- This test can be used for any combination of amplitudes

Example: P-Wave Matrix 2 Pseudoscalars

- The matrix A is shown below for the waveset $[P^{\pm}_{-1}, P^{\pm}_{0}, P^{\pm}_{1}]$
- Its rank is 10 and the number of parameters is 2(6) - 2 = 10

$$H(0,0) = 2|P_1^+|^2 + 2|P_0^+|^2 + 2|P_{-1}^+|^2 + 2|P_1^-|^2 + 2|P_0^-|^2 + 2|P_{-1}^-|^2$$

Positive Reflectivity						Negative Reflectivity					
P_1^2	Re(P_1P_0)	Re(P_1P_{-1})	P_0^2	Re($P_{-1}P_0$)	P_{-1}^2	P_1^2	Re(P_1P_0)	Re(P_1P_{-1})	P_0^2	Re($P_{-1}P_0$)	P_{-1}^2
2	-	-	2	-	2	2	-	-	2	-	2
-	-	-4	2	-	-	-	-	4	-2	-	-
-0.4	-	-	0.8	-	-0.4	-0.4	-	-	0.8	-	-0.4
-	0.7	-	-	-0.7	-	-	0.7	-	-	-0.7	-
-	-	-1	-	-	-	-	-	-1	-	-	-
-	-	0.8	0.8	-	-	-	-	-0.8	-0.8	-	-
-	0.7	-	-	-0.7	-	-	-0.7	-	-	0.7	-
0.5	-	-	-	-	0.5	-0.5	-	-	-	-	-0.5
-	-0.7	-	-	-0.7	-	-	0.7	-	-	0.7	-
-0.5	-	-	-	-	0.5	0.5	-	-	-	-	-0.5

Example: P-Wave Matrix 2 Pseudoscalars

- The reduced row echelon form immediately tell us its rank

Positive Reflectivity						Negative Reflectivity					
P_1^2	$Re(P_1P_0)$	$Re(P_1P_{-1})$	P_0^2	$Re(P_{-1}P_0)$	P_{-1}^2	P_1^2	$Re(P_1P_0)$	$Re(P_1P_{-1})$	P_0^2	$Re(P_{-1}P_0)$	P_{-1}^2
1	-	-	-	-	-	-	-	-	-	-	1
-	1	-	-	-	-	-	-	-	-	-1	-
-	-	1	-	-	-	-	-	-	-	-	-
-	-	-	1	-	-	-	-	-	-	-	-
-	-	-	-	1	-	-	-	-	-	-1	-
-	-	-	-	-	1	-	-	-	-	-	-1
-	-	-	-	-	-	1	-	-	-	-	1
-	-	-	-	-	-	-	1	-	-	-1	-
-	-	-	-	-	-	-	-	1	-	-	-
-	-	-	-	-	-	-	-	-	1	-	-

Example: P-Wave Matrix 2 Pseudoscalars

- The negative refl. P_{-1}^2 depends on $P_{\pm 1}^2$
- The negative refl. $Re(P_0 P_{-1})$ depends on $Re(P_0 P_{\pm 1})$
- The P_0^2 and $Re(P_1 P_{-1})$ are independent on both reflectivities

Hypothesis 1: Ambiguity exists in both reflectivities for m=±1, ±2, etc

Positive Reflectivity						Negative Reflectivity					
P_1^2	$Re(P_1P_0)$	$Re(P_1P_{-1})$	P_0^2	$Re(P_{-1}P_0)$	P_{-1}^2	P_1^2	$Re(P_1P_0)$	$Re(P_1P_{-1})$	P_0^2	$Re(P_{-1}P_0)$	P_{-1}^2
1	-	-	-	-	-	-	-	-	-	-	1
-	1	-	-	-	-	-	-	-	-	-1	-
-	-	1	-	-	-	-	-	-	-	-	-
-	-	-	1	-	-	-	-	-	-	-	-
-	-	-	-	1	-	-	-	-	-	-1	-
-	-	-	-	-	1	-	-	-	-	-	-1
-	-	-	-	-	-	1	-	-	-	-	1
-	-	-	-	-	-	-	1	-	-	-1	-
-	-	-	-	-	-	-	-	1	-	-	-
-	-	-	-	-	-	-	-	-	1	-	-

Example: P-Wave Matrix 2 Pseudoscalars

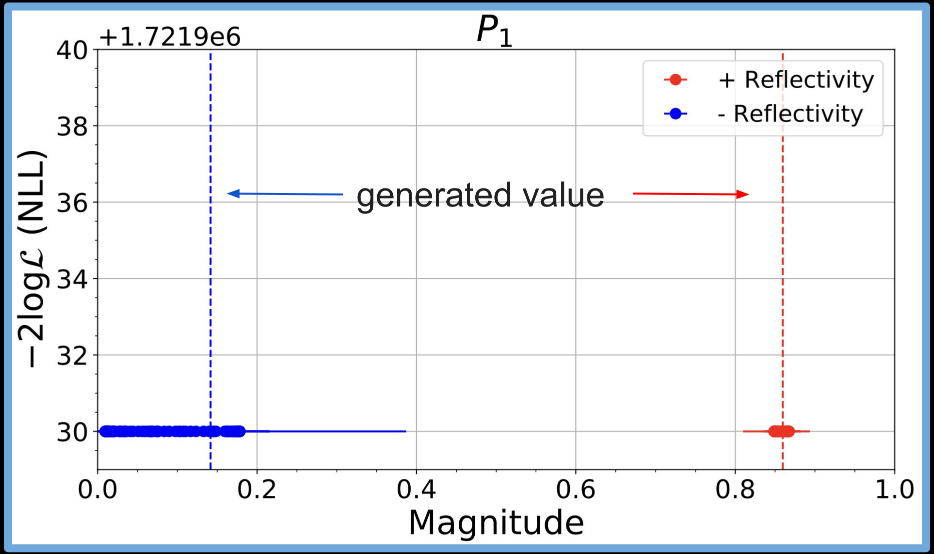
- If nature does not have P_0 waves, then UCS ambiguities must occur
  - rank(A) = 5 < 6 free parameters
- UCS ambiguities may arise from correlation between the $P_{\pm1}$ waves

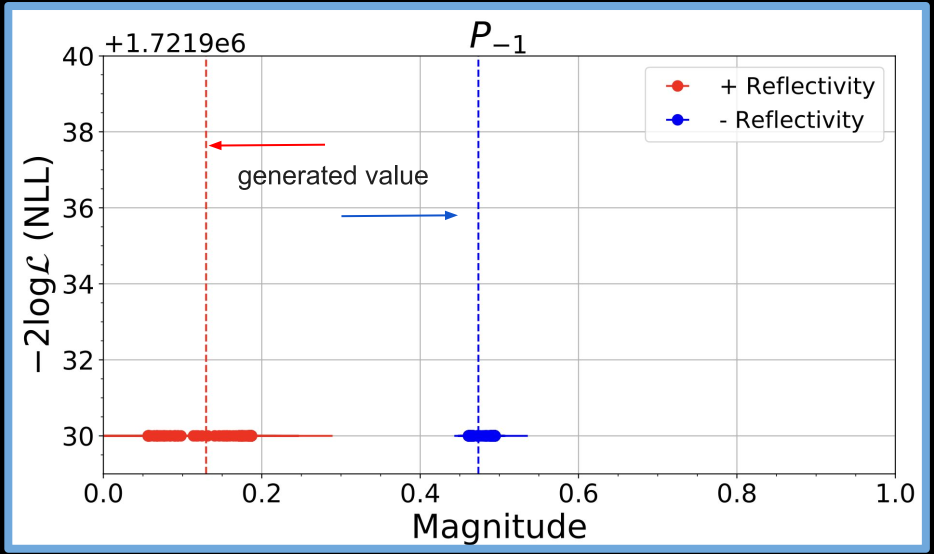
Positive Reflectivity			Negative Reflectivity		
P_1^2	$Re(P_1P_{-1})$	P_{-1}^2	P_1^2	$Re(P_1P_{-1})$	P_{-1}^2
1	-	-	-	-	1
-	1	-	-	-	-
-	-	1	-	-	-1
-	-	-	1	-	1
-	-	-	-	1	-

Hypothesis 2: m=0 provides an "anchor" to the range of ambiguous solutions

Example: MC P-Wave 2 Pseudoscalars

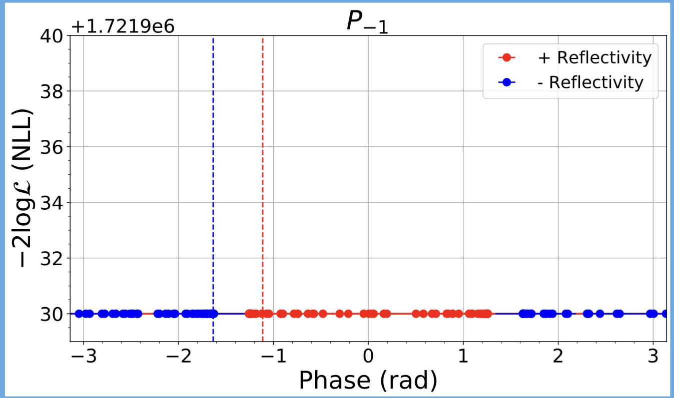
- MC of the waveset [$P^{\pm}_{1}$,$P^{\pm}_{-1}$] shows multiple solutions when fitted

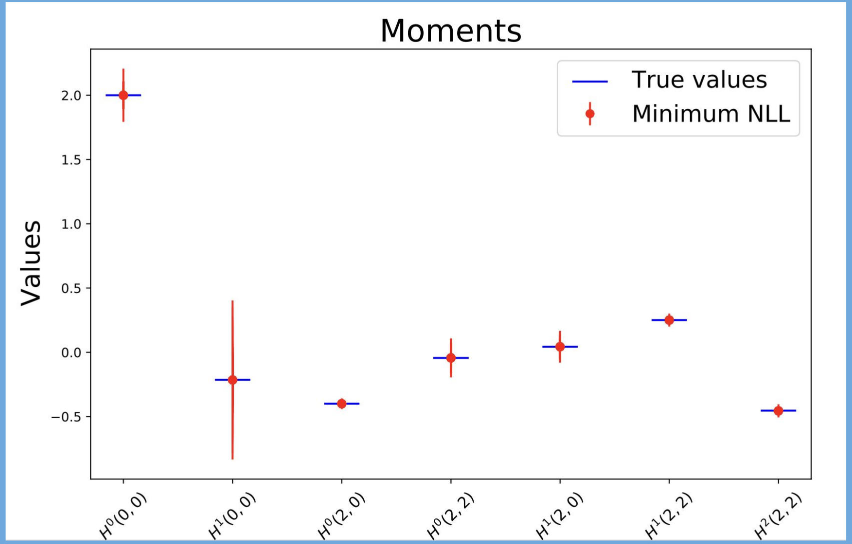


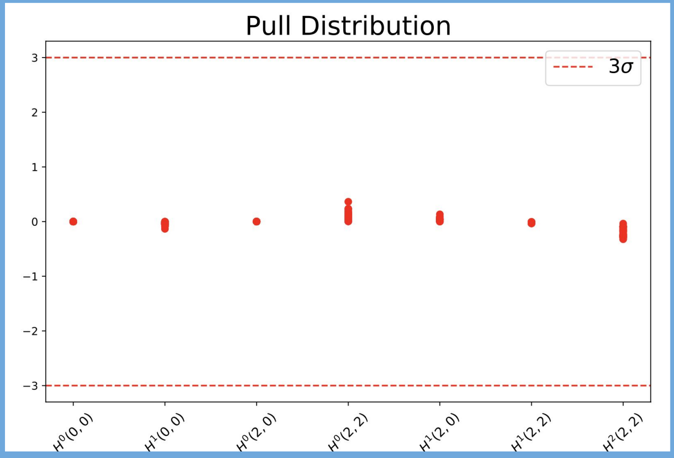


Example: MC P-Wave 2 Pseudoscalars

- Phases behave wildly
- The fits will describe the moments well







Wigner D Functions and m=±1 Correlation

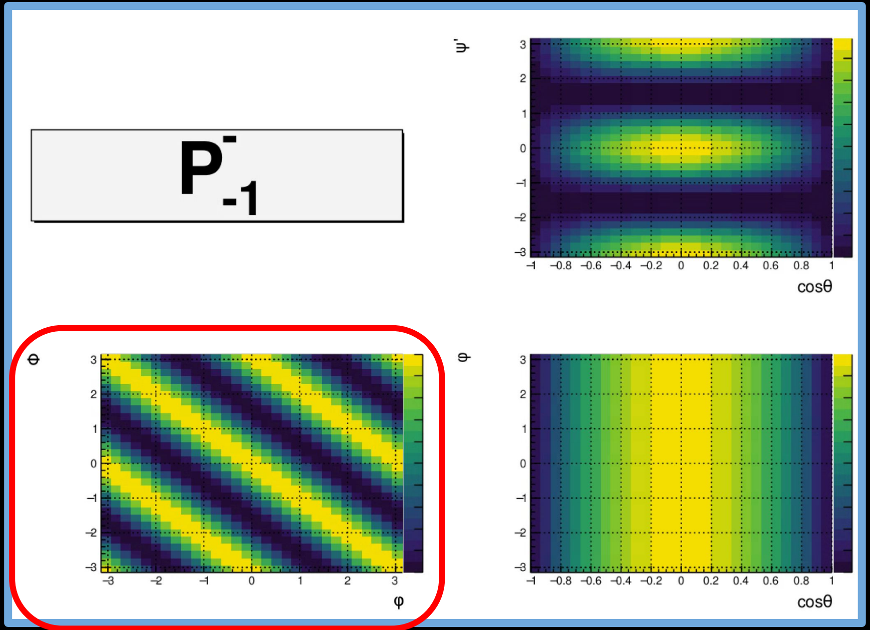
- Amplitudes, like moments, are expanded using Wigner D functions
- Moments are unambiguous by construction
- Amplitudes describe intensity in conjugate Wigner D pairs

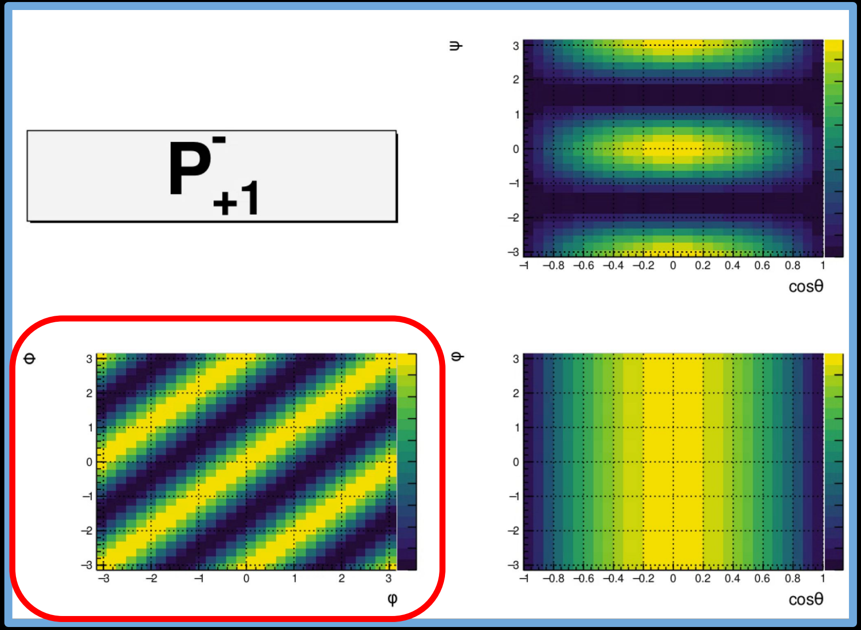
$$D^j_{m'm}(\alpha,\beta,\gamma) = (-1)^{m'-m} D^j_{-m',-m}(\alpha,\beta,\gamma)^*$$

$$D^j_{m'm}(\alpha,\beta,\gamma)^* \; D^j_{m'm}(\alpha,\beta,\gamma) = D^j_{-m',-m}(\alpha,\beta,\gamma)^* \; D^j_{-m',-m}(\alpha,\beta,\gamma)$$

Polarization Adds Distinguishing Power

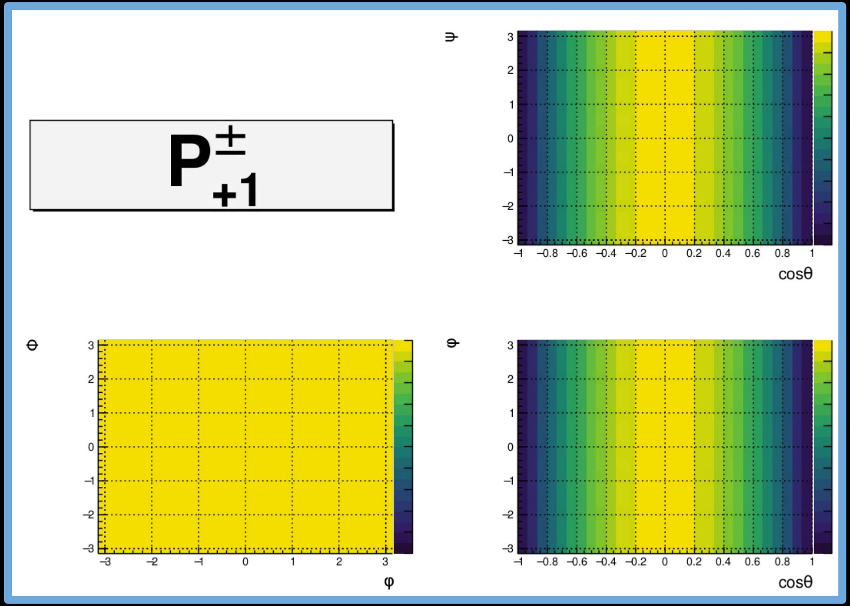
- The photon linear polarization helps distinguish waves
- The amount of both reflectivities affects the distinguishing power

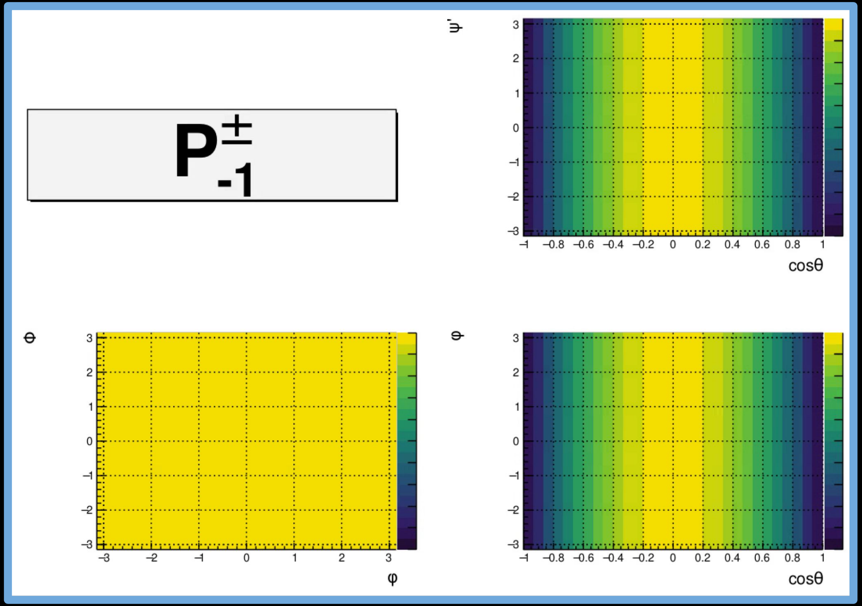




The “Flat Wave” Problem

- When equal amounts of positive and negative reflectivity are present, the m = ±1 waves become indistinguishable



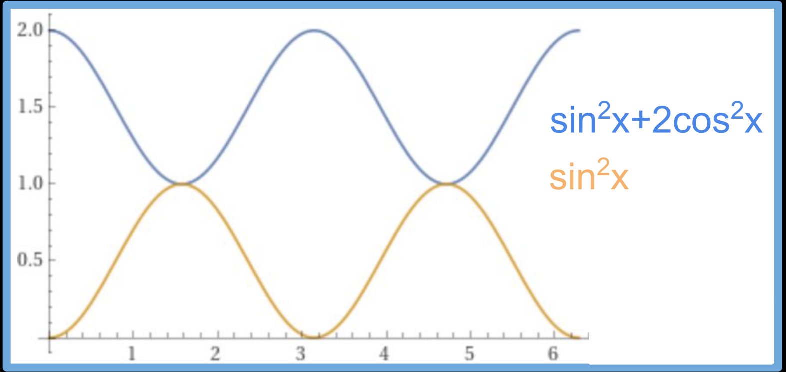


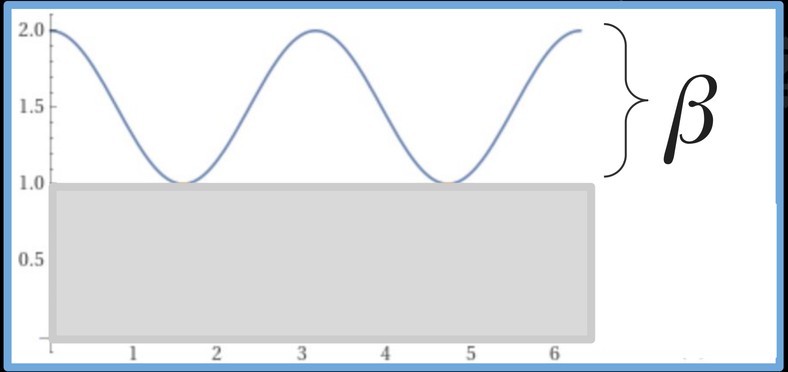
The "Flat Wave" Problem

- Ex: Dominant reflectivity for a single m:

$$J^{+}_{1} \sim \sin^2 x < J^{-}_{1} \sim 2\cos^2 x$$

- When combined with the other m, the "flat" part becomes ambiguous i.e. both m = ±1 flat parts can describe the gray box
- Can unambiguously extract the difference between the reflectivities (β) in an m projection

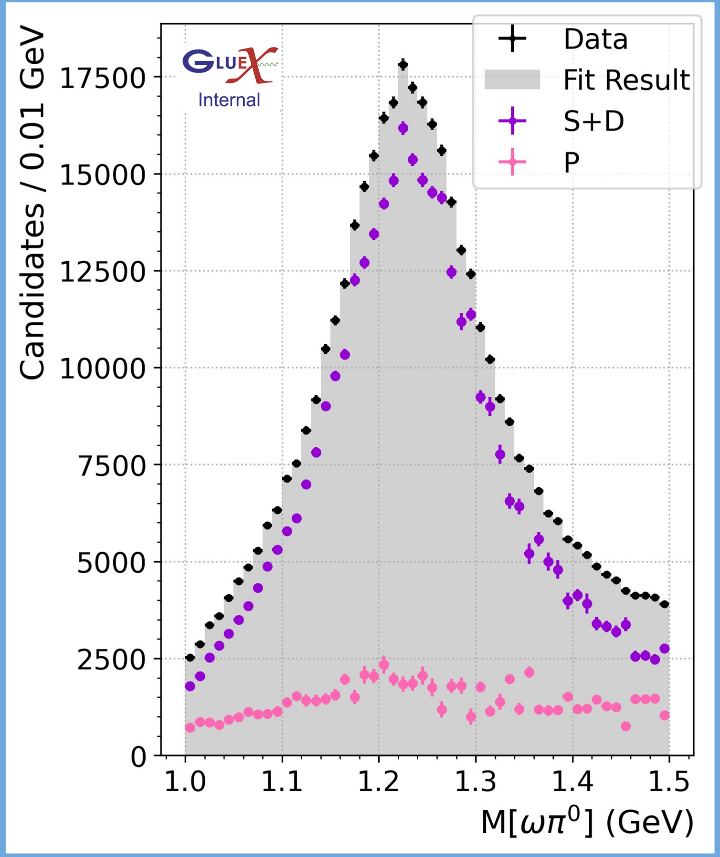


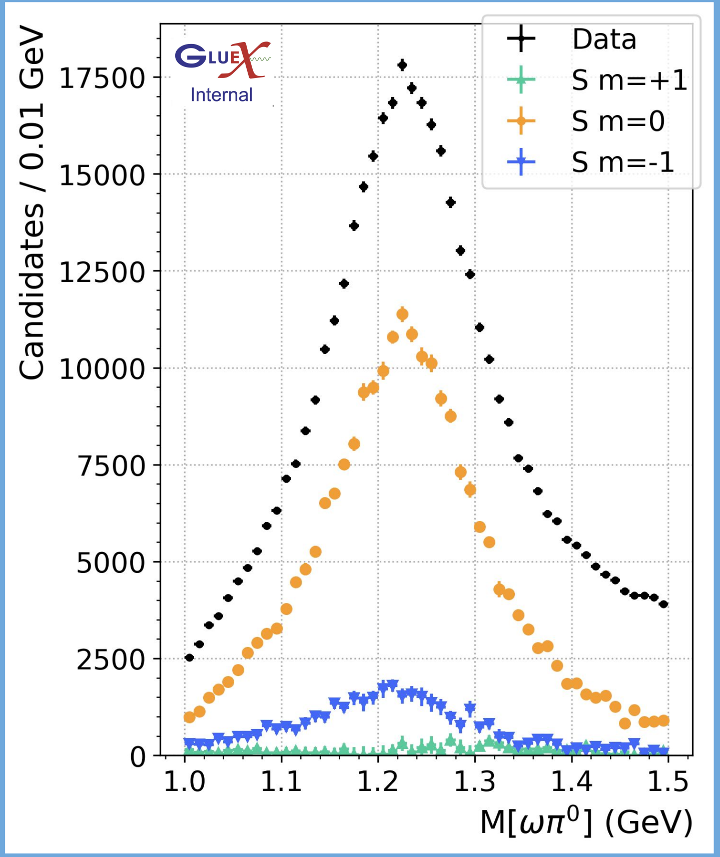


Example Data: Ambiguities in $\omega\pi^0$ and $\omega\eta$

- The current wavesets used in both channels predict no ambiguities:
  - ✓ (1S, 1P, 1D waves): 34 free params < rank = 46
  - ✓ (1S, 1P, 2P, 3F waves): 70 free params < rank = 153
- $\omega\pi^0$ is observed to effectively have no ambiguities, while $\omega\eta$ seems to be influenced by them

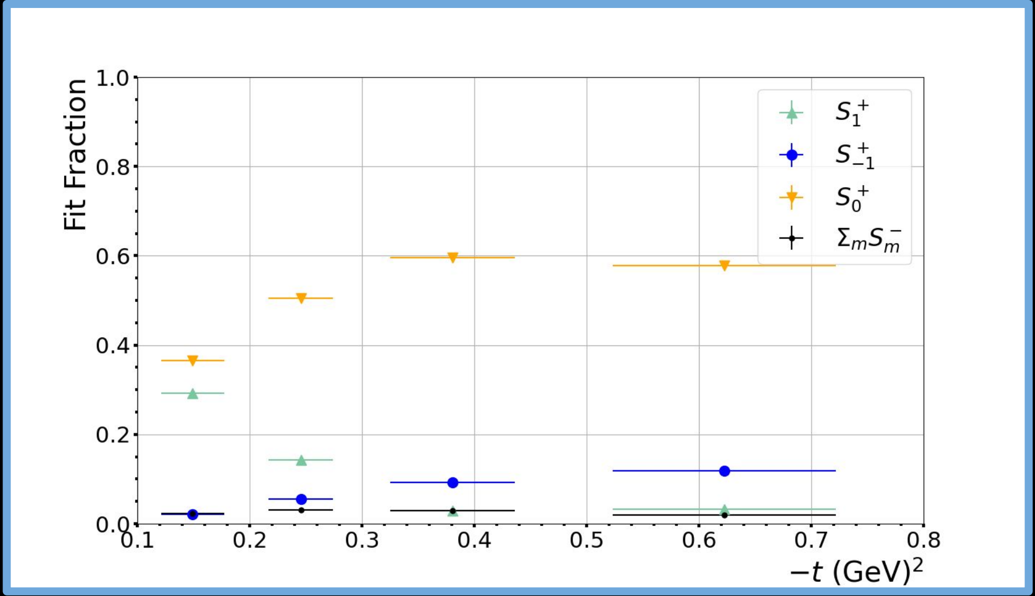
PWA of $\omega\pi^0$ Showing Stability





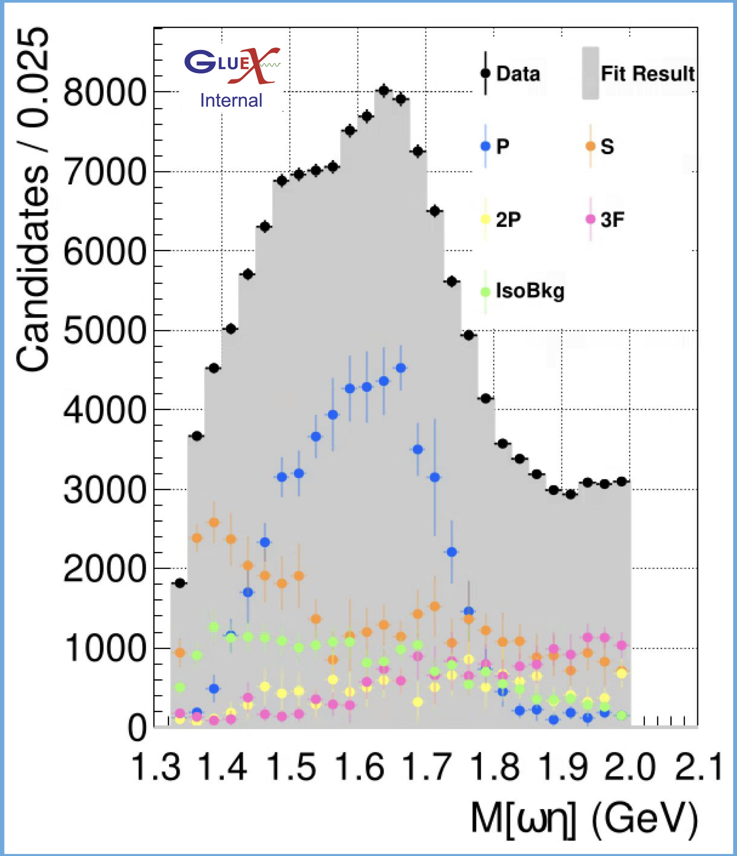
Hypothesis: $\omega\pi^0$ PWA

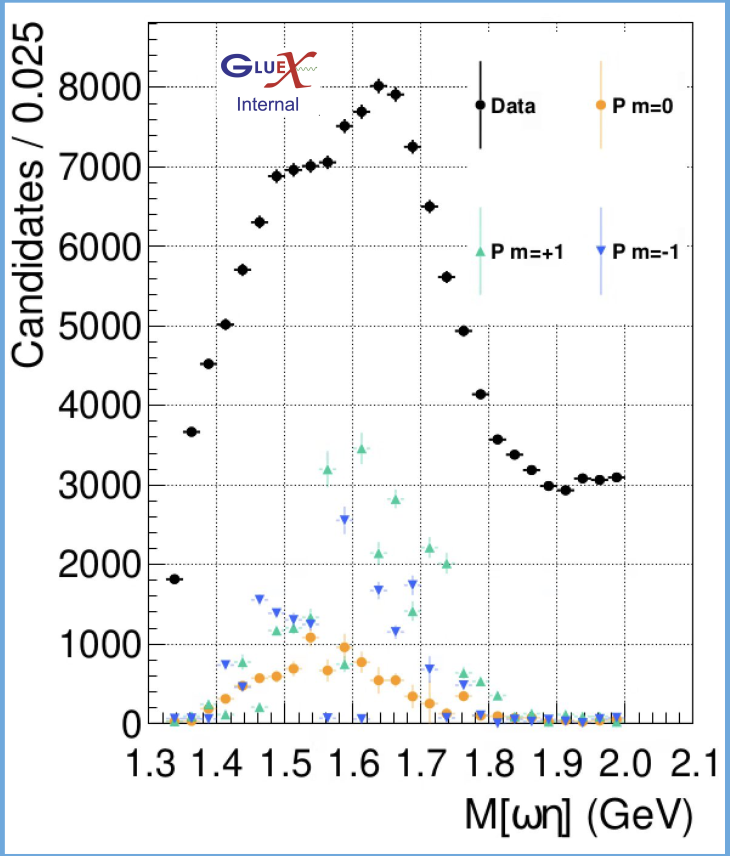
- Fit to GlueX-I
  - Very dominant S-wave
  - ✓ Significant m=0 contribution
  - ✓ Dominance of a reflectivity



Data Hypothesis 1: a strong m = 0 contribution restricts USC ambiguities

PWA of $\omega\eta$ Showing Signs of UCS Ambiguities

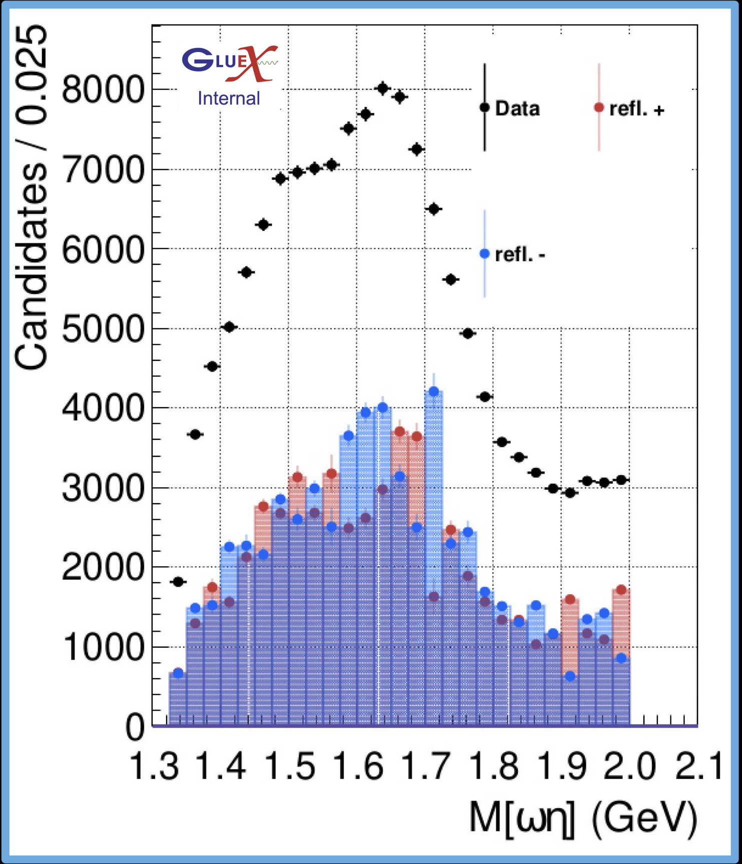




Hypothesis: $\omega\eta$ PWA

- Fit to GlueX-I
  - ~ Small m=0 contribution
  - x No dominant reflectivity

Data Hypothesis 2: the closer the contribution from both reflectivities are, the higher the chances the fit suffers from UCS ambiguities



Paths to Help Fits with UCS Ambiguities

- Amplitudes use an orthogonal basis that is “less” orthogonal when describing the quadratic space in which the intensity lives. This creates a dependency between the pair of $m = \pm 1, \pm 2$, etc waves
- Two paths that could help with UCS ambiguities:
  - Reorganizing amplitudes in the intensity
  - Adding constraints

Solution: Reorganizing Amplitudes in the Intensity

What is it about:

Grouping a set of amplitudes together could help the fit find the global minima more reliably

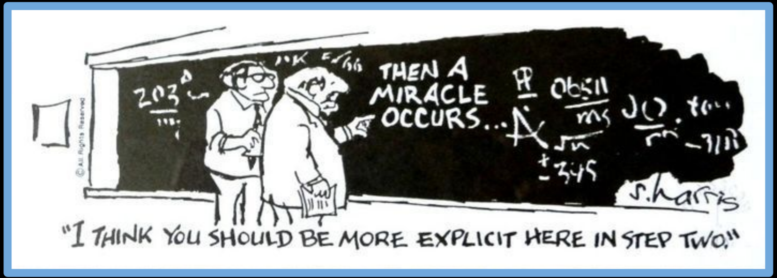
Pros:

A general approach independent of the model

It might be possible to calculate phase differences and help recognize resonances in data

Cons:

Sweeps the ambiguities under the rug, lose distinguishing power between m projections



Unpolarized-"Flat"

$$I(\{c^\epsilon, X\}^i_m) \approx \sum_{i,j,m_i,m_j} \left([c^i]^+_{m_i}[c^j]^{+*}_{m_j} + [c^i]^-_{m_i}[c^j]^{-*}_{m_j}\right)\left(X^i_{m_i}X^{j*}_{m_j} + X^{i*}_{m_i}X^j_{m_j}\right)$$

$$-P_\gamma\Big[\left([c^i]^+_{m_i}[c^j]^{+*}_{m_j} - [c^i]^-_{m_i}[c^j]^{-*}_{m_j}\right) \times \Big(\cos 2\Phi (X^{i*}_{m_i}X^{j*}_{m_j} + X^i_{m_i}X^j_{m_j}) + i \sin 2\Phi (X^{i*}_{m_i}X^{j*}_{m_j} - X^i_{m_i}X^j_{m_j})\Big)\Big]$$

polarized-β

Solution: Constraints

What is it about:

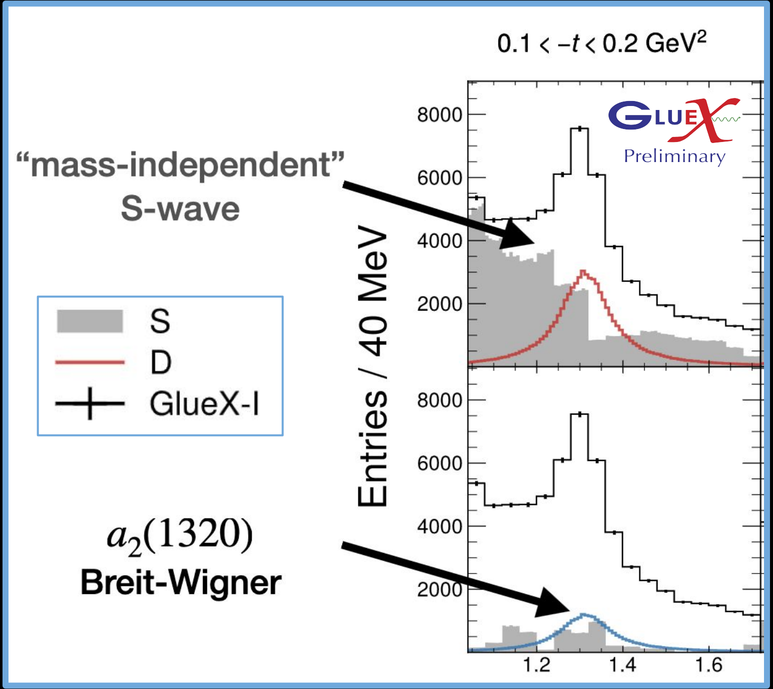
We could look for physical insight that would help us constrain the equations, e.g. the addition of Breit-Wigner amplitude across bins

Pros:

We add extra information to the system

Cons:

The constraints could be a case by case situation and not general



See Malte's talk: Tuesday 10:05 am
See Lawrence's talk: Next talk

Conclusions

- There is a challenge when describing the intensity in terms of amplitudes because of the way Wigner D functions describe the quadratic space
- Data sets might be sensitive to these type of issues even when the model used to fit it has, in principle, no ambiguities
- We are testing ways to circumvent the dependencies created by the math, to find a solution that is practical and can be generalized

GlueX acknowledges the support of several funding agencies and computing facilities: gluex.org/thanks

Backup

Spin Density Matrix

$$
{}^{0}\rho^{\ell,\ell'}_{m,m'}(\omega,t) = \frac{1}{2} \sum_{\substack{\lambda,\lambda'=\pm 1 \\ \lambda_1,\lambda_2=\pm 1/2}} \mathcal{T}^{\ell}_{m\lambda;\lambda_1\lambda_2}(\omega,t)\; \mathcal{T}^{\ell'*}_{m'\lambda';\lambda_1\lambda_2}(\omega,t)
$$

$$
{}^{1}\rho^{\ell,\ell'}_{m,m'}(\omega,t) = \frac{1}{2} \sum_{\substack{\lambda,\lambda'=\pm 1 \\ \lambda_1,\lambda_2=\pm 1/2}} \mathcal{T}^{\ell}_{m-\lambda;\lambda_1\lambda_2}(\omega,t)\; \mathcal{T}^{\ell'*}_{m'\lambda';\lambda_1\lambda_2}(\omega,t)
$$

$$
{}^{2}\rho^{\ell,\ell'}_{m,m'}(\omega,t) = \frac{i}{2} \sum_{\substack{\lambda,\lambda'=\pm 1 \\ \lambda_1,\lambda_2=\pm 1/2}} \lambda\; \mathcal{T}^{\ell}_{m-\lambda;\lambda_1\lambda_2}(\omega,t)\; \mathcal{T}^{\ell'*}_{m'\lambda';\lambda_1\lambda_2}(\omega,t)\,.
$$