Three-Hadron Systems from Lattice QCD

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International Workshop on Partial Wave Analyses

and Advanced Tools for Hadron Spectroscopy (PWA13/ATHOS8)

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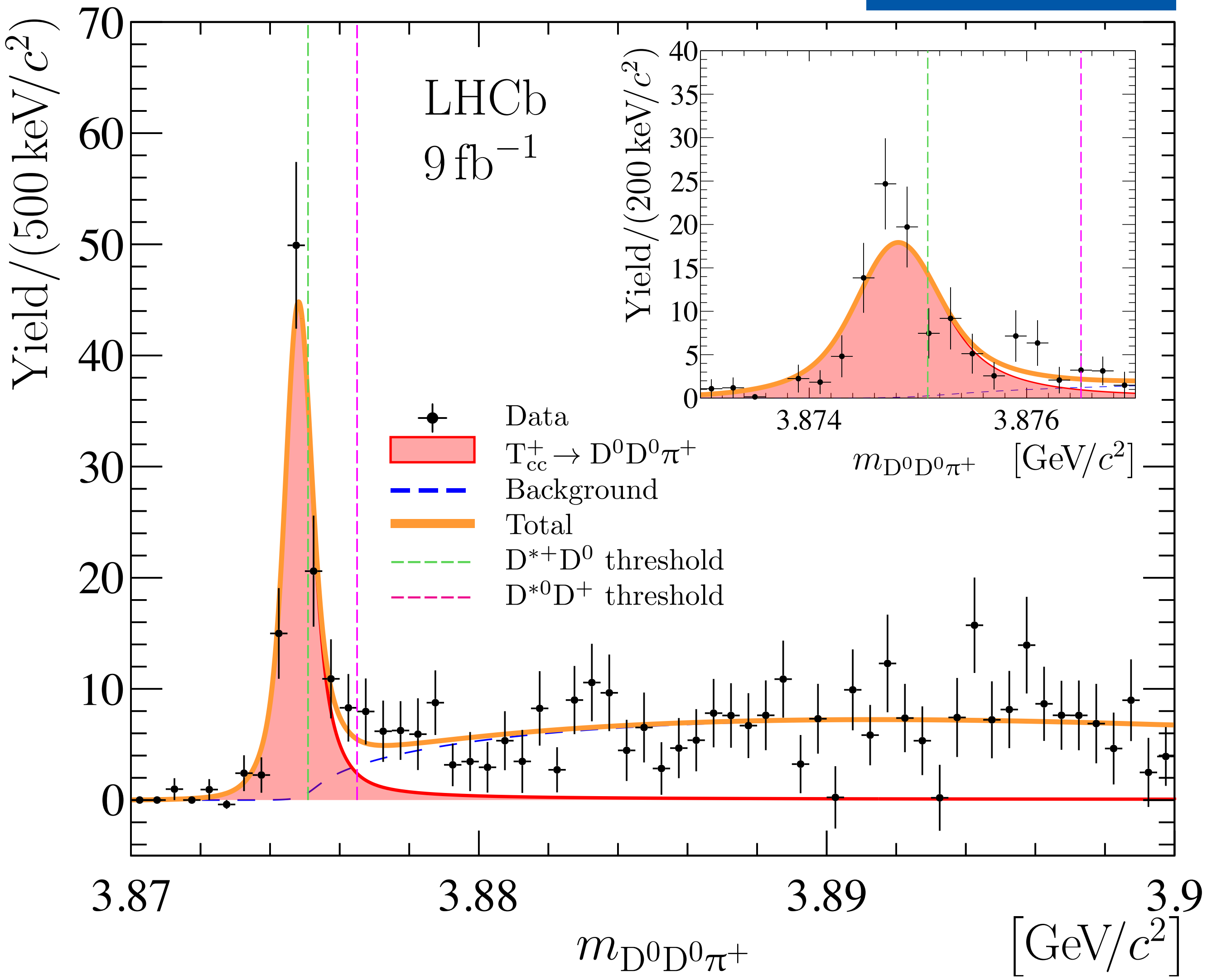
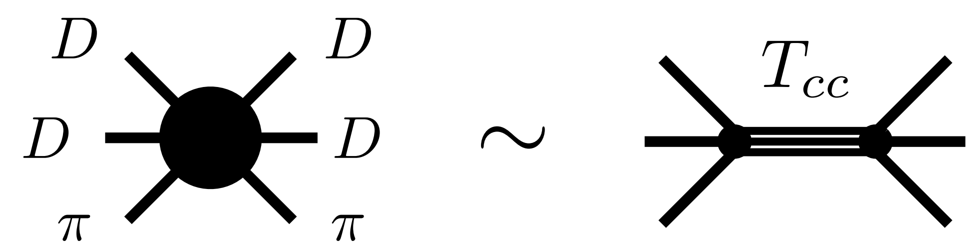
Three-Hadron Spectroscopy

Many hadrons in the excited QCD spectrum decay into three (or more) particles

- e.g., the very exciting T_{cc} tetraquark candidate



R. Aaij et al., [LHCb Collaboration]
Nature Physics **18**, 751–754 (2022)



$$\mathcal{M}_{DD\pi\to DD\pi} \sim -\frac{g^2_{T_{cc}}}{s - s_{T_{cc}}}$$

$$\sqrt{s_{T_{cc}}} = M_{T_{cc}} - i\Gamma_{T_{cc}}/2$$

See talk by S. Dawid today

Hadron Spectroscopy & Lattice QCD

Lattice QCD is a numerical tool to estimate low-energy QCD observables

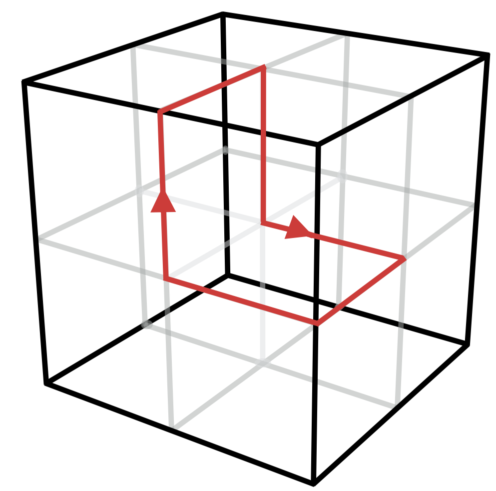
- Formulated on **finite**, discretized, *Euclidean* spacetime
- Extract finite-volume energy levels from correlation functions

hadron / multi-hadron operators

$$\mathcal{C}_L^{(\mathsf{E})}(\tau) = \langle \mathcal{O}(\tau)\mathcal{O}^\dagger(0)\rangle_L$$

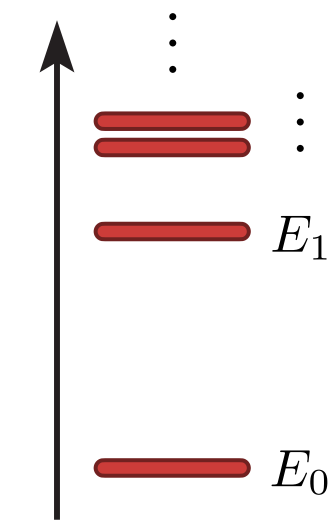
$$= \int \mathcal{D}\psi \int \mathcal{D}\bar{\psi} \int \mathcal{D}U \, e^{-S_{\mathrm{QCD}}^{(\mathsf{E})}} \, \mathcal{O}(\tau)\mathcal{O}^\dagger(0)$$

$$= \sum_{\mathfrak{n}} Z_{\mathfrak{n}} Z_{\mathfrak{n}}^\dagger \, e^{-E_{\mathfrak{n}}\tau}$$



QCD gauge configurations

finite volume spectrum



Very hard problem!
(No time to discuss)

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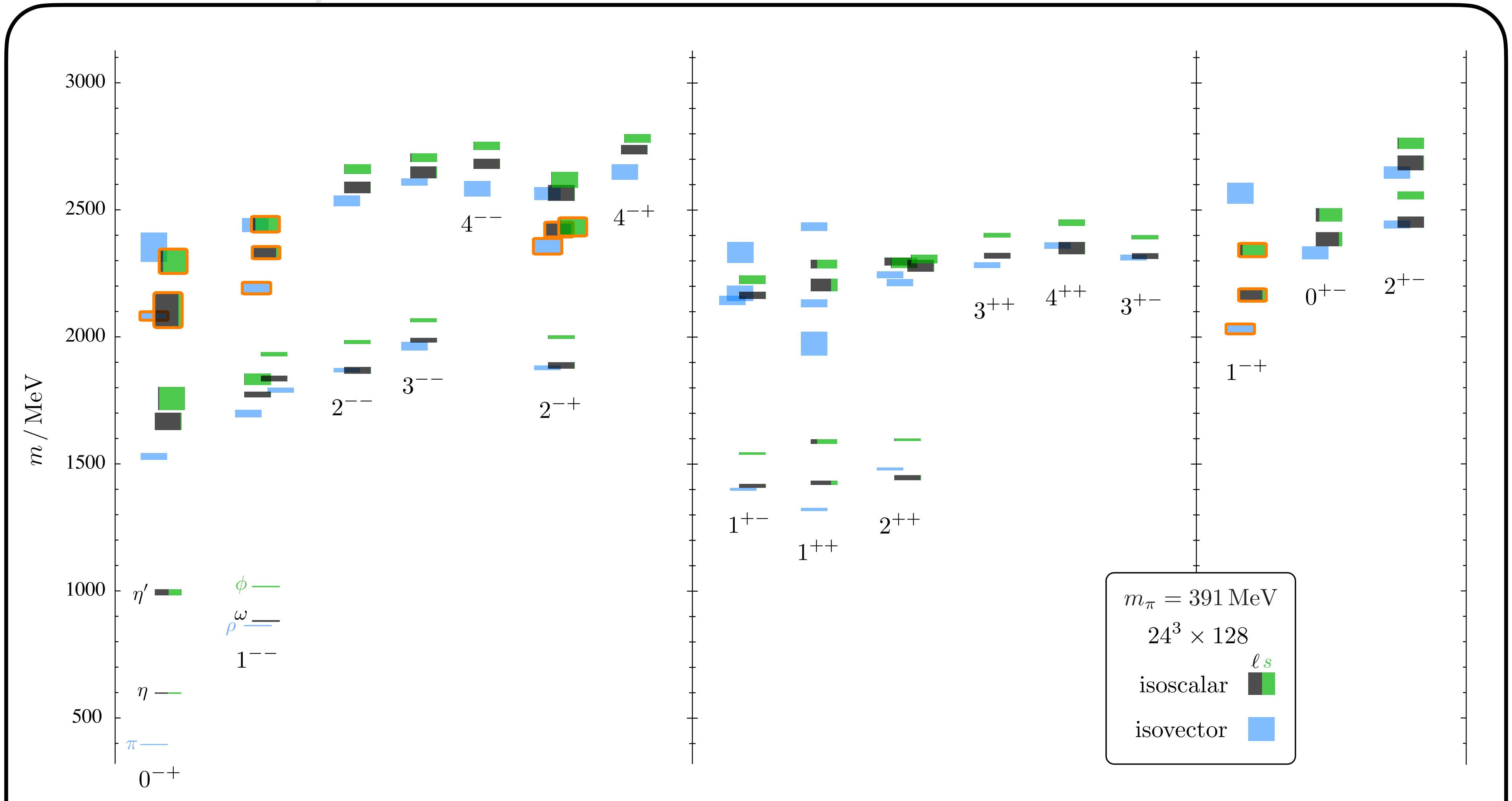

J. Dudek, R. Edwards, P. Guo, and C. Thomas [HadSpec]
Phys.Rev.D **88**, 094505 (2013)

Very hard problem!
(No time to discuss)

…misses multi-hadron couplings (i.e., amplitudes)…

…an incomplete picture…

Hadron Spectroscopy & Lattice QCD

Lattice QCD is a numerical tool to estimate low-energy QCD observables

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- Extract finite-volume energy levels from correlation functions

However, can extract amplitudes from lattice QCD

- Key — Map finite-volume energies to infinite-volume objects (via Lüscher)
- A path toward model independent resonance parameters from QCD

Hadron Spectroscopy & Lattice QCD

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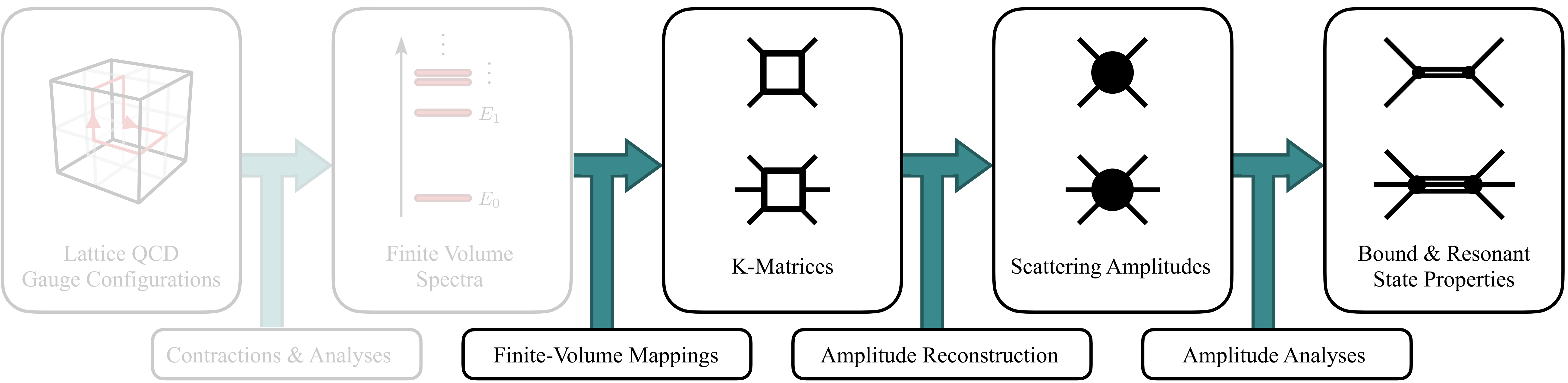
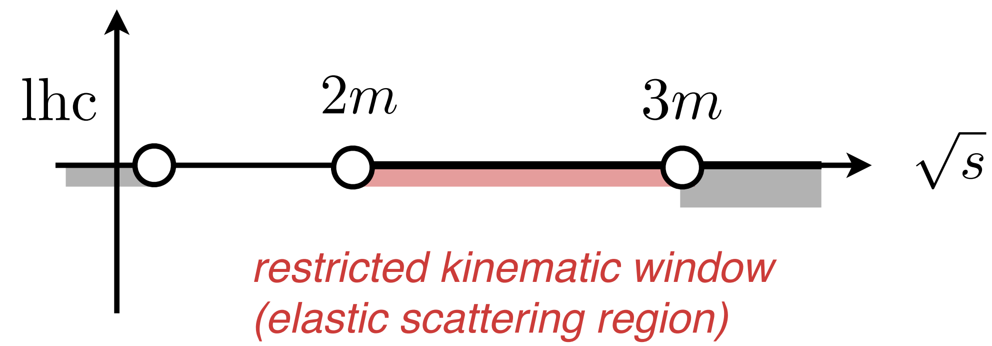


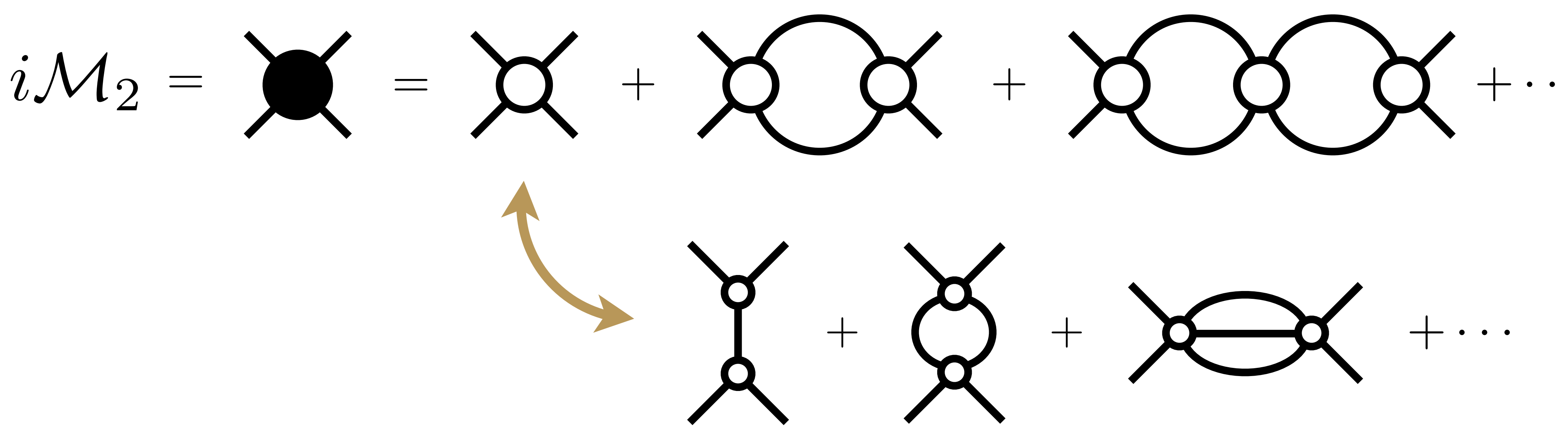
…focus on this part of process

Two-Hadron Systems

Let's review the *two-hadron case* as a template for three-hadron systems

- e.g., elastic scattering of spinless particles, *with* restricted energy region





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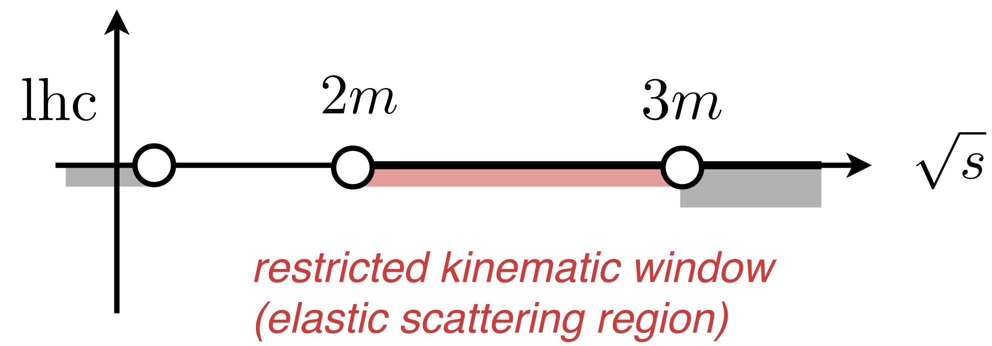
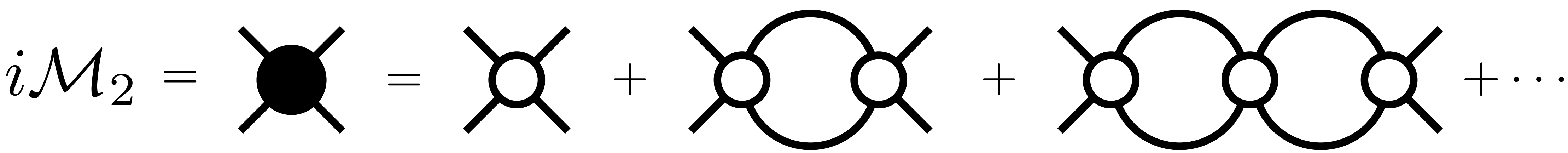
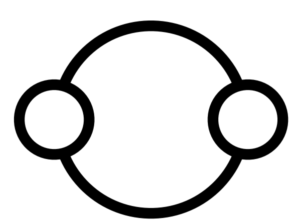




= isolate singular behavior, re-sum short-distance 'stuff' into new real function



$\sim$ real function $- \; i\rho$

two-body phase space

$$\rho \sim \sqrt{1 - \frac{4m^2}{s}}$$

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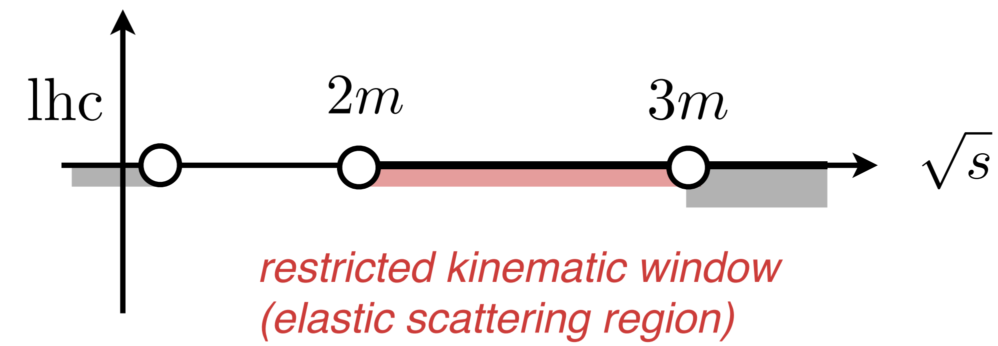

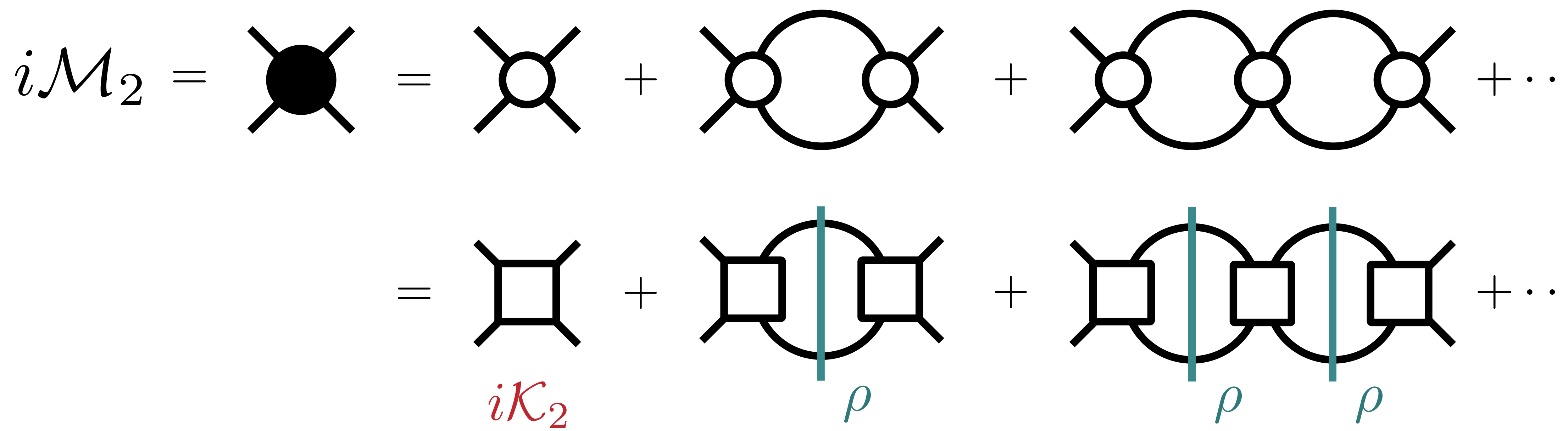

***K*-Matrix**

— short-distance dynamics

— constrained from data (exp. or th.)

Two-Hadron Systems

Let's review the *two-hadron case* as a template for three-hadron systems

- e.g., elastic scattering of spinless particles, *with* restricted energy region

K matrix is unphysical …

… choose convenient parameterizations (must preserve unitarity)

elastic scattering $(\mathcal{K}_2^{-1} \sim q \cot \delta)$

$$\mathcal{K}_2^{-1} \sim -\frac{1}{a} + \frac{1}{2} r q^2 + \cdots$$

near threshold scattering (effective range)

$$\mathcal{K}_2^{-1} \sim \frac{m_0^2 - s}{\sqrt{s}\,\Gamma}$$

Isolated narrow resonance (Breit-Wigner)

generically — polynomials and poles

$$\mathcal{K}_2 \sim \sum_j \frac{g_j^2}{m_j^2 - s} + \sum_j \gamma_j s^j$$

ematic window
ering region)

K-Matrix

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— constrained from data (exp. or th.)

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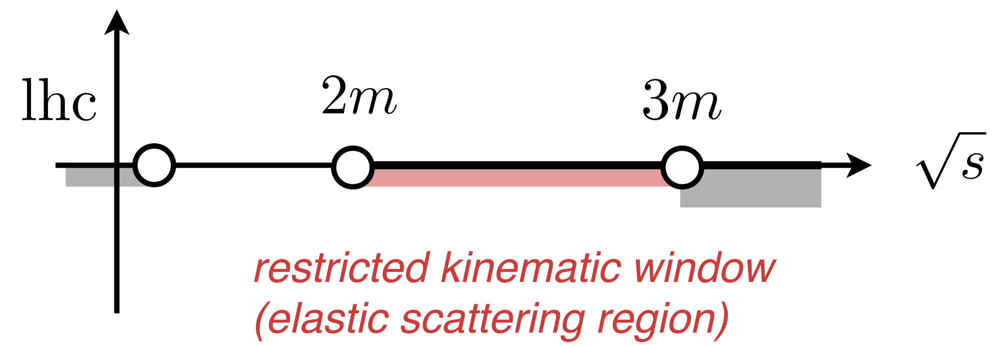
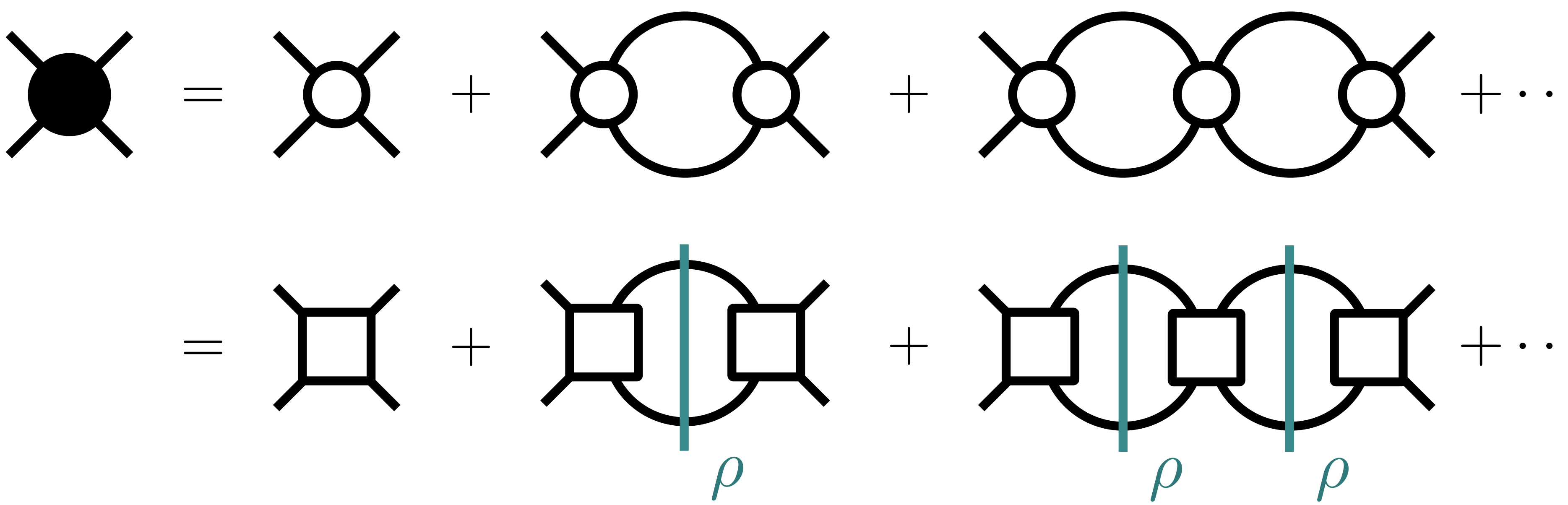


$i\mathcal{M}_2 =$



$$= \quad i\mathcal{K}_2 \, \frac{1}{1 - i\rho\mathcal{K}_2}$$

S matrix unitarity

$$\mathrm{Im}\,\mathcal{M}_2 = \rho|\mathcal{M}_2|^2$$

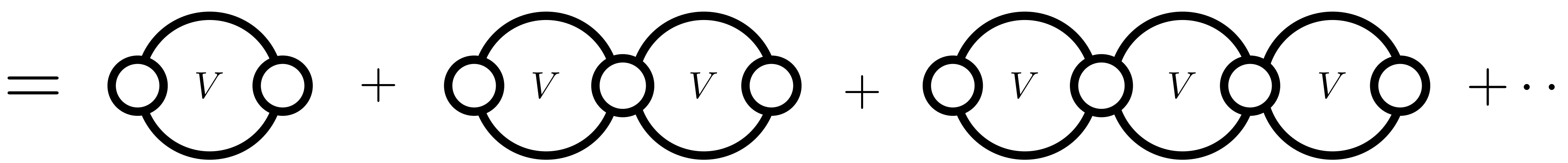
Two-Hadron Systems

Let's review the *two-hadron case* as a template for three-hadron systems

- e.g., elastic scattering of spinless particles, *with* restricted energy region
- Connect K matrix to finite-volume through correlation function

$$\mathcal{C}_L^{(\mathsf{M})}(E) = i \sum_{\mathfrak{n}} \frac{Z_{\mathfrak{n}} Z_{\mathfrak{n}}^{\dagger}}{E - E_{\mathfrak{n}}}$$

$$\mathbf{p} = \frac{2\pi}{L}\mathbf{n}$$



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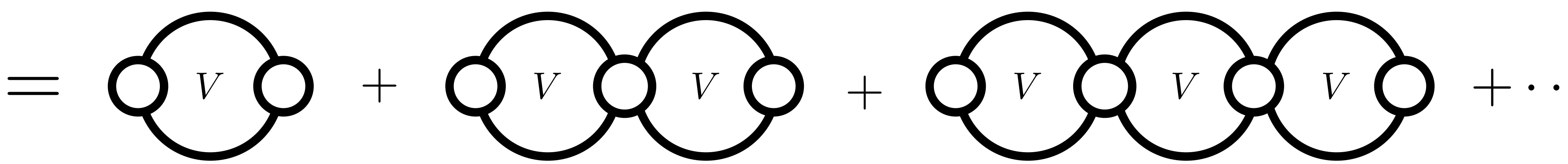
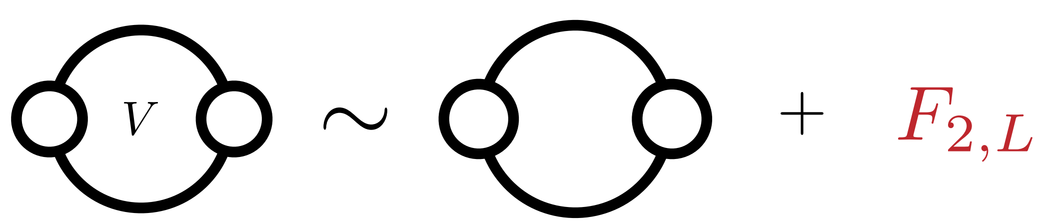


$=$ Follow similar procedure as before …

… additional correction for finite-volume effects



as before

Finite-volume correction (known)

Two-Hadron Systems

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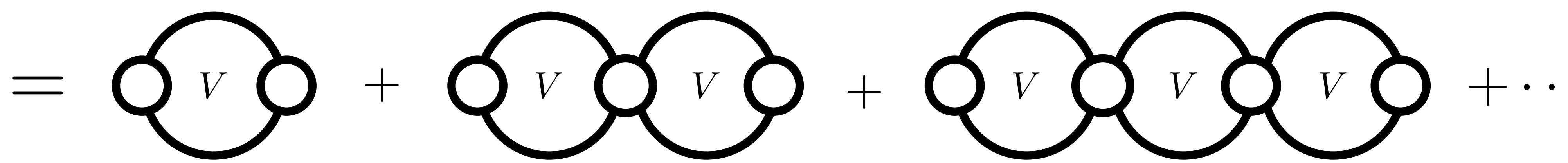
$$\mathcal{C}_L^{(\mathsf{M})}(E) = i \sum_{\mathfrak{n}} \frac{Z_{\mathfrak{n}} Z_{\mathfrak{n}}^\dagger}{E - E_{\mathfrak{n}}}$$

what we compute via lattice QCD



$$\longrightarrow \frac{i\mathcal{R}_L}{1 + \mathcal{K}_2 \cdot F_{2,L}}$$

what we want

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what we compute via lattice QCD

$$\longrightarrow \frac{i\mathcal{R}_L}{1 + \mathcal{K}_2 \cdot F_{2,L}}$$

what we want

Spectrum satisfies

$$\det[1 + \mathcal{K}_2 \cdot F_{2,L}]\Big|_{E=E_{\mathfrak{n}}} = 0$$

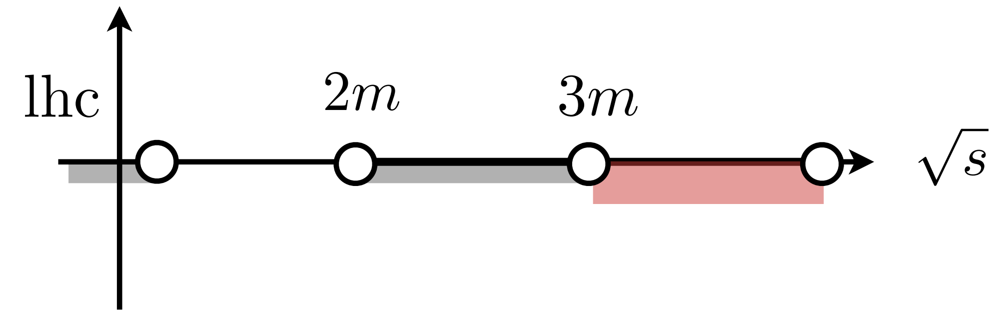
See talks by J. Dudek (Tue.), F. Ortega-Gama (Wed.), A. Rodas (Fri.)

Three-Hadron Systems

Follow recipe for three-hadron scattering

- Find three-body quantization condition (consider elastic 3-body scattering, no 2-3 coupling)

$$\det \left[1 + \mathcal{K}_3 \cdot F_{3,L} \right] \Big|_{E=E_{\mathfrak{n}}} = 0$$



parameterize & constrain

known finite-volume effects

Equivalence proofs

AJ et al. [JPAC]
Phys. Rev. D **100**, 034508 (2019)

T. Blanton and S. Sharpe
Phys. Rev. D **102**, 054515 (2020)

AJ
Phys. Rev. D **108**, 034505 (2023)

M. Hansen and S. Sharpe
Phys. Rev. D **90**, (2014) 116003

and many more…

See talk by Y. Feng next

Three-Hadron Systems

Follow recipe for three-hadron scattering

- Find three-body quantization condition (consider elastic 3-body scattering, no 2-3 coupling)

$$\det\left[1 + \mathcal{K}_3 \cdot F_{3,L}\right]\Big|_{E=E_n} = 0$$

parameterize & constrain

known finite-volume effects

Given a three-body K matrix, e.g., for three pions …

… we need to reconstruct the three hadron scattering amplitude …

… more challenging than 2-body case …

N.B. constraining the K matrix via lattice QCD is *hard*, but I will assume it can be done

ivalence proofs

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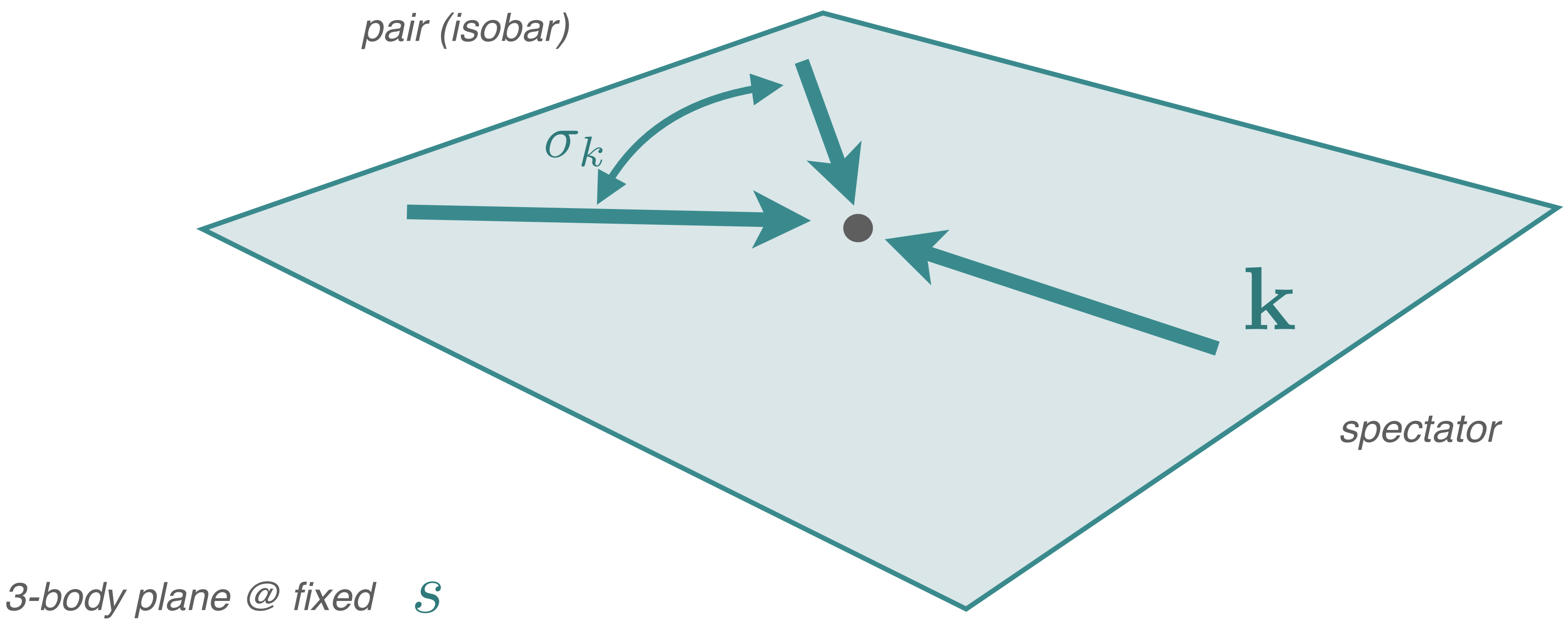
Challenges in the three-hadron scattering

Increased number of degrees of freedom



$\sqrt{\sigma_k}$ isobar ‘mass’

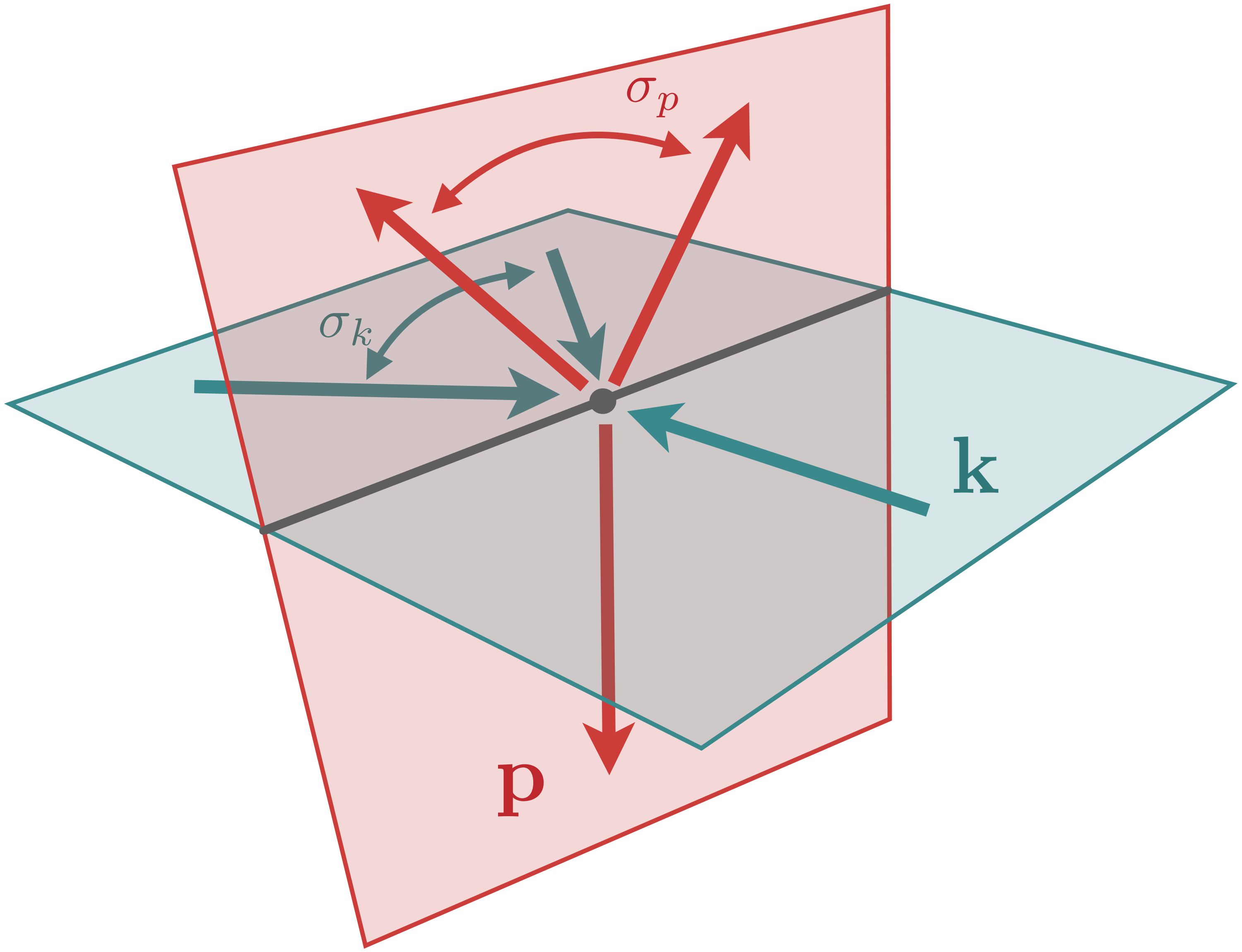
$$3m \leq \sqrt{s} < \infty$$

$$2m \leq \sqrt{\sigma_k} < \sqrt{s} - m$$

Challenges in the three-hadron scattering

Increased number of degrees of freedom

- 8 kinematic variables — after partial wave projection, 3 variables (s, σ_k, σ_p) or (s, k, p)



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Increased number of degrees of freedom

- 8 kinematic variables — after partial wave projection, 3 variables (s, σ_k, σ_p) or (s, k, p)
- Scattering equations are integral equations

$$\mathcal{M}_3 = \mathcal{D} + \int\int \mathcal{L} \cdot \mathcal{T} \cdot \mathcal{R}$$

integrals in k, p

Challenges in the three-hadron scattering

Increased number of degrees of freedom

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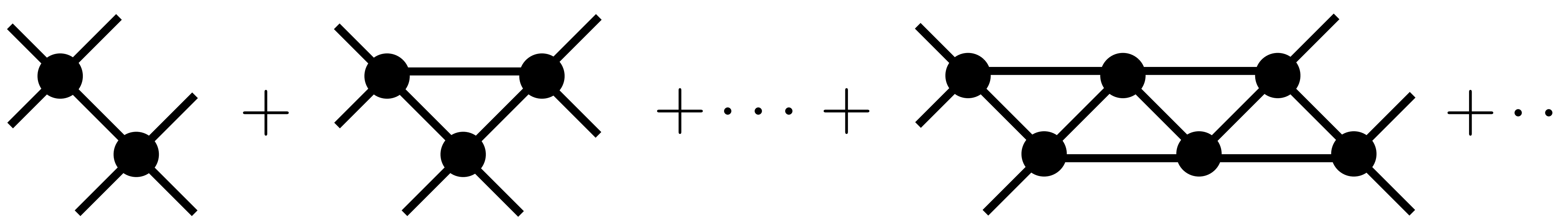
- Scattering equations are integral equations

$$\mathcal{M}_3 = \mathcal{D} + \int\int \mathcal{L} \cdot \mathcal{T} \cdot \mathcal{R}$$

$$\mathcal{D} = \mathcal{M}_2\, \mathcal{G}\, \mathcal{M}_2 + \mathcal{M}_2 \int \mathcal{G}\, \mathcal{D}$$

$=$ 

no 3-body K matrix

Challenges in the three-hadron scattering

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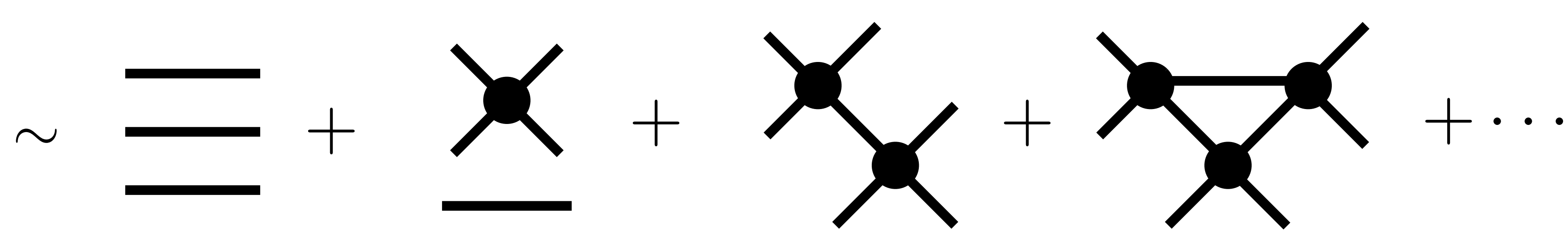
- Scattering equations are integral equations

$$\mathcal{M}_3 = \mathcal{D} + \int\int \mathcal{L} \cdot \mathcal{T} \cdot \mathcal{R}$$

$$\mathcal{T} = \mathcal{K}_3 + \int\int \mathcal{K}_3 \cdot \mathcal{C} \cdot \mathcal{T}$$

rescattering functions

$$\mathcal{L}, \mathcal{R}, \mathcal{C} \sim 1 + \mathcal{M}_2 + \mathcal{D}$$



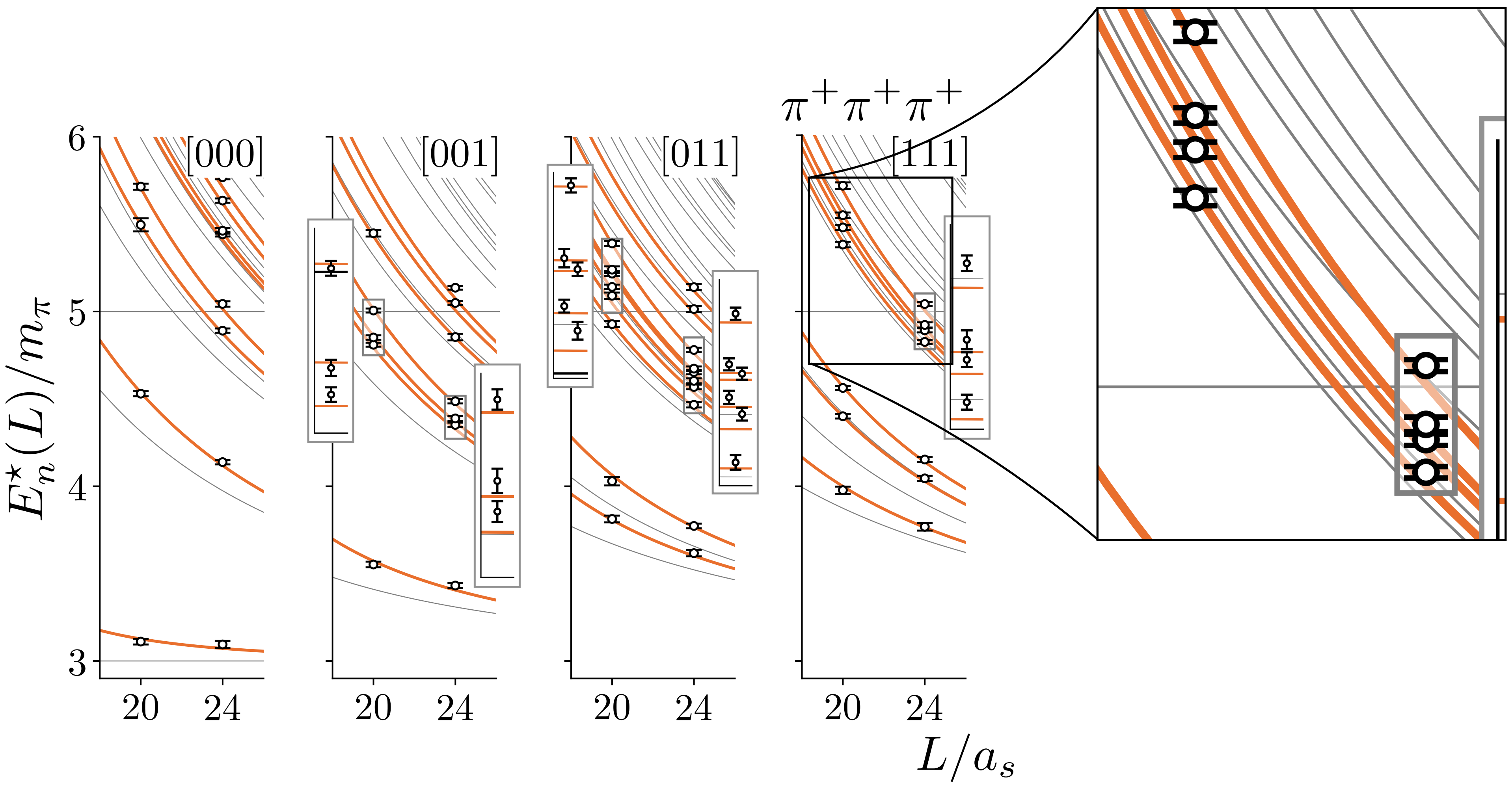
Three-Hadron Systems from Lattice QCD

Follow recipe for three-hadron scattering

- Compute finite-volume spectrum
- Constrain K matrix with three-body quantization condition
- Reconstruct scattering amplitude via integral equations
- Analytically continue to complex energy plane to search for poles

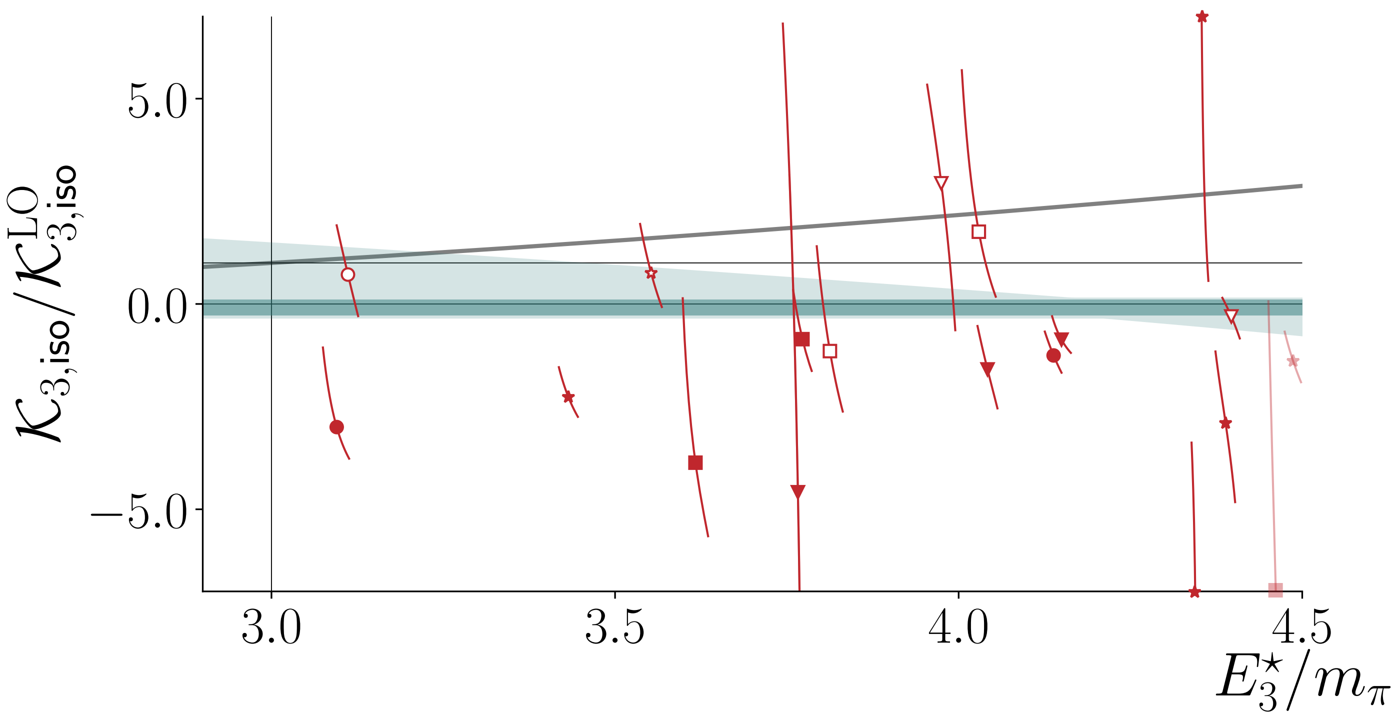
$3\pi^+$ from Lattice QCD

$m_\pi \approx 391$ MeV



103 energy levels described by three numbers

m_π, $a_{\pi\pi}^{I=2}$, $\mathscr{K}_{3,\mathrm{iso}}^{I=3}$



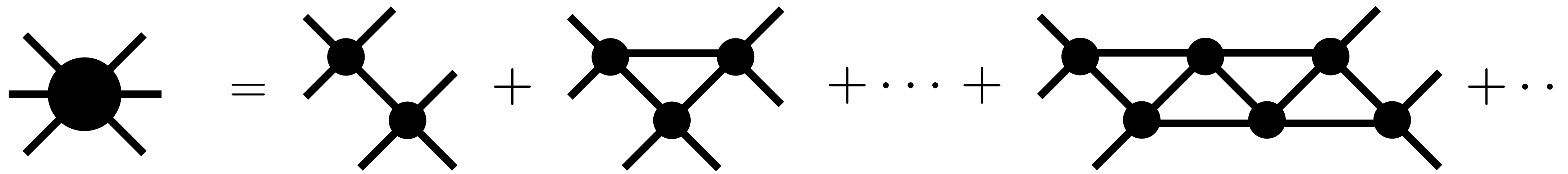
hadspec

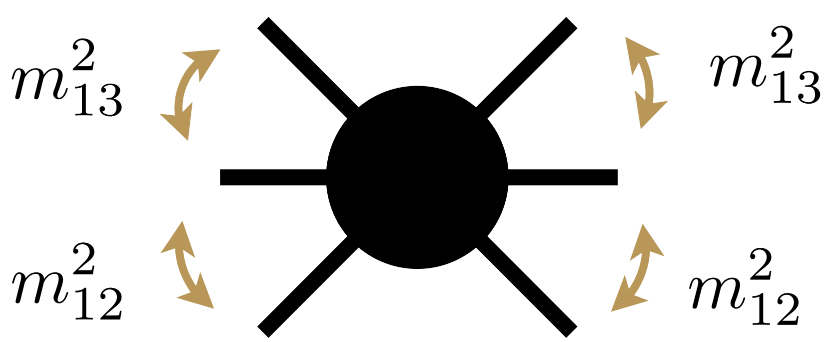
M. Hansen et al. [HadSpec]
Phys. Rev. Lett. **126**, (2021) 012001

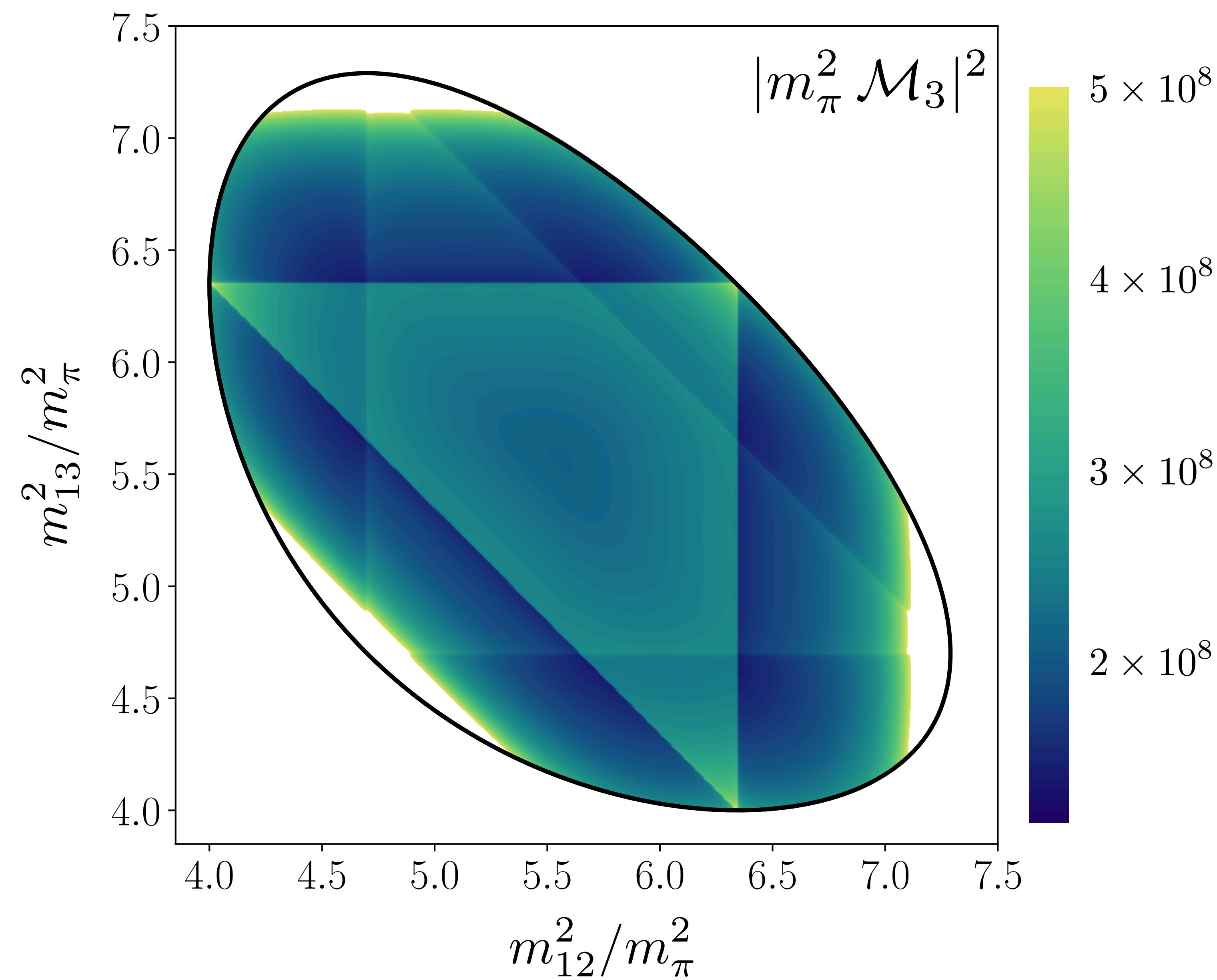
$3\pi^+$ from Lattice QCD

$m_\pi \approx 391$ MeV

Amplitude reconstruction — since $\mathcal{K}_{3,\text{iso}} \approx 0$, amplitude dominated by one-particle exchanges







hadspec

M. Hansen et al. [HadSpec]
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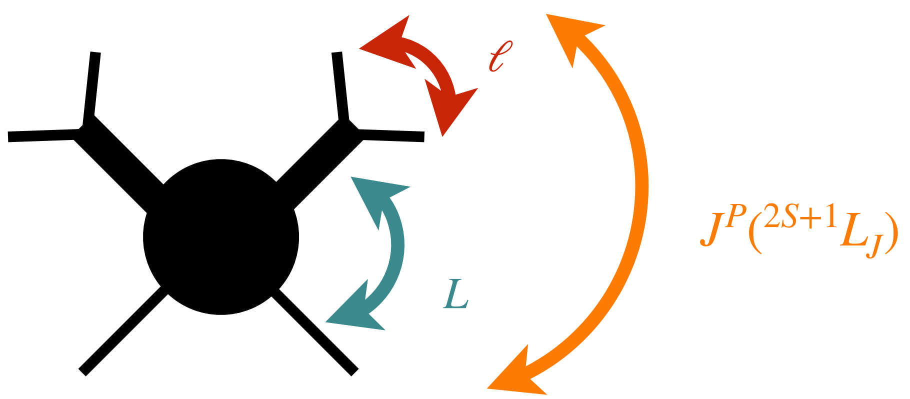
Going Further

Most* work so far has revolved around three-body system in total S wave

- Interesting systems usually have non-zero partial waves
- Need to project integral equations to definite J^P amplitudes



$I^G_{3\pi}$	J^{PC}	$([\pi\pi]^I_\ell \pi)_L$
3^-	0^{-+}	$([\pi\pi]^2_S\pi)_S$
	1^{-+}	none
	1^{++}	$([\pi\pi]^2_S\pi)_P$
2^-	0^{--}	$([\pi\pi]^2_S\pi)_S$, $([\pi\pi]^1_P\pi)_P$
	1^{--}	$([\pi\pi]^1_P\pi)_P$
	1^{+-}	$([\pi\pi]^2_S\pi)_P$, $([\pi\pi]^1_P\pi)_S$, $([\pi\pi]^1_P\pi)_D$
1^-	0^{-+}	$([\pi\pi]^{0,2}_S\pi)_S$, $([\pi\pi]^1_P\pi)_P$
	1^{-+}	$([\pi\pi]^1_P\pi)_P$
	1^{++}	$([\pi\pi]^{0,2}_S\pi)_P$, $([\pi\pi]^1_P\pi)_S$, $([\pi\pi]^1_P\pi)_D$
0^-	0^{--}	$([\pi\pi]^1_P\pi)_P$
	1^{--}	$([\pi\pi]^1_P\pi)_P$
	1^{+-}	$([\pi\pi]^1_P\pi)_S$, $([\pi\pi]^1_P\pi)_D$

Partial Wave Amplitudes

Follow usual procedure to construct partial wave amplitudes

- Couple to definite parity (LS basis) since interested in spectroscopy

Rotational Dependence

— Clebsch-Gordan coefficients

— Wigner D matrix elements

— associated factors

$$\mathcal{M}_3 = \sum_{J^P} \mathcal{M}_3^{J^P} \mathcal{R}_{J^P}$$

Partial Wave Amplitudes

— depends on 'energy' variables only

— Matrix in LS-space for given J^P

$$\mathcal{R}_{J^P} \sim \sum_{m_J} Z_{L'S'}^{Jm_J\,*}(\hat{\mathbf{p}}) Z_{LS}^{Jm_J}(\hat{\mathbf{k}})$$

$$Z_{LS}^{Jm_J}(\hat{\mathbf{k}}) = \sqrt{4\pi(2L+1)} \sum_{\lambda} \langle J\lambda | L0, S\lambda \rangle \; D^{(J)}_{m_J\lambda}(\hat{\mathbf{k}})\, Y^*_{S\lambda}(\hat{\mathbf{a}})$$

Partial Wave Amplitudes

Projection of one-particle exchange

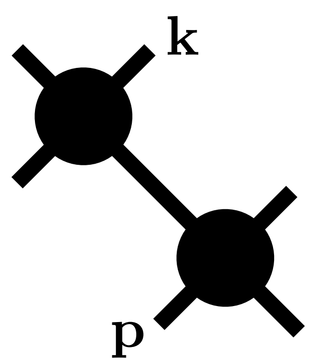
- Most convenient to start in helicity basis (simple Lorentz boosting), then recouple to *LS* basis

$$\mathcal{G}_{\lambda'\lambda}(\mathbf{p},\mathbf{k}) = \frac{\mathcal{H}_{\lambda'\lambda}(\mathbf{p},\mathbf{k})}{u - m^2 + i\epsilon}$$

depends on spins of isobars and their decay angles

exchange propagator

$=$



AJ and R. Briceño
Phys. Rev. D. **109**, (2024) 096030

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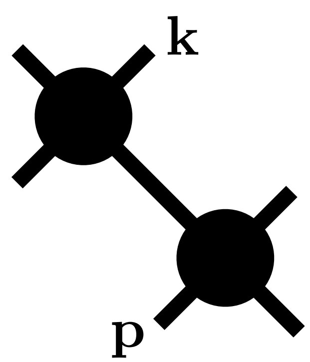

$$= \sum_{J^P} \mathcal{G}^{J^P} \mathcal{R}_{J^P}$$

Known coefficients for particular scattering channel
— could contain spurious singular behavior
— must be compensated by three-body K matrix

$$\Longrightarrow \qquad \mathcal{G}^{J^P} = \mathcal{K}_{\mathcal{G}}^{J^P} + \mathcal{T}^{J^P} Q_0(\zeta)$$

$$Q_0(\zeta) = \frac{1}{2}\log\left(\frac{\zeta+1}{\zeta-1}\right)$$

AJ and R. Briceño
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Partial Wave Amplitudes

Projection of one-particle exchange

- Most convenient to start in helicity basis (simple Lorentz boosting), then recouple to *LS* basis
- Compare to what is usually done in phenomenological studies

$$\mathcal{G}(z) = \frac{\mathcal{N}(z)}{\zeta - z}$$

$$= \frac{\mathcal{N}(\zeta)}{\zeta - z} + \frac{\mathcal{N}(z) - \mathcal{N}(\zeta)}{\zeta - z}$$

$$O\left((\zeta - z)^0\right)$$

Ignore
— parameterizing K matrices anyway, why bother with some *regular* piece?

$$\mathcal{G}^{J^P} \to \int \mathrm{d}z\, \mathcal{R}_{J^P}(z)\, \mathcal{G}(z)$$

$$= \mathcal{N}(\zeta) \int \mathrm{d}z\, \frac{\mathcal{R}_{J^P}(z)}{\zeta - z}$$

$$\sim e^J(\zeta)$$

Lattice FV framework
— quantization conditions rely on systematically controlling finite-volume correction
— K matrix in QC must be consistently defined with scattering equations
— Cannot ignore pieces of the OPE!

AJ and R. Briceño
Phys. Rev. D. **109**, (2024) 096030

Partial Wave Amplitudes

Look at lowest lying waves, e.g., $J^P = 0^-$

$I_{3\pi}^G$	J^{PC}	$([\pi\pi]_\ell^I \pi)_L$
	0^{--}	$([\pi\pi]_S^2\pi)_S$, $([\pi\pi]_P^1\pi)_P$
2^-	1^{--}	$([\pi\pi]_P^1\pi)_P$
	1^{+-}	$([\pi\pi]_S^2\pi)_P$, $([\pi\pi]_P^1\pi)_S$, $([\pi\pi]_P^1\pi)_D$

$^1S_0$ $\quad$ $^3P_0$

$$\mathcal{G}^{J^P=0^-} = -\frac{1}{2pk}\begin{pmatrix} 0 & \frac{\sqrt{3}}{q_p^\star}\gamma_p k \\ \frac{\sqrt{3}}{q_k^\star}\gamma_k p & \frac{3}{q_k^\star q_p^\star}\, pk g_{pk} \end{pmatrix} + \frac{1}{2pk}\begin{pmatrix} 1 & \frac{\sqrt{3}}{q_p^\star} k f_{pk} \\ \frac{\sqrt{3}}{q_k^\star} p f_{kp} & \frac{3}{q_k^\star q_p^\star}\, pk f_{pk} f_{kp} \end{pmatrix} Q_0(\zeta_{pk})$$

$$f_{pk} = \gamma_p\left(\frac{\beta_p\omega_k}{k} + \zeta_{pk}\right)$$

$$g_{pk} = \gamma_p\gamma_k\left(\frac{\beta_p\omega_k}{k} + \frac{\beta_k\omega_p}{p} + \zeta_{pk}\right)$$

$$f_{kp} = f_{pk}(k \leftrightarrow p)$$

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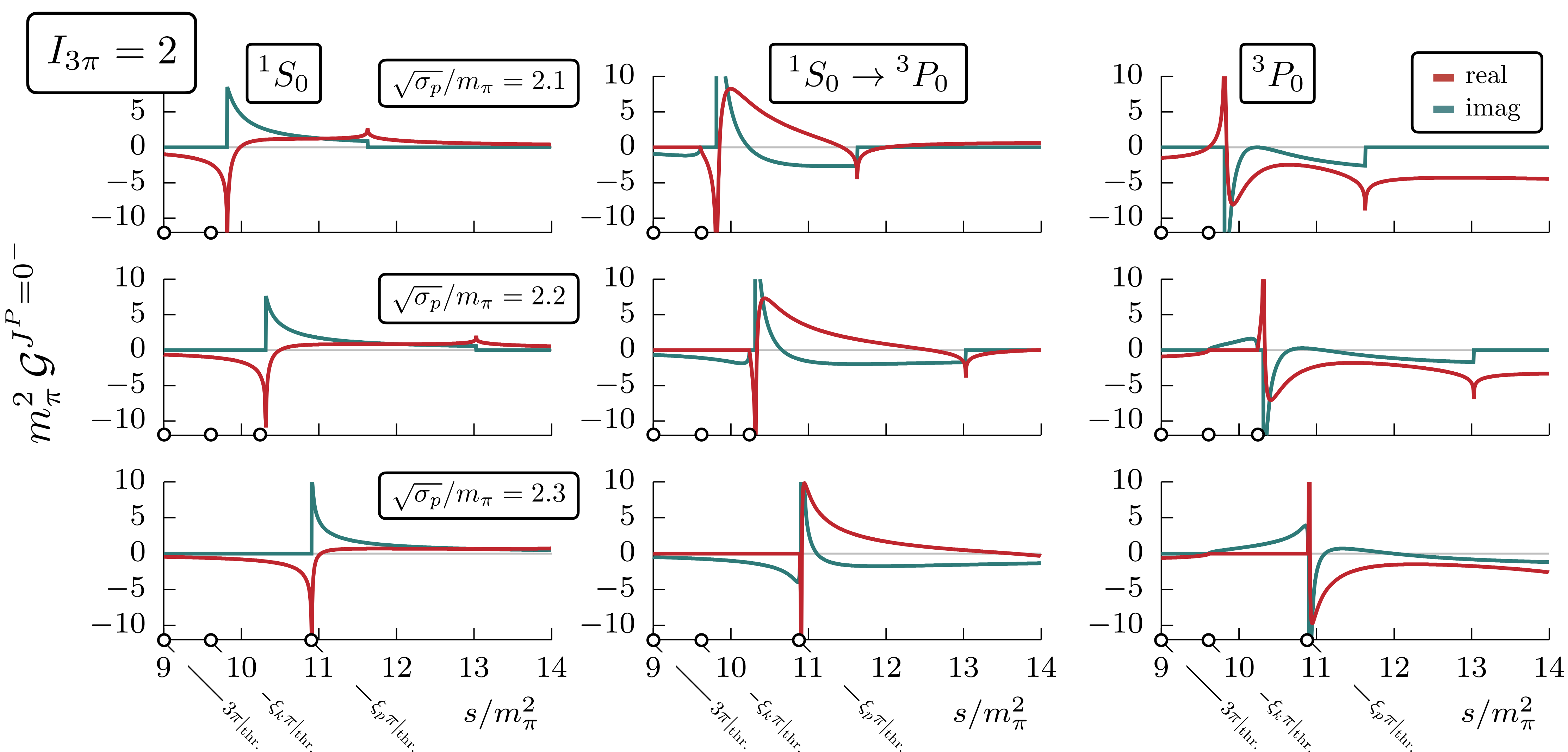

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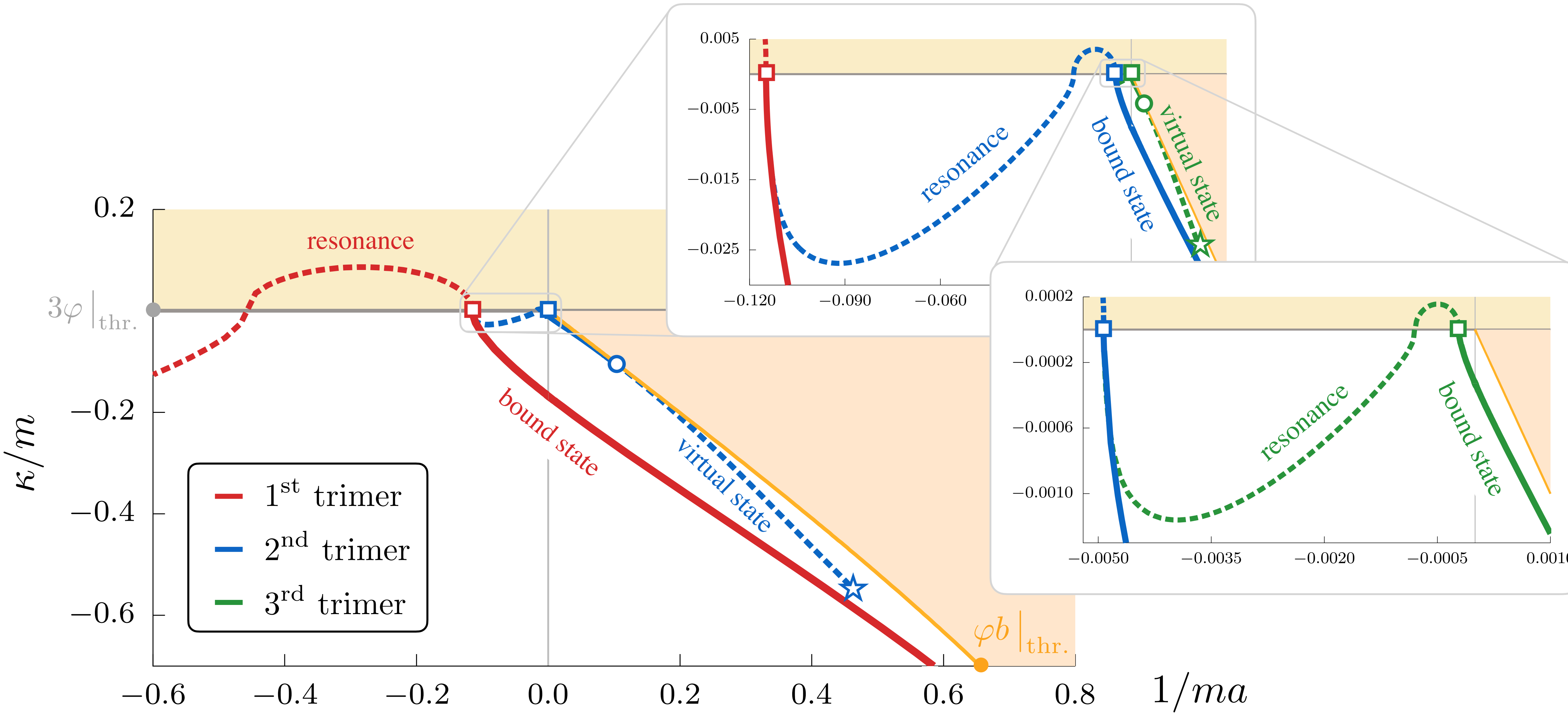
Exploring Analytic Continuations

Spectral analyses require continuing to pole positions

- Community is building confidence in robust solution strategies
- e.g., exploring Efimov physics with relativistic three-body framework



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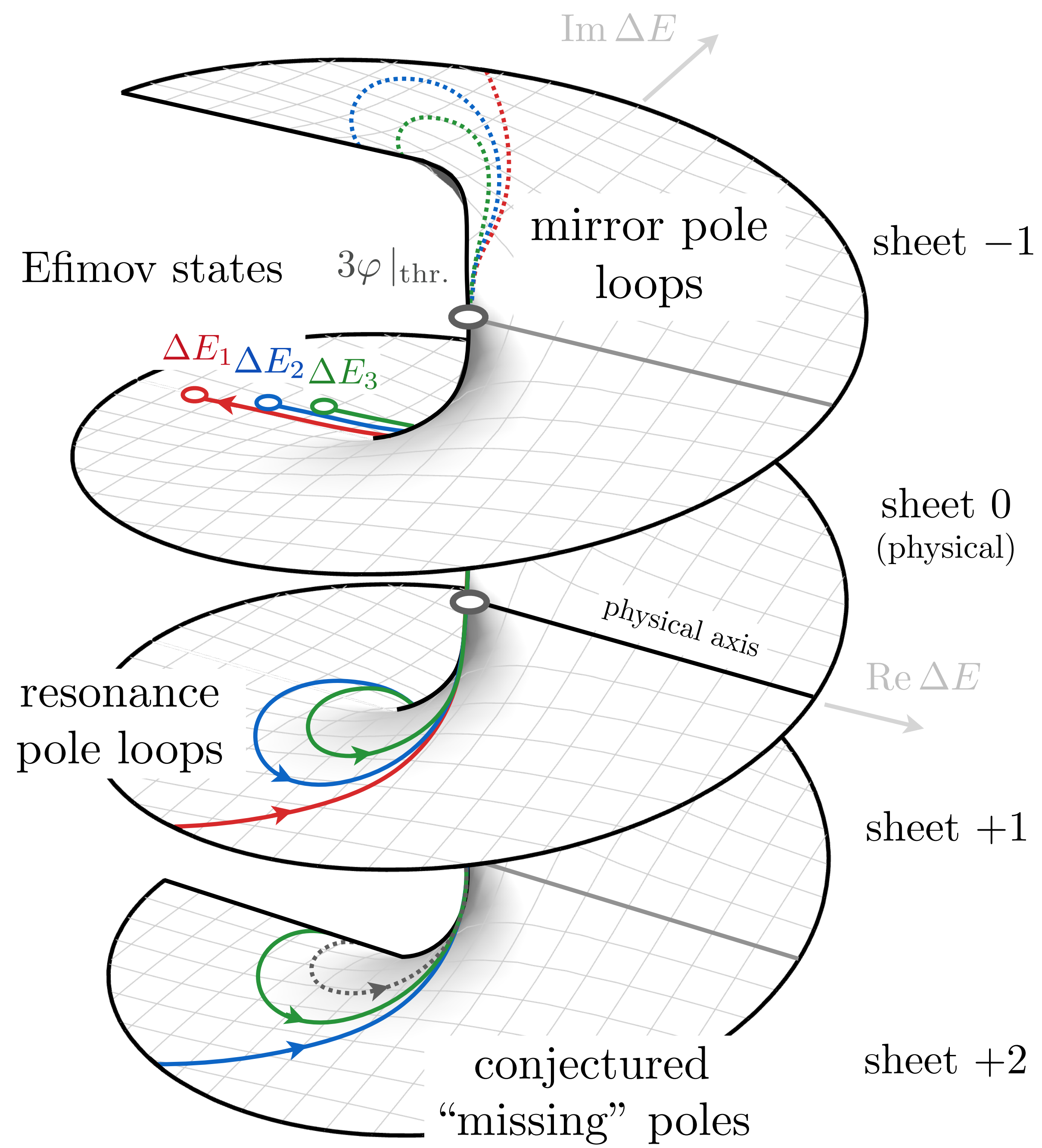
Exploring Analytic Continuations

Spectral analyses require continuing to pole positions

- Community is building confidence in robust solution strategies
- e.g., exploring Efimov physics with relativistic three-body framework
- Gain deeper understanding on analytic structure



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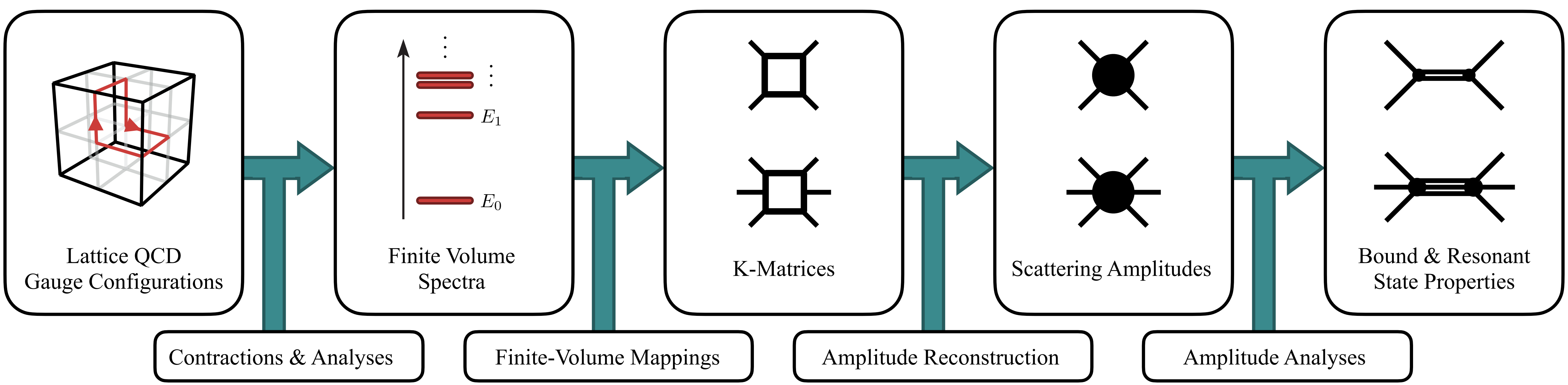
Summary

Lots of progress on 3-body spectroscopy from QCD

- Finite-volume framework has been in place for a while, now going through use cases
- Applications of numerical solutions for the on-shell integral equations
- Formal/numerical extensions to non-zero partial waves
- Gaining insight into analytic continuation of amplitudes

More results on the horizon!



K Matrix Parameterizations

To constrain *K* matrix, parameterize partial wave components

- Saturate sum to include enough freedom for FV irrep

$$\mathcal{K}_3 = \sum_{J^P} \mathcal{K}_3^{J^P} \, \mathcal{R}_{J^P}$$

Energy-dependent parameterizations

— polynomials & rational functions

— fit parameters for Quantization Condition

$$\mathcal{K}_3^{J^P} = \mathcal{B}_{LL'} \left(\mathcal{P}_\gamma(E) + \frac{\mathcal{P}_\alpha(E)}{\mathcal{P}_\beta(E)} \right)$$

Angular momentum barrier factors

$$\det\left[1 + \mathcal{K}_3 \cdot F_{3,L}\right]\Big|_{E=E_\mathfrak{n}} = 0 \quad \Longrightarrow \quad \mathcal{M}_3^{J^P} = \mathcal{M}_3^{J^P}\left[\mathcal{K}_2, \mathcal{K}_3^{J^P}, \mathcal{I}, \mathcal{G}^{J^P}\right]$$