Partial Wave Analyses and Advanced Tools for Hadron Spectroscopy
PWA13/ATHOS8

Progress in Coupled-Channel Three-Pion Scattering for (In)Finite volume

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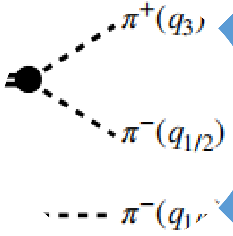
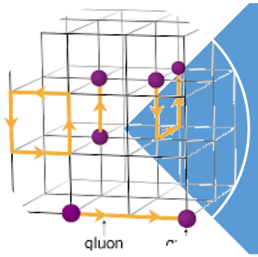
EXOHAD
EXOTIC HADRONS TOPICAL COLLABORATION

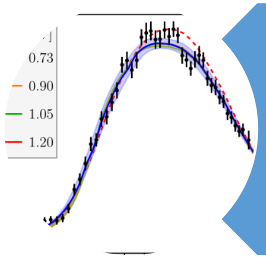
Outline



3-Body Dynamics



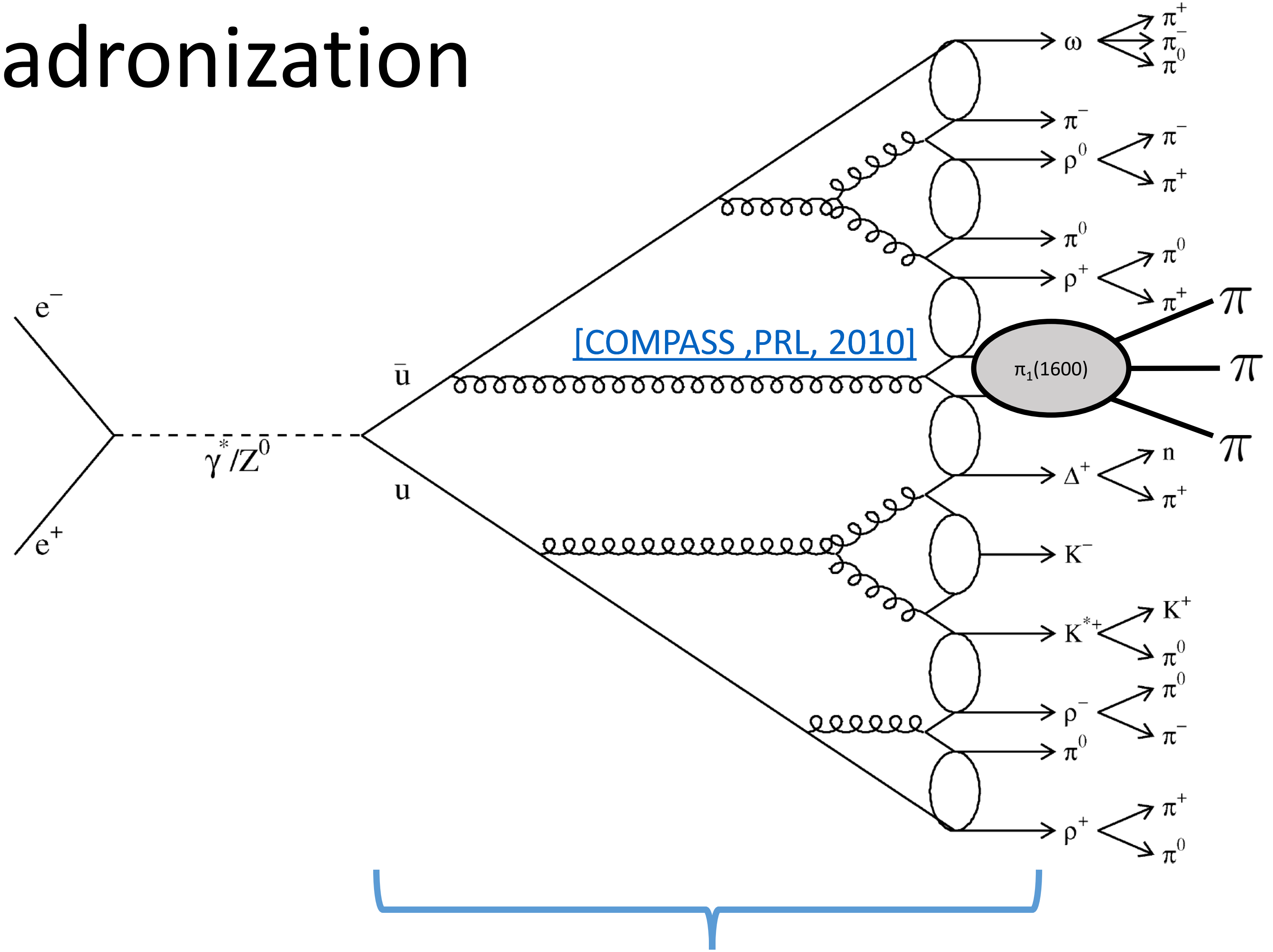
Finite-Volume Calculations



Infinite-Volume Calculations

3-Body Dynamics

Hadronization



Non-perturbative quark-gluon dynamics in QCD

- Confinement into color-neutral hadrons
- Mass generation of hadrons
- **Resonances**: key phenomena connecting QCD and experiment

Experiments

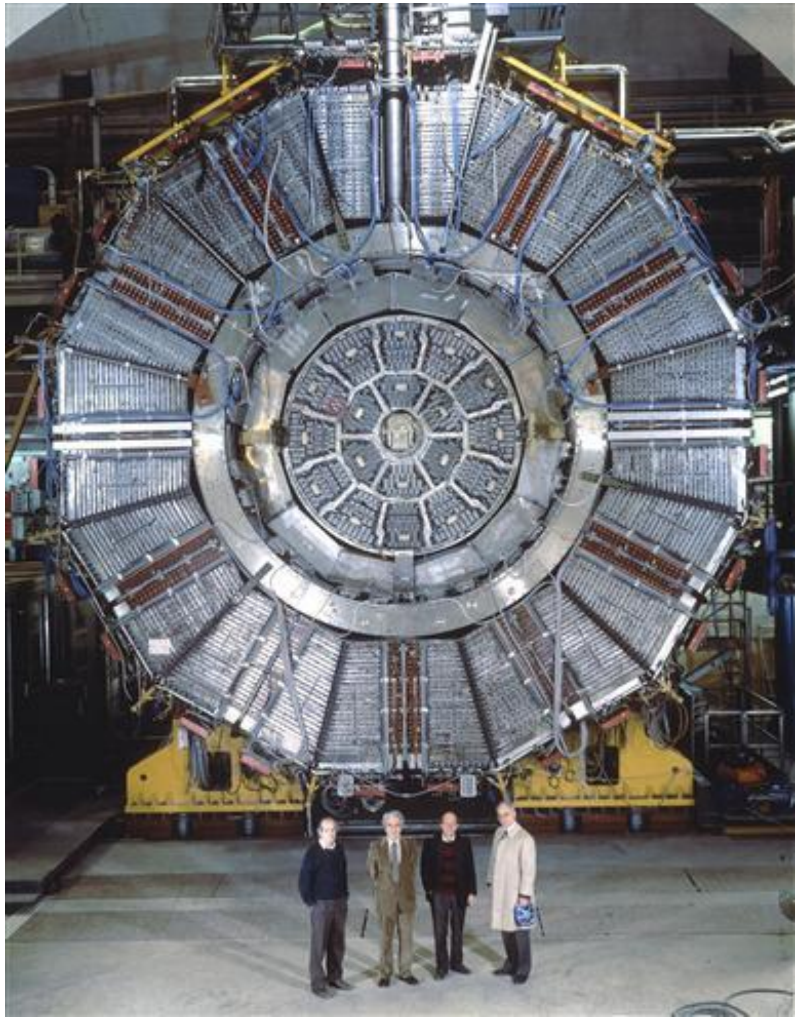
Decay mode	Resonance	$\mathcal{B}$ (%)	
Leptonic decays		35.2	
$\tau^- \to e^- \overline{\nu}_e \nu_\tau$			17.8
$\tau^- \to \mu^- \overline{\nu}_\mu \nu_\tau$			17.4
Hadronic decays		64.8	
$\tau^- \to h^- \nu_\tau$			11.5
$\tau^- \to h^- \pi^0 \nu_\tau$	$\rho(770)$		25.9
$\tau^- \to h^- \pi^0 \pi^0 \nu_\tau$	$a_1(1260)$		9.5
$\tau^- \to h^- h^+ h^- \nu_\tau$	$a_1(1260)$		9.8
$\tau^- \to h^- h^+ h^- \pi^0 \nu_\tau$			4.8
Other			3.3

***$h^{\pm}$* refers to charged pions**

a1(1260) decay involves three-body dynamics

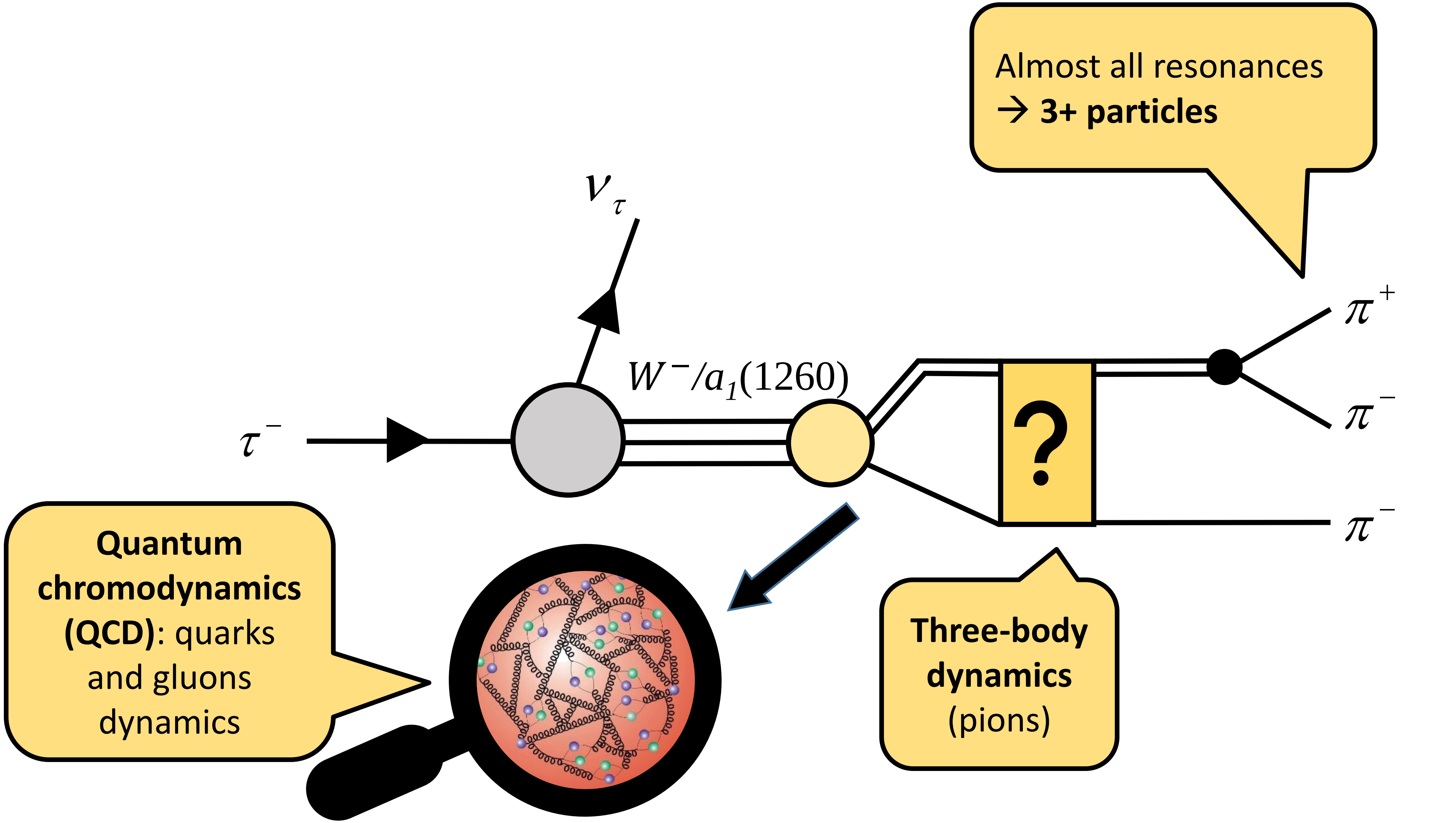
$$a_1(1260) \quad I^G(J^{PC}) = 1^-(1^{++})$$

[PDG]



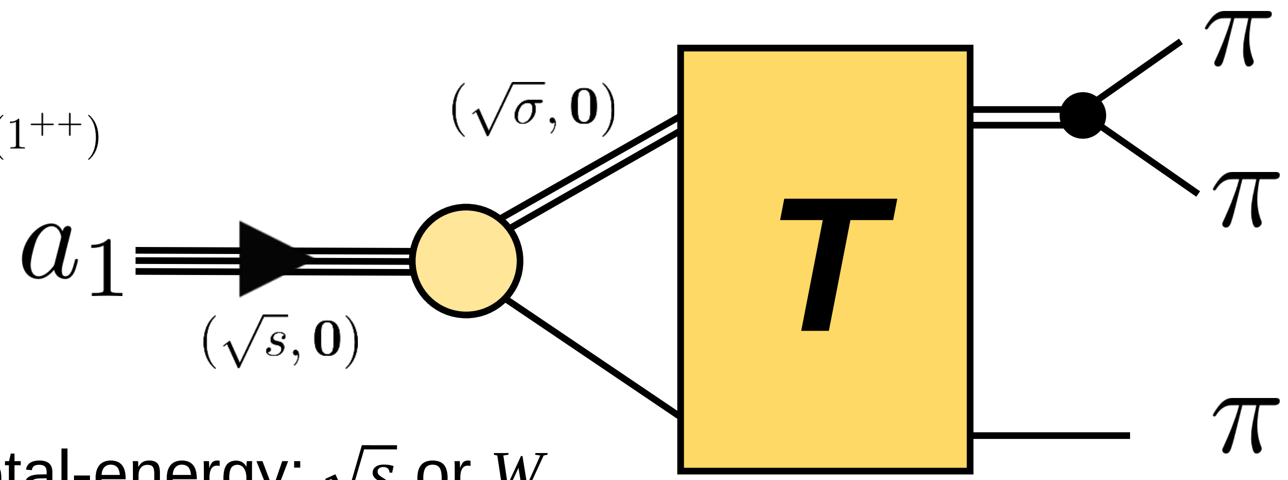
ALEPH Detector

τ-decay


Almost all resonances
→ **3+ particles**
ν_τ
$W^-/a_1(1260)$
τ^-
?
π^+
π^-
π^-
Quantum chromodynamics (QCD): quarks and gluons dynamics
Three-body dynamics (pions)

Kinematics

$a_1(1260) \quad I^G(J^{PC}) = 1^-(1^{++})$



Total-energy: $\sqrt{s}$ or W
Sub-energy: $\sqrt{\sigma}$

IS-SP Interaction → Lots of channels

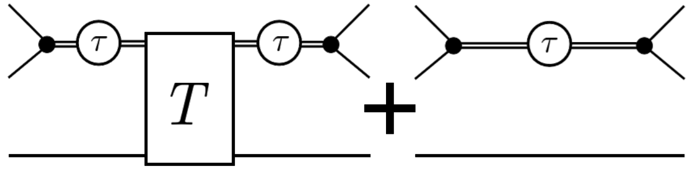
═ Isobar (Two stable particles interact)
— Spectator (The third particle)

Three-body Unitarity

"Conservation of probability"

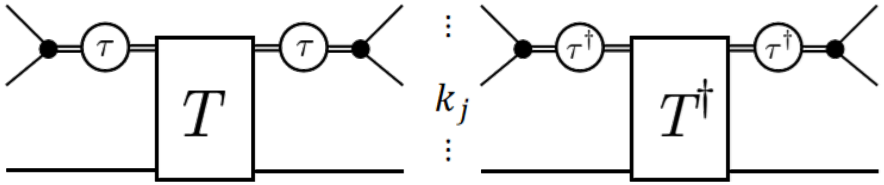
$$SS^\dagger = \mathbb{1} \qquad \langle q_1,q_2,q_3|(\hat{T}-\hat{T}^\dagger)|p_1,p_2,p_3\rangle = i\int_P \langle q_1,q_2,q_3|\hat{T}^\dagger|k_1,k_2,k_3\rangle\langle k_1,k_2,k_3|\hat{T}|p_1,p_2,p_3\rangle$$

$\hat{T}$: 3 → 3 amplitude



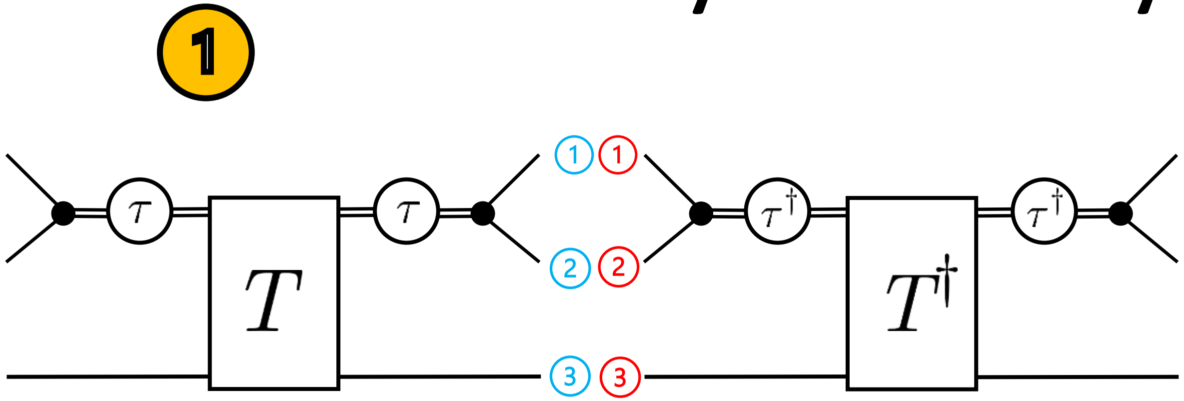
Connected Disconnected

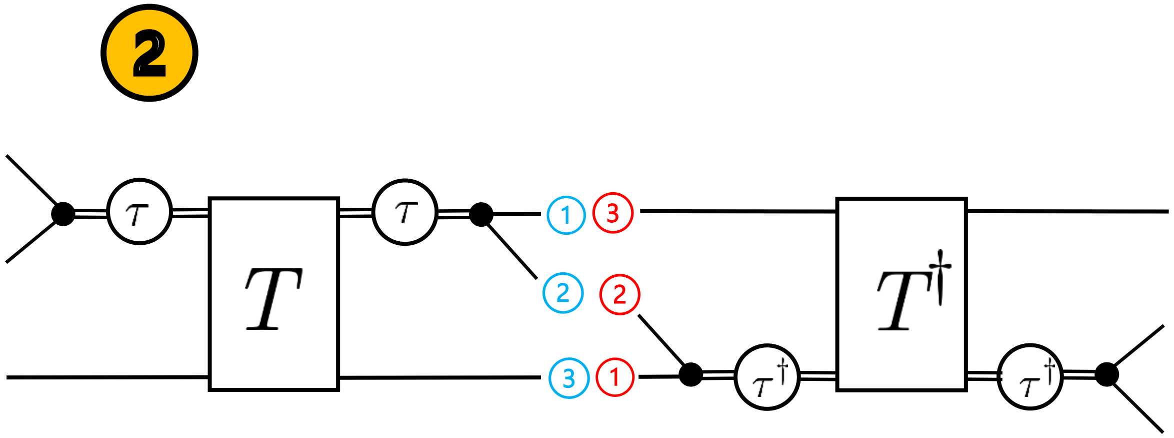
$$\int \prod_{j=1}^{3} d^4 k_j$$



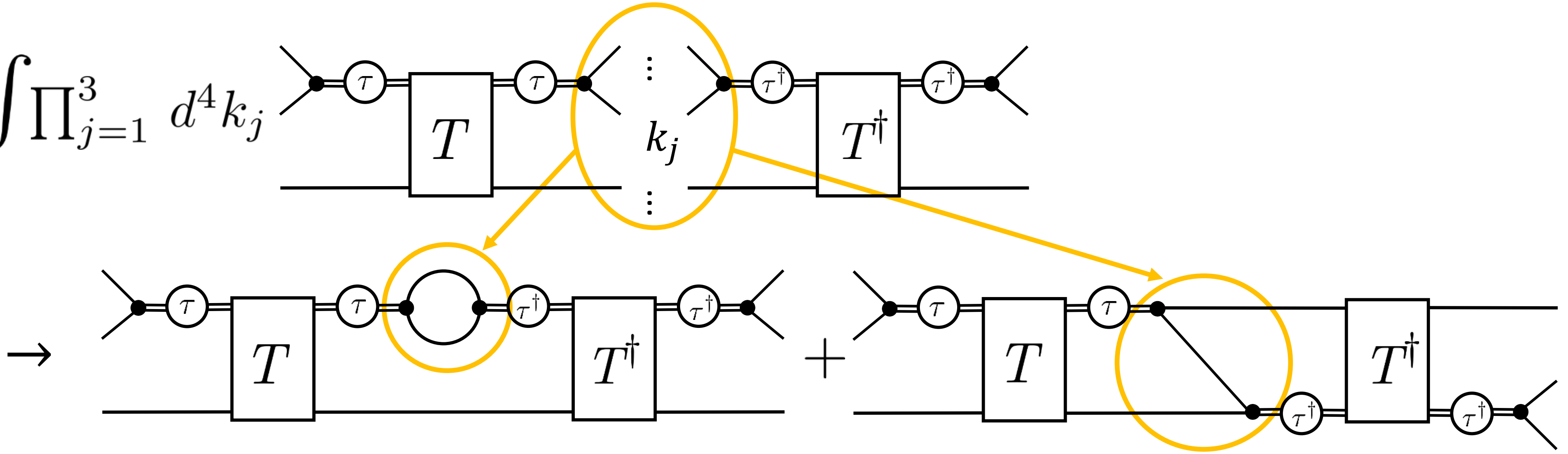
[Mai 2017]

Three-body Unitarity





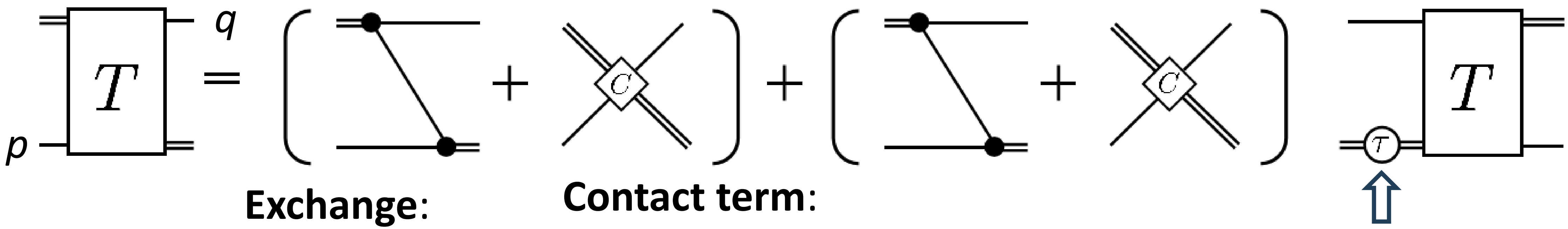
$$\int \prod_{j=1}^{3} d^4k_j$$



Scattering Equation

$$\boxed{T}\quad \langle q|\,T(s)\,|p\rangle = \langle q|\,B(s)\,|p\rangle + \langle q|\,C(s)\,|p\rangle + \int \frac{d^4k}{(2\pi)^4}\,\langle q|\,(B(s)+C(s))\,|k\rangle\,\tau(\sigma(k))\,\langle k|\,T(s)\,|p\rangle\ ,$$

- General Ansatz Bethe-Salpeter equation (BSE)
- Derive manifestly unitary amplitude (similar to Lippman-Schwinger equation (LSE))



Exchange:

- Complex
- Required by unitarity

Contact term:

- Does not destroy unitarity
- Free parametrization: fit to data

Isobar-spectator Green's functions:

- 2-body scattering input

Partial-wave decomposition

- Plane-wave basis
  (Finite volume)

$$T_{\lambda'\lambda}(\boldsymbol{p},\boldsymbol{q}_1)=(B_{\lambda'\lambda}(\boldsymbol{p},\boldsymbol{q}_1)+C)+\sum_{\lambda''}\int\frac{d^3\boldsymbol{l}}{(2\pi)^3 2E_l}(B_{\lambda'\lambda''}(\boldsymbol{p},\boldsymbol{l})+C)\,\tau(\sigma(l))T_{\lambda''\lambda}(\boldsymbol{l},\mathbf{q_1})$$

- *JLS* basis:
  (Infinite volume)

$$T^J_{LL'}(q_1,p)=(B^J_{LL'}(q_1,p)+C_{LL'}(q_1,p))+\int\limits_0^\Lambda\frac{\mathrm{d}l\,l^2}{(2\pi)^3 2E_l}\left(B^J_{LL''}(q_1,l)+C_{LL''}(q_1,l)\right)\tau(\sigma(l))T^J_{L''L'}(l,p)$$

Upgrade to 4-channel model

$\Gamma(\, a_1(1260) \to (\rho\pi)_{S\text{-wave}}\, , \rho \to \pi\pi\,)/\Gamma_{\text{total}}$

VALUE (10^{-2})	*EVTS*	*DOCUMENT ID*		*TECN*	*COMMENT*
• • We do not use the following data for averages, fits, limits, etc. • •					
60.19	37k	[1] ASNER	2000	CLE2	10.6 $e^+ e^- \to \tau^+\tau^-$, $\tau^- \to \pi^-\pi^0\pi^0\nu_\tau$

Old: isobar in *P*-wave [Sadasivan 2021 PRD]

New: isobars in *S*-wave

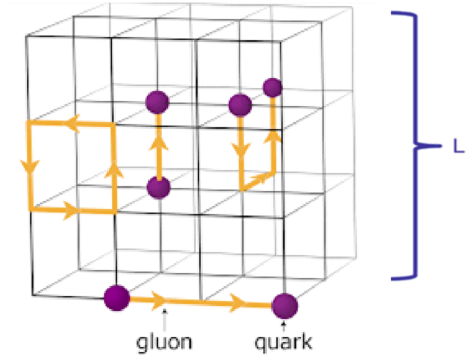
$\pi\rho \xrightarrow{S\to S} \pi\rho$	$\pi\rho \xrightarrow{D\to S} \pi\rho$	$\pi\sigma \xrightarrow{P\to S} \pi\rho$	$\pi(\pi\pi)_2 \xrightarrow{P\to S} \pi\rho$
$\pi\rho \xrightarrow{S\to D} \pi\rho$	$\pi\rho \xrightarrow{D\to D} \pi\rho$	$\pi\sigma \xrightarrow{P\to D} \pi\rho$	$\pi(\pi\pi)_2 \xrightarrow{P\to D} \pi\rho$
$\pi\rho \xrightarrow{S\to P} \pi\sigma$	$\pi\rho \xrightarrow{D\to P} \pi\sigma$	$\pi\sigma \xrightarrow{P\to P} \pi\sigma$	$\pi(\pi\pi)_2 \xrightarrow{P\to P} \pi\sigma$
$\pi\rho \xrightarrow{S\to P} \pi(\pi\pi)_2$	$\pi\rho \xrightarrow{D\to P} \pi(\pi\pi)_2$	$\pi\sigma \xrightarrow{P\to P} \pi(\pi\pi)_2$	$\pi(\pi\pi) \xrightarrow{P\to P} \pi(\pi\pi)_2$

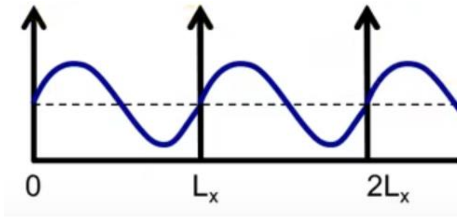
- σ: $f_0(500)$; $(\pi\pi)_2$: repulsive isospin I=2 channel
- Systematic inclusion of all possible isobars up to *P*-wave
- More channels will be included

To Finite Volume

LQCD: Quantization Condition (QC)

- Impose periodic Boundary Condition (Lattice)





$$\psi(\boldsymbol{r}) = \psi(\boldsymbol{r} + L\hat{\boldsymbol{e}}_i) = e^{\overbrace{iL\,\hat{\boldsymbol{e}}_i\cdot\boldsymbol{p}_{\mathrm{op}}}^{2n\pi i}}\psi(\boldsymbol{r})$$

Infinite volume

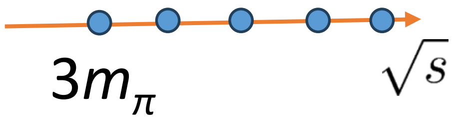
Continuous momenta

Finite volume

$$\boldsymbol{p} = 2\pi\boldsymbol{n}/L \text{ for } \boldsymbol{n} \in \mathbb{Z}^3$$

Energy spectrum: 

$$\int \frac{d^3p}{(2\pi)^3} f(\boldsymbol{p}) \to \frac{1}{L^3} \sum_{\boldsymbol{p}\in 2\pi/L\,\mathbb{Z}^3} f(\boldsymbol{p})$$



- Phase shifts
- Pole positions

QC ← Discrete energy spectrum (from LQCD)

Results

$$T = (B + C) + (B + C)\tau T$$

$$T = \frac{B+C}{\mathbb{1}-(B+C)\tau}$$

Quantization Condition (QC)

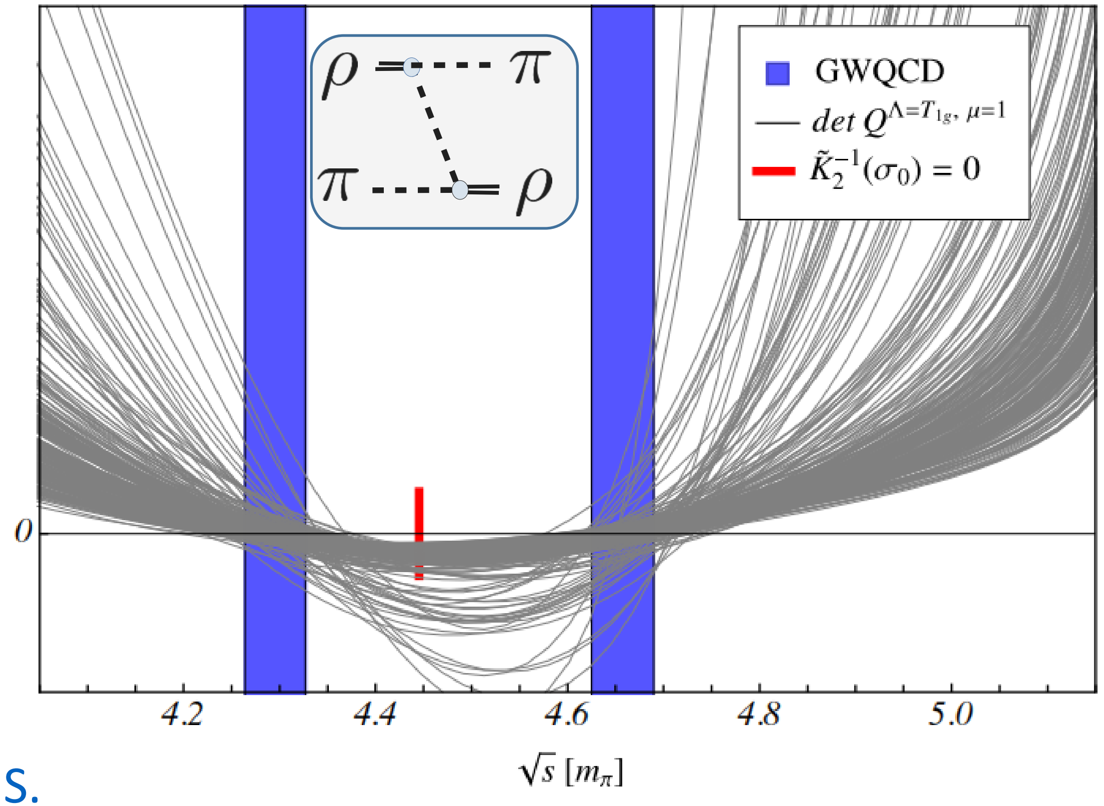
$$0 = \det\left[B(s) + \boxed{C(s)} - E_L\left(\tilde{K}_2^{-1}(s) - \Sigma_2^L(s)\right)\right]_{\substack{(\lambda'\lambda)\\(\boldsymbol{p}'\boldsymbol{p})}}$$

fit parameters

- Generalization of QC to 4-channels achieved,
  All isobars up to *P*-wave are included:
  $a_1 \leftrightarrow (\pi\rho)_S \leftrightarrow (\pi\rho)_D \leftrightarrow (\pi\sigma)_P \leftrightarrow (\pi(\pi\pi)_{S,I=2})$
- Analysis is on the way [YF et al., in preparation]

First extraction 3-body resonance from lattice QCD (2 channels)

$a_1(1260) \to \pi\rho$ [Mai et al., PRL 2021]



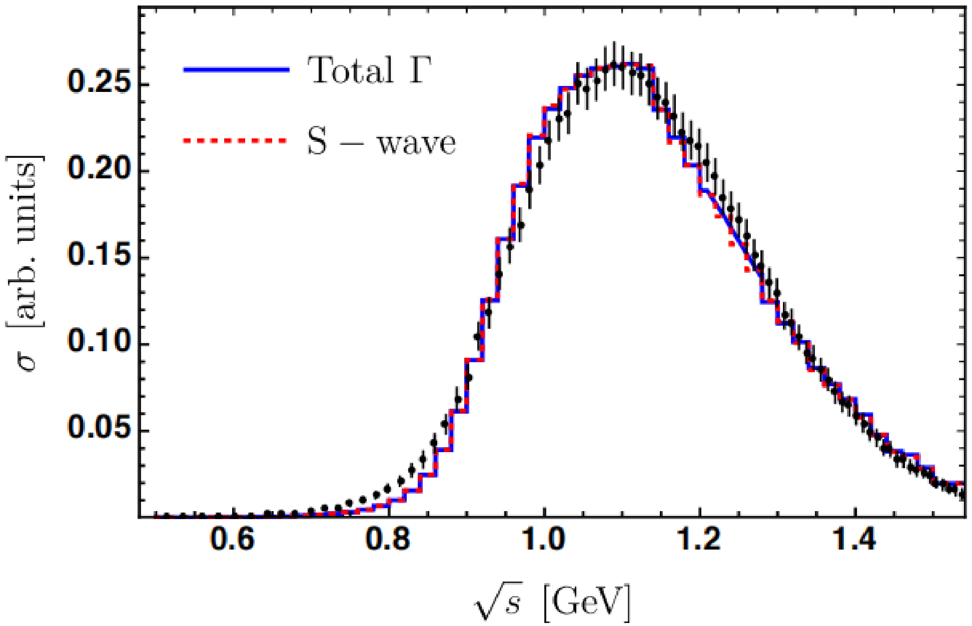
- Extension to $\eta\pi\pi + K\bar{K}\pi$ in finite volume: [Z. Draper, S. Sharpe 2403.20064 [hep-ph]]

To Infinite Volume

Production amplitude

$$T_{L'L}(q_{\text{out}}, p_{\text{in}}) \to \breve{\Gamma}_{L'}(q_{\text{out}})$$

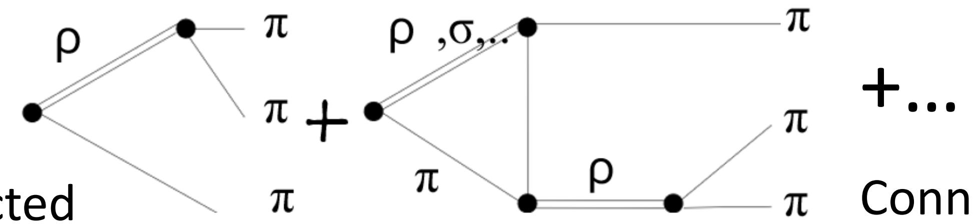
[Sadasivan et al., 2020 PRD]



- An example of a1(1260) lineshape (ALEPH data) from the decay
  $\tau^- \to \pi^-\pi^-\pi^+\nu_\tau$

Disconnected part = Traditional Isobar model

Disconnected



Connected

$$\breve{\Gamma}_{L'} = \breve{\Gamma}^{\text{disc}}_{L'} + \breve{\Gamma}^{\text{con}}_{L'}$$

Lineshape

$$\mathcal{L}(\sqrt{s}) = \frac{1}{\sqrt{s}} \int \frac{d^3\boldsymbol{q}_1}{(2\pi)^3} \frac{d^3\boldsymbol{q}_2}{(2\pi)^3} \frac{d^3\boldsymbol{q}_3}{(2\pi)^3} \frac{1}{2E_{q_1} 2E_{q_2} 2E_{q_3}} \times (2\pi)^4 \delta^4(P_3 - q_1 - q_2 - q_3) \overline{|\Gamma(\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3)|}^2 . \quad (18)$$

Dalitz plots

- CLEO data [Phys.Rev.D 61 (2000) 012002]
- Channel dynamics mostly visible in Dalitz plots, not line shape
- Large data set for τ Dalitz decays

- Need more channels for combined analysis of Dalitz plots.
- Analyze future data with different final states $\pi\pi\pi / \pi K \bar{K}$

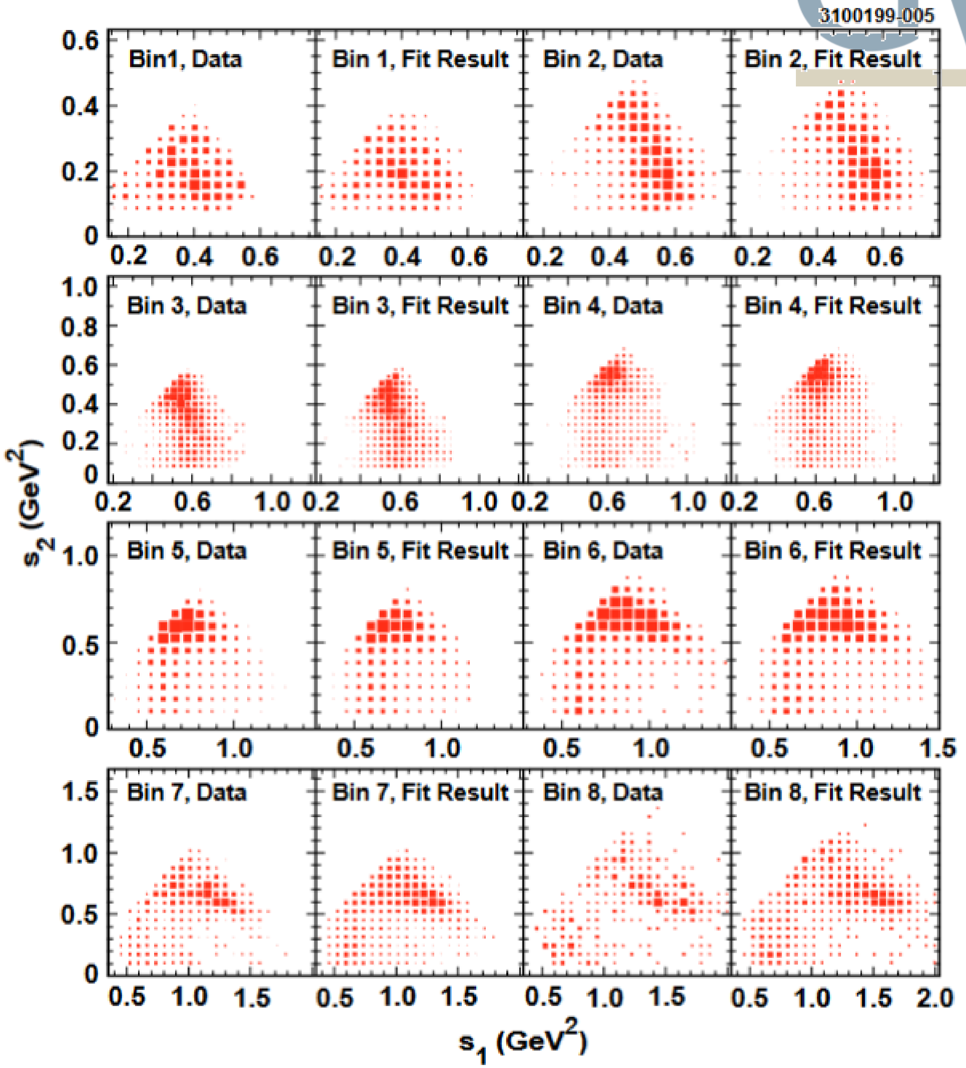
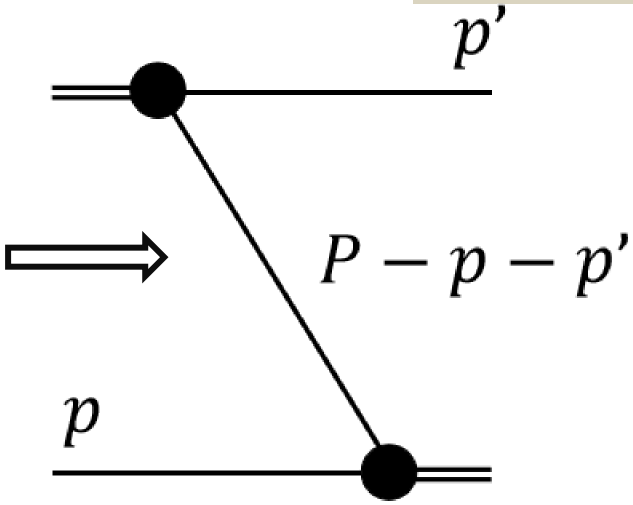


FIG. 5. Dalitz plot distributions for data and fit result. Here s_1 is taken to be the higher of the two values of $M^2_{\pi^-\pi^0}$ in each event. Bins 1 through 8 correspond to slices in $\sqrt{s}$ = 0.6-0.9, 0.9-1.0, 1.0-1.1, 1.1-1.2, 1.2-1.3, 1.3-1.4, 1.4-1.5, 1.5-1.8 GeV.

Partial-wave projection

$$B_{L'L}(q_{\rm out}, p_{\rm in}) \to T_{L'L}(q_{\rm out}, p_{\rm in}) \to \breve{\Gamma}_{L'}(q_{\rm out})$$

$$B_{\lambda'\lambda}(\boldsymbol{p}', \boldsymbol{p}) = \frac{I_f v^*_{\lambda'}(P-p-p',p) v_\lambda(P-p-p',p')}{2E_{p'+p}} f(\boldsymbol{p}', \boldsymbol{p})$$



$$x = \cos(\boldsymbol{p}', \boldsymbol{p}),\ E^2_{p+p'} = m^2_\pi + p^2 + p'^2 + 2pp'x,$$

$$f(\boldsymbol{p}', \boldsymbol{p}) = \frac{1}{\sqrt{s} - E_p - E_{p'} - E_{p+p'} + i\epsilon}$$

Forward propagator

Logarithmic Singularities

$$B_{\lambda'\lambda}(p', p) = 2\pi \int_{-1}^{+1} dx\, d^J_{\lambda'\lambda}(x) B_{\lambda'\lambda}(\boldsymbol{p}', \boldsymbol{p})$$

$$B_{L'L}(p', p) = U_{L'\lambda'} B_{\lambda'\lambda}(p', p) U_{\lambda L}$$

Projection: Plane-wave → Partial-wave

Projection: Helicity → *JLS*

See also [Jackura Phys.Rev.D 109 (2024)]

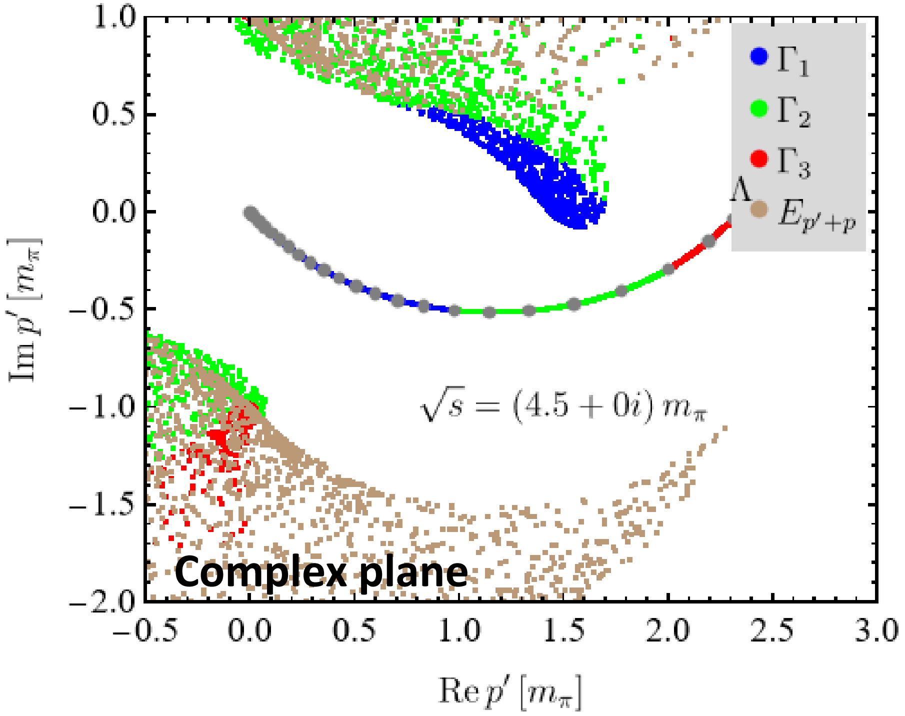
Solving scattering equation for complex momenta

$$f(\boldsymbol{p}', \boldsymbol{p}) = \frac{1}{\sqrt{s} - E_p - E_{p'} - E_{p+p'} + i\epsilon}$$

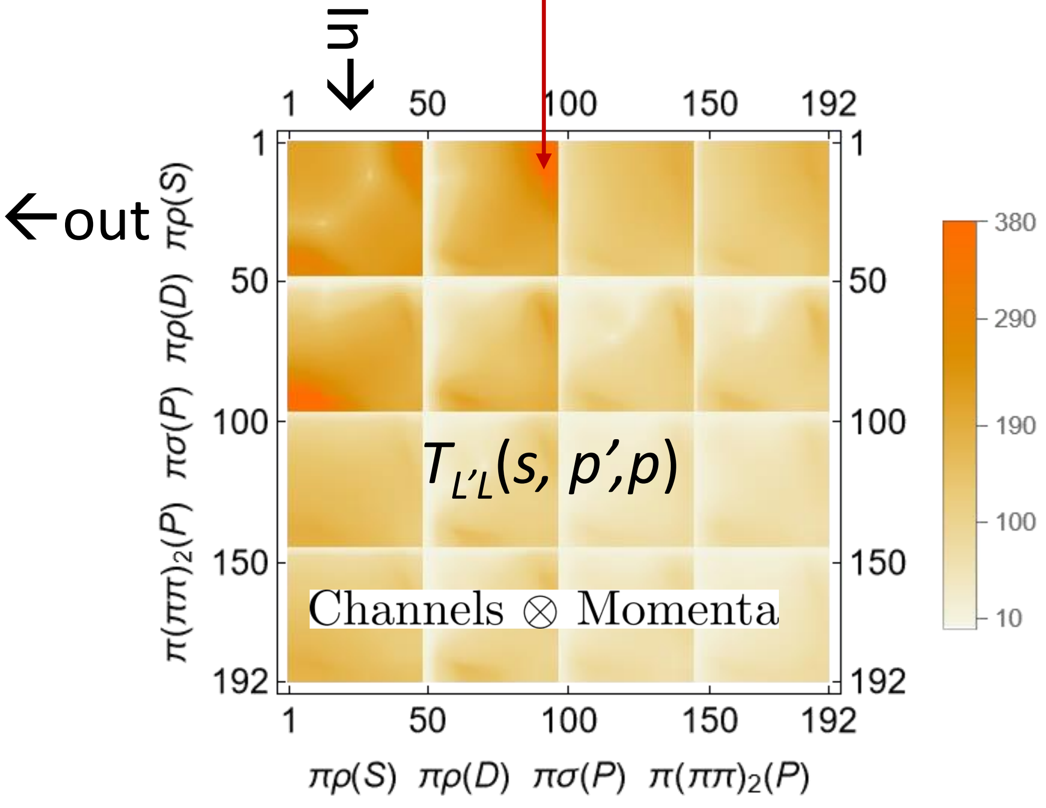
- Avoid vanishing denominator at

$$p'_{\pm} = \frac{px(p^2 - \alpha^2) \pm \alpha\sqrt{\left(\beta + p^2\left(x^2 - 1\right)\right)^2 - 4m_\pi^2\beta}}{2\beta},$$

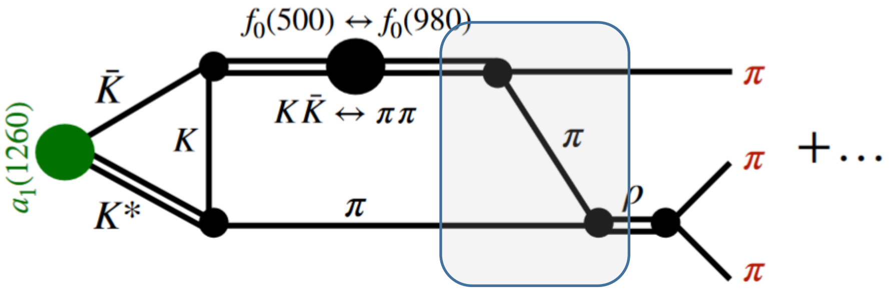
$$\alpha(p) = \sqrt{s} - E_p\,, \quad \beta(p, x) = \alpha^2(p) - p^2x^2\,. \quad (23)$$

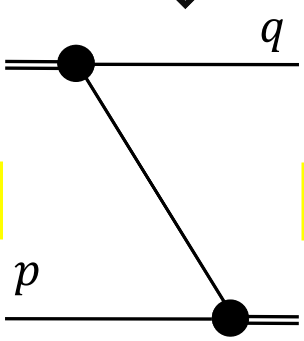


$\pi\rho \xrightarrow{S\to S} \pi\rho$	$\pi\rho \xrightarrow{D\to S} \pi\rho$	$\pi\sigma \xrightarrow{P\to S} \pi\rho$	$\pi(\pi\pi)_2 \xrightarrow{P\to S} \pi\rho$
$\pi\rho \xrightarrow{S\to D} \pi\rho$	$\pi\rho \xrightarrow{D\to D} \pi\rho$	$\pi\sigma \xrightarrow{P\to D} \pi\rho$	$\pi(\pi\pi)_2 \xrightarrow{P\to D} \pi\rho$
$\pi\rho \xrightarrow{S\to P} \pi\sigma$	$\pi\rho \xrightarrow{D\to P} \pi\sigma$	$\pi\sigma \xrightarrow{P\to P} \pi\sigma$	$\pi(\pi\pi)_2 \xrightarrow{P\to P} \pi\sigma$
$\pi\rho \xrightarrow{S\to P} \pi(\pi\pi)_2$	$\pi\rho \xrightarrow{D\to P} \pi(\pi\pi)_2$	$\pi\sigma \xrightarrow{P\to P} \pi(\pi\pi)_2$	$\pi(\pi\pi) \xrightarrow{P\to P} \pi(\pi\pi)_2$



Critical Region

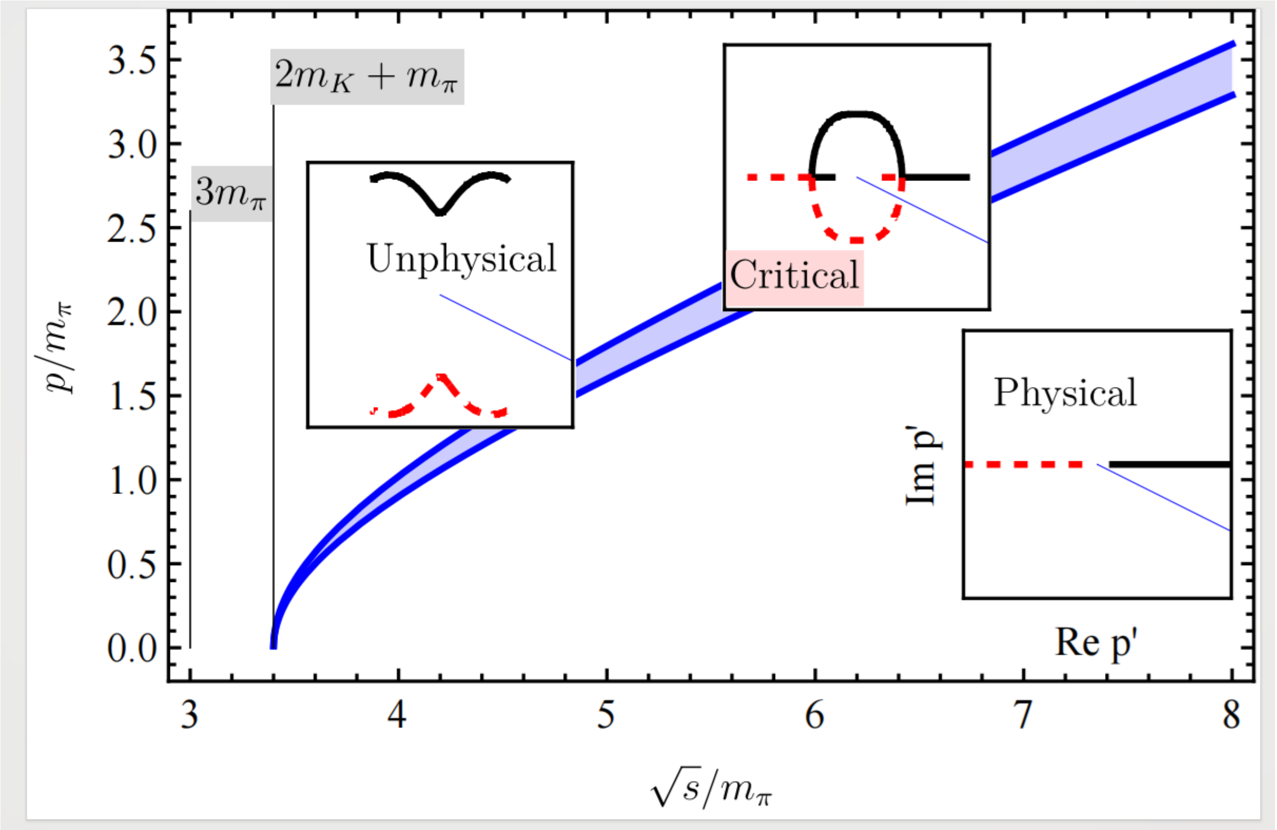




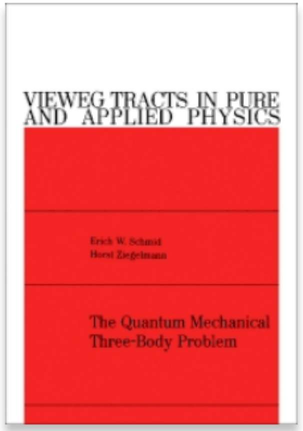
Complex momenta

Real momenta

$$p_\pm = \frac{qx(q^2-\alpha^2) \pm \alpha\sqrt{(\beta + q^2(x^2-1))^2 - 4m_\pi^2\beta}}{2\beta} \quad (1)$$



Heatherington and Schick method



The Quantum Mechanical Three-Body Problem

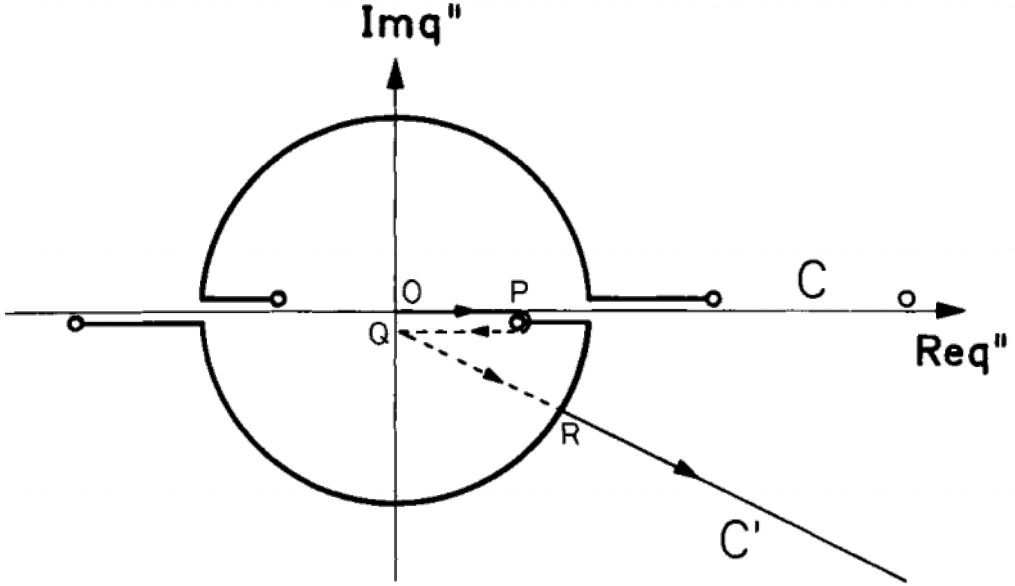
Vieweg Tracts in Pure and Applied Physics

1st Edition - January 1, 1974

Authors: Erich W. Schmid, Horst Zieġelmann

Editor: H. Stumpf • Language: English

eBook ISBN: 9781483160788



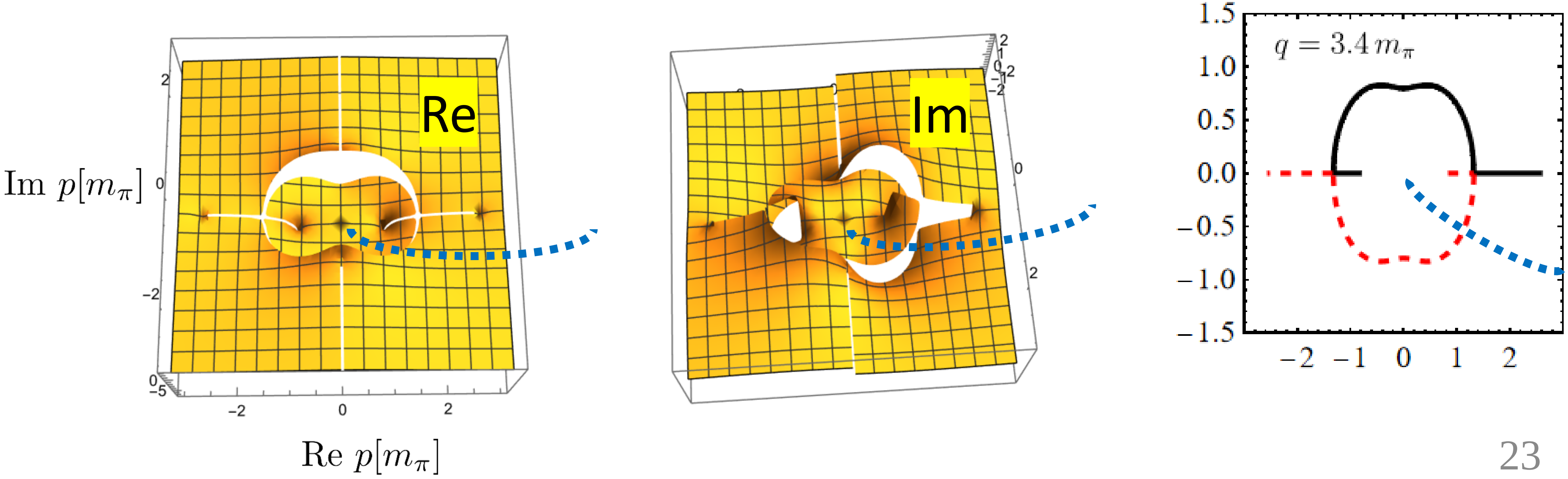
Heatherington and Schick method

B~ $f(\mathbf{p}',\mathbf{p}) = \dfrac{1}{\sqrt{s} - E_p - E_{p'} - E_{p+p'} + i\epsilon}$

$$B_{\lambda'\lambda}(p',p) = 2\pi \int_{-1}^{+1} dx\, d^J_{\lambda'\lambda}(x) B_{\lambda'\lambda}(\mathbf{p}',\mathbf{p})$$

3D plot: Integrate over the angle

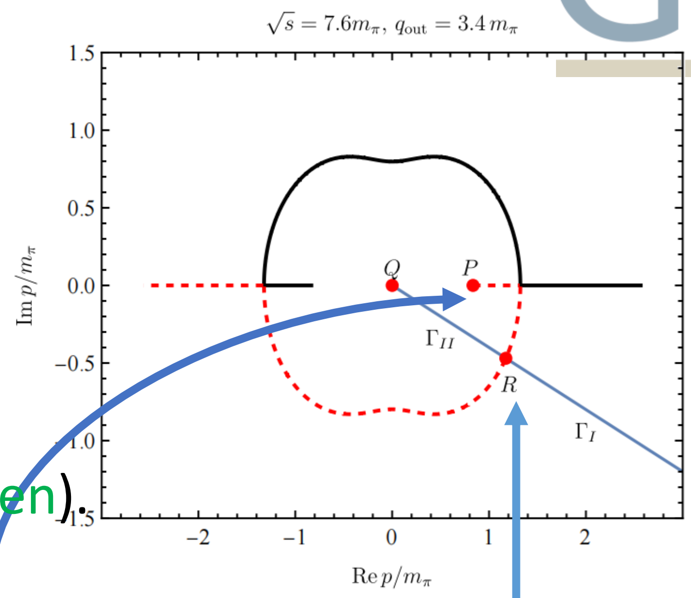
$\sqrt{s} = 7.6 m_\pi$, $q = 3.4 m_\pi$



Connection between sheets

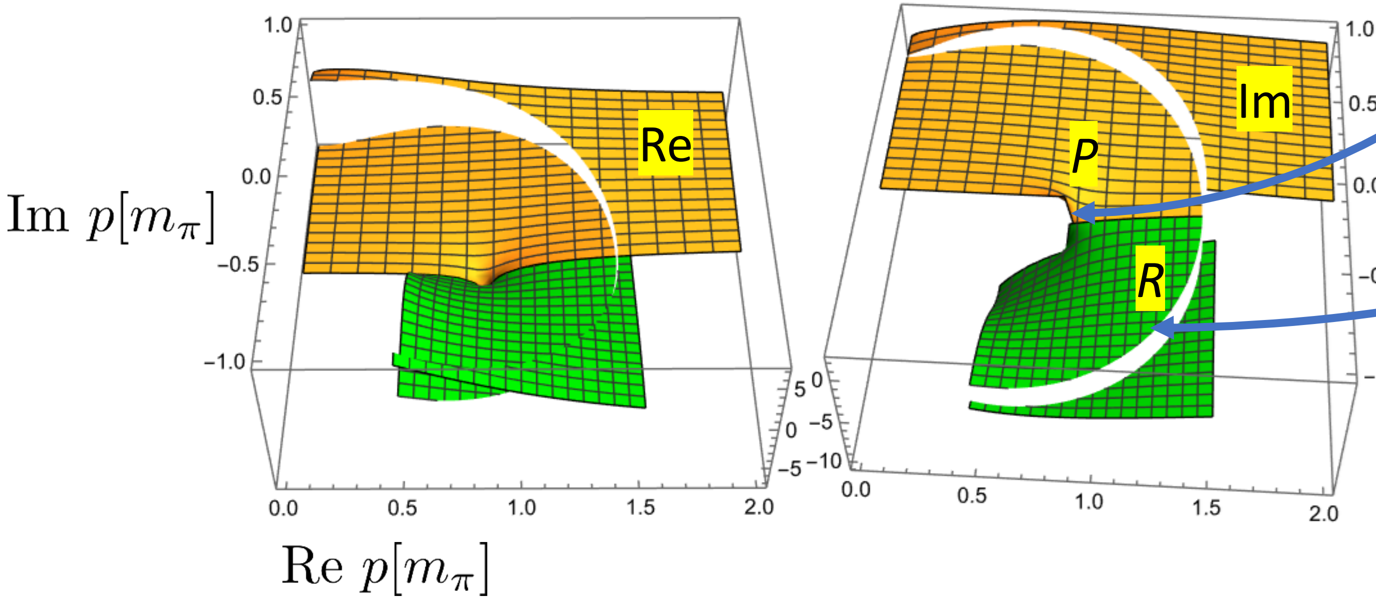
Discontinuity: 2 times of

$$d_f = \mathrm{Im} \int_{-1}^{1} dx\, f(\boldsymbol{q}', \boldsymbol{q}). \qquad f(\boldsymbol{p}', \boldsymbol{p}) = \frac{1}{\sqrt{s} - E_p - E_{p'} - E_{p+p'} + i\epsilon}$$

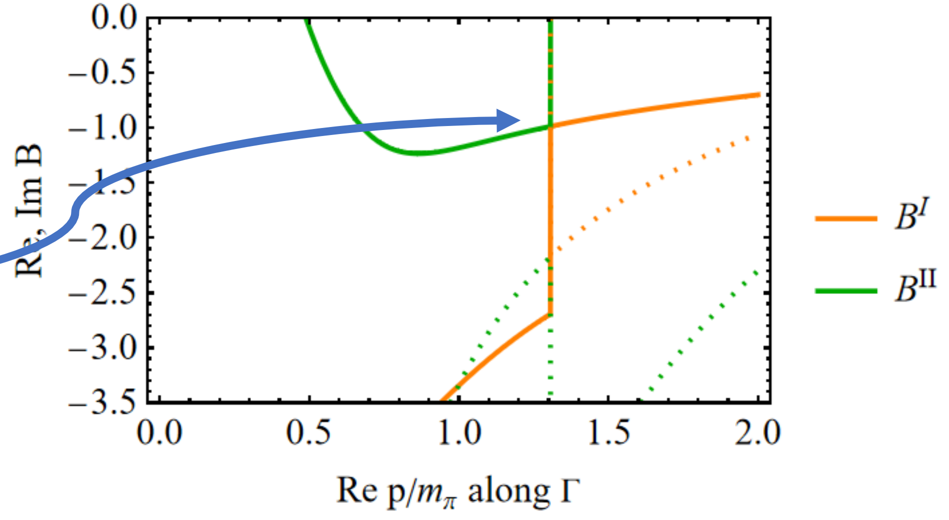


Smooth transition between 1st sheet (orange) and 2nd sheet (green).

$\sqrt{s} = 7.6 m_\pi,\ q = 3.4 m_\pi$



Cross Section

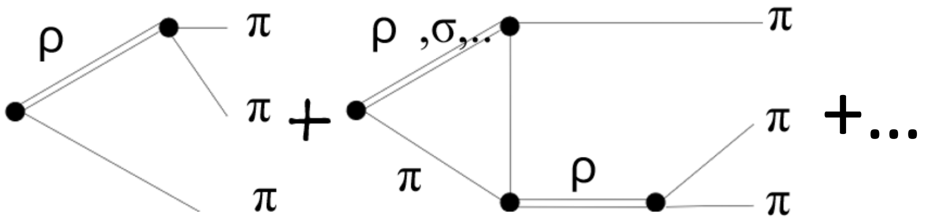


Preliminary results

[YF, F. Gil, M. Doering, R. Molina, M. Mai, V. Shastry, A. Szczepaniak]

Lineshape Modification

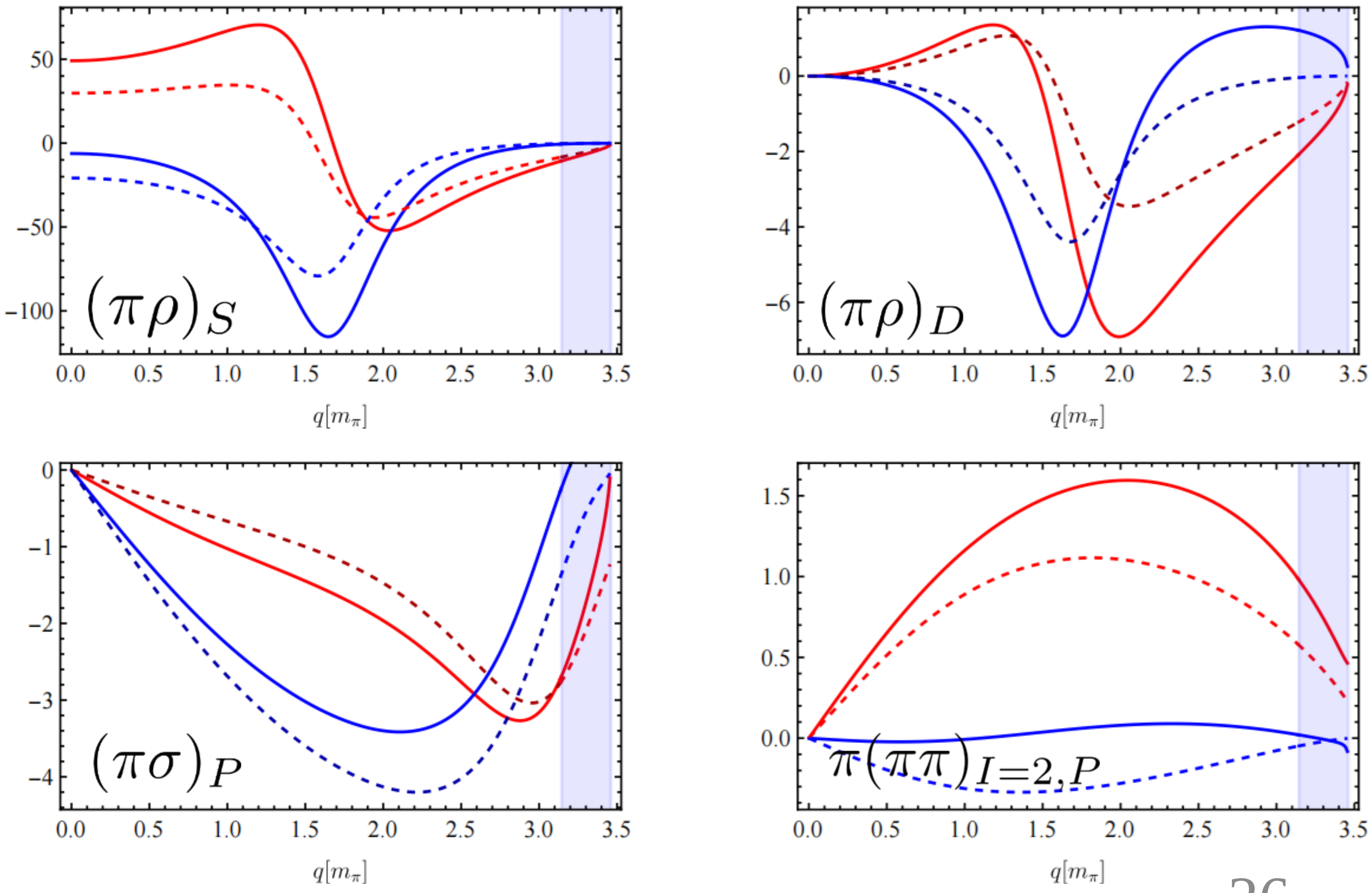
YF, M. Doering,
V. Shastry, A. Szczepaniak



$$\breve{\Gamma}_{L'} = \breve{\Gamma}^{\rm disc}_{L'} + \breve{\Gamma}^{\rm con}_{L'}$$

$$\breve{\Gamma}_{L'}(s,q) = \breve{v}_{L'}(q)\tau_{L'}(\sigma(q))\left[\tilde{\Gamma}_{L'}(s,q) + D_{L'}(s,q)\right]$$

$$a_1 \leftrightarrow (\pi\rho)_S \leftrightarrow (\pi\rho)_D \leftrightarrow (\pi\sigma)_P \leftrightarrow (\pi(\pi\pi)_{S,I=2})$$

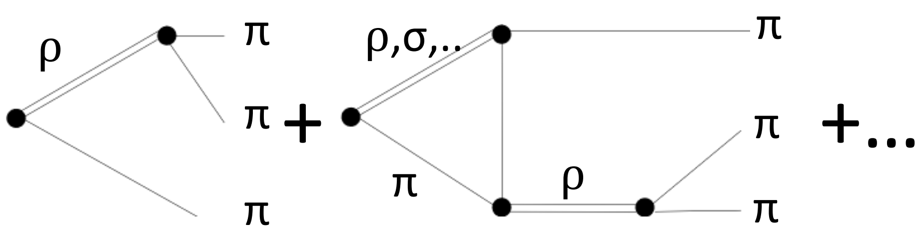


Solid line: Full
Dashed line: Disc

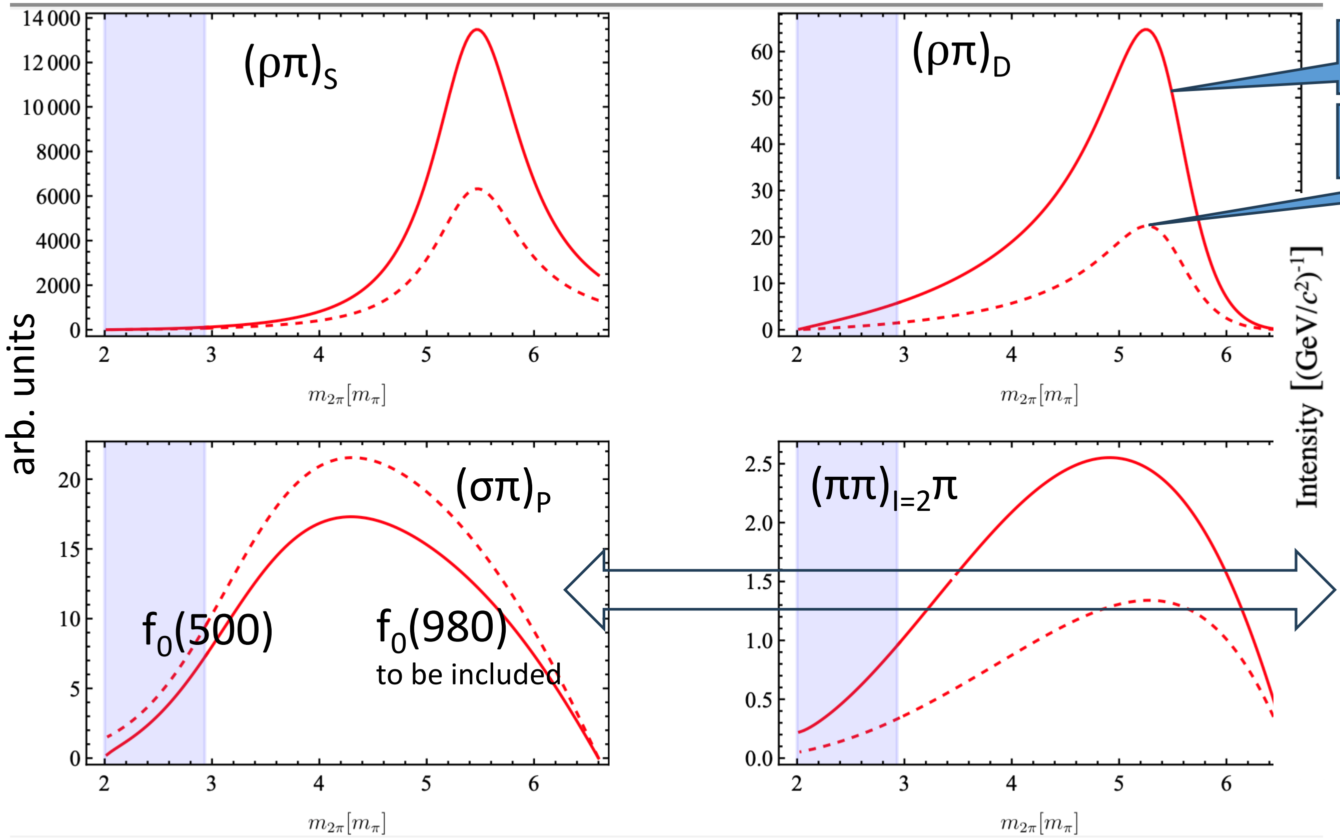
Red: *Re*
Blue: *Im*

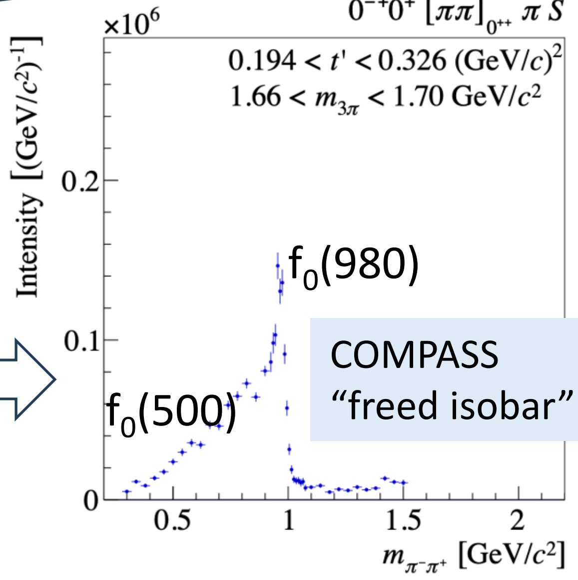
Lineshape Modification

- Isobar lineshapes modification through 3-body effects



$I^G(J^{PC}) = 1^-(1^{++}) \rightarrow$
$(\rho\pi)_S$, $(\rho\pi)_D$,
$(\sigma\pi)_P$, $(\pi\pi)_{I=2}\pi$,…
$\rightarrow \pi\ \pi\ \pi$





COMPASS "freed isobar"

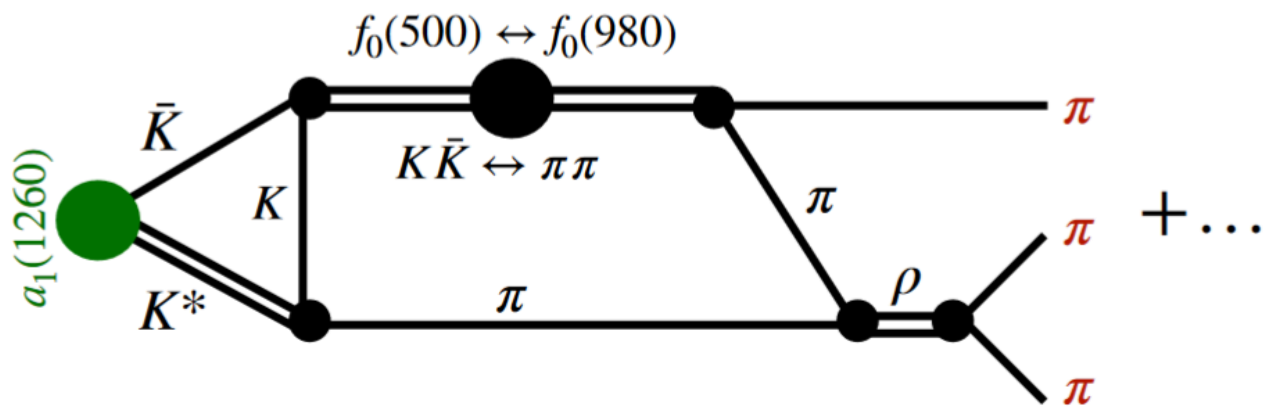
2b lineshape extracted from 3b experiment

[Phys. Rev. D 95, 032004 (2017)]

Extension to strangeness

Isobar (S_I, I_I)	$(1,1)$		$(1,1/2)$		$(0,0)$		$(0,2)$	$(0,1/2)$	$(0,3/2)$
HB basis (11 Ch.)	$\pi\rho_{\lambda=\pm1,0}$		$KK^*_{\lambda=\pm1,0}$		$\pi\sigma$	$\pi(K\bar{K})_S$	$\pi\pi_2$	$K\kappa$	$K(\pi K)_S$
JLS basis (9 Ch.)	$(\pi\rho)_S$	$(\pi\rho)_D$	$(KK^*)_S$	$(KK^*)_D$	$(\pi\sigma)_P$	$(\pi(K\bar{K})_S)_P$	$(\pi\pi_2)_S$	$(K\kappa)_S$	$(K(\pi K)_S)_P$

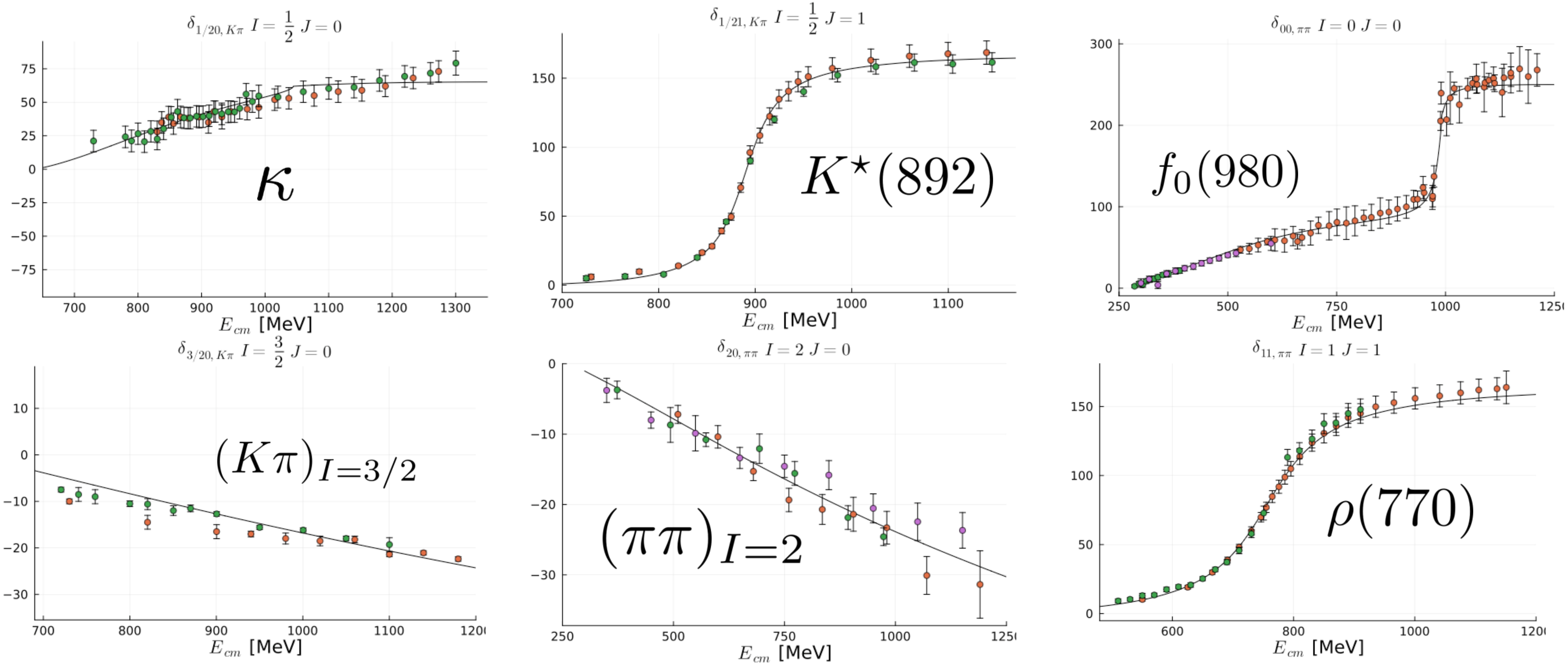
$\Downarrow$ $K_0^*(700)$



$$T_{ji}(s,p',p) = \tilde{B}_{ji}(s,p',p) + \tilde{C}_{ji}(s,p',p) + \int_0^\Lambda \frac{\mathrm{d}l\, l^2}{(2\pi)^3\, 2E_l} \left(\tilde{B}_{jk}(s,p',l) + \tilde{C}_{jk}(s,p',l)\right) \tilde{\tau}_k(\sigma_l)\, \tilde{T}_{kj}(s,l,p)$$

2-body input

More 2-body inputs: η^{00} $\delta^{00}_{K\bar{K}\to\pi\pi}$

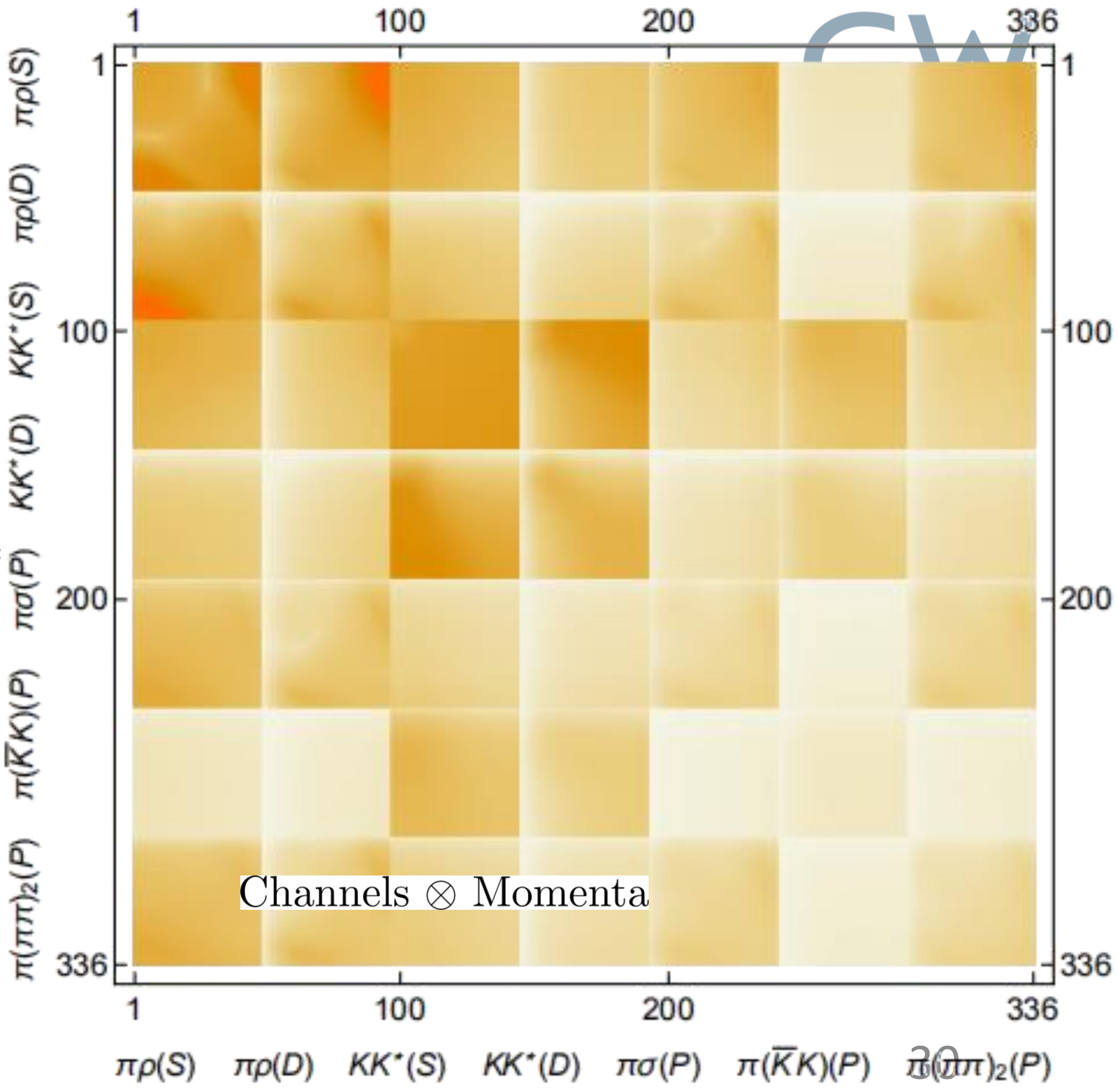


Find data resources from [Doering JHEP 01 (2012) 009]

Prelim Results

7-channel model T-matrix

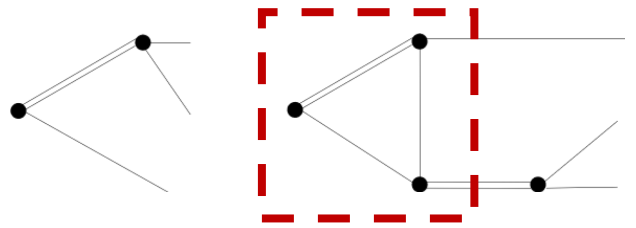
$\pi\rho_{\lambda=\pm1,0}$		$KK^*_{\lambda=\pm1,0}$		$\pi\sigma$	$\pi(K\bar{K})_S$	$\pi\pi_2$
$(\pi\rho)_S$	$(\pi\rho)_D$	$(KK^*)_S$	$(KK^*)_D$	$(\pi\sigma)_P$	$(\pi(K\bar{K})_S)_P$	$(\pi\pi_2)_S$



Prelim results 2

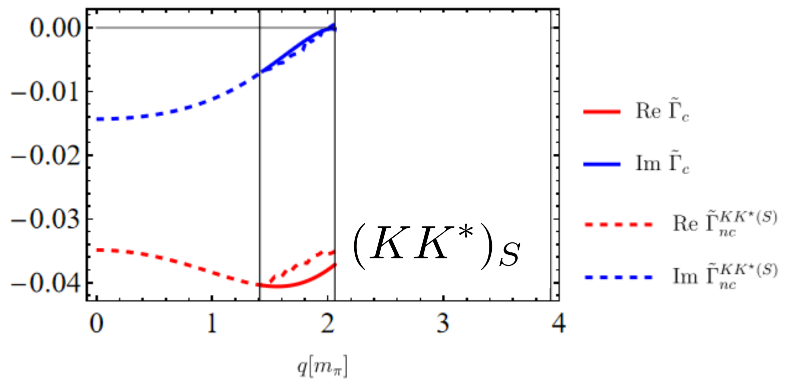
Production amplitude (strangeness): Only the (non-trivial) re-scattering piece

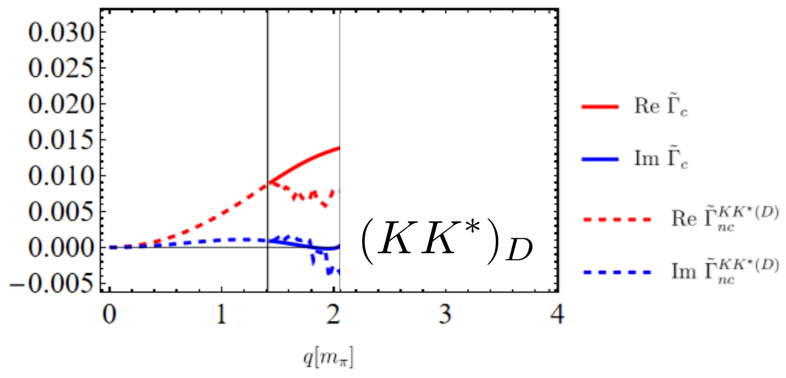
Without final isobar decay



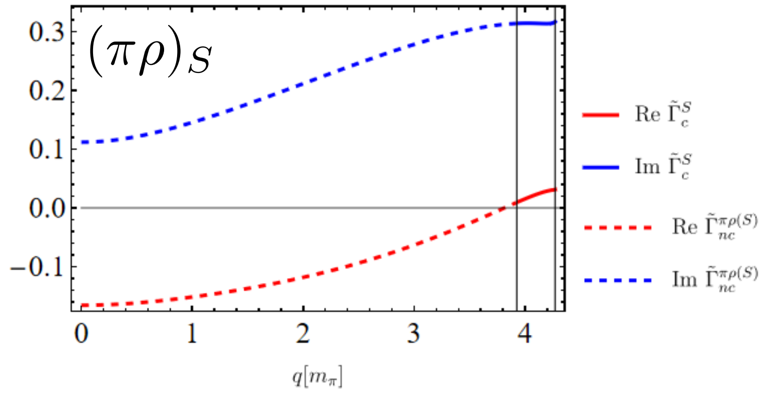
$$\breve{\Gamma}_{L'}(s,q) = \breve{v}_{L'}(q)\tau_{L'}(\sigma(q))\left[\tilde{\Gamma}_{L'}(s,q) + D_{L'}(s,q)\right]$$

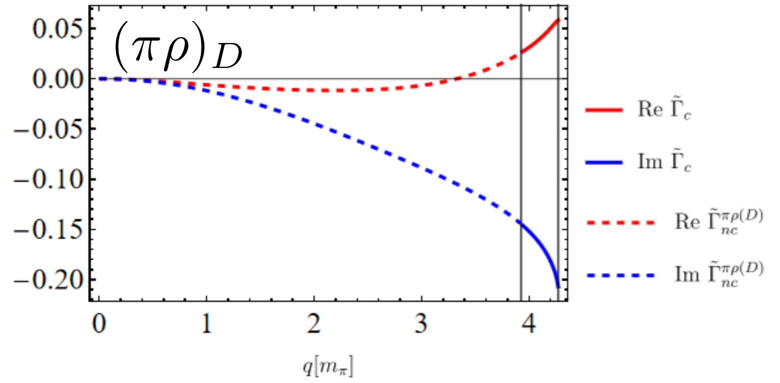
$$W > 2m_K + m_\pi$$





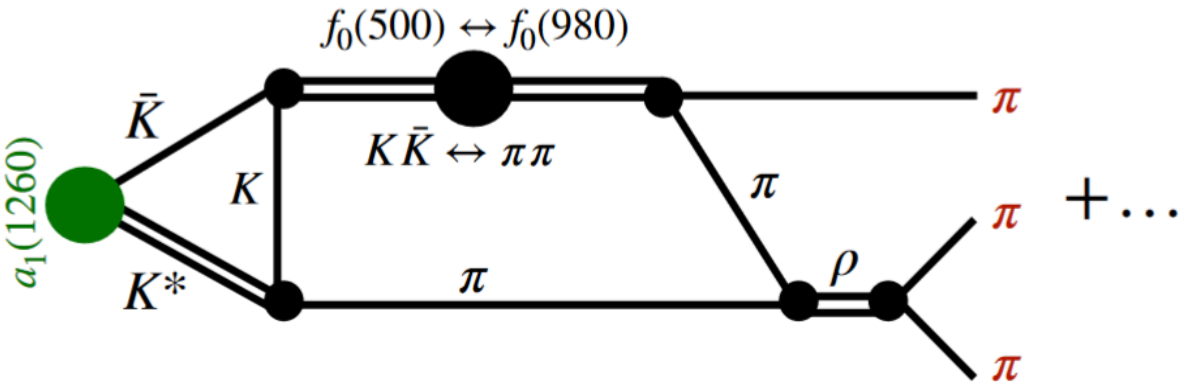
Rescattering generates phase motion of the three-body amplitude





Summary and outlook

- **Finite volume**: Four-channel quantization condition allows for better determination of a_1 amplitude and other spins/isospins (in progress).
- **Infinite volume**:
  - Quantify line-shape modification and phase motion induced by three-body effects beyond traditional isobar model
  - Inclusion of kaons (+eta) will allow to analyze different final states (help needed!)
  - Allows for consistent inclusion of triangles as coupled-channel transition (a_1(1420))
  - Can some resonances be generated by nonlinear meson dynamics from three-body unitarity?



π

For questions

Delta00

$$S = \begin{bmatrix} \eta e^{2i\delta_1} & i(1-\eta^2)^{1/2}\, e^{i(\delta_1+\delta_2)} \\ i(1-\eta^2)^{1/2} e^{i(\delta_1+\delta_2)} & \eta e^{2i\delta_2} \end{bmatrix}$$

$$\left[\tilde{C}(s)\right]_{(\boldsymbol{p}',j)(\boldsymbol{p},i)} = \frac{3}{4\pi} \sum_{M=-1}^{1} \mathfrak{D}^{1*}_{-M-\lambda(j)}(\phi_{-\boldsymbol{p}'}, \theta_{-\boldsymbol{p}'}, 0) \tilde{C}_{ji}(s, p', p) \mathfrak{D}^{1}_{-M-\lambda(i)}(\phi_{-\boldsymbol{p}}, \theta_{-\boldsymbol{p}}, 0)$$

$$\text{with} \quad \tilde{C}_{ji}(s,p',p) = U_{jL'} \tilde{C}_{L'L}(s,p',p) U_{Li} \quad \text{for} \quad U_{Lj} = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{1}{\sqrt{6}} & -\sqrt{\frac{2}{3}} & \frac{1}{\sqrt{6}} & 0 \end{pmatrix}_{Lj}$$

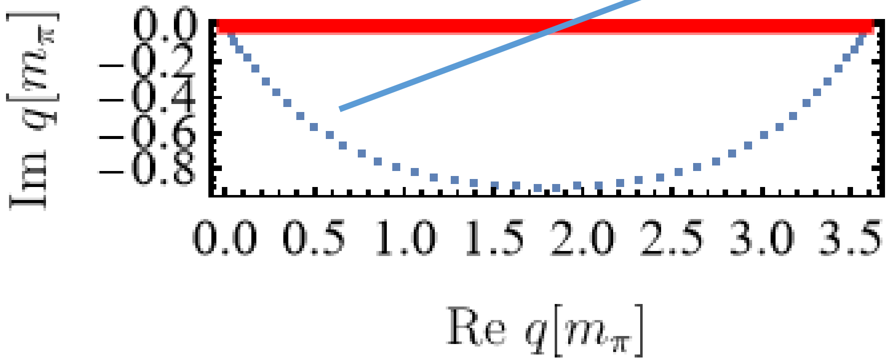
$$\text{with} \quad \tilde{C}_{L'L}(s,p',p) = \sum_{i=0}^{\infty} (p')^{L'} \cdot \tilde{c}^{(i)}_{L'L} (s)^{i} \cdot (p)^{L} \ .$$

Extrapolation to real momenta

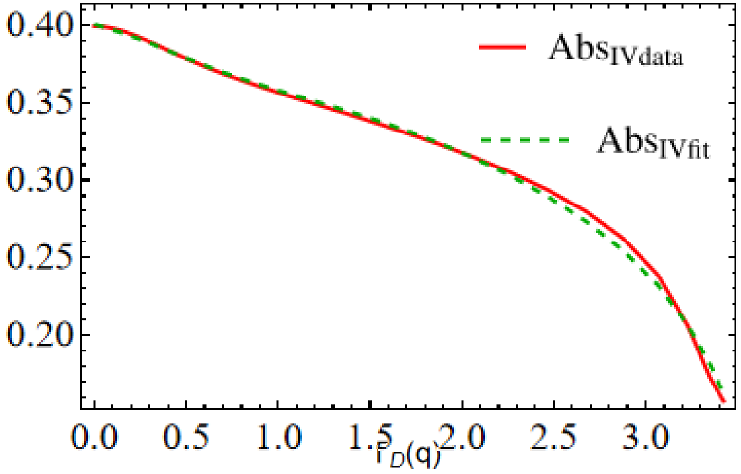
$$T_{L'L}(q_{\rm out}, p_{\rm in}) \to \breve{\Gamma}_{L'}(q_{\rm out})$$

First try: Pade fit

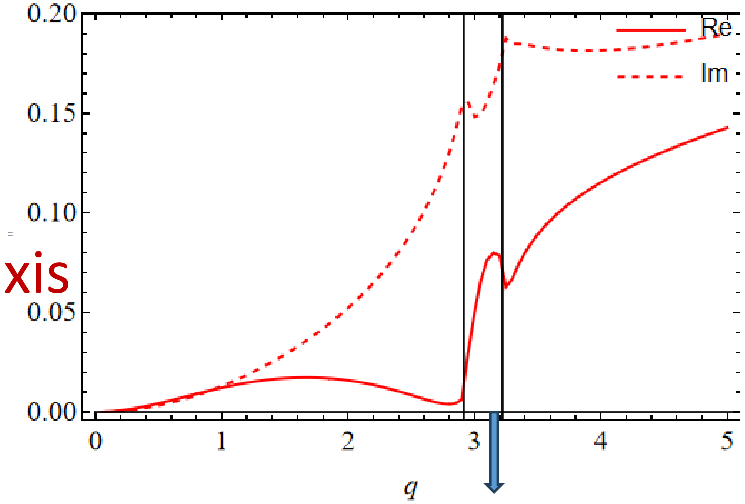
$$\breve{\Gamma}_L(q_1) = \left(\frac{q_1}{m_\pi}\right)^L H(q_1) \frac{\sum_{j=0}^{m} a_j^L (q_1/m_\pi)^j}{\sum_{k=0}^{n} b_k^L (q_1/m_\pi)^k}$$



Fit in complex plane



Extrapolate to real-axis



Not good in “critical region” for *D*-wave

Finite volume 2-body input

$$\Sigma^L_{n,\lambda'\lambda}(s,\boldsymbol{p}) = \frac{J(\boldsymbol{p})}{L^3}\sum_{\boldsymbol{k}\in\mathcal{S}_L}\frac{\sigma_p^n}{(4E^2_{k^\star})^n}\frac{\epsilon^{\star\nu *}_{\lambda'}(\boldsymbol{P}_3-\boldsymbol{p})(P^\star_\nu - p^\star_\nu - 2k^\star_\nu)\epsilon^{\star\mu}_{\lambda}(\boldsymbol{P}_3-\boldsymbol{p})(P^\star_\mu - p^\star_\mu - 2k^\star_\mu)}{2E_{k^\star}(\sigma_p - 4E^2_{k^\star})}\,.$$

$$\boldsymbol{k}^\star(s,\boldsymbol{k},\boldsymbol{p}) = \boldsymbol{k} + \boldsymbol{p}\left(\frac{\boldsymbol{k}\cdot\boldsymbol{p}}{\boldsymbol{p}^2}\left(J(s,\boldsymbol{p})-1\right) + \frac{1}{2}J(s,\boldsymbol{p})\right)\,, \quad J(s,\boldsymbol{p}) = \frac{\sqrt{\sigma_p}}{\sqrt{s}-E_{\boldsymbol{p}}}\,,$$

$$\tau^{-1}_{\lambda'\lambda}(\sigma_p) = \delta_{\lambda'\lambda}\tilde{K}^{-1}_n(s,\boldsymbol{p}) - \Sigma_{n,\lambda'\lambda}(s,\boldsymbol{p})\,, \qquad (4)$$

$$\tilde{K}^{-1}_n(s,\boldsymbol{p}) = \sum_{i=0}^{n-1} a_i\sigma_p^i \quad \text{and} \quad \Sigma_{n,\lambda'\lambda}(s,\boldsymbol{p}) =$$

$$\int\frac{d^3k}{(2\pi)^3}\frac{\sigma_p^n}{(4E_k^2)^n}\frac{\hat{v}^*_{\lambda'}(P-p-k,k)\hat{v}_\lambda(P-p-k,k)}{2E_k(\sigma_p-4E_k^2+i\epsilon)}\,.$$

[Mai:2021nul]

LQCD DATA

Table I. Details of the GWQCD $N_f = 2$ ensemble parameters used in this work. Here a is the lattice spacing, $N_{\rm cfg}$ the number of Monte-Carlo configurations for each ensemble. The pion and kaon masses are aM_π and aM_K, respectively. The errors on every value are purely stochastic except the lattice spacing which includes an estimated 2% systematic uncertainty.

Ensemble	$N_t \times N^3$	a/fm	$N_{\rm cfg}$	aM_π	af_π	aM_K	af_K
2448	48×24^3	0.1210(2)(24)	300	0.1931(4)	0.0648(8)	0.3236(3)	0.1015(2)
2464	64×24^3	0.1215(3)(24)	400	0.1378(6)	0.0600(10)	0.3132(3)	0.0980(2)

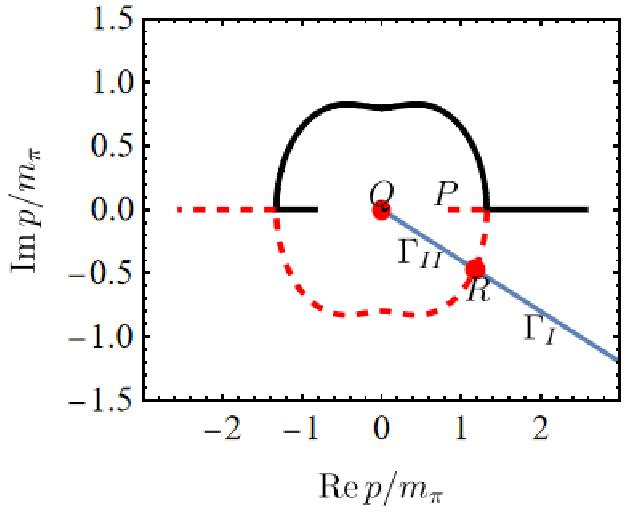
Heatherington and Schick method

$$T(s,q,p) = B^I(s,q,p) - \int_0^{P(q)} \frac{dq''\,(q'')^2}{(2\pi)^3 2E_{q''}} \,\mathrm{Im}\, B^{II}(s,q,q'')\,\tau(\sigma(q''))T(s,q'',p)$$
$$+ \int_{\Gamma_{II}} \frac{dl\,l^2}{(2\pi)^3\,2E_l} B^{II}(s,q,l)\tau(\sigma(l))T(s,l,p) + \int_{\Gamma_I} \frac{dl\,l^2}{(2\pi)^3\,2E_l} B^{I}(s,q,l)\tau(\sigma(l))T(s,l,p)\ ,$$

$$\mathrm{Im}\ B^I = 0\,(0 < q'' < P) \qquad \mathrm{Im}\ B^{II} \to 2d(s,q,q'')$$

Same method for production amplitude

$$\tilde{\Gamma}_{L'}(s,q) = -2\int_0^{P(q)} \frac{dq''\,(q'')^2}{(2\pi)^3 2E_{q''}} d_{L'L}(s,q,q'')\,\tau_L(\sigma(q''))\left(\tilde{\Gamma}_L(s,q'') + D_L(s,q'')\right)$$
$$+ \int_{\Gamma} \frac{dl\,l^2}{(2\pi)^3\,2E_l} \left(B^{II\to I}_{L'L}(s,q,l) + C_{L'L}(s,q,l)\right)\tau_L(\sigma(l))\left(\tilde{\Gamma}_L(s,l) + D_L(s,l)\right)$$



$$D_L(s,p) = D_{fL}(s,p) B_L(\lambda p)$$

3. Blatt-Weisskopf barrier-penetration factors

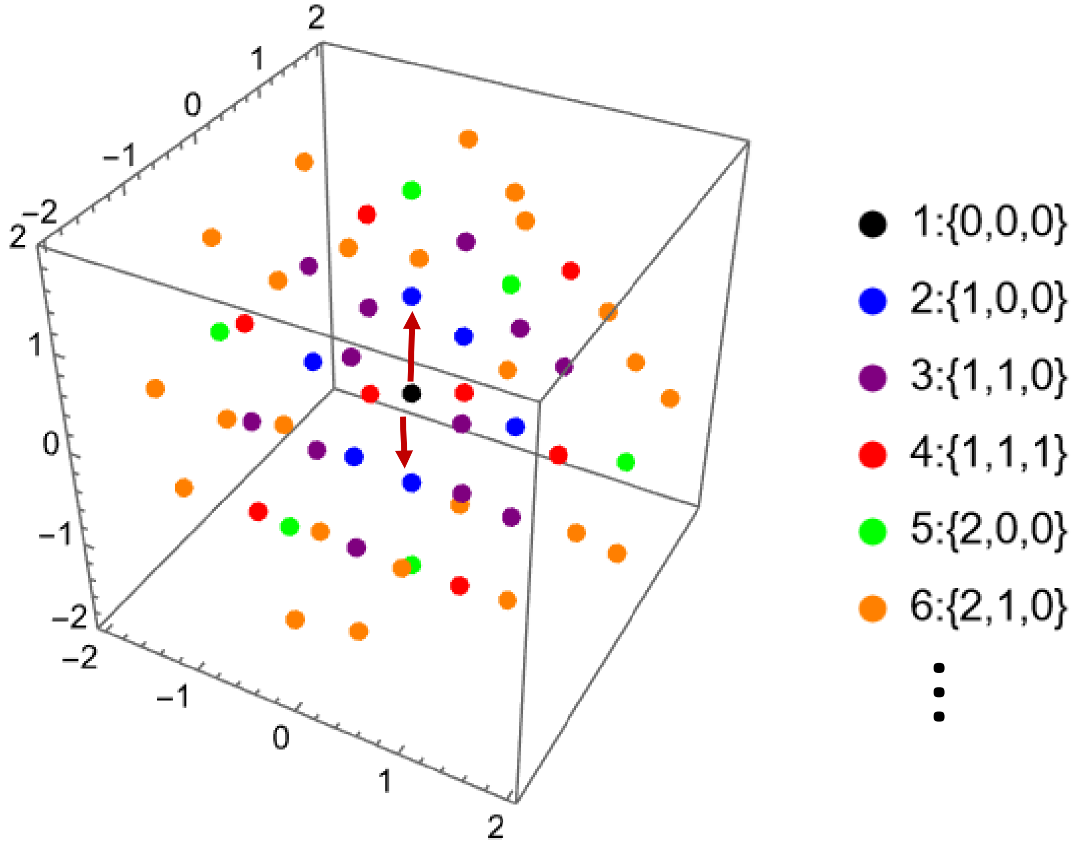
For $\ell = 0, ..., 5$ the Blatt-Weisskopf barrier-penetration factors [167, 168] are explicitly given by

$$\begin{aligned}
B_0(r) &= 1\,, \\
B_1(r) &= r/\sqrt{1+r^2}\,, \\
B_2(r) &= r^2/\sqrt{9+3r^2+r^4}\,, \\
B_3(r) &= r^3/\sqrt{225+45r^2+6r^4+r^6}\,, \\
B_4(r) &= r^4/\sqrt{11025+1575r^2+135r^4+10r^6+r^8}\,, \\
B_5(r) &= r^5/\sqrt{893025+99225r^2+6300r^4+315r^6+15r^8+r^{10}}\,.
\end{aligned}$$

Spectator momentum

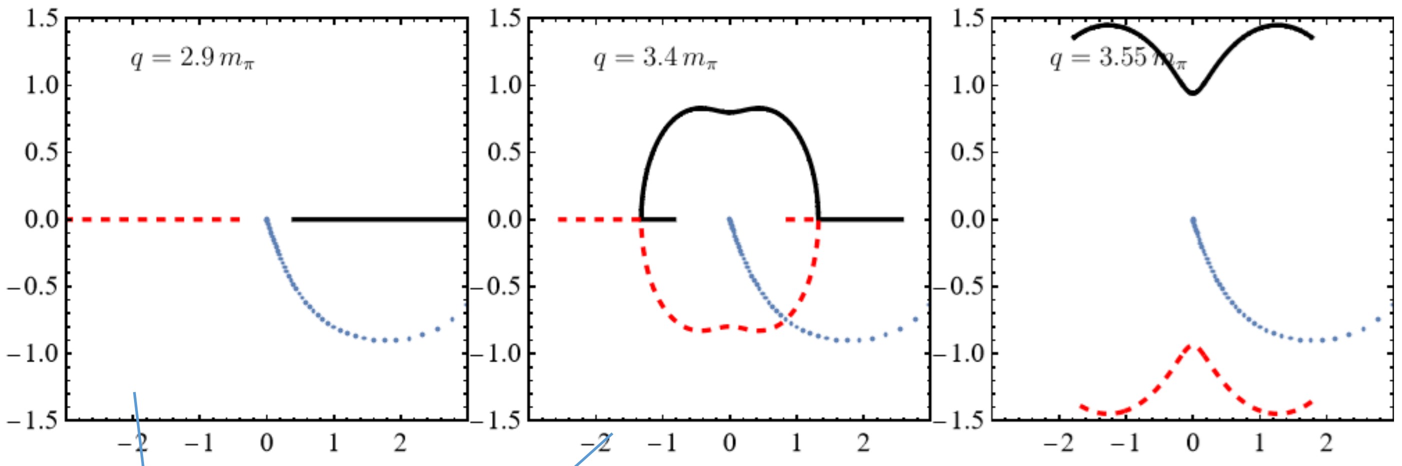
$$\Rightarrow \boldsymbol{p} = 2\pi \boldsymbol{n}/L \text{ for } \boldsymbol{n} \in \mathbb{Z}^3$$

Spectator momentum shells



$\frac{2\pi}{L}$

e.g. Shell 2: we have 2*3 =6 plane-wave momenta: (1,0,0), (1,0,0),(0,1,0), (0,1,0), (0,1,0), (0,0,1), (0,0,-1)



$$\tilde{\Gamma}_{L'}(s,q) = \int_\Gamma \frac{\mathrm{d}p\, p^2}{(2\pi)^3\, 2E_p} T_{L'L}(s,q,p)\tau_L(\sigma(p))D_L(s,p)\ ,$$

for the non-crictical and

$$\tilde{\Gamma}_{L'}(s,q) = -2\int_0^{P(q)} \frac{dq''\,(q'')^2}{(2\pi)^3 2E_{q''}} d_{L'L}(s,q,q'')\,\tau_L(\sigma(q''))\left(\tilde{\Gamma}_L(s,q'') + D_L(s,q'')\right)$$
$$+ \int_\Gamma \frac{dl\, l^2}{(2\pi)^3\, 2E_l} \left(B^{II\to I}_{L'L}(s,q,l) + C_{L'L}(s,q,l)\right)\tau_L(\sigma(l))\left(\tilde{\Gamma}_L(s,l) + D_L(s,l)\right)$$

for the critical region.