# Dispersive analysis of $\eta(1405/1475)$ on the recent BESIII $J/\psi \to \gamma K_S^0 K_S^0 \pi^0$


Lin Qiu (邱林)

Institute of High Energy Physics, CAS
University of Chinese Academy of Sciences

In collaboration with Dr. Yin Cheng (程茵), Dr. MengChuan Du (杜蒙川) and Prof. Qiang Zhao (赵强)



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   - The inhomogeneities from crossed-channel scattering
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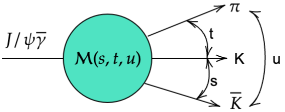
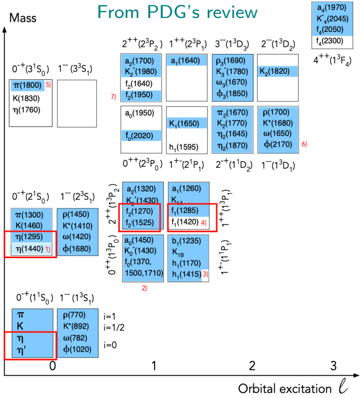


# Introduction

# Status of Light meson spectroscopy

- The first radially-excited states of $\eta - \eta'$ are assigned to $\eta(1295)$ and $\eta(1405/1475)$, among which one of the states is regarded as $0^{-+}$ glueball
- The BES-III collaboration and theoretical groups are making joint efforts to understand these states $(0^{-+}, 1^{++}, 2^{++})$ from $K\bar{K}\pi, \eta\pi\pi, 3\pi$ etc!



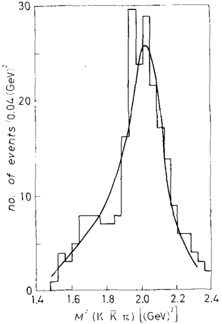

## Puzzles

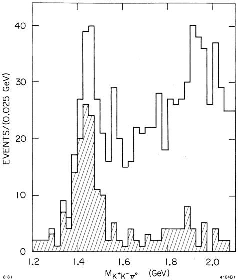
- Controversial observations in experiments: one or two $\eta(1405/1475)$ states observed in different modes $(K\bar{K}\pi, \eta\pi\pi, 3\pi, \gamma V$ etc$)$
- Supernumerary problem: excited states, glueballs, dynamically-generated states, kinematic singularities?
- LQCD: the masses of $0^{-+}$ glueballs are simulated to be above 2GeV on LQCD

A better understanding of $0^{-+}$ spectrum in $1.2 \sim 1.5$GeV is strongly desired!

# Controversial observations: one-state scenario

The iso-scalar $\eta(1405/1475)$ (E-$\iota$ meson) was observed in $p\bar{p}$ annihilation, radiative $J/\psi$ decays (into $K\bar{K}\pi, \gamma\rho, \eta\pi\pi$ etc) and $\gamma\gamma$ collisions (in low-statistics).



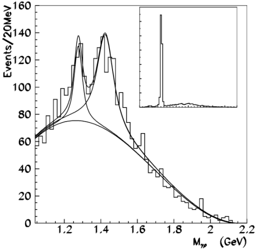
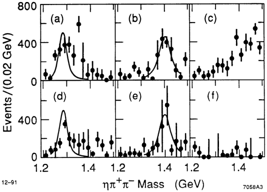
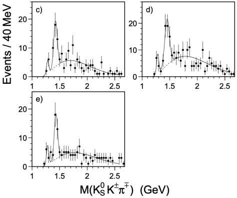




Fig. 2. The $\gamma\rho$ invariant mass distribution. The insert shows the full mass scale where the $\eta(958)$ is clearly observed.




P.H.Baillon et al., Nuovo Cim. A50, 393 (1967).

C. Edwards et al., Phys. Rev. Lett. 49, 259 (1982)

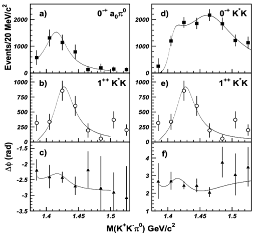
J. Z. Bai et al. (BES), Phys. Lett. B594, 47 (2004)

T. Bolton et al., Phys. Rev. Lett. 69, 1328 (1992).

P. Achard et al. (L3), JHEP 03, 018 (2007).

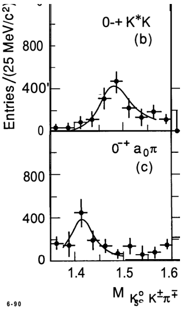
# Controversial observations: two-state scenario

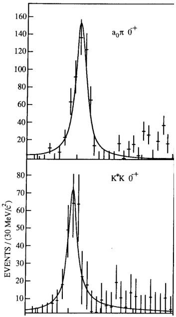
However, the simultaneous observation of two $\eta(1405/1475)$ was also reported from radiative $J/\psi$ decay, $\pi^- p$ and $p\bar{p}$ annihilations (only in $K\bar{K}\pi$ final states).




G. S. Adamset al.(E852), Phys. Lett.B516, 264 (2001)

M. G. Rathet al., Phys. Rev.D40, 693 (1989).




J. E. Augustinet al.(DM2), Phys. Rev.D46, 1951 (1992).

Z. Baiet al.(MARK-III), Phys. Rev. Lett.65, 2507 (1990).

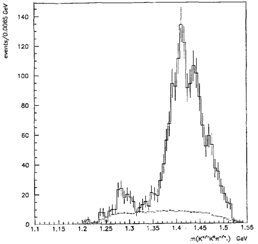



Fig. 3. The $K\bar{K}\pi_1$ mass spectrum for $M(K\bar{K}\pi_2) < 1260$ MeV, together with the contaminating background.

A. Bertinet al.(OBELIX), Phys. Lett.B361, 187 (1995); A. Bertinet al.(OBELIX), Phys. Lett.B400, 226 (1997);

C. Cicaloet al.(OBELIX), Phys. Lett.B462, 453 (1999); F. Nichitiuet al.(OBELIX), Phys. Lett.B545, 261 (2002)

# Theoretical interpretations before BES-III data

## $\eta(1405)$ as a $0^{-+}$ glueball candidate?

- Favored $(< 2.0\text{GeV})$

L. Faddeev et al., PRD70, 114033 [Fluxtube model]
J.F.Donoghue et al.,PLB99,416-420; T.Barnes et al.,Phys.Lett.110B,159 [MIT bag]
H.Y.Cheng et al.,PRD79,014024; B.A.Li,PRD81, 114002; Y.D.Tsai et al.,PRD85,034002 [U(1) anomaly]

G.Li et al.,JPG35,055002; T.Gutsche et al.,PRD80,014014 [mixing with quark states]

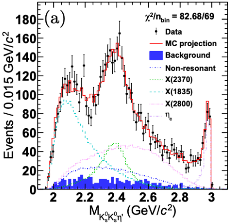
- Disfavored $(> 2.0\text{GeV})$

G.S.Bali et al.(UKQCD), PLB309,378; Y.Chen et al.,PRD73,014516;C.J. Morningstar et al.,PRD60,034509 [LQCD]
W.Qin, Q.Zhao and X.H.Zhong,PRD.97.096002 [U(1) anomaly, mixing with quark states]

V.Mathieu and V.Vento, PRD.81.034004

BESIII,PRL.132.181901

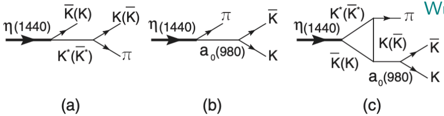
$0^{-+}$ $X(2370)$?




## Other possible explanations

- Does $\eta(1295)$ exit? node effect?
- Triangle singularity (TS) mechanism (the first implementation of Landau singularities)

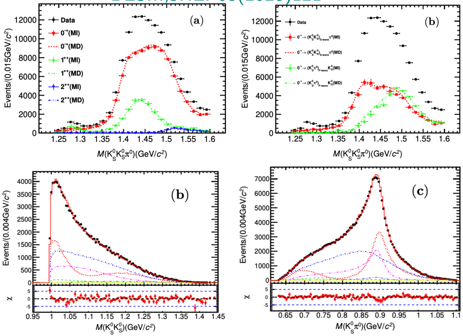
Wu,PRL.108.081803



# High-statistics BES-III data $J/\psi \to \gamma K_S^0 K_S^0 \pi^0$

- Recently, BES-III reported their partial wave analyses of $J/\psi \to \gamma K_S^0 K_S^0 \pi^0$ in mass-(in)dependent ways (bin-by-bin analysis: $24$ bins within $1.24 \sim 1.6$GeV);
- The two- and three-body $0^{-+}$ spectra can only be fitted well with two states in both experimental and theoretical sides (even in the three-body unitarity)

BESIII,JHEP03(2023)121




S.X.Nakamura,PRD.109.014021;PRD.107.L091505

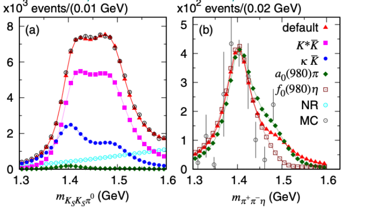



FIG. 4. The $E$ distributions for $J/\psi \to \gamma\eta(1405/1475) \to$ (a) $\gamma(K_SK_S\pi^0)$ and (b) $\gamma(\pi^+\pi^-\eta)$. The default, final $Rc$, and nonresonant (NR) contributions, and MC outputs [Ref. [27] in (a), Ref. [7] in (b)] are shown. Lines connecting the points are guides to the eye.

☺ 3&2-body spectra & Dalitz plots

☹ 2-body spectra (PWA)

**Is the one-state scenario then ruled out?**

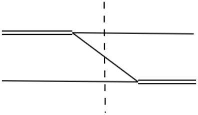
# Motivation

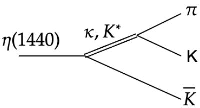
## Isobar model

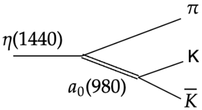
- ✓ Unitarity, Analyticity, Crossing sym.
- ✗ may fail to describe two-body interactions properly
- ✗ hard to deal with off-shell cuts when truncated finitely

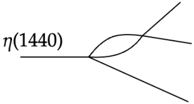
## Dispersion theory

- ✓ Unitarity, Analyticity, Crossing sym.
- ✓ sub-channel interactions determined from scattering data
- ✓ to gain the maximal analyticity and continuation






## Our goals

1. to provide the generic $K\bar{K}\pi, \eta\pi\pi, 3\pi$ FSIs below $1.6$GeV
2. to understand the effect of the three-body unitarity over the two-body one and the triangle singularity mechanism
3. to provide systematic prescriptions for iso-scalar pseudo-scalar spectra

# Muskhelishvili–Omnès Framework

# Amplitudes on the Mandelstam plane

The LO amplitude for $J/\psi \to \gamma 0^{-+} \to \gamma K_0 \bar{K}_0 \pi$ can be factorized as,

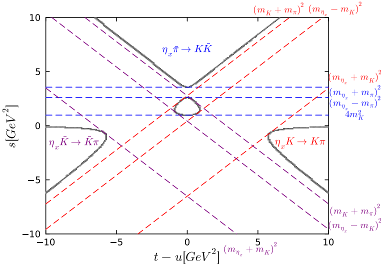
$$\mathcal{M}_{J/\psi\to\gamma K_0\bar{K}_0\pi} \propto \epsilon_{\mu\nu\alpha\beta}\epsilon^{\mu}_{J/\psi}P^{\nu}\epsilon^{*,\alpha}_{\gamma}q_1^{\beta}\cdot\mathcal{M}_{0^{-+}\to K_0\bar{K}_0\pi^0}$$

The physical region is determined by,

$$\text{Gram}(s,t,m^2_{\bar{K}},m^2_{\eta_x},m^2_{\pi},m^2_K)\leq 0$$

In the isospin limit, the main FSIs in the decay process $J/\psi\bar{\gamma}\to\pi(p_1)K(p_2)\bar{K}(p_3)$ are:

- $K\pi$: $(I,J)=(\frac{1}{2},0\&1)\Rightarrow\kappa\&K^*(892)$
- $K\bar{K}$: $(I,J)=(1,0)\Rightarrow a_0(980)$

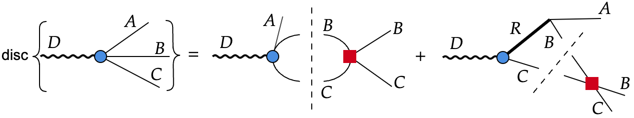



According to the crossing symmetry and reconstruction theorem, J.Stern,PRD.47.3814;J.Kambor,NPB465(1996)215

$$\mathcal{M}(s,t,u)=\frac{1}{\sqrt{2}}\mathcal{F}^1_0(s)+[(-\frac{1}{\sqrt{3}})\mathcal{F}^{1/2}_0(t)+(-\frac{1}{\sqrt{3}})(t(s-u)-\Delta))\mathcal{F}^{1/2}_1(t)]+[t\leftrightarrow u]$$

with $\Delta=(m^2_{\eta_x}-m^2_K)(m^2_\pi-m^2_K)$ and $\mathcal{F}^I_J$ the single-variable amplitudes.

# Dispersion relation in sub-channel scattering




*Dispersion relation* for two-body scattering,

$$\text{disc}\,\mathcal{F}(s) = 2iT^*(s+i\epsilon)\Sigma(s)(\mathcal{F}(s+i\epsilon) + \hat{\mathcal{F}}(s+i\epsilon))$$

R.Omnès,Nuovo Cim.8,316(1958)

- Omnès matrices: $\hat{\mathcal{F}}(s) = 0$ & $\Omega(0) = \mathbf{1}$ $\Rightarrow$ $\Omega(s)$
- Homogeneous Omnès problem: $\hat{\mathcal{F}}(s) = 0$

$$\mathcal{F}(s) = P_n(s)\Omega(s)$$

- Inhomogeneous Omnès problem: $\hat{\mathcal{F}}(s) \neq 0$

$$\mathcal{F}(s) = \Omega(s)(a_0 + a_1 s + \cdots + a_n s^n + \frac{s^{n+1}}{\pi}\int_{s_{\text{th}}}^{\infty}\frac{ds'}{s'^{n+1}}\frac{\Omega^{-1}(s')T^*(s')\Sigma(s')\hat{\mathcal{F}}(s')}{s'-s-i\epsilon})$$

# $(I, J) = (\frac{1}{2}, 0\&1)$ $K\pi$ scattering

The $K\pi$ scattering has the following features:

- Mixing between $I = \frac{1}{2}$ and $I = \frac{3}{2}$ is only significant for S-wave

$$f_S = f_0^{1/2} + f_0^{3/2}/2$$

- elastic up til $K\eta'$ threshold L.von Detten et al., Eur.Phys.J.C81(2021)5,420

For single-channel elastic scattering,

$$T(s+i\epsilon) = \frac{e^{i2\delta(s)} - 1}{2i\sigma(s)}, \sigma(s) = \frac{2q}{\sqrt{s}}\Theta(s - s_{\text{th}})$$

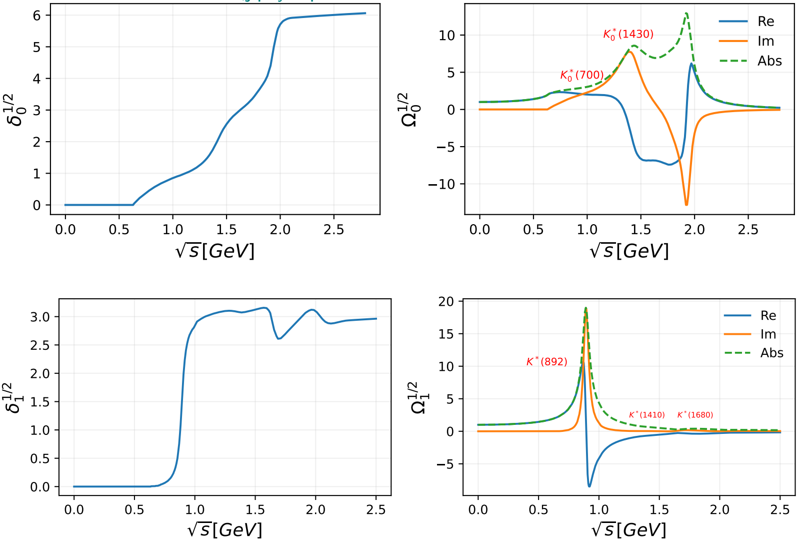
Thus (*Watson's final-state theorem*), K.M.Watson,Phys.Rev.95,228(1954)

$$\text{disc}\,\mathcal{F}(s) = 2i\sin\delta(s)e^{-i\delta(s)}\mathcal{F}(s+i\epsilon) \quad \Rightarrow \quad \mathcal{F}(s+i\epsilon) = P_n(s)\Omega(s+i\epsilon)$$

with an analytical solution $\Omega(s+i\epsilon) = \exp(\frac{s}{\pi}\int_{s_{\text{th}}}^{\infty}\frac{ds'\delta(s')}{s'(s'-s-i\epsilon)})$.

# $(I, J) = (\frac{1}{2}, 0\&1)$ $K\pi$ scattering

F.Niecknig et al., JHEP10(2015)142;
J.R.Pelaeź et al.,j.physrep.2022.03.004



# $(I, J) = (1, 0)\ \pi\eta - K\bar{K}$ scattering

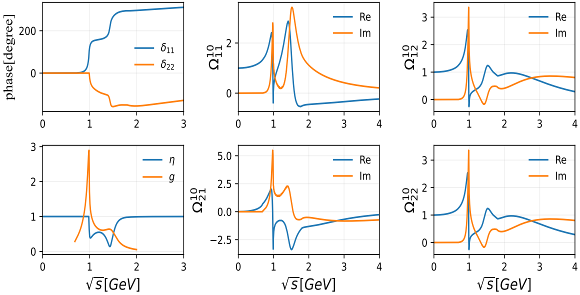
The isovector $\pi\eta - K\bar{K}$ coupling has a significant inelastic effect due to the onset of $a_0(980)$ and $a_0(1450)$. We adopt the following $\delta, \eta, g$ which satisfies the most $(5 \sim 6)$ chiral constraints,

$$\Omega(s) = \frac{1}{\pi}\int_{s_{\text{th}}}^{\infty} ds' \frac{T^*(s')\Sigma(s')\Omega(s')}{s' - s - i\epsilon}$$

B.Moussallam, EPJC14,111–122(2000)
J.F.Donoghue, NPB343(1990)
M.Doring,JHEP10(2013)011

M.Albaladejo et al.,EPJC(2015)75:488
M.Albaladejo et al., EPJC(2017)77:508

$$T(s) = \begin{pmatrix} \frac{\eta e^{2i\delta_{11}} - 1}{2i\sigma_1} & g e^{i\phi_{12}} \\ g e^{i\phi_{12}} & \frac{\eta e^{2i\delta_{22}} - 1}{2i\sigma_2} \end{pmatrix}$$




The form factors $F_S^{\eta\pi, K\bar{K}}(s)$ are evaluated to be the same with that in the Ref.

# "Effective" elastic $K\bar{K}$ scattering

The above Omnès matrix describes the coupling between the production amplitudes $J/\psi\bar{\gamma}\bar{\pi}\to\pi\eta$ and $J/\psi\bar{\gamma}\bar{\pi}\to K\bar{K}$,

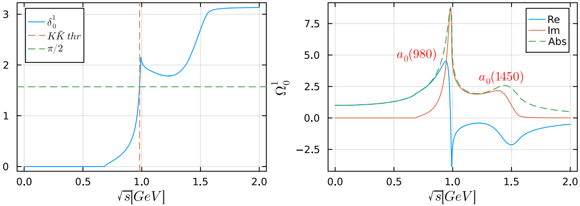
T.Isken et al., EPJC(2017)77:489
E.Kou et al., JHEP12(2023)177

$$\begin{pmatrix}\mathcal{M}^{\pi\eta}\\ \mathcal{M}^{K\bar{K}}\end{pmatrix}=\begin{pmatrix}\Omega_0^1(s)_{\pi\eta\to\pi\eta} & \Omega_0^1(s)_{K\bar{K}\to\pi\eta}\\ \Omega_0^1(s)_{\pi\eta\to K\bar{K}} & \Omega_0^1(s)_{K\bar{K}\to K\bar{K}}\end{pmatrix}\begin{pmatrix}\mathcal{M}^{\chi,\pi\eta}\\ \mathcal{M}^{\chi,K\bar{K}}\end{pmatrix}$$

To simplify the problem, we adopt the idea of "effective phase shift",

$$\begin{aligned}\mathcal{M}^{K\bar{K}}&=\Omega_0^1(s)_{\pi\eta\to K\bar{K}}P_1(s)+\Omega_0^1(s)_{K\bar{K}\to K\bar{K}}P_2(s)\\ &\to(\xi\cdot\Omega_{21}(s)+\Omega_{22}(s))P_{eff}(s)=\Omega_{eff}(s)P_{eff}(s)\end{aligned}$$

When $\xi=1$ ($F_S^{\pi\eta}(0)=0.816$, $F_S^{K\bar{K}}(0)=1$ @NLO therein),




$\delta_0^1$ below $K\bar{K}\Rightarrow a_0(980)$
$\xi\Rightarrow$ high-energy behaviour

By contour deformation, the elastic information can be maximally utilized for angular averages!

# Analytical continuation in angular averages

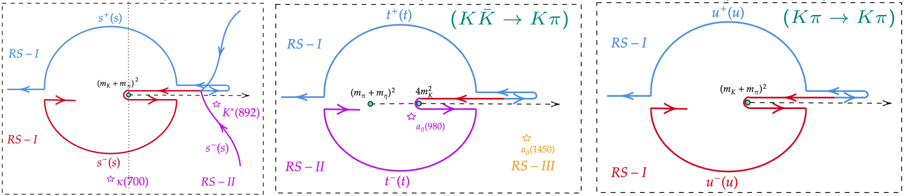
One transforms the cosines $z_x^{\pm} \to x^{\pm}$ in the PWA and following the prescription $M_{i,f}^2 \to M_{i,f}^2 + i\epsilon$:

$$\langle z^n f\rangle_{t_s} = \frac{1}{\kappa_{K\bar{K}}(s)} \int_{s^-(s)}^{s^+(s)} ds' (\frac{2s' - \Sigma_0 + s}{\kappa_{K\bar{K}}(s)})^n f(s')$$

$$\langle z^n f\rangle_{s_t} = \frac{1}{\kappa_{\pi\bar{K}}(t)} \int_{t^-(t)}^{t^+(t)} dt' (\frac{2t't - \Sigma_0 t + t^2 + \Delta_{\eta_x K}\Delta_{\pi\bar{K}}}{t\kappa_{\pi\bar{K}}(t)})^n f(t')$$

$$\langle z^n f\rangle_{t_u} = \frac{(-1)^n}{\kappa_{\pi K}(u)} \int_{u^-(u)}^{u^+(u)} du' (\frac{2u'u - \Sigma_0 u + u^2 - \Delta_{\eta_x\bar{K}} \cdot \Delta_{\pi K}}{u\kappa_{\pi K}(u)})^n f(u')$$

(different from the elastic cases)    (elastic case)




$(K\pi \to K\bar{K})$

$s \in [m_{\pi\eta}^2, 3.5^2\,\text{GeV}^2], t/u \in [m_{K\pi}^2, 3.5^2\,\text{GeV}^2].$

Analytical continuation of $\mathcal{F}_J^I(x) \propto \Omega_J^I(x)$ is required!

# Model-independent methods for analytical continuation

$2^N$ Riemann sheets for $N$-channels,

$$(\mathcal{F}_l^{(n)}(s))_i = \sum_{k=1}^{N} [1 + 2iT_l(s)\hat{\sigma}^{(n)}(s)]^{-1}_{ik} (\mathcal{F}_l(s))_k,$$

L.A.Heuser,arXiv.2403.15539 → talk this afternoon!

1. Resonances are uniquely characterized by their poles and residues
2. The Omneś matrices have only RHCs but t-matrices have both RHCs and LHCs (✗ DR)

Conformal expansion:

J.R.Pelaeź et al.,PRD.93.074025

$$T_J^I(s) = \frac{1}{\sigma_1(s)} \frac{1}{\cot \delta_J^I(s) - i}$$

$$\cot \delta_J^I(s) = \frac{\sqrt{s}}{2q^{2J+1}} F(s) \sum_n B_n \omega(s)^n$$

$$F(s) = \begin{cases} \frac{1}{s - s_{\text{Adler}}}, & \text{scalar PWA} \\ (s - m_r^2), & \text{a narrow resonance} \end{cases}$$

L.Schlessinger,PR167,1411(1968)

Schlessinger(continued) fraction method

$$C_N(s) = F_1(s)/(1 + \frac{a_1(s-s_1)}{1 + \frac{a_2(s-s_2)}{\cdots a_{N-1}(s-s_{N-1})}})$$

P.Masjuan,EPJC73,2594(2013),1306.6308

Pade´series:

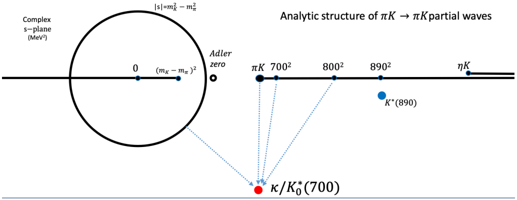
$$P_M^N(s, s_0) = \frac{Q_N(s, s_0)}{R_M(s, s_0)}$$

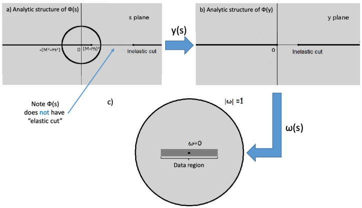
The above methods ⇒ consistent analytical results!

# Analytical structure of $K\pi$ and $\pi\eta - K\bar{K}$ scatterings

J.R.Pelaeź et al.,j.physrep.2022.03.004

The conformal expansion gains its maximal analytical region until:
LHCs (segments, circular cuts etc), inelastic cuts, Alder zeros, spurious poles etc.



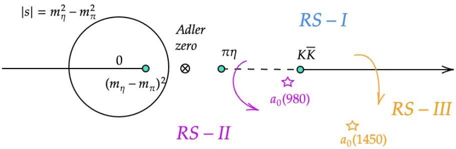




$$\omega(y) = \omega(y(s)) = \frac{\sqrt{y(s)} - \alpha\sqrt{y_0 - y(s)}}{\sqrt{y(s)} + \alpha\sqrt{y_0 - y(s)}},$$

$$y(s) = \left(\frac{s - \Delta_{k\pi/\eta\pi}}{s + \Delta_{k\pi/\eta\pi}}\right)^2, \quad y_0 = y(s)$$

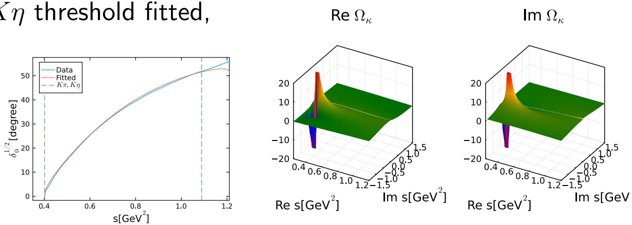
$$\mathcal{F}^{II}(s) = \frac{\mathcal{F}^{I}(s)}{1 + 2it(s)\hat{\sigma}(s)} = \frac{\Omega(s)}{1 + 2it\hat{\sigma}(s)} P_n(s)$$

M.Albaladejo et al.,EPJC(2015)75:488



# Analytical continuation of $K\pi$ scattering

With phase shifts below $K\eta$ threshold fitted,

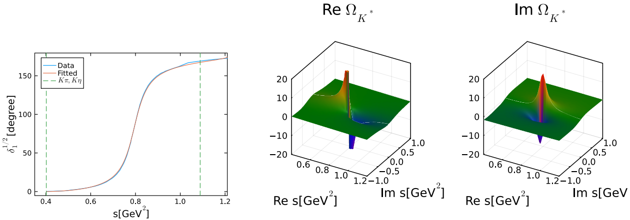



Pole positions (MeV):

$\kappa$ : $(672 - i340)$

$K^*$ : $(891.7 - i27.1)$

The pole of $\kappa$ is far away from $s^-(s; m_{\eta_x})$



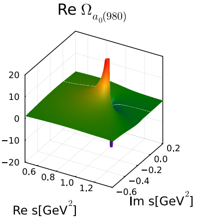

The pole of $K^*(892)$ is encompassed by $s^-(s; m_{\eta_x}) \rightarrow$ discontinuity

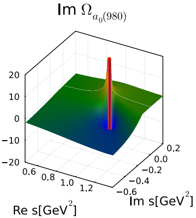
# Analytical continuation of $K\bar{K}$ scattering

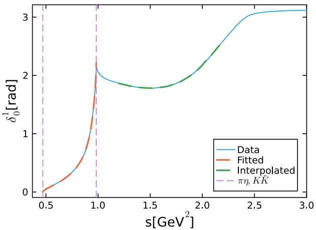
With phase shifts below $K\bar{K}$ fitted, the pole position (MeV) of $a_0(980)$ is :

M.Albaladejo et al.,EPJC(2015)75:488

$$\sqrt{s_{\text{pole}}}(a_0(980)) : (997.1 - i26.1) \qquad \sqrt{s^{II}_{a_0(980)}} = (994 - i25.4) \quad \text{(Ref.)}$$

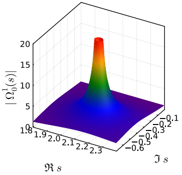



J.R.Peláez,PRL.130.051902

With phase shifts around $1.2 \sim 2.2\text{GeV}^2$(41 points evenly) interpolated, the Schlessinger fraction gives the pole (MeV) of $a_0(1450)$,




$$\sqrt{s_{\text{pole}}}(a_0(1450)) : (1465 - i137)$$

$$\sqrt{s^{III}_{a_0(1450)}} = (1474 \pm 19 - i(133 \pm 7)) \quad \text{(Ref.)}$$

# Khrui-Trieman Framework

# Scattering amplitudes with crossed-channel effects

Due to the LHCs, the single-variable amplitude is,

$$\mathcal{F}(s) = \Omega(s)(P_n(s) + \frac{s^{n+1}}{\pi}\int_{s_{\text{th}}}^{\infty}\frac{ds'}{s'^{n+1}}\frac{\Omega^{-1}(s')T^*(s')\Sigma(s')\hat{\mathcal{F}}(s')}{s'-s-i\epsilon})$$

M.Froissart, Phys.Rev.123 1053(1961).

Concerning the Froissart-Martin bound $\mathcal{M}(s) \preccurlyeq s\log^2(s)$ and $\Omega(\infty) \asymp s^{-\delta_L^I(\infty)/\pi}$,

$$\delta_0^{\frac{1}{2}}(\infty) = 2\pi,\ \delta_1^{\frac{1}{2}}(\infty) = \pi,\ \delta_0^1(\infty) = \pi$$

We parameterize these $\mathcal{F}_J^I$'s as,

$$\mathcal{F}_0^1(s) = \Omega_0^1(s)\{a_0 + a_1\cdot s + a_2\cdot s^2 + \frac{s^3}{\pi}\int_s^{\infty}\frac{ds'}{s'^3}\frac{\hat{\mathcal{F}}_0^1(s')\sin\delta_0^1(s')}{|\Omega_0^1(s')|(s'-s)}\}$$

$$\mathcal{F}_0^{1/2}(t) = \Omega_0^{1/2}(t)\{b_0 + b_1\cdot t + b_2\cdot t^2 + b_3\cdot t^3 + \frac{t^4}{\pi}\int_{t_{th}}^{\infty}\frac{dt'}{t'^4}\frac{\hat{\mathcal{F}}_0^{1/2}(t')\sin\delta_0^{1/2}(t')}{|\Omega_0^{1/2}(t')|(t'-t)}\}$$

$$\mathcal{F}_1^{1/2}(t) = \Omega_1^{1/2}(t)\{c_0 + \frac{t}{\pi}\int_{t_{th}}^{\infty}\frac{dt'}{t'}\frac{\hat{\mathcal{F}}_1^{1/2}(t')\sin\delta_1^{1/2}(t')}{|\Omega_1^{1/2}(t')|(t'-t)}\} \quad [t(s-u)-\Delta]\sim\mathcal{O}(t^2)$$

The #subtractions may be reduced due to the "invariance group" but oversubtraction is always allowed!

# The inhomogeneities from crossed-channel scattering

The general partial-wave expansion is:

$$\mathcal{M}(s,t,u) = \sum_l \mathcal{M}_l(x) P_l(z_x) = \sum_l a_l f_l(x) P_l(z_x)$$

And by construction $f_l(x) = \tilde{\kappa}_x^l(\mathcal{F}(x) + \hat{\mathcal{F}}(x))$, the inhomogeneities read,

$$\hat{\mathcal{F}}_0^1(s) = (-\sqrt{\frac{2}{3}})2\langle \mathcal{F}_0^{1/2} \rangle_{t_s} + (-\sqrt{\frac{2}{3}})[\frac{1}{2}(\Sigma_0 - s)(3s - \Sigma_0) - 2\Delta]\langle \mathcal{F}_1^{1/2} \rangle_{t_s}$$
$$+ (-\sqrt{\frac{2}{3}})s \cdot \kappa_{K\bar{K}} \langle z_s \mathcal{F}_1^{1/2} \rangle_{t_s} + (-\sqrt{\frac{2}{3}})\frac{\kappa_{K\bar{K}}^2}{2}\langle z_s^2 \mathcal{F}_1^{1/2} \rangle_{t_s}$$

$$\hat{\mathcal{F}}_0^{1/2}(t) = \cdots, \qquad \hat{\mathcal{F}}_1^{1/2}(t) = \cdots$$

F.Niecknig,JHEP10(2015)142

The solution set is linearly-independent with $a_i, b_i, c_i$, i.e., ⇒ Generic & Reusable

$$\mathcal{M}(s,t,u;m_{\eta_x}) = \sum_i C_i \mathcal{M}_i(s,t,u;m_{\eta_x})$$

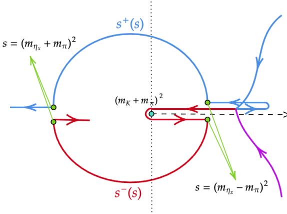
The subtractions are calculated from more fundamental theories ($\chi$PT etc) or fitted from experimental data.

# Discussions and Results

# Pseudo-threshold singularity and its analytical nature

$$\kappa_{K\bar{K}}(s) = \frac{\sqrt{\lambda(s,m_K^2,m_K^2)}\sqrt{(m_{\eta_x}-m_\pi)^2-s+i\epsilon}\sqrt{(m_{\eta_x}+m_\pi)^2-s+i\epsilon}}{s}$$

$$\kappa_{\pi K}(t) = \frac{\sqrt{\lambda(t,m_\pi^2,m_K^2)}\sqrt{(m_{\eta_x}-m_K)^2-t+i\epsilon}\sqrt{(m_{\eta_x}+m_K)^2-t+i\epsilon}}{t}$$

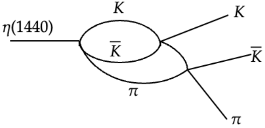



The singular behaviour of $\hat{\mathcal{F}}_J^I(x)$ at pseudo-threshold is $\frac{\tilde{\mathcal{F}}_J^I(x)}{\kappa^{2J+1}(x)} \propto \frac{1}{\sqrt{a_x-x}^{2J+1}}$:

1. manifests both when solving $\mathcal{F}_J^I(x)$ and $\hat{\mathcal{F}}_J^I(x)$
2. S-wave ($J=0$) $\Rightarrow$ integrable numerically
3. above S-wave ($J>0$) $\Rightarrow$ very hard to integrate numerically

The integral $H(x) = \frac{x^n}{\pi}\int \frac{dx'}{x'^n}\frac{\hat{\mathcal{F}}(x')\sin\delta(x')}{|\Omega(x')|(x'-x)}$ J.Gasser,NPB850(2011)96-147

1. is finite on physical sheet, i.e., $H(a_x+i\epsilon)$
2. disc $H(a_x) = H(a_x+i\epsilon) - H(a_x-i\epsilon) = \infty$
3. can be evaluated both analytically and numerically



# Avoiding the pseudo-threshold singularity

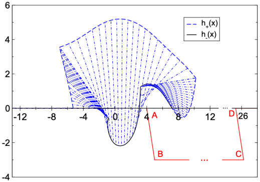
$$H(x+i\epsilon)=\frac{x^n}{\pi}\int\frac{dx'}{x'^n}\frac{\tilde{\mathcal{F}}^I_J(x')}{\kappa^{2J+1}(x')(x'-x-i\epsilon)}\frac{\sin\delta^I_J(x')}{|\Omega^I_J(x')|}$$

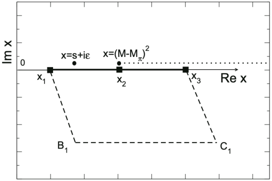
- Analytical approach    G.Colangelo et al.,EPJC(2018)78:947

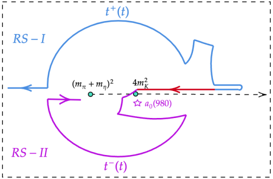
$$\mathcal{M}^H_1(s)=\Omega_1(s)\{\int_{s_1}^{s_3}ds'\frac{\bar{\phi}(s')H_1(s')-h(s')\bar{\phi}(s_2)H_1(s_2)}{(s'-s-i\epsilon)(s_2-s')^{3/2}}+\bar{\phi}(s_2)H_1(s_2)G(s)\}$$

J.Gasser and A.Rusetsky,EPJC(2018)78:906

- Contour deformation without crossing the pole positions ($\delta^I_J$ diverges at pole)



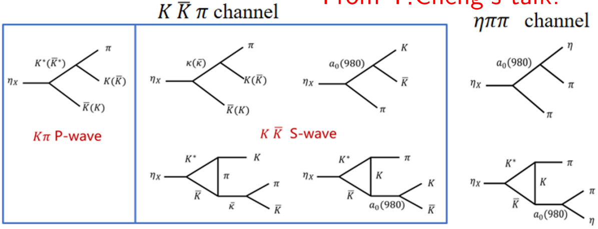

- Contour deformation even crossing the pole positions:

$\frac{\sin\delta^I_J(x')}{|\Omega^I_J(x')|}$ is free of the singularities $\Rightarrow$ { the singularity is avoided; integrate on elastic complex region now! }

# Tree- and one-loop contributions to $J/\psi \to \gamma K_S^0 K_S^0 \pi^0$

From Y.Cheng's talk!

- $\eta_X$ strong decay via :$\boldsymbol{K^*\bar{K}}$, $\boldsymbol{\kappa\,\bar{K}}$, $\boldsymbol{a_0\,\pi}$ dominantly




- Effective Lagrangians

$$\mathcal{L}_{VPP} = ig_{VPP}Tr[(P\partial_\mu P - \partial_\mu PP)V^\mu]$$

$$\mathcal{L}_{SPP} = g_{SPP}\mathrm{Tr}[SPP]$$

- Effective couplings

| $\tilde{G}_{\eta_X K^{*0}\bar{K}^0}$ | $\tilde{G}_{\eta_X \kappa^0\bar{K}^0}$ | $\tilde{G}_{g_{\eta_X} a_0\pi}$ | $\tilde{G}_{K^{*0}K^0\pi^0}$ | $\tilde{G}_{\kappa^0 K^0\pi^0}$ | $\tilde{G}_{a_0 K^0\bar{K}^0}$ |
|---|---|---|---|---|---|
| $g_{\eta_X K^{*0}\bar{K}^0}e^{i\theta_{\eta_X K^{*0}\bar{K}^0}}$ | $g_{\eta_X \kappa^0\bar{K}^0}e^{i\theta_{\eta_X \kappa^0\bar{K}^0}}$ | $g_{\eta_X a_0\pi^0}e^{i\theta_{\eta_X a_0\pi^0}}$ | $g_{K^{*0}K^0\pi^0}e^{i\theta_{K^{*0}K^0\pi^0}}$ | $g_{\kappa^0 K^0\pi^0}e^{i\theta_{\kappa^0 K^0\pi^0}}$ | $g_{a_0 K^0\bar{K}^0}e^{i\theta_{a_0 K^0\bar{K}^0}}$ |

Magnitude Phase

Their magnitudes can be constrained by other processes, while the phases are fitted

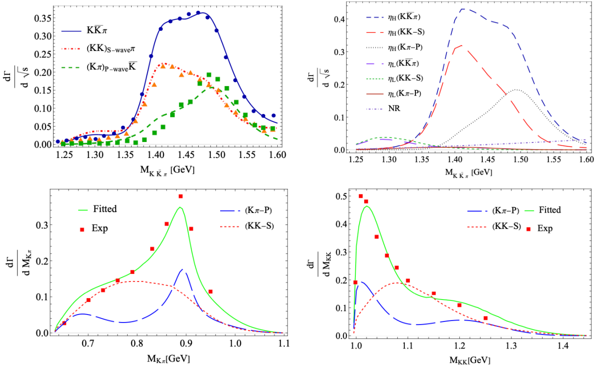
- Form factor for $\eta_X$ couple vertices

$$\exp\left(\frac{-(s-m_{\eta_X}^2)^2}{\lambda^4}\right) \qquad \lambda_{K^*},\ \lambda_\kappa,\ \lambda_{a_0}$$

X.H.Liu,Q.Zhao et al.,PLB753(2016)
X.G.Wu,Q.Zhao et al.,PRD.87.014023
M.C.Du,Q.Zhao et al.,PRD.100.036005
M.C.Du,Q.Zhao et al.,PRD.106.054019
Y.Cheng,Q.Zhao et al.,arXiv:2302.01210

These diagrams can also be evaluated in dispersive manner (see backup slides) and the present knowledge is consistent with previous works **to some extent**!

# Preliminary results up to TS mechanism




☺ 3-body spectrum
☺ 2-body spectra (PWA)
☹ Dalitz plots

1. The $\eta(1440)$ plays the dominant role while the $\eta(1295)$ plays the negligible role (via interference);
2. there are visible peaks of $K^*(892)$ and $a_0(980)$ in the 2-body spectra;
3. the triangle-singularity mechanism shall be responsible for the $\eta(1405/1475)$ puzzles.

# Summary

- The nature of iso-scalar pseudo-scalar states and their dynamics involved are still beyond our knowledge
- The 2-body $K\bar{K}\pi$ FSIs have been established dispersively (almost model-independently) and the 3-body ones are on the way $\Rightarrow \eta\pi\pi, 3\pi$ etc
- The above treatment proceeds similarly for generic 3-body scatterings in a modern & sophisticated perspective $\Rightarrow f_1(1285),\ f_1(1420),\ a_1(1260)$ etc
- The comprehensive understanding of those states relies on the inclusions of more robust experimental data (upcoming) and more fundamental theories such as $\chi$PT (setting up)

# Thank you!

# Analytical continuation of $\sin\delta_J^I(s)/|\Omega_J^I(s)|$ (1)

For 2-body elastic scattering,

$$f_l(s) = \frac{e^{i2\delta_l(s)} - 1}{2i\sigma(s)} = \frac{1}{\sigma(s)} \cdot \frac{1}{\cot\delta_l(s) - i}$$

with $\cot\delta_l(s)$ real and satisfying Schwartz reflection theorem and can be expanded by conformal polynomials on a certain analytical region.
The S-matrix is then,

$$\hat{S}(s) = \begin{cases} 1 + 2i\sigma f_l(s) = \frac{\cot\delta_l(s)+i}{\cot\delta_l(s)-i}, & \Im s \geq 0 \\ \left[\frac{\cot\delta_l(s^*)+i}{\cot\delta_l(s^*)-i}\right]^* = \frac{\cot\delta_l(s)-i}{\cot\delta_l(s)+i}, & \Im s < 0. \end{cases}$$

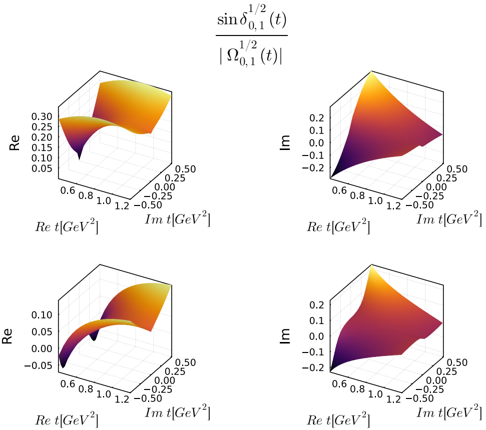
By utilizing $\frac{\sin\delta_l(s)}{|\Omega_l(s)|} = \frac{e^{i\delta_l(s)}\sin\delta_l(s)}{\Omega_l(s)} = \frac{1}{\Omega_l(s)} \cdot \frac{1}{\cot\delta_l(s)-i}$ and $\Omega_l^{(II)}(s) = \frac{\Omega^{(I)}(s)}{\hat{S}(s)}$, one derives,

$$\frac{\sin\delta_l(s)}{|\Omega_l(s)|} = \begin{cases} \frac{1}{\Omega_l^{(I)}(s)} \frac{1}{\cot\delta_l(s)-i}, & \Im s \geq 0 \\ \frac{1}{\Omega_l^{(I)}(s)} \frac{1}{\cot\delta_l(s)+i}, & \Im s < 0 \end{cases}.$$

The convention of $\cot\delta_l(s)$ may differentiate from the literature by an extra minus sign on the lower half plane but the conclusion shall not change!

# Analytical continuation of $\sin\delta_J^I(s)/|\Omega_J^I(s)|$ (2)

The complex function $\frac{\sin\delta_{0,1}^{1/2}(s)}{|\Omega_{0,1}^{1/2}(s)|}$ of $K\pi$ scatterings are plotted below,




The dispersive integral on any deformed integral-path on the lower half plane (even crossing the pole position) has been checked to be consistent with that integrated from the real axis!

# $\eta_x \to K^*\bar{K} \to$ 3-body

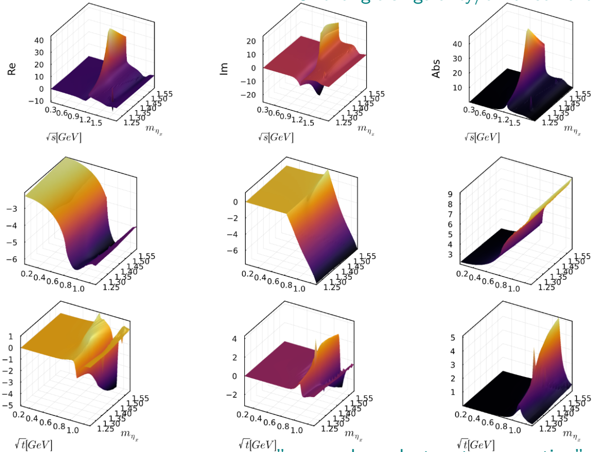
$$\Omega_J^I(x)\frac{x^n}{\pi}\int_{x_{th}}^{\Lambda^2}\frac{dx'}{x'^n}\frac{\hat{\mathcal{F}}_J^I(x')\sin\delta_J^I(x')}{|\Omega_J^I(x')|(x'-x)},\quad \hat{\mathcal{F}}_J^I(x)\propto\langle P_m(y)\mathcal{F}_{J'}^{I'}(y)\rangle_{y_x}$$

"LO: triangle singularity/enhancement"

$\eta_x \to K^*\bar{K} \to \pi a_0(K\bar{K})$

$\eta_x \to K^*\bar{K} \to K\bar{\kappa}(\bar{K}\pi)$

$\eta_x \to K^*\bar{K} \to K\bar{K}^*(\bar{K}\pi)$




"energy-dependent vertex correction"

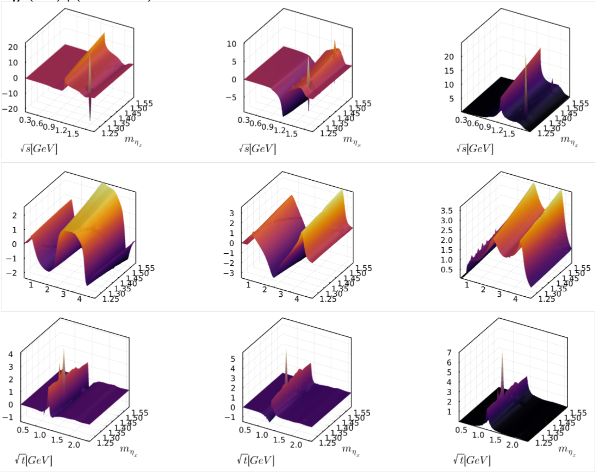
# $\eta_x \to \kappa\bar{K} \to$ 3-body

$$\Omega_J^I(x)\frac{x^n}{\pi}\int_{x_{th}}^{\Lambda^2}\frac{dx'}{x'^n}\frac{\hat{\mathcal{F}}_J^I(x')\sin\delta_J^I(x')}{|\Omega_J^I(x')|(x'-x)}, \quad \hat{\mathcal{F}}_J^I(x)\propto\langle P_m(y)\mathcal{F}_{J'}^{I'}(y)\rangle_{y_x}$$

$\eta_x \to \kappa\bar{K} \to \pi a_0(K\bar{K})$

$\eta_x \to \kappa\bar{K} \to K\bar{\kappa}(\bar{K}\pi)$

$\eta_x \to \kappa\bar{K} \to K\bar{K}^*(\bar{K}\pi)$

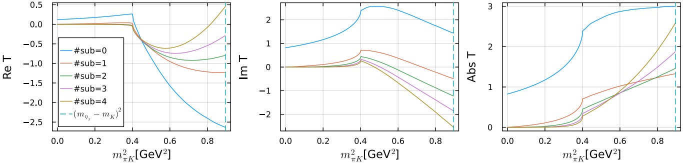


# $\eta_x \to a_0\pi \to K\bar{\kappa}, K\bar{K}^*$

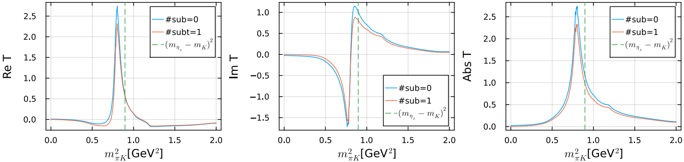
$$\Omega_{0,1}^{1/2}(t)\frac{t^n}{\pi}\int_{t_{th}}^{\Lambda^2}\frac{dt'}{t'^n}\frac{\hat{\mathcal{F}}_{0,1}^{1/2}(t')\sin\delta_{0,1}^{1/2}(t')}{|\Omega_{0,1}^{1/2}(t')|(t'-t)}, \quad \hat{\mathcal{F}}_{0,1}^{1/2}(t) \propto \langle P_m(s)\Omega_0^1(s)\rangle_{s_t}$$

When $\eta_x = 1.44$GeV (14-th bin),

- $\eta_x \to a_0\pi \to K\bar{\kappa}$ (up to 4 subtractions)




- $\eta_x \to a_0\pi \to K\bar{K}^*$ (up to 1 subtractions)




Such diagrams shall be at NLO up to one-loop.