Spin density matrix elements of the $\Delta^{++}(1232)$

Vanamali Shastry

Indiana University

with V. Mathieu, G. Montaña, A. Szczepaniak, and others

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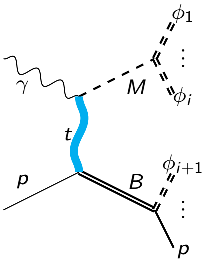
Understanding the photoproduction of mesonic and baryonic resonances

- Single π-photoproduction is well understood (Gloria's talk, Tue)
- 2π-photoproduction can be a ρ production or $\pi\Delta$, or Deck process.
- $\pi\Delta$ production
  - π exchange process is to be understood (gauge invariance).
  - What is the physics behind the photoproduction of $\pi\Delta$?
  - Understand the lower vertex.
- Available observables: $\frac{d\sigma}{dt}$, Σ, $\rho^{0,1,2}$.



- SDMEs are coefficients of angular distributions (of final proton) and hence can be extracted from the intensity distribution [1].
- Theoretically, SDMEs give insights into the production process.

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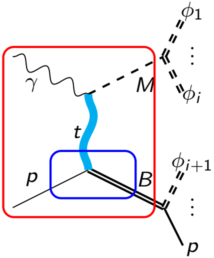

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- Theoretically, SDMEs give insights into the production process.

The $\vec{\gamma} p \to \pi\Delta \to \pi(p\pi)$ amplitude is given by,

$$A_{\lambda_\gamma,\lambda_1,\lambda_2}(\Omega) = \sum_{\lambda_\Delta} T_{\lambda_\gamma,\lambda_1,\lambda_\Delta}(s,t) D^{3/2*}_{\lambda_\Delta,\lambda_2}(\Omega) \tag{1}$$

The intensity is given by,

$$I(\Omega,\Phi,P_\gamma) = \frac{\kappa}{2} \sum_{\lambda_\gamma^{(\prime)},\lambda_1,\lambda_2} A_{\lambda_\gamma,\lambda_1,\lambda_2}(\Omega)\ \hat{\rho}_{\lambda_\gamma,\lambda'_\gamma}\ A^*_{\lambda'_\gamma,\lambda_1\lambda_2}(\Omega) \tag{2}$$

where κ is the phase space factor. $\Omega = (\theta,\phi)$ are the Δ-decay angles.

$$\begin{aligned}
I(\Omega,\Phi) = 2N\Bigg\{ & \rho^0_{33}\sin^2\theta + \rho^0_{11}\left(\frac{1}{3}+\cos^2\theta\right) - \frac{2}{\sqrt{3}}\,\mathrm{Re}\,\rho^0_{31}\sin 2\theta\cos\phi - \frac{2}{\sqrt{3}}\,\mathrm{Re}\,\rho^0_{3-1}\sin^2\theta\cos 2\phi \\
& - P_\gamma\cos 2\Phi\left[\rho^1_{33}\sin^2\theta + \rho^1_{11}\left(\frac{1}{3}+\cos^2\theta\right) - \frac{2}{\sqrt{3}}\,\mathrm{Re}\,\rho^1_{31}\sin 2\theta\cos\phi - \frac{2}{\sqrt{3}}\,\mathrm{Re}\,\rho^1_{3-1}\sin^2\theta\cos 2\phi\right] \\
& - P_\gamma\sin 2\Phi\left[\frac{2}{\sqrt{3}}\,\mathrm{Im}\,\rho^2_{31}\sin 2\theta\sin\phi + \frac{2}{\sqrt{3}}\,\mathrm{Im}\,\rho^2_{3-1}\sin^2\theta\sin 2\phi\right]\Bigg\}.
\end{aligned} \tag{3}$$

where the SDMEs are defined as,

$$\rho^0_{\lambda_\Delta \lambda'_\Delta} = \frac{1}{2N} \sum_{\lambda_\gamma \lambda_1} T_{\lambda_\gamma \lambda_1 \lambda_\Delta} T^*_{\lambda_\gamma \lambda_1 \lambda'_\Delta} \tag{4}$$

$$\rho^1_{\lambda_\Delta \lambda'_\Delta} = \frac{1}{2N} \sum_{\lambda_\gamma \lambda_1} T_{\lambda_\gamma \lambda_1 \lambda_\Delta} T^*_{-\lambda_\gamma \lambda_1 \lambda'_\Delta} \tag{5}$$

$$\rho^2_{\lambda_\Delta \lambda'_\Delta} = \frac{i}{2N} \sum_{\lambda_\gamma \lambda_1} \lambda_\gamma T_{\lambda_\gamma \lambda_1 \lambda_\Delta} T^*_{-\lambda_\gamma \lambda_1 \lambda'_\Delta} \tag{6}$$

$$\mathrm{Tr}(\rho^0) = 1 \qquad \mathrm{Tr}(\rho^1) = \Sigma \qquad \mathrm{Tr}((\rho^{0,1,2})^2) \leq (\mathrm{Tr}(\rho^{0,1,2}))^2 \tag{7}$$

$$\rho^0_{\lambda_\Delta \lambda_\Delta} > 0 \qquad 0 < \rho^0_{11} < 0.5 \tag{8}$$

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- GlueX operates at $E_\gamma \sim 9$ GeV

- Large-s $\rightarrow$ exponential fall-off
- Residual (polynomial) behavior

$$\frac{d\sigma}{dt} = \beta^R(t)s^{2\alpha_{\text{eff}}-2} \tag{9}$$

Hallmark of Regge exchange

- Two distinct trajectories (small-t and large-t).
- Similar cross-section as single pion photoproduction at large-t

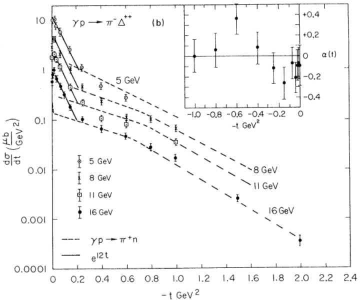


Figure: $\pi\Delta$ photoproduction cross section as reported in Ref. [2].

- Simple model; exchange of π, ρ, b_1, a_2
- Upper and lower vertices factorize
- The phases of the amplitudes are fixed by the Regge theory.
- Poor man's absorption (PMA) model for π-exchange [3].

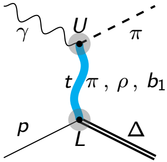


Figure: The t-channel photoproduction process of $\pi\Delta$. U and L are the upper and lower vertices.

General form of the helicity amplitude is [4]

$$T_{\lambda_\gamma,\lambda_1,\lambda_\Delta}(s,t) = \sum_{\times}\left[\xi_{\lambda_\gamma\lambda_1\lambda_\Delta}\, T^{\times}_{\lambda_\gamma,\lambda_1,\lambda_\Delta}(s,t)\right] \quad ; \qquad \times \in \{\pi,\rho,b_1,a_2\} \tag{10}$$

$$T^{\times}_{\lambda_\gamma,\lambda_1,\lambda_\Delta}(s,t) = \sqrt{-t}^{\;|\lambda_\gamma|}\sqrt{-t}^{\;|\lambda_1-\lambda_\Delta|}\;\hat{\beta}^{\times,U}_{\lambda_\gamma}(t)\;\hat{\beta}^{\times,L}_{\lambda_1,\lambda_\Delta}(t)\;\mathcal{P}^{\times}_{R}(s,t)\;\mathcal{S}_{\times}(t) \tag{11}$$

- $\xi_{\lambda_\gamma\lambda_1\lambda_\Delta}$ is the half-angle factor.
- Like exchanges take the same form of the vertex (exchange degeneracy):

$$\hat{\beta}^{\pi,L}_{\lambda_1,\lambda_\Delta}(t) = \hat{\beta}^{b_1,L}_{\lambda_1,\lambda_\Delta}(t); \qquad \hat{\beta}^{\rho,L}_{\lambda_1,\lambda_\Delta}(t) = \hat{\beta}^{a_2,L}_{\lambda_1,\lambda_\Delta}(t)$$

- Overall (exponential + polynomial) suppression factors for each exchange (also removes wrong signature zeroes)

- Factorization leads to separation of parity relations

$$\beta^{\pi,U}_{-\lambda_\gamma} = -\beta^{\pi,U}_{\lambda_\gamma}, \quad \beta^{b_1,U}_{-\lambda_\gamma} = -\beta^{b_1,U}_{\lambda_\gamma}, \quad \beta^{\rho,U}_{-\lambda_\gamma} = \beta^{\rho,U}_{\lambda_\gamma}, \quad \beta^{a_2,U}_{-\lambda_\gamma} = \beta^{a_2,U}_{\lambda_\gamma} \tag{12}$$

$$\beta^{\pi,L}_{-\lambda_1,-\lambda_\Delta} = (-1)^{\lambda_1-\lambda_\Delta}\beta^{\pi,L}_{\lambda_1,\lambda_\Delta}, \quad \beta^{\rho,L}_{-\lambda_1,-\lambda_\Delta} = -(-1)^{\lambda_1-\lambda_\Delta}\beta^{\rho,L}_{\lambda_1,\lambda_\Delta} \tag{13}$$

JPAC model [4]

- Residues are polynomials in t
- Residues constructed phenomenologically using covariant Lagrangians (possible sign ambiguity)
- Upper vertex coupling constants fixed using decay widths
- Lower vertex is the same for all exchanges of a given naturality because of exchange degeneracy
- Lower vertex: $\pi p\Delta$ coupling constant is fixed from the $\Delta \to p\pi$ decay width, $\rho p\Delta$ coupling constants fitted to the scattering data

Irving & Worden Model (IWM) [5]

- Residues are constants
- All vertices fixed phenomenologically

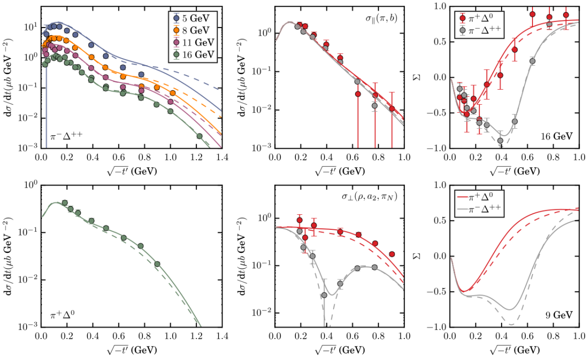
Absorption

- Additional corrections to Regge exchange
- Experimental evidences in the form of
  - Non-zero polarization of $\pi N \to \pi N$ scattering
  - Forward scattering cross section of π-photoproduction (Gloria's talk on day 1)), $np \to pn$, etc
  - Peripheriality of π-exchange reactions
  - SDMEs and BSA of $\pi\Delta$ photoproduction (more later)
- "Effective" way of taking care of multiple exchanges (FSI, etc).

$$T_{\lambda_\gamma,\lambda_1,\lambda_\Delta}(s,t) = \sum_{\times} \left[\xi_{\lambda_\gamma\lambda_1\lambda_\Delta} T^{\times}_{\lambda_\gamma,\lambda_1,\lambda_\Delta}(s,t) \right] \; ; \qquad \times \in \{\pi, \rho, b_1, a_2\} \tag{14}$$

$$T^{\times}_{\lambda_\gamma,\lambda_1,\lambda_\Delta}(s,t) = \sqrt{-t}^{\,|\lambda_\gamma|} \sqrt{-t}^{\,|\lambda_1-\lambda_\Delta|} \; \hat{\beta}^{\times,U}_{\lambda_\gamma}(t) \; \hat{\beta}^{\times,L}_{\lambda_1,\lambda_\Delta}(t) \; \mathcal{P}^{\times}_{R}(s,t) \; \mathcal{S}_{\times}(t) \tag{15}$$

- The $\sqrt{-t}$ factors arise from angular mometum and the model
- The model factors for unnatural exchange are evaluated at $t = m_\pi^2 \to$ PMA.



Cross sections for $\pi\Delta$ photoproduction from JPAC model compared to data (from [4]).

Reflectivity basis

- Reflectivity operation involves 180° rotation about the "y-axis" + parity inversion ⇒ inversion of the "y-axis" [6, 7].
- The amplitude in the reflectivity basis can be defined as (valid for $\gamma p \to \pi\Delta$):

$$T^{(\epsilon)}_{\lambda_1,\lambda_\Delta}(s,t) = \frac{1}{2}\left(T_{1,\lambda_1,\lambda_\Delta}(s,t) + \epsilon T_{-1,\lambda_1,\lambda_\Delta}(s,t)\right) \tag{16}$$

- The SDMEs take the form,

$$\rho^0_{\lambda_\Delta\lambda'_\Delta} = \frac{1}{N}\sum_{\lambda_1}\left[T^{(+)}_{\lambda_1,\lambda_\Delta} T^{(+)^*}_{\lambda_1,\lambda'_\Delta} + T^{(-)}_{\lambda_1,\lambda_\Delta} T^{(-)*}_{\lambda_1,\lambda'_\Delta}\right] \tag{17}$$

$$\rho^1_{\lambda_\Delta\lambda'_\Delta} = \frac{1}{N}\sum_{\lambda_1}\left[T^{(+)}_{\lambda_1,\lambda_\Delta} T^{(+)^*}_{\lambda_1,\lambda'_\Delta} - T^{(-)}_{\lambda_1,\lambda_\Delta} T^{(-)*}_{\lambda_1,\lambda'_\Delta}\right] \tag{18}$$

- The $\epsilon = (-)+$ amplitudes are dominated by (un)natural parity meson exchange.

Natural and Unnatural amplitudes

$$N_{-1} = T^{(+)}_{\frac{1}{2}\frac{3}{2}} \qquad N_0 = T^{(+)}_{\frac{1}{2}\frac{1}{2}} \qquad N_1 = T^{(+)}_{\frac{1}{2}-\frac{1}{2}} \qquad N_2 = T^{(+)}_{\frac{1}{2}-\frac{3}{2}} \tag{19}$$

$$U_{-1} = T^{(-)}_{\frac{1}{2}\frac{3}{2}} \qquad U_0 = T^{(-)}_{\frac{1}{2}\frac{1}{2}} \qquad U_1 = T^{(-)}_{\frac{1}{2}-\frac{1}{2}} \qquad U_2 = T^{(-)}_{\frac{1}{2}-\frac{3}{2}} \tag{20}$$

$\lambda_1 = -\frac{1}{2}$ amplitudes are related via parity.
Eg:

$$N_0 = (\beta_1^\rho S_\rho \mathcal{P}_\rho - \beta_1^{a_2} S_{a_2} \mathcal{P}_{a_2})\sqrt{-t}\beta^\rho_{\frac{1}{2}\frac{1}{2}} \tag{21}$$

$$N_1 = (\beta_1^\rho S_\rho \mathcal{P}_\rho - \beta_1^{a_2} S_{a_2} \mathcal{P}_{a_2})(-t)\beta^\rho_{\frac{1}{2}-\frac{1}{2}} + \frac{1}{2}(-\beta_1^\pi S_\pi \mathcal{P}_\pi + \beta_1^{b_1}\sqrt{-t}\, S_{b_1}\mathcal{P}_{b_1})(-m_\pi^2)\left(1 - \frac{t}{m_\pi^2}\right)\beta^\pi_{\frac{1}{2}-\frac{1}{2}} \tag{22}$$

$$U_0 = (-\beta_1^\pi S_\pi \mathcal{P}_\pi + \beta_1^{b_1}\sqrt{-t}\, S_{b_1}\mathcal{P}_{b_1})\sqrt{-t}\beta^\pi_{\frac{1}{2}\frac{1}{2}} \tag{23}$$

$$U_1 = \frac{1}{2}(-\beta_1^\pi S_\pi \mathcal{P}_\pi + \beta_1^{b_1}\sqrt{-t}\, S_{b_1}\mathcal{P}_{b_1})(-m_\pi^2)\left(1 + \frac{t}{m_\pi^2}\right)\beta^\pi_{\frac{1}{2}-\frac{1}{2}} \tag{24}$$

- Absorption correction to natural exchanges – No pole contributions
  - Negative reflectivity amplitudes are purely π and b_1 exchanges
  - Positive reflectivity amplitudes are ρ and a_2 exchanges, and get absorption corrections from π and b_1 exchanges.

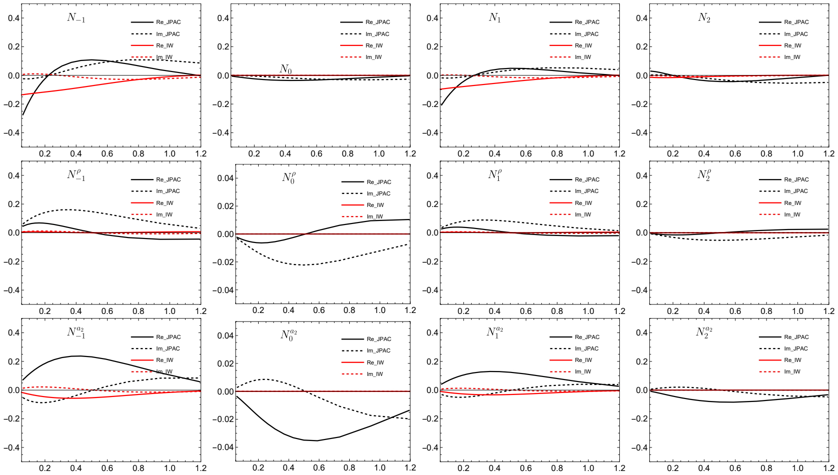


Figure: The natural exchange amplitudes and their ρ and a_2 components from the JPAC model compared with IWM.

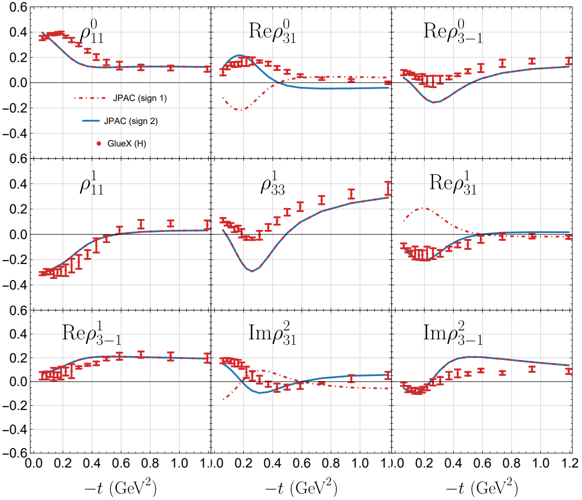


Figure: SDMEs in the helicity frame compared to the GlueX data (F. Afzal (GlueX Collaboration), private communication. Publication in preparation).

Sign 1: $\beta^{\times,L}_{\frac{1}{2},\frac{3}{2}}$ and $\beta^{\times,L}_{\frac{1}{2},-\frac{1}{2}}$ as given in [4] Sign 2: Signs of $\beta^{\times,L}_{\frac{1}{2},\frac{3}{2}}$ and $\beta^{\times,L}_{\frac{1}{2},-\frac{1}{2}}$ flipped

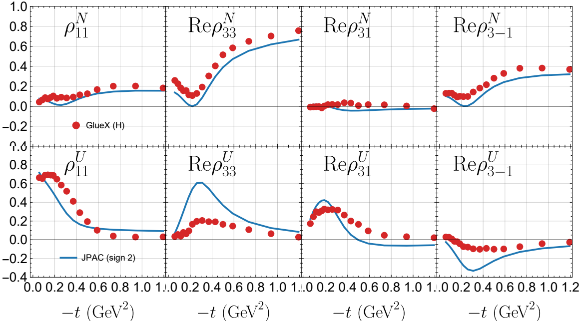


Figure: Natural and unnatural cominations of SDMEs in the helicity frame.

The non-zero vaule of ρ^N_{33} at small-t indicates the presence of absorption corrections.

- A simplistic model that assumes Regge exchange and factorization
- Explains the general features the SDMEs of $\pi\Delta$ photoproduction; needs fine tuning to mathc the data.
- Diagonal elements of $\rho^{0,1}$ can be interpreted as sums of production probabilities
- Δ is produced dominantly in the helicity $\pm 1/2$ ($\pm 3/2$) configurration at small-t (large-t)
- Relative phases of helicity amplitudes can be fixed from SDMEs
- All amplitudes except the natural spin-nonflip amplitude experience absorption corrections
- Absorption is evident in the SDMEs and the BSA
- π-exchange dominates small-t region; a_2 exchange dominates the large-t region.

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Thank you!

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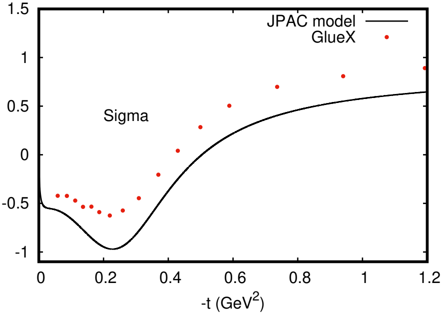
$\hat{\beta}^{e,if}_{\mu_i\mu_f}$	Expression
$\hat{\beta}^{\pi,\gamma\pi}_{+1}(t)$	$\sqrt{2}e$
$\hat{\beta}^{\rho,\gamma\pi}_{+1}(t)$	$\frac{g_{\rho\pi\gamma}}{2m_\rho}$
$\hat{\beta}^{b_1,\gamma\pi}_{+1}(t)$	$\frac{g_{b_1\pi\gamma}}{2m_{b_1}}$
$\hat{\beta}^{a_2,\gamma\pi}_{+1}(t)$	$\frac{g_{a_2\pi\gamma}}{2m^2_{a_2}}$
$\hat{\beta}^{\pi,N\Delta}_{+\frac{1}{2}+\frac{3}{2}}(t)$	$\frac{g_{\pi N\Delta}(m_N+m_\Delta)}{\sqrt{2}m_\Delta}$
$\hat{\beta}^{\pi,N\Delta}_{-\frac{1}{2}+\frac{1}{2}}(t)$	$\frac{g_{\pi N\Delta}(-m_N^2+m_N m_\Delta+2m_\Delta^2+t)}{\sqrt{6}m_\Delta^2}$
$\hat{\beta}^{\pi,N\Delta}_{+\frac{1}{2}+\frac{1}{2}}(t)$	$\frac{-g_{\pi N\Delta}(-m_N^3-m_N^2 m_\Delta+m_\Delta^3+2m_\Delta t+m_N(m_\Delta^2+t))}{\sqrt{6}m_\Delta^2}$
$\hat{\beta}^{\pi,N\Delta}_{-\frac{1}{2}+\frac{3}{2}}(t)$	$\frac{-g_{\pi N\Delta}}{\sqrt{2}m_\Delta}$
$\hat{\beta}^{\rho,N\Delta}_{+\frac{1}{2}+\frac{3}{2}}(t)$	$\frac{-(2m_\Delta g^{(1)}_{\rho N\Delta}+g^{(2)}_{\rho N\Delta}(m_N-m_\Delta))}{2m_\Delta^2}$
$\hat{\beta}^{\rho,N\Delta}_{-\frac{1}{2}+\frac{1}{2}}(t)$	$\frac{-(2m_N m_\Delta g^{(1)}_{\rho N\Delta}+g^{(2)}_{\rho N\Delta}(-m_N m_\Delta+m_\Delta^2+2t)+2t g^{(3)}_{\rho N\Delta})}{2\sqrt{3}m_\Delta^3}$
$\hat{\beta}^{\rho,N\Delta}_{+\frac{1}{2}+\frac{1}{2}}(t)$	$\frac{-(2m_\Delta g^{(1)}_{\rho N\Delta}+g^{(2)}_{\rho N\Delta}(2m_N-3m_\Delta)+2g^{(3)}_{\rho N\Delta}(m_N-m_\Delta))}{2\sqrt{3}m_\Delta^3}(-t)$
$\hat{\beta}^{\rho,N\Delta}_{-\frac{1}{2}+\frac{3}{2}}(t)$	$\frac{g^{(2)}_{\rho N\Delta}}{2m_\Delta^2}$

Residues from the JPAC model.



BSA from the JPAC model compared to the GlueX data (F. Afzal (GlueX Collaboration), private communication. Publication in preparation).

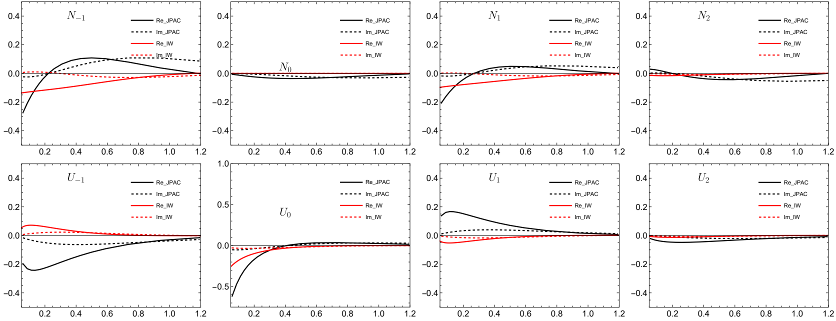


Figure: Natural and Unnatural amplitudes from the JPAC model compared with IWM.

$$\rho^0_{\frac{1}{2}\frac{1}{2}} + \rho^1_{\frac{1}{2}\frac{1}{2}} = \frac{2}{N}\left(|N_0|^2 + |N_1|^2\right) \qquad \mathrm{Re}\left(\rho^0_{\frac{3}{2}\frac{1}{2}} + \rho^1_{\frac{3}{2}\frac{1}{2}}\right) = \frac{2}{N}\,\mathrm{Re}\left(N_{-1}N_0^* - N_1N_2^*\right) \tag{25}$$

$$\rho^0_{\frac{1}{2}\frac{1}{2}} - \rho^1_{\frac{1}{2}\frac{1}{2}} = \frac{2}{N}\left(|U_0|^2 + |U_1|^2\right) \qquad \mathrm{Re}\left(\rho^0_{\frac{3}{2}\frac{1}{2}} - \rho^1_{\frac{3}{2}\frac{1}{2}}\right) = \frac{2}{N}\,\mathrm{Re}\left(U_{-1}U_0^* - U_1U_2^*\right) \tag{26}$$

$$\rho^0_{\frac{3}{2}\frac{3}{2}} + \rho^1_{\frac{3}{2}\frac{3}{2}} = \frac{2}{N}\left(|N_{-1}|^2 + |N_2|^2\right) \qquad \mathrm{Re}\left(\rho^0_{\frac{3}{2}-\frac{1}{2}} + \rho^1_{\frac{3}{2}-\frac{1}{2}}\right) = \frac{2}{N}\,\mathrm{Re}\left(N_0N_2^* + N_1N_{-1}^*\right) \tag{27}$$

$$\rho^0_{\frac{3}{2}\frac{3}{2}} - \rho^1_{\frac{3}{2}\frac{3}{2}} = \frac{2}{N}\left(|U_{-1}|^2 + |U_2|^2\right) \qquad \mathrm{Re}\left(\rho^0_{\frac{3}{2}-\frac{1}{2}} - \rho^1_{\frac{3}{2}-\frac{1}{2}}\right) = \frac{2}{N}\,\mathrm{Re}\left(U_0U_2^* + U_1U_{-1}^*\right) \tag{28}$$

$$\mathrm{Im}\,\rho^2_{\frac{3}{2}\frac{1}{2}} = \frac{1}{N}\,\mathrm{Re}\left(N_{-1}U_0^* + N_2U_1^* - N_1U_2^* - N_0U_{-1}^*\right) \tag{29}$$

$$\mathrm{Im}\,\rho^2_{\frac{3}{2}-\frac{1}{2}} = \frac{1}{N}\,\mathrm{Re}\left(N_{-1}U_1^* - N_2U_0^* - U_{-1}N_1^* + U_2N_0^*\right) \tag{30}$$

$$N = 2\left(|N_{-1}|^2 + |N_0|^2 + |N_1|^2 + |N_2|^2 + |U_{-1}|^2 + |U_0|^2 + |U_1|^2 + |U_2|^2\right) \tag{31}$$

$$\Sigma = 2\sum_{\sigma=-1,0,1,2}\left(|N_\sigma|^2 - |U_\sigma|^2\right) \tag{32}$$