New developments for complete experiments

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Introduction: Importance of spin-reactions

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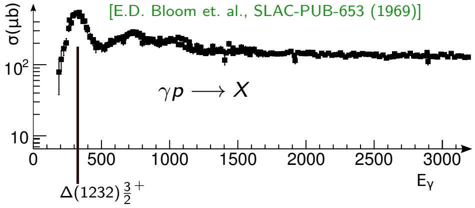

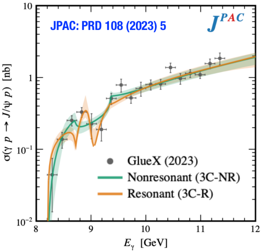

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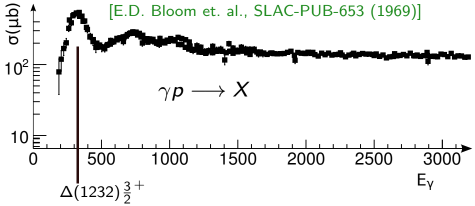

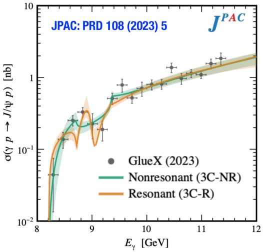

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$*$) 'T-matrix' $\mathcal{T}_{fi}$ parameterized by $N_{\mathcal{A}}$ spin-amplitudes $\{b_i, i = 1, \ldots, N_{\mathcal{A}}\}$

$*$) The usual reactions under study are:

- Pion-Nucleon (πN-) scattering: $\pi N \longrightarrow \pi N$ (2 spin-amplitudes)
- Pion photoproduction: $\gamma N \longrightarrow \pi N$ (4 spin-amplitudes)
- Pion electroproduction: $eN \longrightarrow e' \pi N$ (6 spin-amplitudes)
- 2-Pion photoproduction: $\gamma N \longrightarrow \pi \pi N$ (8 spin-amplitudes)
- ...

Spin amplitudes

Generic case:

Photoproduction:

$*$) General meson-production reaction:

$$\mathcal{P} N \rightarrow \{\varphi_i\}\, B, \text{ with:}$$

- $\mathcal{P}$: 'probe'-particle $(\pi, \gamma, \gamma^{*}, \ldots)$,
- N: target-nucleon,
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$$\mathcal{T}_{fi} = \chi_B^\dagger \left[\sum_{k=1}^{N_\mathcal{A}} \boldsymbol{\kappa}_k b_k \left(\Omega_2^{n_f}\right) \right] \chi_N ,$$

- $\kappa_i \left(\{p_j\}, \{\sigma_i\} \right)$: spin-kinem. operators,
- $b_i \left(\Omega_2^{n_f}\right)$: spin ('transversity') ampl.'s,
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$$N_{\mathcal{A}} = \boldsymbol{n}_{\mathcal{P}} \boldsymbol{n}_N \boldsymbol{n}_{\varphi_1} \ldots \boldsymbol{n}_{\varphi_{\tilde{n}}} \boldsymbol{n}_B,$$

with additional factor of (1/2) in case of a $2 \rightarrow 2$ reaction (parity!)

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∗) Produce one pseudoscalar meson φ:

$$\gamma N \rightarrow \varphi B .$$

∗) Then: $N_{\mathcal{A}} = \frac{1}{2} \left(\boldsymbol{n}_{\gamma} \boldsymbol{n}_N \boldsymbol{n}_{\varphi} \boldsymbol{n}_B \right) = \frac{1}{2} \left(2 \times 2 \times 1 \times 2 \right) = \underline{\underline{4}}$.

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$*$) $\{\boldsymbol{\kappa}_1, \ldots, \boldsymbol{\kappa}_4\}$ are complicated, e.g.:

$$\boldsymbol{\kappa}_1 = \frac{(\hat{q} \cdot \hat{\epsilon})}{\sqrt{2} \sin^2 \theta} \left[e^{-i \frac{\theta}{2}} \left(\hat{k} \cdot \hat{\boldsymbol{\sigma}} \right) - e^{i \frac{\theta}{2}} \left(\hat{q} \cdot \hat{\boldsymbol{\sigma}} \right) \right]$$

$\boldsymbol{\kappa}_2 = \ldots$, where:

- $\hat{k}, \hat{q}$: photon- and meson momentum,
- $\hat{\epsilon}$: γ-polarization,
- $\hat{\boldsymbol{\sigma}} = (\sigma_x, \sigma_y, \sigma_z)^T$: Pauli-matrices,
- W: CMS-energy,
- θ: CMS scattering-angle.

Polarization observables

<u>Generic case:</u>
Observables defined as *bilinear forms*:

$\mathcal{O}^{\alpha} = \boldsymbol{c}^{\alpha} \sum_{i,j=1}^{N_{\mathcal{A}}} b_i^{*} \tilde{\Gamma}_{ij}^{\alpha} b_j,$

for $\alpha = 1, \ldots, N_{\mathcal{A}}^2$.
($\boldsymbol{c}^{\alpha}$: normaliz. factors)

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Photoproduction observable	Class
$\sigma_0 = \frac{1}{2}\left(\lvert b_1\rvert^2 + \lvert b_2\rvert^2 + \lvert b_3\rvert^2 + \lvert b_4\rvert^2\right)$	
$-\check{\Sigma} = \frac{1}{2}\left(\lvert b_1\rvert^2 + \lvert b_2\rvert^2 - \lvert b_3\rvert^2 - \lvert b_4\rvert^2\right)$	$\mathcal{S}$
$-\check{T} = \frac{1}{2}\left(-\lvert b_1\rvert^2 + \lvert b_2\rvert^2 + \lvert b_3\rvert^2 - \lvert b_4\rvert^2\right)$	
$\check{P} = \frac{1}{2}\left(-\lvert b_1\rvert^2 + \lvert b_2\rvert^2 - \lvert b_3\rvert^2 + \lvert b_4\rvert^2\right)$	
$\mathcal{O}_{1+}^{a} = \lvert b_1\rvert \lvert b_3\rvert \sin\phi_{13} + \lvert b_2\rvert \lvert b_4\rvert \sin\phi_{24} = \mathrm{Im}\left[b_3^* b_1 + b_4^* b_2\right] = -\check{G}$	
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$\mathcal{O}_{2+}^{a} = \lvert b_1\rvert \lvert b_3\rvert \cos\phi_{13} + \lvert b_2\rvert \lvert b_4\rvert \cos\phi_{24} = \mathrm{Re}\left[b_3^* b_1 + b_4^* b_2\right] = -\check{E}$	
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$\mathcal{O}_{1+}^{b} = \lvert b_1\rvert \lvert b_4\rvert \sin\phi_{14} + \lvert b_2\rvert \lvert b_3\rvert \sin\phi_{23} = \mathrm{Im}\left[b_4^* b_1 + b_3^* b_2\right] = \check{O}_{z'}$	
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The complete-experiment analysis (CEA)

CEA: What are the minimal subsets of the observables $\mathcal{O}^{\alpha}$, which allow for the unique extraction of the amplitudes b_i up to one unknown overall phase $\phi\left(\Omega_2^{n_f}\right)$?

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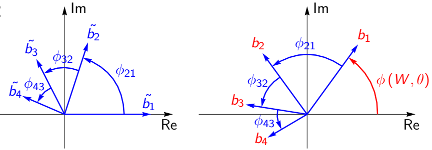

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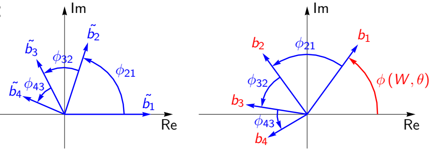

- $*$) Analysis operates on each 'bin' in $\Omega_2^{n_f}$ individually.
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- $*$) Initial standard assumption: the moduli $|b_1|, |b_2|, \ldots, |b_N|$ are already known from a certain subset of $N_{\mathcal{A}}$ 'diagonal' observables.
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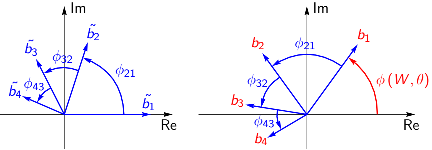

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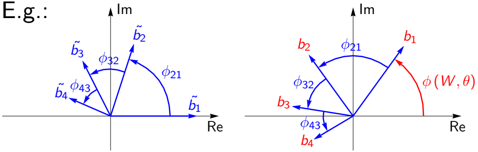

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- ∗) Simplest (formal) solution: $\mathcal{O}^{\alpha} = \boldsymbol{c}^{\alpha} \sum_{i,j=1}^{N} b_i^{*} \tilde{\Gamma}_{ij}^{\alpha} b_j$ can be 'inverted' (using the *completeness* of the $\tilde{\Gamma}$-matrices):

$$b_i^{*} b_j = \frac{1}{\tilde{N}} \sum_{\alpha=1}^{N_{\mathcal{A}}^2} \left(\tilde{\Gamma}_{ij}^{\alpha}\right)^{*} \left(\frac{\mathcal{O}^{\alpha}}{\boldsymbol{c}^{\alpha}}\right) .$$

  $\Rightarrow$ Obtain moduli from $|b_i| = \sqrt{b_i^{*} b_i}$ and rel.-phases from a 'minimal' set of $b_i^{*} b_j$
  $\Rightarrow$ Obtain (quite large) over-complete set $\{\mathcal{O}^{\alpha}\}$ determined via the RHS

Truncated partial-wave analysis (TPWA)

Generic case: partial-wave exp. for $2 \rightarrow 2$ spin-reaction in helicity-formalism:

$$\mathcal{T}_{\mu_1\mu_2,\lambda_1\lambda_2}(s,t) = e^{i(\lambda-\mu)\phi} \sum_{j=\max(|\lambda|,|\mu|)}^{\infty} (2j+1) \mathcal{T}^{j}_{\mu,\lambda}(s)\, d^{j}_{\mu,\lambda}(\theta),$$

where $\lambda := \lambda_1 - \lambda_2$, $\mu := \mu_1 - \mu_2$ and $\{b_i\} \Leftrightarrow \{H_i\} = \{\mathcal{T}_{\pm\pm,\pm\pm}\}$.

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E.g.: for photoproduction ($N_{\mathcal{A}} = 4$), one has the multipole-series ($x = \cos\theta$):

$$F_1(W,\theta) = \sum_{\ell=0}^{\not\infty\, \ell_{\max}} \left[\ell M_{\ell+}(W) + E_{\ell+}(W)\right] P^{'}_{\ell+1}(x) + \left[(\ell+1) M_{\ell-}(W) + E_{\ell-}(W)\right] P^{'}_{\ell-1}(x),$$

$$\vdots$$

$$F_4(W,\theta) = \sum_{\ell=2}^{\not\infty\, \ell_{\max}} \left[M_{\ell+}(W) - E_{\ell+}(W) - M_{\ell-}(W) - E_{\ell-}(W)\right] P^{''}_{\ell}(x), \text{ where } \{F_i\} \Leftrightarrow \{b_i\},$$

∗) $4\ell_{\max}$ complex multipoles present in every truncation-order $\ell_{\max} \geq 1$:

$$\mathcal{M}_{\ell} = \{E_{0+}, E_{1+}, M_{1+}, M_{1-}, E_{2+}, E_{2-}, \ldots, M_{\ell_{\max}-}\}.$$

(Generic case: $N_{\mathcal{A}} * \ell_{\max}$ waves for every order $\ell_{\max} \geq 1$.)

∗) $\mathcal{M}_{\ell}$ determined up to 1 overall phase $\Rightarrow$ $8\ell_{\max} - 1$ real par.'s in TPWA.

Complete experiments

Complete experiments: 'amplitude formulation'

A *complete experiment* is a minimum subset selected from entire set of $N_{\mathcal{A}}^2$ polarization observables that allows for an unambiguous extraction of the complex amplitudes describing the process (either b_i or $\mathcal{M}_\ell$), up to one unknown overall phase ($\phi(W, \theta)$ for the CEA, $\phi(W)$ for the TPWA).

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A *complete experiment* is a set of measurements that is sufficient to predict all other possible experiments. For polarization experiments, this means a subset of all existing polarization observables that is capable of determining all the remaining observables. cf.: [L. Tiator, AIP Conf. Proc. **1432**, no.1, 162-167 (2012)]

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$\hookrightarrow$ $\exists$ solution-methods for both CEA and TPWA

CEA-solution: Moravcsik's Theorem

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'Geometrical (graphical) analog': Represent each amplitude $b_1, \ldots, b_{N_\mathcal{A}}$ by a *point* and every product $b_j^* b_i$, or rel.-phase ϕ_{ij}, by a *line connecting points 'i' and 'j'*.

Furthermore: $\hookrightarrow$ Represent every $\mathrm{Re}\left[b_i^* b_j\right] \propto \cos\phi_{ij}$ by a *solid line*,

$\hookrightarrow$ Represent every $\mathrm{Im}\left[b_i^* b_j\right] \propto \sin\phi_{ij}$ by a *dashed line*.

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Moravcsik's Theorem (modified): The thus constructed graph is *fully complete*, i.e. it allows for neither any continuous nor any discrete ambiguities, if it satisfies:

CEA-solution: Moravcsik's Theorem

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'Geometrical (graphical) analog': Represent each amplitude $b_1, \ldots, b_{N_\mathcal{A}}$ by a *point* and every product $b_j^* b_i$, or rel.-phase ϕ_{ij}, by a *line connecting points* 'i' and 'j'.

Furthermore: $\hookrightarrow$ Represent every $\mathrm{Re}\left[b_i^* b_j\right] \propto \cos\phi_{ij}$ by a *solid line*,

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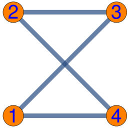
Example:



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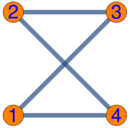

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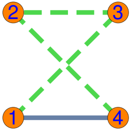
(ii) the graph has to have an *odd* number of dashed lines, as well as *any* number of solid lines:

- all discrete ambiguities are resolved.

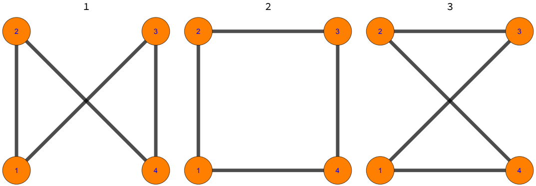
E.g.:



is complete ✓

Moravcsik's Th. applied to photoproduction ($N_{\mathcal{A}} = 4$)

For $N_{\mathcal{A}} = 4$, one gets $\frac{(N_{\mathcal{A}}-1)!}{2} = \frac{3!}{2} = 3$ possible graph-topologies :



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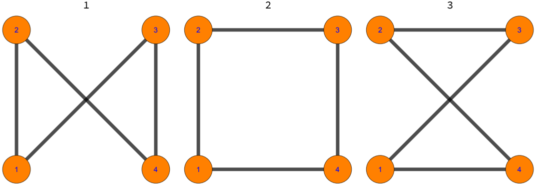

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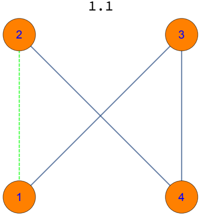

$\longrightarrow \{\sin\phi_{12}, \cos\phi_{24}, \cos\phi_{34}, \cos\phi_{13}\}$

$\Rightarrow$ Map relative-phases to observables:

$|b_1|\,|b_2| \sin\phi_{12} = (1/2)\left[-\check{L}_{x'} - \check{T}_{z'}\right]$, $|b_2|\,|b_4| \cos\phi_{24} = (1/2)\left[-\check{E} - \check{H}\right]$,
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$\Rightarrow$ Extract 'Moravcsik-complete' set: $\left\{\sigma_0, \check{\Sigma}, \check{T}, \check{P}, \check{E}, \check{H}, \check{L}_{x'}, \check{T}_{z'}, \check{L}_{z'}, \check{T}_{x'}\right\}$.

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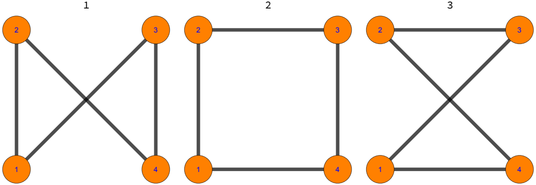

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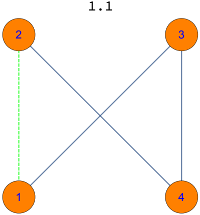

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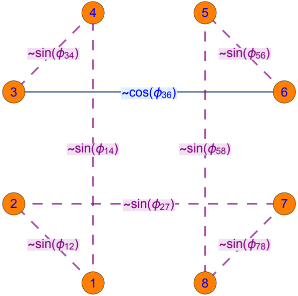
$\Rightarrow$ For $\underline{N_{\mathcal{A}} = 4}$: obtain 12 Moravcsik-complete sets of length $10 > 2N_{\mathcal{A}} = 8$.

Cases with larger numbers of $N_{\mathcal{A}} > 6$ amplitudes

Two-meson photoproduction

*) 8 amplitudes vs. 64 observables

*) Typical complete graph:



[P. Kroenert et al. (2021)]

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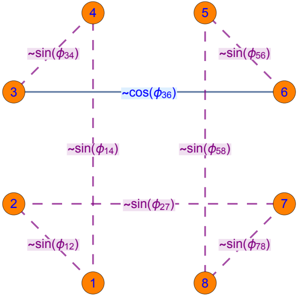

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Vector-meson photoproduction

∗) 12 amp.'s vs. 144 observables

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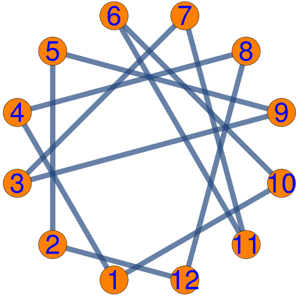

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$\frac{(N_{\mathcal{A}}-1)!}{2} = 19\,958\,400$, (demanding!)

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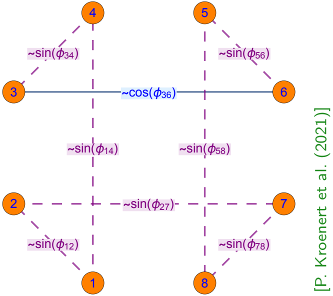

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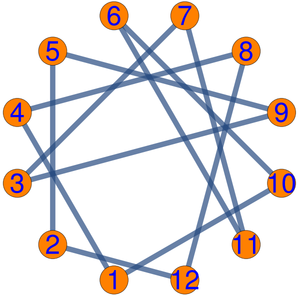

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<u>Observation:</u> Moravcsik-complete sets tend to be slightly over-complete, i.e. to contain *more than* $2N_{\mathcal{A}}$ observables, for problems with $N_{\mathcal{A}} \geq 4$ amplitudes

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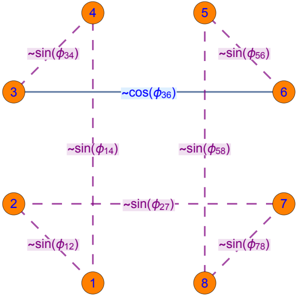

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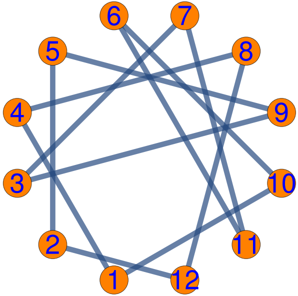

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↪ Improvement using new graphs, containing additional *directional information*.

cf.: [YW, Phys. Rev. C **104**, no.4, 045203 (2021)]

Complete experiments for the TPWA

$\ast$) Example: photoproduction ($N_{\mathcal{A}} = 4$). Consider group $\mathcal{S}$ $\left\{\sigma_0, \check{\Sigma}, \check{T}, \check{P}\right\}$, i.e. 'diagonal' observables: $\mathcal{O}^{\alpha s} \propto \pm \left|b_1\right|^2 \pm \left|b_2\right|^2 \pm \left|b_3\right|^2 + \left|b_4\right|^2$.

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$$b_1(\theta) \propto \frac{\exp\left(-i\frac{\theta}{2}\right)}{\left(1+t^2\right)^{\ell_{\mathrm{max}}}} \prod_{j=1}^{2\ell_{\mathrm{max}}} \left(t + \beta_j\right), \; b_3(\theta) \propto \frac{\exp\left(-i\frac{\theta}{2}\right)}{\left(1+t^2\right)^{\ell_{\mathrm{max}}}} \prod_{i=1}^{2\ell_{\mathrm{max}}} \left(t + \alpha_i\right),$$

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[A. S. Omelaenko, Sov. J. Nucl. Phys. **34**, 406 (1981)]

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$\hookrightarrow$ Study discrete ambiguities of group $\mathcal{S}$, generated by ($i \in \left\{1, \ldots, 2\ell_{\mathrm{max}}\right\}$):

$$\left(t - \alpha_i^*\right)\left(t - \alpha_i\right) \xrightarrow{\alpha_i \rightarrow \alpha_i^*} \left(t - \left[\alpha_i^*\right]^*\right)\left(t - \alpha_i^*\right) = \left(t - \alpha_i^*\right)\left(t - \alpha_i\right).$$

$\Rightarrow$ Surprise: *all* ambiguites can be resolved using less than $2N_{\mathcal{A}} = 8$ observables!

[YW, R. Beck and L. Tiator PRC **89**, no.5, 055203 (2014)]

[R. L. Workman, *et al.*, PRC **95**, no.1, 015206 (2017)]

[YW, arXiv:2008.00514 [nucl-th]]

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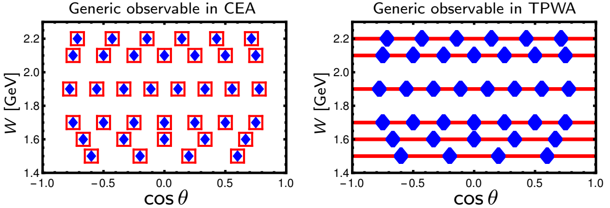
Comment: number of *observables* needed for completeness has been reduced, but not (!) the number of *datapoints*!



Testing completeness of photoproduction TPWA

$*)$ I-O studies using model-data (MAID2007, $\gamma p \to \pi^0 p$), set: $\left\{ \sigma_0, \check{\Sigma}, \check{T}, \check{P}, \check{F} \right\}$

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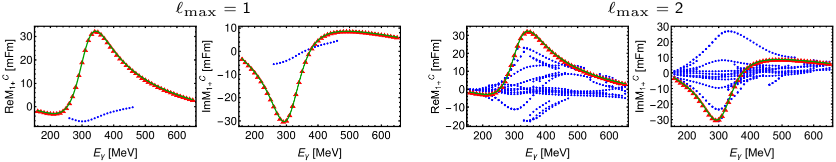

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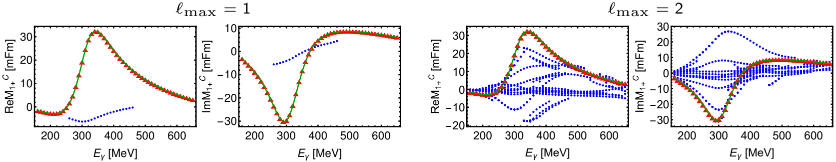
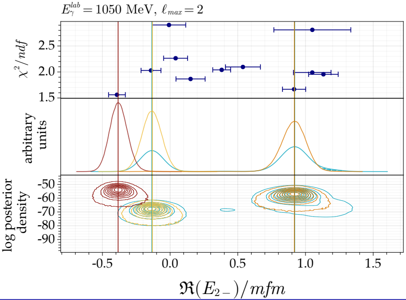
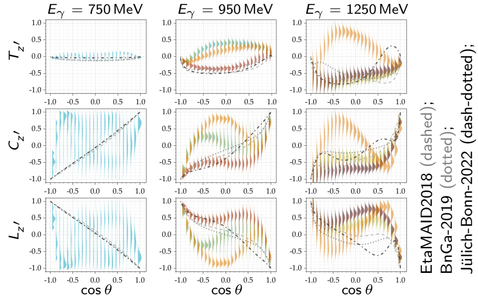


$*$) Bayesian inference for (real) $\gamma p \to \eta p$ data, set $\{\sigma_0, \check{\Sigma}, \check{T}, \check{E}, \check{F}, \check{G}\}$

cf.: [P. Kroenert, YW, F. Afzal and A. Thiel, Phys. Rev. C **109**, no.4, 045206 (2024)]





TPWA for 2-meson photoproduction (preliminary!!) - I

- **5-dimensional** phase-space:

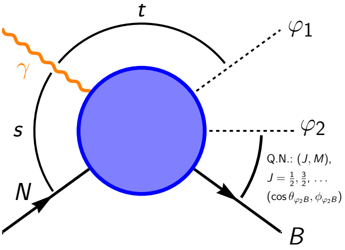
$$\left\{ s, t, m^2_{\varphi_2 B}, \Omega_{\varphi_2 B} = \left(\theta_{\varphi_2 B}, \phi_{\varphi_2 B}\right) \right\}$$

- 8 helicity-configurations:

$$\lambda_\gamma = \pm 1,\ \lambda_N = \pm \tfrac{1}{2},\ \lambda_B = \pm \tfrac{1}{2}.$$

- $\varphi_2 B$ angular-momentum Q.N.'s:

$J = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots, \infty$ and $M = -J, \ldots, +J$.



[cf. Talk by V. Mathieu (Monday)]

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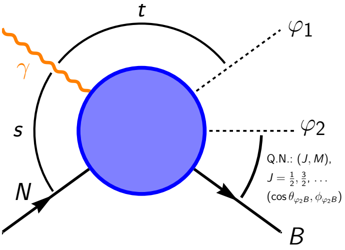

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$\Rightarrow$ TPWA:

$$\mathcal{A}_{\lambda_\gamma;\lambda_N \lambda_B} \left(s, t, m^2_{\varphi_2 B}, \Omega_{\varphi_2 B} \right) = \sum_{J=\frac{1}{2},\frac{3}{2},\ldots}^{J_{\mathrm{max}}} \sum_{M=-J}^{+J} \mathcal{T}^{(J)}_{\lambda_\gamma, M; \lambda_N \lambda_B} \left(s, t, m^2_{\varphi_2 B} \right) D^{*J}_{M, -\lambda_B} \left(\Omega_{\varphi_2 B} \right).$$

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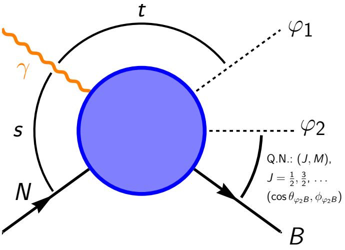

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$*$) Assume: 8 functions $\left|\mathcal{A}_{\lambda_\gamma;\lambda_N\lambda_B}\right|^2$ uniquely fixed from **8 pol.-measurements**:

$$\left\{ \left|\mathcal{A}_{+;++}\right|^2, \left|\mathcal{A}_{+;+-}\right|^2, \ldots, \left|\mathcal{A}_{-;--}\right|^2 \right\} \Leftrightarrow \left\{ I_0, P_z, P_{z'}, \mathcal{O}_{zz'}, I^{\odot}, P_z^{\odot}, P_{z'}^{\odot}, \mathcal{O}_{zz'}^{\odot} \right\}.$$

$\rightarrow$ Forget dependence on $(\lambda_\gamma; \lambda_N\lambda_B)$ from now on (fix $\lambda_B = 1/2$ in D-fct.)

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- $\varphi_2 B$ angular-momentum Q.N.'s:

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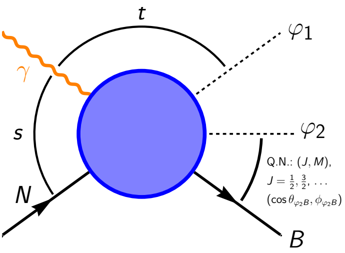

$\Rightarrow$ TPWA:

[cf. Talk by V. Mathieu (Monday)]

$$\mathcal{A}_{\lambda_\gamma;\lambda_N\lambda_B}\left(s, t, m_{\varphi_2 B}^2, \Omega_{\varphi_2 B}\right) = \sum_{J=\frac{1}{2},\frac{3}{2},\ldots}^{J_{\max}} \sum_{M=-J}^{+J} \mathcal{T}_{\lambda_\gamma, M;\lambda_N\lambda_B}^{(J)}\left(s, t, m_{\varphi_2 B}^2\right) D_{M,-\lambda_B}^{*J}\left(\Omega_{\varphi_2 B}\right).$$

$*$) Assume: 8 functions $\left|\mathcal{A}_{\lambda_\gamma;\lambda_N\lambda_B}\right|^2$ uniquely fixed from **8 pol.-measurements**:

$$\left\{\left|\mathcal{A}_{+;++}\right|^2, \left|\mathcal{A}_{+;+-}\right|^2, \ldots, \left|\mathcal{A}_{-;--}\right|^2\right\} \Leftrightarrow \left\{I_0, P_z, P_{z'}, \mathcal{O}_{zz'}, I^{\odot}, P_z^{\odot}, P_{z'}^{\odot}, \mathcal{O}_{zz'}^{\odot}\right\}.$$

$\rightarrow$ Forget dependence on $(\lambda_\gamma; \lambda_N\lambda_B)$ from now on (fix $\lambda_B = 1/2$ in D-fct.)

$\Rightarrow$ Consider discrete partial-wave ambiguities for the squared-modulus function

$$\mathcal{A}\left(\Omega_{\varphi_2 B}\right)\mathcal{A}^*\left(\Omega_{\varphi_2 B}\right).$$

TPWA for 2-meson photoproduction (preliminary!!) - II

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$\Rightarrow$ 2-meson TPWA is **much better constrained, most likely unique!**

cf.: [I. S. Stefanescu, J. Math. Phys. **26** (9), 2141-2160 (1985)]

& [W. A. Smith *et al.* [JPAC], Phys. Rev. D **108**, no.7, 076001 (2023)]

Thank You!

Additional Slides

CEA-solution using new 'directional' graphs

↪ One can improve the situation using new kind of graphs, containing additional *directional information*. [YW, Phys. Rev. C **104**, no.4, 045203 (2021)]

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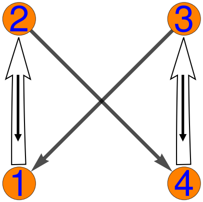

$\Leftrightarrow$ complete photoproduction-set ($2N = 8$ obs.'s in combination with 4 'diagonal' obs.'s $\left\{ \sigma_0, \check{\Sigma}, \check{T}, \check{P} \right\}$):
$\left\{ \mathcal{O}^a_{2+}, \mathcal{O}^a_{2-}, \mathcal{O}^c_{1+}, \mathcal{O}^c_{2-} \right\} = \left\{ \check{E}, \check{H}, \check{L}_{x'}, \check{T}_{x'} \right\}$.

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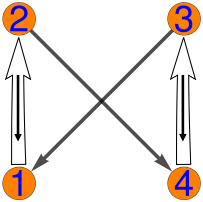

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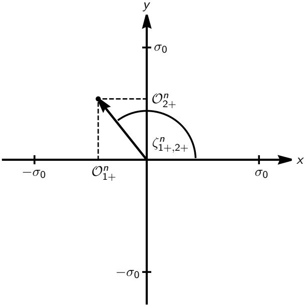
- Single-lined arrows: same as in Moravcsik's Theorem
- Double-lined arrows: 'crossed' selection $\mathcal{O}^c_{1\pm} \oplus \mathcal{O}^c_{2\pm}$
- 'Outer' direction ⇔ 'directional convention' for consistency rel.: $\phi_{12} + \phi_{24} + \phi_{43} + \phi_{31} = 0$.
- Direction of 'inner' arrows: sign of 'ζ-angle' (cf. Figure on the right) in discrete-ambiguity formulas



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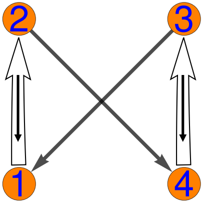

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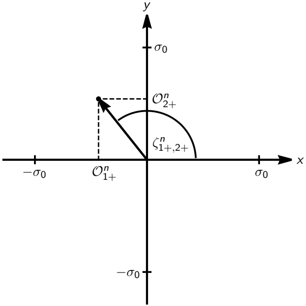

↪ Confirm min. sets of $2N_{\mathcal{A}} = 8$ for photoproduction ; Obtain new sets of $2N_{\mathcal{A}} = 12$ for e^--production! For $N_{\mathcal{A}} > 6$, sets still (slightly) over-complete ...

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We have new *standard-approaches* in place to determine complete experiments, and thus $|b_i|$ and ϕ_{ij}, for reactions with any number of amplitudes $N_{\mathcal{A}}$!

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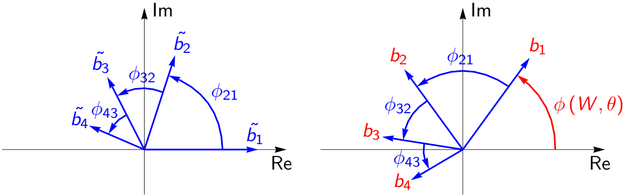
However: the unknown phase $\phi\left(\Omega_2^{n_f}\right)$ remains a problem, because:



$$\mathcal{T}_{fi}\left(\Omega_2^{n_f}\right) = \chi_B^{\dagger}\left[\sum_{k=1}^{N_{\mathcal{A}}} \boldsymbol{\kappa}_k b_k\left(\Omega_2^{n_f}\right)\right]\chi_N = \chi_B^{\dagger}\left[\sum_{k=1}^{N_{\mathcal{A}}} \boldsymbol{\kappa}_k e^{i\phi_{fi}\left(\Omega_2^{n_f}\right)}\tilde{b}_k\left(\Omega_2^{n_f}\right)\right]\chi_N$$

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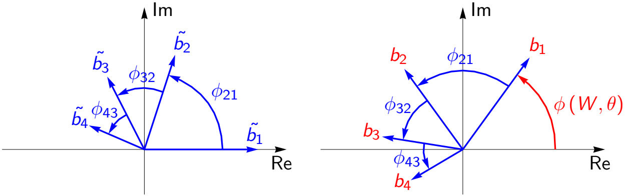

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The phase has *physical significance*, because:

- (Perfect) knowledge of the phase $\Rightarrow$ project-out partial waves to all orders!
- *Analytically continue* part. waves away from physical region $\Rightarrow$ *unique* resonance-poles!

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Measure the overall phase in scattering experiments using vortex beams
$\equiv$ beams of particles with intrinsic orbital angular momentum $\langle L_z \rangle = \hbar\ell$ along the axis of propagation (i.e. z-axis) cf. [Ivanov, Phys. Rev. D **85**, 076001 (2012)], [Ivanov, arXiv:2205.00412 [hep-ph] (2022)]

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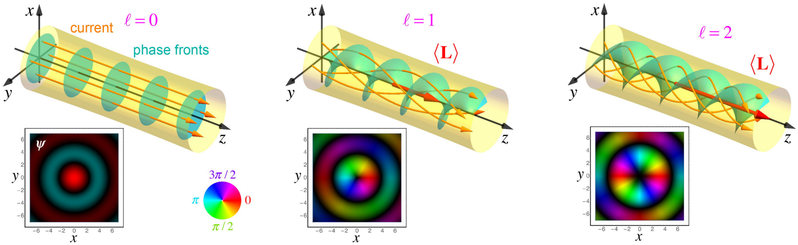

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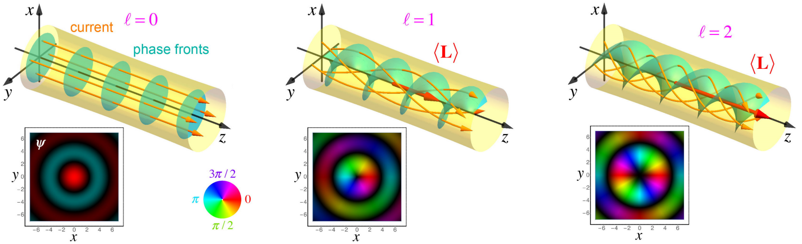

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Proposal [Ivanov, Phys. Rev. D **85**, 076001 (2012)]: for the example $\gamma p \to \pi p$, consider double-twisted γp-collision (i.e. both γ and p are in a [Bessel] vortex-state)

$\Rightarrow$ Measure *azimuthal asymmetry* $A = \frac{\Delta\sigma}{\sigma}$ ('sine-weighted' c.s. $\Delta\sigma$; non-weighted σ)

$\Rightarrow$ Then, one has $\boxed{A = \frac{d\phi\left(\theta_{\gamma\pi}^{\mathrm{LAB}}\right)}{d\theta_{\gamma\pi}^{\mathrm{LAB}}} \cdot P}$, with an 'analyzing power' P.

$\Rightarrow$ For insanely good accuracy and statistics, integration yields: $\underline{\phi\left(\theta_{\gamma\pi}^{\mathrm{LAB}}\right) + \mathcal{C}}$.

Vortex-beams at the GeV-scale maybe feasible within 10-20 years [Ivanov, priv. comm. (2022)]

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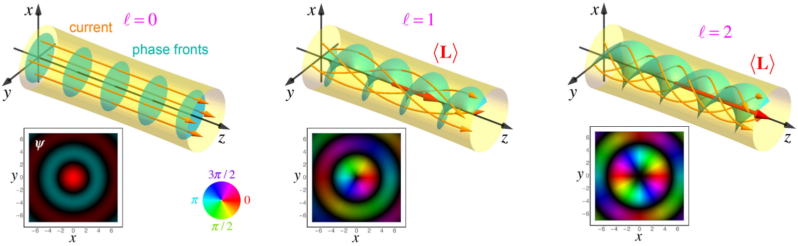

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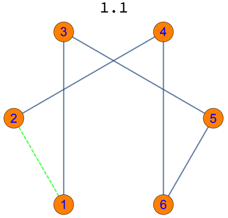
Alternative: Hanbury-Brown and Twiss experiment [Goldberger *et al.* Phys. Rev. 132, 2764 (1963)]

Moravcsik's Th. applied to electroproduction ($N_{\mathcal{A}} = 6$)

$*$) From the first of 60 topologies, construct example (1.1) (fully complete):



$\longrightarrow \quad \left\{ \sin\phi_{12}, \cos\phi_{24}, \cos\phi_{46}, \cos\phi_{56}, \cos\phi_{35}, \cos\phi_{13} \right\}$

$\hookrightarrow$ In the same way as before, extract the 'Moravcsik-complete' set (combined with 'diagonal' observables $\left\{ R_{T}^{00},\ {}^{c}R_{TT}^{00}, R_{T}^{0y}, R_{T}^{y'0}, R_{L}^{00}, R_{L}^{0y} \right\}$):

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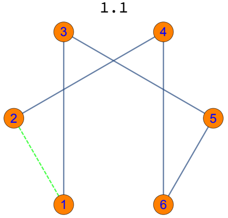

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∗) In total, we obtain for the first time (!):

- 64 non-redundant Moravcsik-complete sets composed of 13 observables
  ↪ Only one observable more than the minimal number of $2N_{\mathcal{A}} = 12$ observables!

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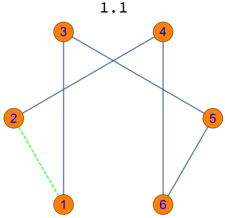
$*$) From the first of 60 topologies, construct example (1.1) (fully complete):



$\longrightarrow \quad \{\sin\phi_{12}, \cos\phi_{24}, \cos\phi_{46}, \cos\phi_{56}, \cos\phi_{35}, \cos\phi_{13}\}$

$\hookrightarrow$ In the same way as before, extract the 'Moravcsik-complete' set (combined with 'diagonal' observables $\left\{R_{T}^{00},\ {}^{c}R_{TT}^{00}, R_{T}^{0y}, R_{T}^{y'0}, R_{L}^{00}, R_{L}^{0y}\right\}$):

$$\left\{\mathcal{O}_{2+}^{a}, \mathcal{O}_{2-}^{a}, \mathcal{O}_{1+}^{c}, \mathcal{O}_{1-}^{c}, \mathcal{O}_{2}^{d}, \mathcal{O}_{2+}^{h}, \mathcal{O}_{2-}^{h}\right\}$$
$$\equiv \left\{R_{TT'}^{0z},\ {}^{s}R_{TT}^{0x}, R_{T}^{x'z}, R_{T}^{z'x}, R_{L}^{x'x},\ {}^{c}R_{LT}^{x'x},\ {}^{s}R_{LT'}^{z'x}\right\}.$$

$*$) In total, we obtain for the first time (!):

- 64 non-redundant Moravcsik-complete sets composed of 13 observables
  $\hookrightarrow$ Only one observable more than the minimal number of $2N_{\mathcal{A}} = 12$ observables!

$\hookrightarrow$ What about problems with large numbers of amplitudes (i.e. $N_{\mathcal{A}} > 6$)?

CEA-solution using Fierz identities

Clifford algebra $\left\{ \tilde{\Gamma}^{\alpha} \right\}$ implies so-called 'Fierz-identities':

$$\mathcal{O}^{\alpha} \mathcal{O}^{\beta} = \sum_{\delta,\eta} C_{\delta\eta}^{\alpha\beta} \mathcal{O}^{\delta} \mathcal{O}^{\eta}, \text{ with } C_{\delta\eta}^{\alpha\beta} := \frac{1}{N_{\mathcal{A}}^{2}} \mathrm{Tr} \left[\tilde{\Gamma}^{\delta} \tilde{\Gamma}^{\alpha} \tilde{\Gamma}^{\eta} \tilde{\Gamma}^{\beta} \right].$$

$\Rightarrow$ Use these to solve for complete experiments! ('measurement formulation')

cf.: [Chiang & Tabakin, Phys. Rev. C **55**, 2054-2066 (1997)]

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E.g.: πN-scattering, i.e. $N_{\mathcal{A}} = 2$, $\{b_1, b_2\}$ vs. $N_{\mathcal{A}}^2 = 4$ obs.'s $\left\{ \sigma_0, \check{P}, \check{R}, \check{A} \right\}$;
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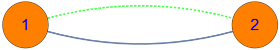

$\Leftrightarrow \{\sin \phi_{21}, \cos \phi_{21}\} \Leftrightarrow$ all 4 obs.'s needed ✓

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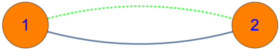

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∗) There exists one relevant Fierz-identity:

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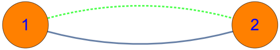

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$\hookrightarrow$ Consistent, but cumbersome, alternative solution-method

$\Rightarrow$ Maybe easier to automate in the future ... (?)

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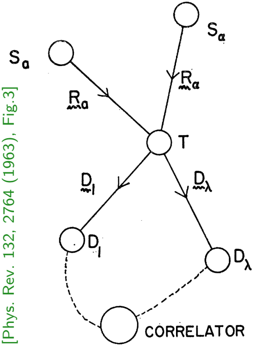
Measure the overall phase via intensity correlations in a *Hanbury-Brown and Twiss-type* experiment [Goldberger, Lewis & Watson, Phys. Rev. 132, 2764 (1963)]



[Phys. Rev. 132, 2764 (1963), Fig.3]

∗) Two sources, S_a and S_α, emitting beam-particles

∗) One single irradiated target T

∗) Two spatially separated detectors, D_l and D_λ

∗) A CORRELATOR, which registers only in case D_l and D_λ count in *coincidence*

↪ The correlator counting-rate contains an isolatable term, which is proportional to:

$$\mathrm{Re}\,[\mathcal{T}_{\lambda\leftarrow\alpha}\mathcal{T}^*_{l\leftarrow\alpha}\mathcal{T}_{l\leftarrow a}\mathcal{T}^*_{\lambda\leftarrow a}].$$

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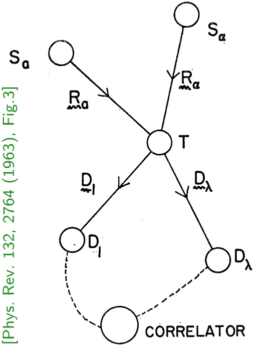

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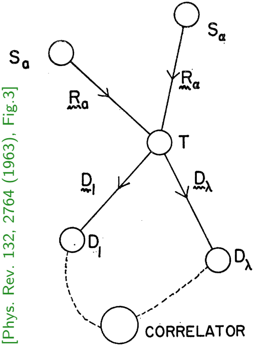

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∗) *Varying positions* of detectors and sources, do measurements for *many* angles:

$$\left\{\boldsymbol{g}^{(1)}_{la}, \boldsymbol{g}^{(2)}_{la}, \ldots\right\}, \left\{\boldsymbol{g}^{(1)}_{l\alpha}, \boldsymbol{g}^{(2)}_{l\alpha}, \ldots\right\}, \left\{\boldsymbol{g}^{(1)}_{\lambda\alpha}, \boldsymbol{g}^{(2)}_{\lambda\alpha}, \ldots\right\}, \left\{\boldsymbol{g}^{(1)}_{\lambda a}, \boldsymbol{g}^{(2)}_{\lambda a}, \ldots\right\}.$$

↪ Extract: $\phi\left(g^{(\nu+1)}_{la}\right) - \phi\left(g^{(\nu)}_{la}\right) \equiv \delta\phi(\nu) \longrightarrow$ <u>Overall phase:</u> $\phi(\boldsymbol{g})$.