Triangle Singularities in a Hilbert's House

NSTAR24, York

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Presentation Outline

Introduction

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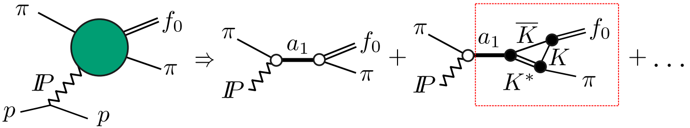

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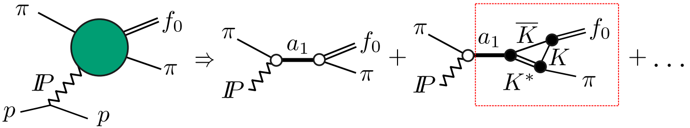

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- Relevance to lattice QCD — Korpa et al. (10.1103/PhysRevD.107.L031505), Isken et al. (10.1103/PhysRevD.109.034032)

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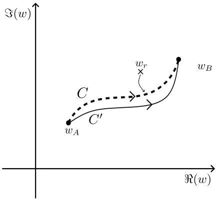

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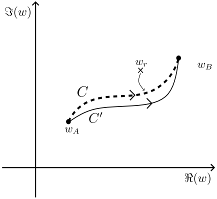

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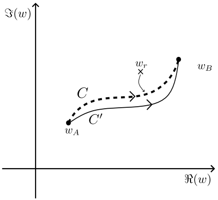

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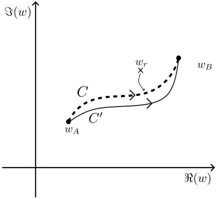

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- For some real parameters α_i, the Landau equations read,

$$\alpha_i (q_i^2 - m_i^2) = 0 \implies \alpha_i = 0 \text{ or } q_i^2 = m_i^2,$$

$$\sum_{i \in \text{loop}} \alpha_i \, q_i^{\mu}(k_j) = 0.$$

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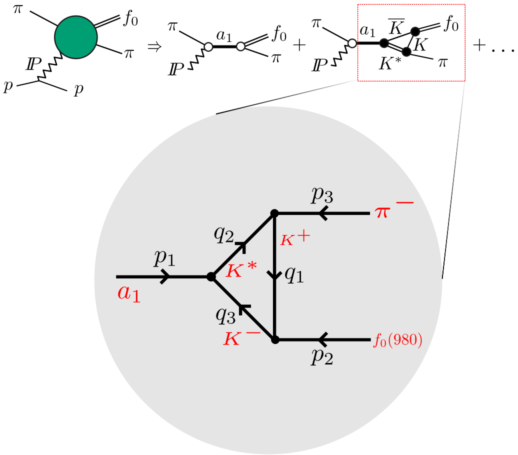
- Interesting analogy: Feynman diagrams $\rightarrow$ electrical circuits; propagators $\rightarrow$ wires, with current q_i and resistance α_i. Then the Landau equations are identical to Kirchhoff's law![2]

[2] $\Delta V = \sum IR$, for a triangle circuit with $\Delta V = 0$.

- The leading Landau singularity associated with the graph:



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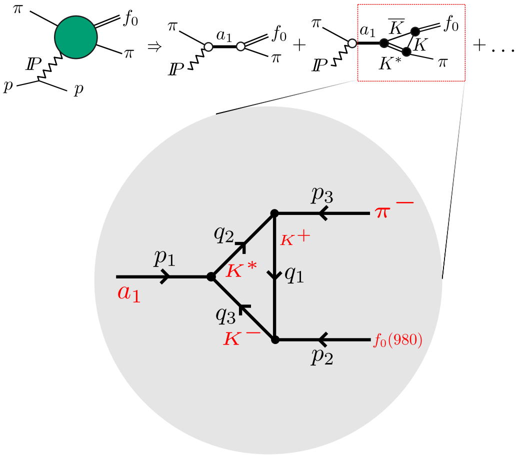

- Landau singularity for $p_3 = m_\pi$: $p_2/\mathrm{GeV} \equiv \sqrt{\sigma}/\mathrm{GeV} \in [0.986, 1.024]$ and $p_1/\mathrm{GeV} \equiv \sqrt{s}/\mathrm{GeV} \in [1.385, 1.436]$.

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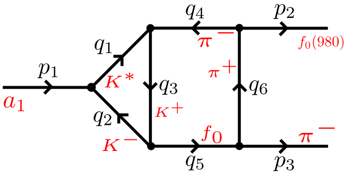

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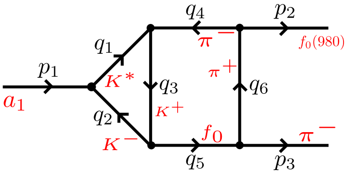

- Check for leading and subleading singularities. . .

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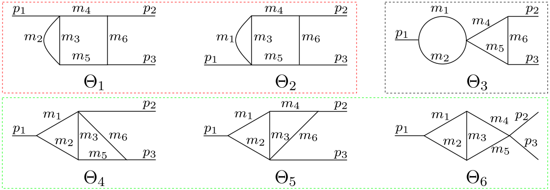

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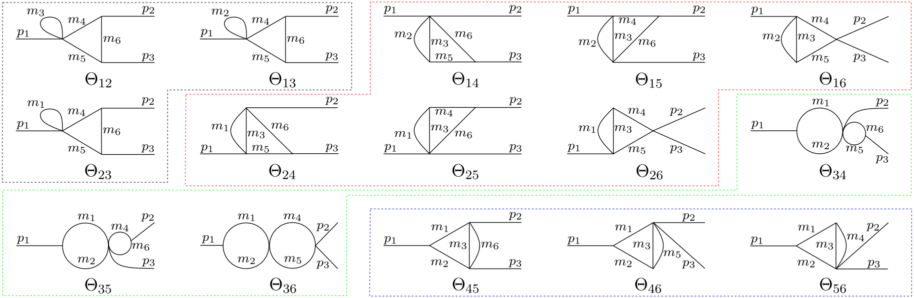

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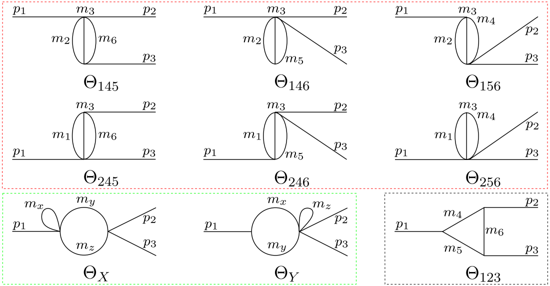

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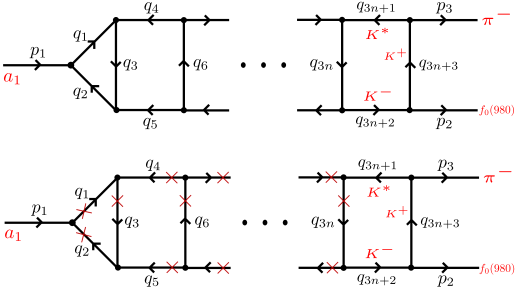
- The triangle singularity is present and no other singularities.



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- One still needs to evaulate the corresponding Feynman integrals.

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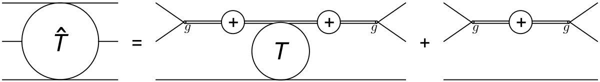

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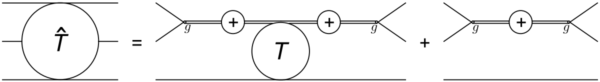
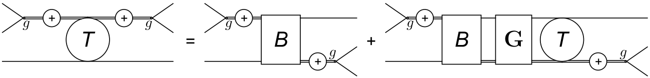


- One considers the Bethe-Salpeter ansatz, $T = B + BGT$, to determine T.[4]



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- The equation to be solved,

$$T(p,q) = B(p,q) + \int \frac{d^3\mathbf{l}}{(2\pi)^3}\frac{1}{2E_l}B(p,l)\tau(\sigma(l))T(l,q).$$

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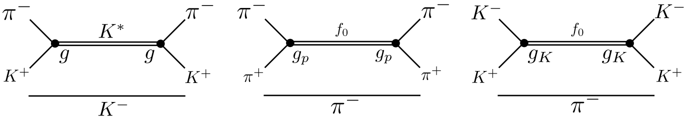
- In this work, we consider scalar propagators, in relative s-wave.

- Need to carry out a coupled channel analysis.



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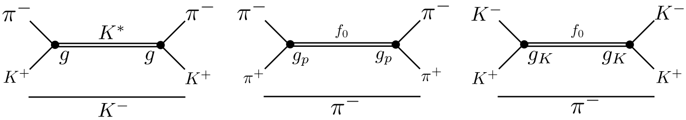

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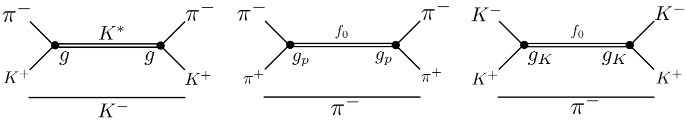

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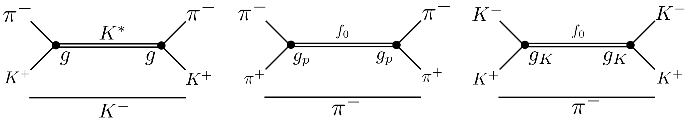

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- For f_0 we also make use of g_K/g_p ratio from BaBaR collaboration.

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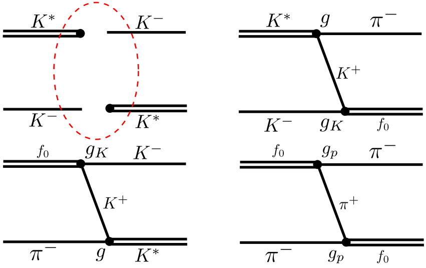
- Diagrammatically,



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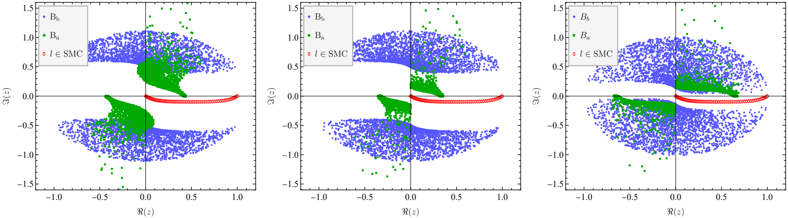

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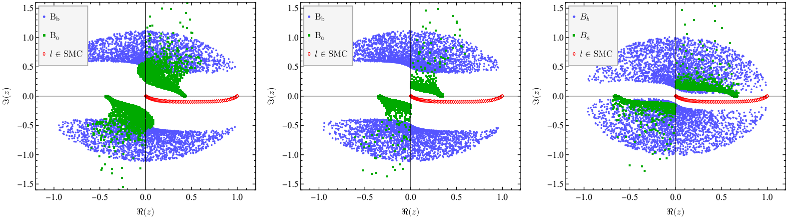
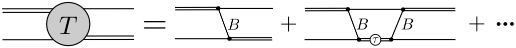


- But, we need the amplitudes for $q \in \mathbb{R}$ → can be done in different ways!

- The solution of the Bethe-Salpeter equation can be given in form of the Born series.



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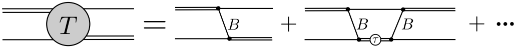

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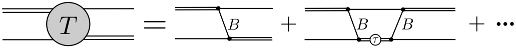

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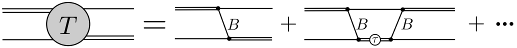

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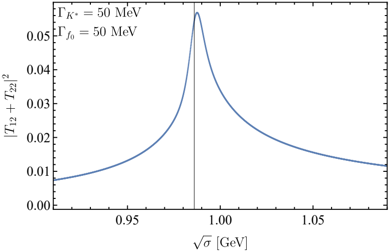
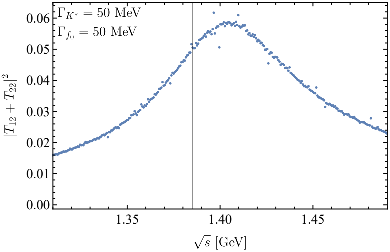


Amplitude squared vs. $\sqrt{\sigma}$, for $\sqrt{s} = 1.42$ GeV.



Amplitude squared vs. $\sqrt{s}$, for $\sqrt{\sigma} = 0.99$ GeV.

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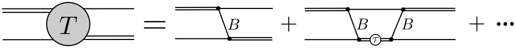

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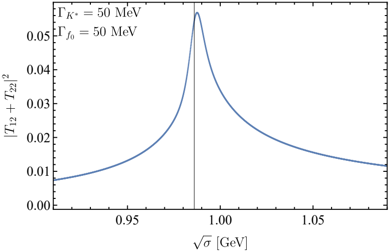

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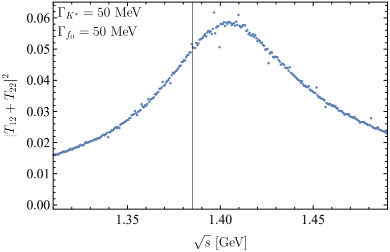
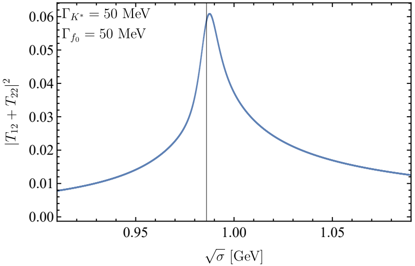


Amplitude squared vs. $\sqrt{s}$, for $\sqrt{\sigma} = 0.99$ GeV.

- We see the (leading) Landau singularity in both the plots.

- At $1+\infty$-loop level,



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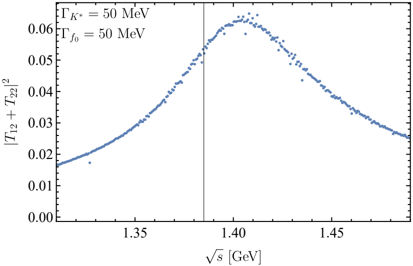

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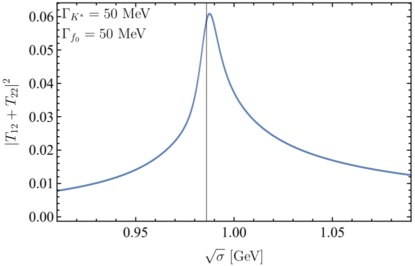

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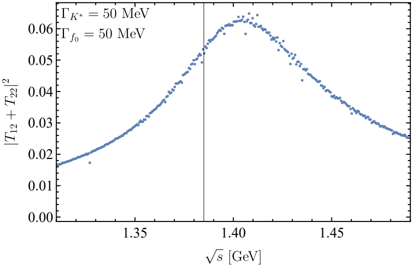

Amplitude squared vs. $\sqrt{s}$, for $\sqrt{\sigma} = 0.99$ GeV.

- The magnitutes are quite similar!

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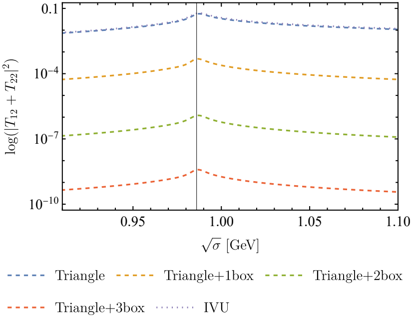
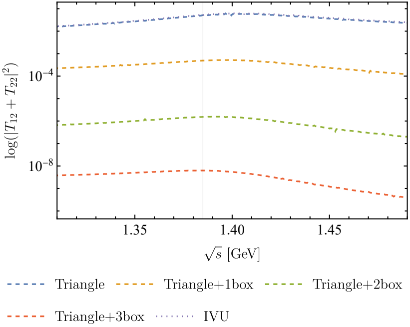


Amplitude squared vs. $\sqrt{\sigma}$, for $\sqrt{s} = 1.42$ GeV.



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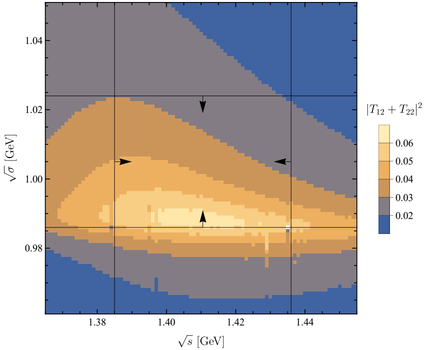
- Cherecteristic of triangle singularities: very sensitive to invariant masses!



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  2. $a_1(1420)$ on the lattice? Implement finite volume unitarity.[5]

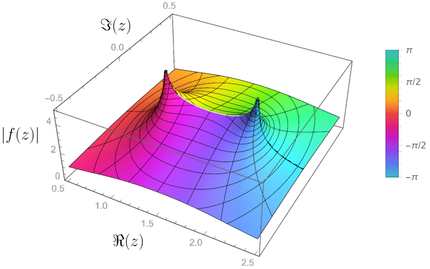
[5] 10.1140/epja/i2017-12440-1

Thanks for listening!

Landau singularities

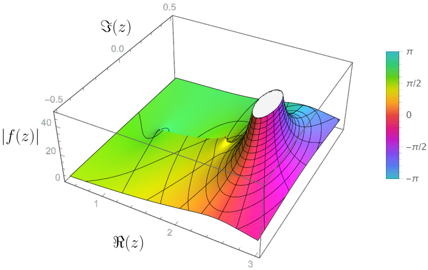
Endpoint singularities: singularities in g, hitting the contour C. E.g.,

$$f(z) = \int_1^2 dw \ \frac{1}{w - z}. \qquad (1)$$

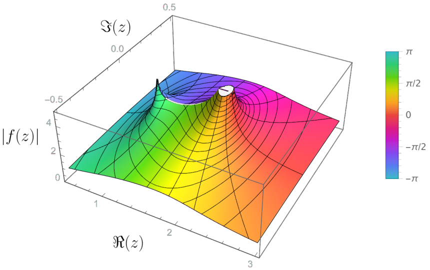


Pinch singularities: Contour C, gets trapped between two singularities in g. E.g.,

$$f(z) = \int_1^2 dw \; \frac{1}{(w-z)(w-5/2)}. \quad (2)$$



Pinch singularity avoided when the singularities approach from the same side of the contour.



The Other Method

- **Complex Contour on Riemann Surfaces (Cahill & Sloan)**[6]: Instead of a continued fraction, use the knowledge of the location of the branch cuts to analytically continue the amplitude for $q \in \mathbb{R}$.[7]

[6] 10.1016/0375-9474(71)90156-4

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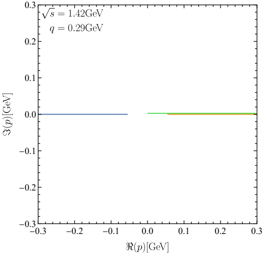
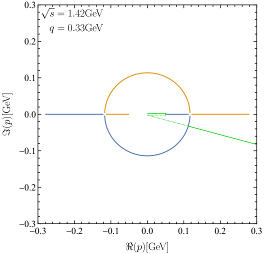
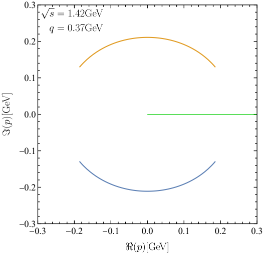


Branch cuts of B for small q.



Branch cuts of B for intermediate q.

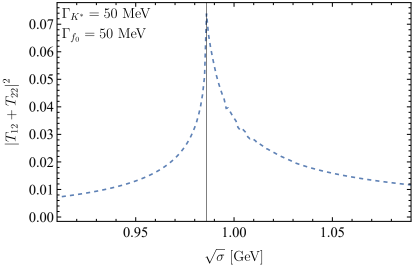


Branch cuts of B for large q.

[6] 10.1016/0375-9474(71)90156-4

[7] For a comprehensive review: *The Quantum Mechanical Three-Body Problem*, Schmid & Ziegelmann.

- At $1 + \infty$-loop level,



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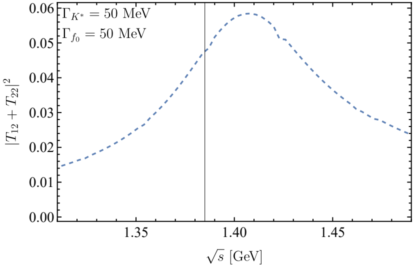

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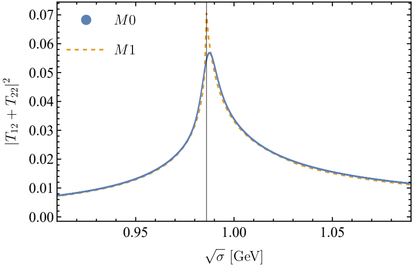
- Matches with the other method.



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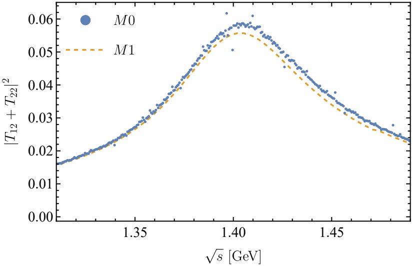

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