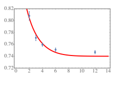

Pseudo-PDFs and extraction of PDFs from lattice

A.V. Radyushkin (ODU/Jlab)

SBS Collaboration Meeting
July 18, 2023

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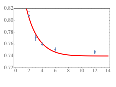
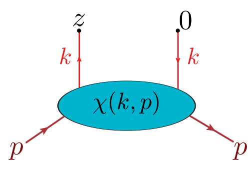


Hadrons and Partons

- Experimentally, we work with hadrons
- Theoretically, we works with quarks

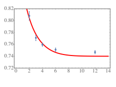
Hadron to Parton Transition



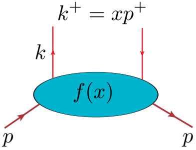
- Can be described in coordinate or momentum space

$$\langle p|\phi(0)\phi(z)|p\rangle \equiv M(z,p) = \frac{1}{\pi^2}\int d^4k\, e^{-ikz}\,\chi(k,p)$$

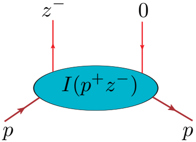
- Concept of PDFs does not rely on spin complications



Light-cone PDFs



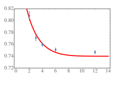
- In momentum representation:
  PDF $f(x)$ gives probability that parton carries fraction xp^+ of hadron momentum component p^+



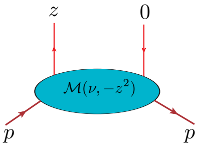
- In coordinate representation:
  PDF $f(x)$ is given by Fourier transform of matrix element $M(z,p)$ on the light cone $z^2 = 0$
- By Lorentz invariance, $M(z,p)$ is a function of (zp) and z^2, i.e. (zp) only when $z^2 = 0$
- Ioffe time ν: taking $z = z^-$ we have $(zp) = p^+z^- \equiv -\nu$

$$f(x) = \frac{p^+}{2\pi}\int_{-\infty}^{\infty} dz^-\, e^{ixp^+z^-}\, M(z^-,p) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\nu\, e^{-ix\nu}\,\mathcal{I}(\nu)$$

- Ioffe-time distribution $\mathcal{I}(\nu)$
- Observation: ν-dependence governs x-dependence



Pseudodistributions

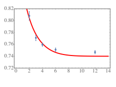


- Lattice is Euclidean: no lightcone separations
- Take z off the light cone: $z^2 < 0$
- By Lorentz invariance $M(z,p) = \mathcal{M}(-(pz), -z^2)$
- Ioffe time $\nu = -(pz)$
- $\mathcal{M}(\nu, -z^2)$: pseudo-ITD
- Pseudo $\equiv$ off the light cone, $z^2 \neq 0$

- Using Schwinger's α-representation, it is possible to show that, for any contributing Feynman diagram, for arbitrary z^2 and arbitrary p^2

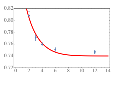
$$\mathcal{M}(\nu, -z^2) = \int_{-1}^{1} dx\, e^{ix\nu}\, \mathcal{P}(x, -z^2)$$

- $\mathcal{P}(x, -z^2)$ = pseudo-PDF, or PDF off the light cone
- $e^{ix\nu} = e^{-ix(pz)}$: decomposition over plane waves with momentum $k = xp$
- "Canonical" limits $-1 \leq x \leq 1$
- Negative x correspond to anti-particles
- Note: x is Lorentz invariant: same "on" and "off" LC

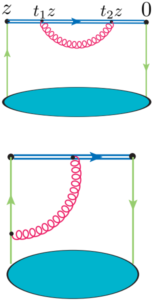


Pseudodistributions on the lattice

- On the lattice: cannot take "z" on the light cone
  Need to take it off the light cone!
- Take $z = \{0, 0, 0, z_3\}$ (X. Ji (2013), quasi-PDF approach, $p_3 \to \infty$)
- Pseudo-PDF approach is based on key observation:
  It does not matter if ν was obtained as $-(p_+ z_-)$ or as $p_3 z_3$:
  the function $\mathcal{M}(\nu, -z^2)$ is the same!
- For $z = z_3$, we have $\nu = p_3 z_3$ and $-z^2 = z_3^2$
- Analogy with DIS structure functions $W(\omega, Q^2)$
- $\omega = 1/x$ and
- $1/Q$ characterizes "probing distance"
- In pseudo-PDFs, z_3 is the "probing distance" literally
- Important to realize: dependence of $M(z, p)$ on z
  comes (1) through dependence on (pz)
  and (2) remaining dependence on z for a fixed (pz)
- Pseudo-PDF strategy: map lattice data on $M(z_3, p)$
  in terms of ν and z_3^2 and extrapolate $\mathcal{M}(\nu, z_3^2)$ to $z_3^2 = 0$
- Need to understand various types of z^2-dependence of $\mathcal{M}(\nu, z_3^2)$



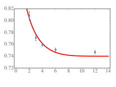
Link-related z_3^2-dependence



- Specific source of z^2-dependence in QCD: gauge link $\hat{E}(0, z; A) = P \exp\{ig \int_0^z A^\mu dx_\mu\}$
- It comes together with ultraviolet divergences: linear $\sim z_3/a$ and logarithmic $\ln(z_3^2/a^2)$, where $a \sim$ UV cut-off, e.g. lattice spacing a_L
- At one loop, UV terms have been calculated in lattice perturbation theory (Ji et al., 2016)
- Result close to that obtained using Polyakov regularization $1/z^2 \to 1/(z^2 - a^2)$ for gluon propagator in coordinate space, with $a = a_L/\pi$

$$\Gamma_{\rm UV}(z_3, a) \sim -\frac{\alpha_s}{2\pi} C_F \left[2\frac{|z_3|}{a} \tan^{-1}\left(\frac{|z_3|}{a}\right) - \ln\left(1 + \frac{z_3^2}{a^2}\right)\right]$$

- 1-loop result exponentiates in higher orders, producing $\sim e^{-2\alpha_s z_3/3a}$ factor for large z_3
- Vertex corrections produce extra $\frac{\alpha_s}{2\pi} C_F \ln\left(1 + z_3^2/a^2\right)$ term exponentiating in higher orders

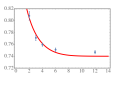


Renormalization

- Link-related UV divergences have the same structure as in HQET
- They are multiplicatively renormalizable
  (Qiu et al. , Ji et al. , Green et al. 2017)
- UV regulator a appears only in the combination z_3/a
- UV-sensitive terms form a factor $Z(z_3^2/a^2)$
- This factor is an artifact of having a non-lightlike z: $Z = 1$ on the light cone
- It has nothing to do with the usual PDFs
- We should build modified function $Z^{-1}(z_3^2/a^2)\mathcal{M}(\nu, z_3^2; a)$
- To do this, one should know the $Z(z_3^2/a^2)$ factor
- Easier way out: consider reduced pseudo-ITD

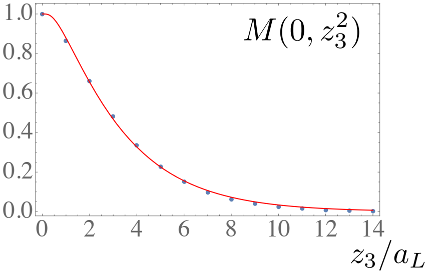
$$\mathfrak{M}(\nu, z_3^2) \equiv \frac{\mathcal{M}(\nu, z_3^2)}{\mathcal{M}(0, z_3^2)} = \lim_{a \to 0} \frac{\mathcal{M}(\nu, z_3^2; a)}{\mathcal{M}(0, z_3^2; a)}$$

- $Z(z_3^2/a^2)$ factors cancel, and $\mathfrak{M}(\nu, z_3^2)$ has finite $a \to 0$ limit

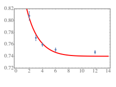


Rest-frame density and Z factor

- Exploratory study in quenched approximation (Orginos et al. 2017), is still the most precise pPDF calculation
- Allows to study basic aspects of hadron dynamics on the lattice
- Rest-frame density $\mathcal{M}(0, z_3^2)$ is produced by data at $p_3 = 0$

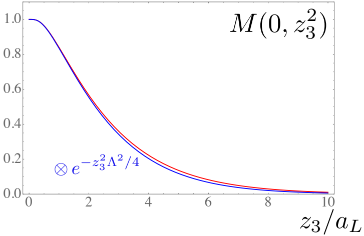


- $\mathcal{M}(0, z_3^2)$ serves as the UV renormalization factor
- Red line is exponential of 1-loop result for link self-energy and vertex corrections with $\alpha_s = 0.19$
- Very strong effect from $Z(z_3^2) \sim e^{-c|z_3|/a}$

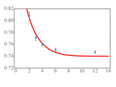


Higher-twist effects

- From phenomenology: $f(x, k_\perp) \sim e^{-k_\perp^2/\Lambda^2} f(x)$, with $\Lambda \sim 300$ MeV
- Reflects finite hadron size
- Translates into $\mathcal{P}(x, z_3^2) \sim e^{-z_3^2\Lambda^2/4} f(x)$ for pPDF
- Translates into $\mathcal{M}(\nu, z_3^2) \sim e^{-z_3^2\Lambda^2/4} I(\nu)$ for pITD



- Small correction compared to $Z(z_3^2)$
- Also: cancels in the $\mathcal{M}(\nu, z_3^2)/\mathcal{M}(0, z_3^2)$ ratio
- If $\mathcal{M}(\nu, z_3^2) \sim e^{-z_3^2\Lambda^2/4} I(\nu)$ is not perfect, some residual HT terms $\sim z_3^2\lambda^2$ may remain, with $\lambda \lesssim 100$ MeV
- Strategy: fit residual HT from data



Pseudo-PDF strategy in action

- Exploratory lattice study of reduced pseudo-ITD $\mathfrak{M}(\nu, z_3^2)$ for the valence $u_v - d_v$ parton distribution in the nucleon [Orginos et al. 2017]
- Lattice QCD calculations in quenched approximation
- $32^3 \times 64$ lattices, lattice spacing $a = 0.093$ fm
- Pion mass $601(1)$ MeV and nucleon mass $1411(4)$MeV
- Six lattice momenta $p_i\ (2\pi/L)$, with 2.5 GeV maximal momentum
- Relation between PDF and ITD involves $e^{ix\nu} = \cos x\nu + i \sin x\nu$

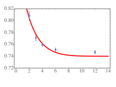
$$\mathcal{I}(\nu) = \int_{-\infty}^{\infty} d\nu\, e^{ix\nu}\, f(x)$$

- Real part of ITD $\mathcal{I}(\nu)$ corresponds to cosine Fourier transform of $q_v(x) = u_v(x) - d_v(x)$

$$\mathcal{R}(\nu) \equiv \mathrm{Re}\, \mathcal{I}(\nu) = \int_0^1 dx\, \cos(\nu x)\, q_v(x)$$

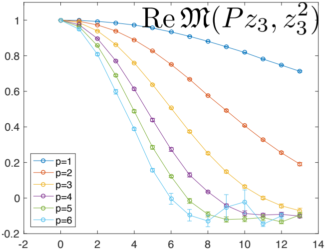
- On the lattice, we extract the reduced pseudo-ITD

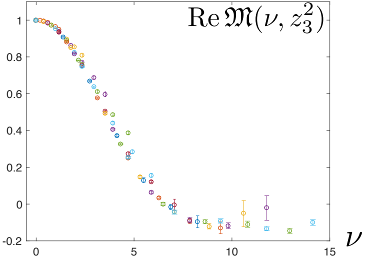
$$\mathfrak{M}(\nu, z_3^2) \equiv \frac{\mathcal{M}(\nu, z_3^2)}{\mathcal{M}(0, z_3^2)}$$



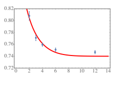
Reduced Ioffe-time distributions

- Left: Real part of the ratio $\mathcal{M}(Pz_3, z_3^2)/\mathcal{M}(0, z_3^2)$ as a function of z_3
- Taken at six values of P $\Rightarrow$ curves have Gaussian-like shape
- $\Rightarrow$ $Z(z_3^2)$ link factor cancels in the ratio





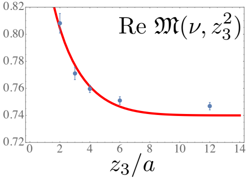
- Right: Same data, as functions of $\nu = Pz_3$ (z_3^2 varies from point to poiint)
- Data practically fall on the same universal curve
- Data show no polynomial z_3-dependence for large z_3 though z_3^2/a^2 changes from 1 to ~ 200
- Apparently no higher-twist terms in the reduced pseudo-ITD



Evolution z_3^2-dependence

- After cancellation of z_3^2-dependence from $Z(z_3^2)$ and (hopefully) HT:
- Remaining z_3^2-dependence corresponds to perturbative (DGLAP) evolution
- At one loop,

$$\mathfrak{M}^{(1)}(\nu, z_3^2) = -\frac{\alpha_s}{2\pi} C_F \ln\left(z_3^2\right) \int_0^1 du\, B(u)\, \mathfrak{M}^{(0)}(u\nu)$$



- Altarelli-Parisi (AP) evolution kernel

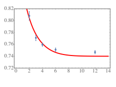
$$B(u) = \left[\frac{1+u^2}{1-u}\right]_+$$

- Example of z_3-dependence for $\nu = 12\pi/16 \approx 2.36$
- “Magic” Ioffe-time pz value:

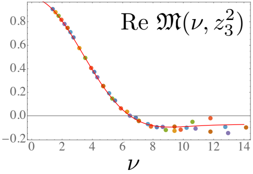
$$12 = 1 \times 12 = 2 \times 6 = 3 \times 4 = 4 \times 3 = 6 \times 2$$

can be obtained for 5 different z's

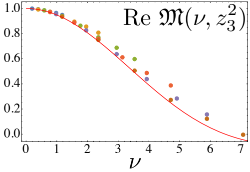
- Shows “perturbative” $\ln\left(1/z_3^2\right)$ for small z_3
- Close to a constant for $z_3 > 6a$
- Finite-size (“HT”) effect in 1-loop terms



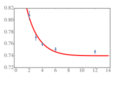
Evolution in lattice data



- Points corresponding to $7a \le z_3 \le 13a$ values
- Some scatter for points with $\nu \gtrsim 10$
- Otherwise, practically all the points lie on a universal curve
- No z_3^2-evolution visible in large-z_3 data



- Points in $a \le z_3 \le 6a$ region
- All points lie higher than the curve based on the $z_3 \ge 7a$ data
- Perturbative evolution increases real part of the pseudo-ITD when z_3 decreases
- Observed higher values of $\mathrm{Re}\,\mathfrak{M}$ for smaller-z_3 points are a consequence of evolution

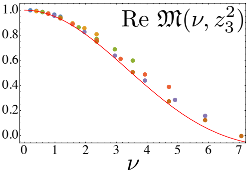


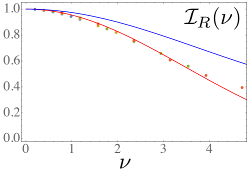
Matching relations

- Matching condition between reduced pseudo-ITD and $\overline{\rm MS}$ ITD (Y. Zhao 2017, A.R. 2017)

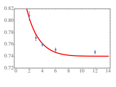
$$\mathfrak{M}(\nu, z_3^2) = \mathcal{I}(\nu, \mu^2) - \frac{\alpha_s(\mu)}{2\pi} C_F \int_0^1 dw\, \mathcal{I}(w\nu, \mu^2) \times \left\{ B(w) \left[\ln\left(z_3^2 \mu^2 \frac{e^{2\gamma_E}}{4} \right) + 1 \right] + \left[4\frac{\ln(1-w)}{1-w} - 2(1-w) \right]_+ \right\}$$

- Building $\overline{\rm MS}$ ITD





- Points in $a \le z_3 \le 4a$ region $\mu = 1/a_L \approx$ 2.15 GeV , $\alpha_s/\pi = 0.1$
- Evolved points have a rather small scatter
- The curve corresponds to the cosine transform of a normalized $\sim x^a(1-x)^b$ distribution with $a = 0.35$ and $b = 3$
- Upper curve: ITD of the CJ15 global fit PDF for $\mu =$2.15 GeV

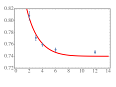


Range of applicability

- Rule of thumb: use perturbation theory when correction is small

$$\mathfrak{M}(\nu, z_3^2) = \mathcal{I}(\nu, \mu^2) - \frac{\alpha_s(\mu)}{2\pi} C_F \int_0^1 dw\, \mathcal{I}(w\nu, \mu^2) \times \left\{ B(w) \left[\ln\left(z_3^2 \mu^2 \frac{e^{2\gamma_E}}{4} \right) + 1 \right] + \left[4 \frac{\ln(1-w)}{1-w} - 2(1-w) \right]_+ \right\}$$

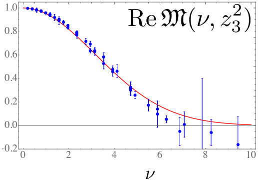
- Factor $e^{2\gamma_E}/4 \approx 1/1.2$ relates scales in $\overline{\rm MS}$ and "z^2" scheme
- Suggesting $\Lambda_{z^2} \approx \Lambda_{\overline{\rm MS}}/1.1$
- Next step: $\mathfrak{M}(\nu, z_3^2) = \mathcal{I}(\nu, \mu^2)$ when α_s correction is zero
- This happens when $\mu \approx 4/z_3$, because of large correction from $\ln(1-w)$
- Numerically: $\mathcal{I}(\nu, (2\,{\rm GeV})^2) \approx \mathfrak{M}(\nu, (0.4\,{\rm fm})^2)$
- Take $\mu = 1$ GeV: $\mathcal{I}(\nu, (1{\rm GeV})^2) \approx \mathfrak{M}(\nu, (0.8\,{\rm fm})^2)$
- $\Rightarrow$ for $a_L \sim 0.1$ fm , PT is formally applicable till $z_3 \sim 8a_L$
- Caution: data show deviation from $\ln(z_3^2)$ for $z_3 \gtrsim 5a_L$
- Finite hadron size effects in $\mathcal{O}(\alpha_s)$ terms



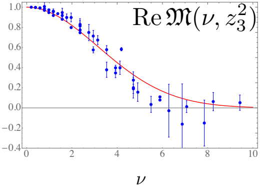
Dynamic fermions (Joo et al., 2019)

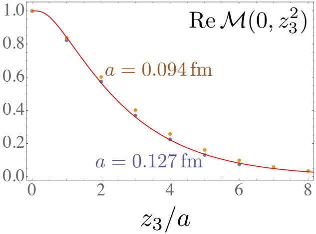
Reduced ITD for two lattice spacings

- $a = 0.127$ fm

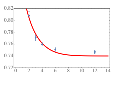


- $a = 0.094$ fm



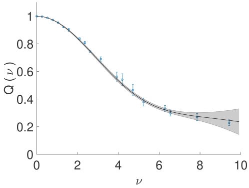


- Z-factor Re $\mathcal{M}(0, z_3^2)$ for two lattice spacings
- Essentially universal function of z/a
- Curve is given by perturbative formula for the link $Z(z/a)$ factor with $\alpha_s = 0.26$
- $a_L = 0.094$ data are described by PT formula with $\alpha_s = 0.24$

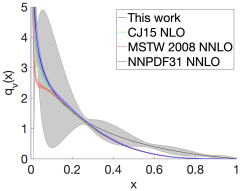


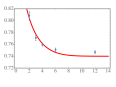
PDF from dynamic fermions (2019)

- Light-cone ITD for $\mu = 2$ GeV extracted from $a = 0.127$ fm data



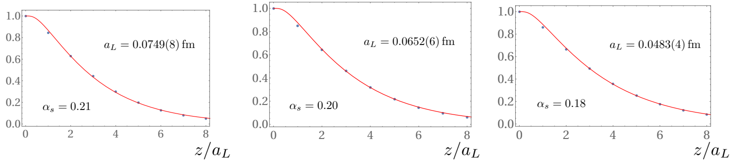
- PDF compared to global fits

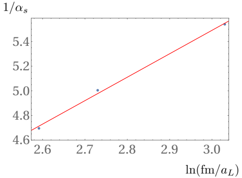


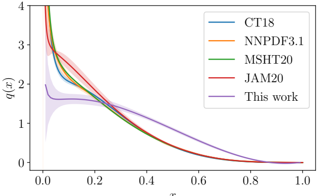


3 lattice spacings (Karpie et al. 2021)

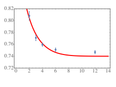
- Z-factor Re $\mathcal{M}(0, z_3^2)$ is clearly a function of z_3/a_L







- α_s decreases with a_L. Check if it is $\alpha_s(1/a_L)$
- Since $\alpha_s(1/a_L) = 2\pi/[b_0 \ln(1/a_L\Lambda)]$, we plot $1/\alpha_s$ versus $\ln(1/a_L)$
- Fit corresponds to $\Lambda = 200$ MeV, and $\beta_0 = 11.4$
- Since $\beta_0 = 11 - 2N_f/3$, contribution of quark loops into α_s in this simulation is not visible
  - Comparison with global fits
  - Lattice result is smaller for small x
  - Pion mass was taken 440 MeV
  - Too large to give realistic PDF for small x
  - Higher twists $\lesssim 0.15\,\Lambda^2_{\mathrm{QCD}} z_3^2$



Gluon PDFs

- Correlator of two gluon fields has 4 indices

$$M_{\mu\alpha;\nu\beta}(z,p) \equiv \langle p|\, G_{\mu\alpha}(z)\, [z,0]\, G_{\nu\beta}(0) |p\rangle$$

- Need 6 invariant amplitudes $\mathcal{M}(\nu, z^2)$

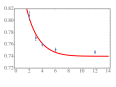
$$\begin{aligned} M_{\mu\alpha;\nu\beta}(z,p) = & \left(g_{\mu\nu}p_\alpha p_\beta - g_{\mu\beta}p_\alpha p_\nu - g_{\alpha\nu}p_\mu p_\beta + g_{\alpha\beta}p_\mu p_\nu\right) \mathcal{M}_{pp}(\nu, z^2) \\ & + \left(g_{\mu\nu}z_\alpha z_\beta - g_{\mu\beta}z_\alpha z_\nu - g_{\alpha\nu}z_\mu z_\beta + g_{\alpha\beta}z_\mu z_\nu\right) \mathcal{M}_{zz}(\nu, z^2) \\ & + \left(g_{\mu\nu}z_\alpha p_\beta - g_{\mu\beta}z_\alpha p_\nu - g_{\alpha\nu}z_\mu p_\beta + g_{\alpha\beta}z_\mu p_\nu\right) \mathcal{M}_{zp}(\nu, z^2) \\ & + \left(g_{\mu\nu}p_\alpha z_\beta - g_{\mu\beta}p_\alpha z_\nu - g_{\alpha\nu}p_\mu z_\beta + g_{\alpha\beta}p_\mu z_\nu\right) \mathcal{M}_{pz}(\nu, z^2) \\ & + \left(p_\mu z_\alpha - p_\alpha z_\mu\right)\left(p_\nu z_\beta - p_\beta z_\nu\right) \mathcal{M}_{ppzz}(\nu, z^2) \\ & + \left(g_{\mu\nu}g_{\alpha\beta} - g_{\mu\beta}g_{\alpha\nu}\right) \mathcal{M}_{gg}(\nu, z^2) \end{aligned}$$

- "Light-cone" gluon distribution $f_g(x)$ is defined through the convolution $g^{\alpha\beta} M_{+\alpha;\beta+}(z,p)$, with z taken in the light-cone "minus" direction, $z = z_-$:

$$g^{\alpha\beta} M_{+\alpha;\beta+}(z_-,p) = p_+^2 \int_{-1}^{1} \mathrm{d}x\, e^{ixp_+z_-}\, x f_g(x)$$

- In terms of invariant amplitudes

$$g^{\alpha\beta} M_{+\alpha;\beta+}(z_-,p) = -2p_+^2 \mathcal{M}_{pp}(\nu, 0)$$



Picking out twist-2 distribution

- Strategy is to choose matrix elements $M_{\mu\alpha;\lambda\beta}$ that contain $\mathcal{M}_{pp}$ in their parametrization – and ideally nothing else!
- Split the "+" components onto sum of space- and time-components
- Due to antisymmetry of $G_{\rho\sigma}$ with respect to its indices, $g^{\alpha\beta} M_{+\alpha;\beta+}(z,p)$ includes summation over transverse indices $i,j = 1,2$ only

$$g^{ij} M_{+i;j+} = -M_{+1;1+} - M_{+2;2+} = M_{0i;0i} + M_{3i;3i} + (M_{0i;3i} + M_{3i;0i})$$

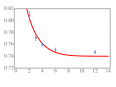
- Decomposition of these matrix elements in invariant amplitudes

$$\begin{aligned} M_{0i;i0} &= 2p_0^2 \mathcal{M}_{pp} + 2\mathcal{M}_{gg} \\ M_{3i;i3} &= 2p_3^2 \mathcal{M}_{pp} + 2z_3^2 \mathcal{M}_{zz} + 2z_3 p_3 \left(\mathcal{M}_{zp} + \mathcal{M}_{pz}\right) - 2\mathcal{M}_{gg} \\ M_{0i;i3} &= 2p_0 \left(p_3 \mathcal{M}_{pp} + z_3 \mathcal{M}_{pz}\right) \\ M_{3i;i0} &= 2p_0 \left(p_3 \mathcal{M}_{pp} + z_3 \mathcal{M}_{zp}\right) \end{aligned}$$

- All contain the $\mathcal{M}_{pp}$, though with different kinematical factors
- Unfortunately, none of them is just $\mathcal{M}_{pp}$
- Fortunately, $M_{ji;ij} = -2\mathcal{M}_{gg}$
- Hence, the combination

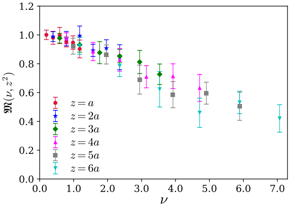
$$M_{0i;i0} + M_{ji;ij} = 2p_0^2 \mathcal{M}_{pp}$$

may be used for extraction of the twist-2 function $\mathcal{M}_{pp}$

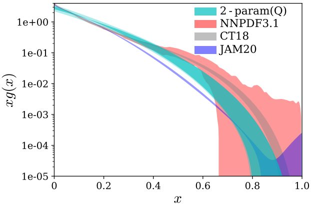


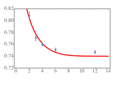
Gluon PDF extraction

- Reduced Ioffe-time pseudo-distribution $\mathfrak{M}(\nu, z^2)$



- Extracted gluon distribution (HadStruc, 2021)





Polarized gluon distribution

- In the polarized gluon case, tensor structures may be built from 3 vectors: z_μ, p_μ and s_μ
- As a result, one deals with 12 invariant amplitudes
- Combination, similar to that used in unpolarized case

$$\widetilde{M}_{0i;0i}(z,p) + \widetilde{M}_{ij;ij}(z,p) = -2p_z p_0 \widetilde{\mathcal{M}}_{sp}^{(+)}(\nu, z^2) + 2p_0^3 z \widetilde{\mathcal{M}}_{pp}(\nu, z^2)$$

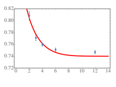
- The polarized gluon PDF is determined by the Ioffe-time distribution

$$-i\widetilde{\mathcal{I}}_p(\nu) \equiv \widetilde{\mathcal{M}}_{ps}^{(+)}(\nu) - \nu \widetilde{\mathcal{M}}_{pp}(\nu)$$

- Matrix element $\widetilde{M}_{0i;0i}(z,p) + \widetilde{M}_{ij;ij}(z,p)$ has a “slightly” different structure

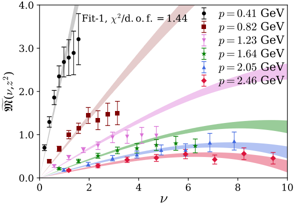
$$\widetilde{\mathfrak{M}}(\nu, z^2) = \left[\widetilde{\mathcal{M}}_{sp}^{(+)}(\nu, z^2) - \nu \widetilde{\mathcal{M}}_{pp}(\nu, z^2)\right] - \frac{m_p^2}{p_z^2} \nu \widetilde{\mathcal{M}}_{pp}(\nu, z^2)$$

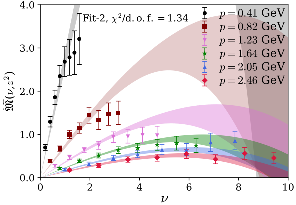
- The goal is to eliminate the $\mathcal{O}(m_p^2/p_z^2)$ contamination term and extract $\widetilde{\mathcal{M}}_{sp}^{(+)}(\nu, z^2) - \nu \widetilde{\mathcal{M}}_{pp}(\nu, z^2)$



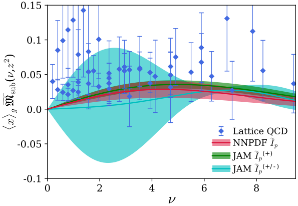
Extraction of polarized gluon PDF

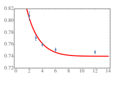
- Reduced Ioffe-time pseudo-distribution $\widetilde{\mathfrak{M}}(\nu, z^2)$





- Comparison with experimental fits converted into ITD (HadStruc, 2022)





In conclusion

- Psedodistribution approach allows to study hadron structure in a way similar to experimental study of DIS
- Instead of structure functions $W(x, Q^2)$, we study Ioffe-time distributions $\mathcal{M}(\nu, z_3^2)$
- Ioffe time ν is Fourier-conjugate to x
- z_3 is probing scale, like $1/Q$ in DIS
- Detailed studies of ν- and z_3^2-dependence decipher subtleties of hadron dynamics
- Existing lattice extractions of PDFs still play exploratory role
- The current goal is to check that lattice methods give reasonable results for PDFs known experimentally
- The future goal is to get the functions which are not directly accessible by experiment: a key example is given by GPDs $H(x, \xi; t)$