Parametrization of PDFs with Bézier Curves

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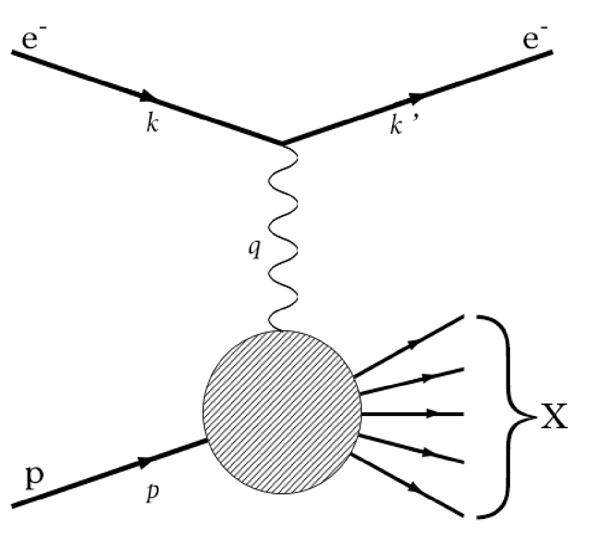
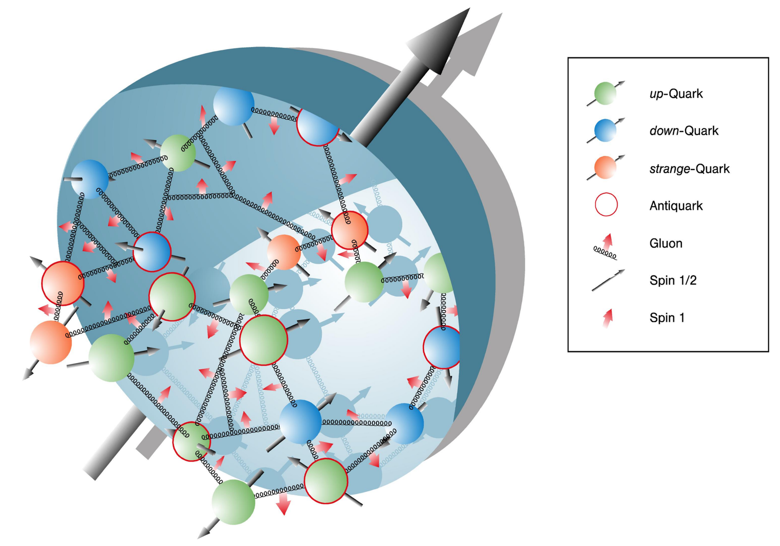
Theoretical Background: DIS + QCD



(a) DIS as an inclusive process. [1]



(b) A hadron containing valence quarks, gluons and quark-antiquark pairs.

Figure 1. DIS and QCD provide a picture of the internal structure of hadrons.

Parton distribution functions (PDFs) are densities arising from DIS, which describe the probability of finding a parton of type i carrying an x-fraction of the proton's momentum p such that, assuming the parton model and Bjorken scaling, the proton's structure functions are single-variable dependent and determined by adding the contribution of all partons and integrating over all values of x [1]:

$$\begin{cases} W_1(x) = \sum_i \int_0^1 f_i(x')\, W_1^i\, dx' = \frac{1}{2M} \sum_i f_i(x)\, e_i^2 & (1) \\ W_2(x) = \sum_i \int_0^1 f_i(x')\, W_2^i\, dx' = \frac{x}{\nu} \sum_i f_i(x)\, e_i^2 & (2) \end{cases}$$

where x represents Bjorken's scaling variable, e_i is the electric charge of an i-parton, M is the proton rest mass and $\nu = (p \cdot q)/M$ is the rest frame energy transfer.

Hadrons are not only constituted by valence quarks which give them their quantum numbers, but by gluons and a sea of quark-antiquark pairs carrying a momentum fraction as well. QCD states that structure functions, in fact, depend logarithmically on Q^2, which is known as *scaling violation*. Still, the PDF x-dependence of none of the flavors is originally known since PDFs are objects inherent to hadrons, implying they cannot be derived from QCD first principles nor can be computed via perturbation theory. Aitchison and Hey label them in DIS as *"fundamental parameters of the proton"* [1].

Determination of PDFs

PDFs are essential to make precise QCD predictions. Moreover, PDFs are universal objects showing up in distinct combinations for different hard scattering processes. However unknown PDFs are, they must comply theoretical constraints.

Knowledge on PDFs is accessed through experimental data of hard-scattering experiments on nucleons at distinct values of Q^2 and x. Global analyses on such data are carried out by different groups: CTEQ, NNPDF, MSHT, among others; each collaboration follows a different procedure and uncertainty assessment. Then, PDFs are fit to the experimental data by using chosen functional forms and determining the set of free parameters which minimize a *goodness-of-fit* χ^2 function. Once PDFs are determined at an initial Q_0 scale, the evolution through Q^2 is determined through DGLAP equations.

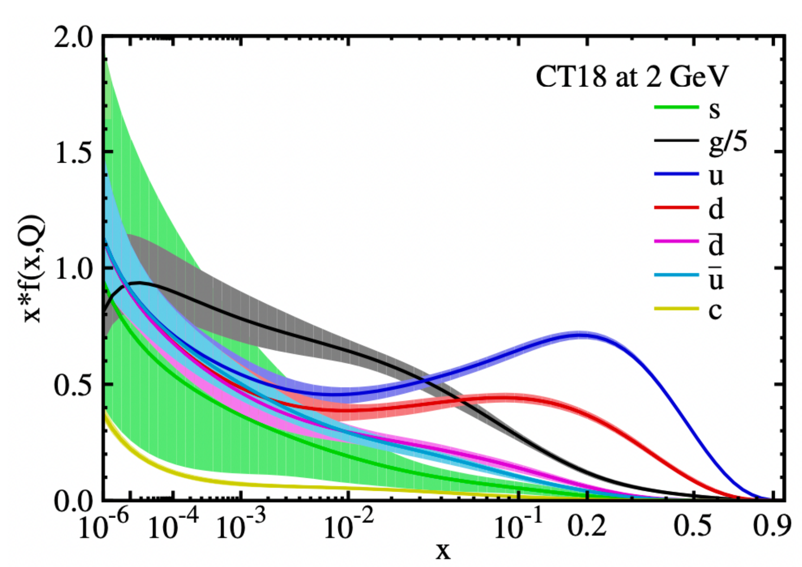


Figure 2. CT18 set of PDFs at $Q = 2$ GeV [5].

Bézier curves and Bernstein polynomials

Bézier curves are a parametric tool originated in Computer Aided Graphic Design: *"a discipline dealing with computational aspects of geometrical objects"* [3]. They have numerous applications such as design and manufacturing, digital animation and scientific visualization [3, 4]. They were created independently by Paul de Casteljau in the early 1960s and revisited by Pierre Bézier years after.

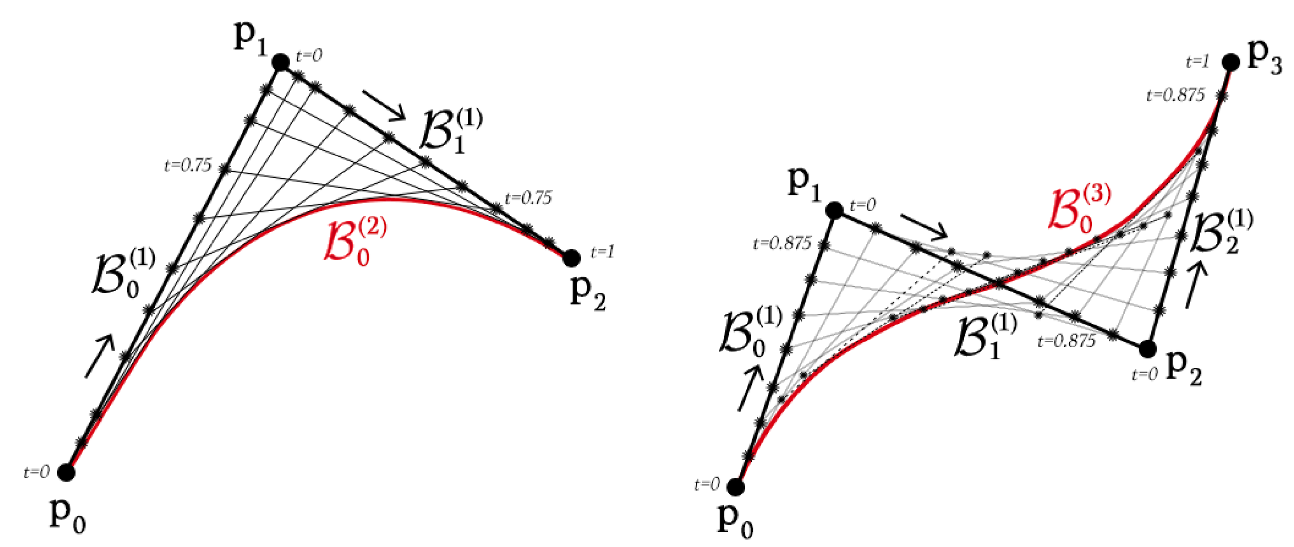


Figure 3. Bézier curves of second and third order, respectively.

The algorithm devised by De Casteljau allows to draw smooth curves by doing recursive linear interpolations between chosen control points in $\mathbb{R}^2$ or $\mathbb{R}^3$. As an example, a second order linear interpolation is given by the equation:

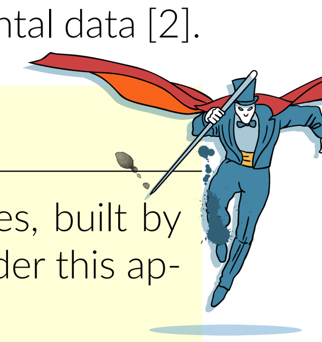
$$\mathcal{B}_0^{(2)} = (1-t)^2\, \mathbf{p_0} + 2t(1-t)\, \mathbf{p_1} + t^2\, \mathbf{p_2} \qquad (3)$$

revealing a binomial structure which is preserved as higher-degree curves are produced. Therefore, it is convenient to make use of the Bernstein basis, such that a Bézier curve of order n is given by:

$$\mathcal{B}^{(\mathbf{n})}(t) = \sum_{l=0}^{n} \mathbf{p}_l\, B_{n,l}(t) = \sum_{l=0}^{n} \mathbf{p}_l \binom{n}{l} t^l (1-t)^{n-l} \qquad (4)$$

where points' coordinates serve as weights to modulate the curvature along the line. Nonetheless, our approach is to employ control points for PDF fitting while maintaining the Bernstein basis as a functional form. Our goal is to take advantage of the flexibility of Bézier curves to mimic a variety of behaviours of PDFs and produce new fits agreeing with experimental data [2].

*Fantômas4QCD**



Fantômas4QCD is the name of a module made for PDF fitting using Bézier curves, built by physicists from UNAM's Physics Institute and Southern Methodist University. Under this approach, a PDF is given the following functional form:

$$xf(x) = \mathbf{F_c}\,(x; \{A\}) \times \mathbf{P}\,(x^{\alpha_x}; \{c\}) \qquad (5)$$

with

$$\mathbf{F_c}\,(x; \{A\}) = A_f x^{B_f} (1-x)^{C_f}, \qquad \mathbf{P}\,(x^{\alpha_x}; \{c\}) = \sum_{l=0}^{n} c_l\, B_{n,l}\,(x^{\alpha_x}) \qquad (6)$$

where $\mathbf{F_c}$ and $\mathbf{P}$ are the *carrier* and *modulator* functions, respectively. The carrier function with three free parameters determines the normalization and asymptotic behaviour as $x \to 0, 1$, while the modulator is a Bernstein polynomial of degree n with a set of c_l coefficients which must be determined. Additionally, the modulator x-dependence allows scaling with a power $\alpha_x > 0$ to stretch the argument and attain smooth curves. To compute the set of coefficients, the modulator is expressed as a matrix product:

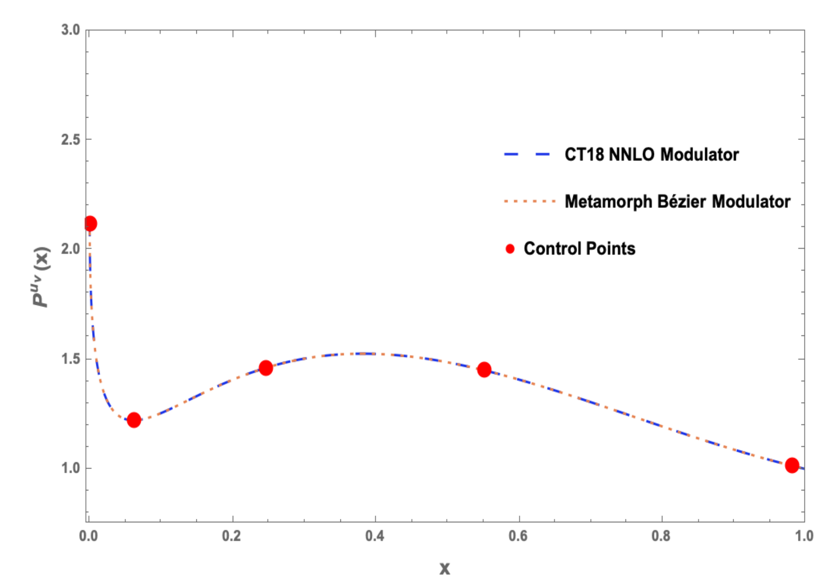
$$\mathbf{P} = \boldsymbol{T} \cdot \boldsymbol{M} \cdot \boldsymbol{C} \Rightarrow \boldsymbol{C} = \boldsymbol{M}^{-1} \cdot \boldsymbol{T}^{-1} \cdot \boldsymbol{P} \qquad (7)$$

becoming a linear system which is solved by providing a grid of experimental (x_i, f_i) control points to be fitted. $\boldsymbol{C}$ is the coefficient vector, the $\boldsymbol{M}$ matrix contains combinations of binomial coefficients, the $\boldsymbol{T}$ matrix contains increasing powers of x_i and the $\boldsymbol{P}$ vector is comprised of f_i coordinates. An extensive derivation of this method is provided in [6]. We call the Bézier-based computed fits *Metamorphs*.

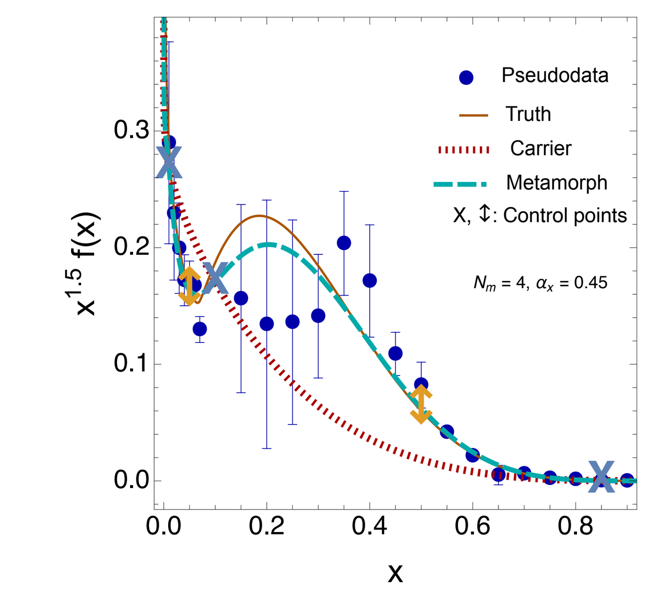
**The module is named after the french antihero Fantômas, who flies a winged Citröen DS, a car designed by means of Bézier curves.*

Preliminary Results and Future Plans

Once the initial setup was completed, *Fantômas4QCD*'s functioning was cross-checked by reproducing the shapes of CT18 fits [5] with *Metamorphs*, selecting different sets of given control points and polynomial degrees:



(a) Fourth degree *Metamorph* fit for $\mathbf{P}_{u_v}(\sqrt{x})$.



(b) Fourth degree *Metamorph* with minimization.

Figure 4. *Metamorph* fits

Fantômas4QCD was able to reproduce the behaviour of the original functions (fig. **4a**). As depicted in fig. **4b**, the carrier function successfully determines the behaviour at endpoints, while the modulator makes shifts in between. The *Metamorph* environment allows to either fix or vary the carrier parameters as well. After several runs, the development team conceived a set of criteria on how to implement the *Metamorph* module to take advantage of the flexibility of Bézier curves. Three parameters are of special importance: control-point spacing, power scaling and polynomial degree. Additionally, control points must be chosen in an optimal way according to the problem, so that *Fantômas4QCD* works in conjunction with the *xFitter* package to find the set of parameters minimizing the χ^2 function using the Hessian methodology [5].

Bézier curves have proven to be suitable for PDF fitting by providing sufficient flexibility to mimic a variety of behaviours. On the other hand, *Fantômas4QCD* is fully functional now and ready to produce Bézier fits. The *C++* module and *Mathematica* notebook will be released soon, along with the set of guidelines aforementioned and analysis on error propagation.

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