Lattice QCD calculations of Transverse Momentum Dependent Parton Distribution Functions (TMDs)

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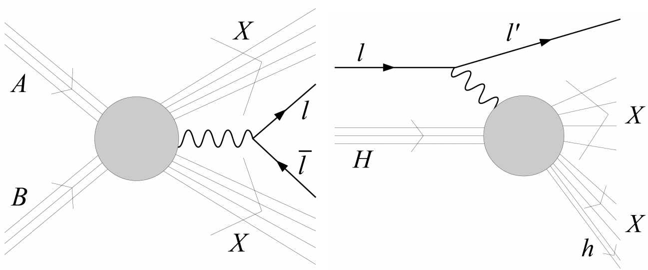
Introduction: TMDs

The **intrinsic motion of quarks and gluons** inside the proton or neutron, specifically **with respect to the transverse momentum**, can be described in terms of Transverse Momentum Dependent Parton Distribution Functions (TMDs)



(a) The Drell-Yan process. (b) The SIDIS process.

Figure 1: Two examples of processes sensitive to TMD PDFs. We draw the leading contributions, in which a single electroweak gauge boson (wiggled lines) is exchanged.

Introduction: TMDs

In the SIDIS cross section

$$\frac{d\sigma}{d^3 P_h d^3 P_{l'}} \propto L_{\mu\nu} W^{\mu\nu} \tag{1}$$

the lepton tensor $L_{\mu\nu}$ is calculable in perturbation theory. All the non-perturbative information related to hadron structure is encoded in the hadron tensor

$$W^{\mu\nu}(P,q,P_h) = \int \frac{d^4 l}{(2\pi)^4} e^{iq\cdot l} \sum_X \langle N(P,S)| J^\mu(-b) |Xh(P_h,S_h)\rangle \times \langle Xh(P_h,S_h)| J^\nu(0) |N(P,S)\rangle$$

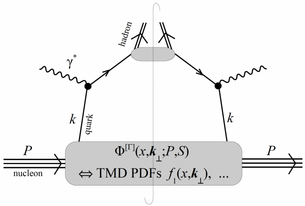


Figure 2: Simplifed factorized tree level diagram of the hadron tensor in SIDIS. *arXiv:0907.2381*

Definition of TMDs

- Let's consider a frame of reference where the nucleon has large momentum in z-direction, i.e., $P^+ \gg m_N$, $\boldsymbol{P}_\mathrm{T} = 0$. In light cone coordinates, the components $\boxed{\mathrm{k}^+ : \boldsymbol{k}_\mathrm{T} : k^- \sim P^+/m_N : 1 : m_N/P^+}$, under boosts along the z-axis.

The starting point for our discussion of TMDs are the correlator of the general form

$$\Phi^{[\Gamma]}(k, P, S; \ldots) \equiv \int \frac{d^4 b}{(2\pi)^4} e^{ik\cdot b} \frac{\overbrace{\frac{1}{2} \langle P, S| \ \bar{q}(0)\, \Gamma\, \mathcal{U}[\mathcal{C}_b]\ q(b)\ |P, S\rangle}^{\equiv \widetilde{\Phi}^{[\Gamma]}_{\text{unsubtr.}}(b, P, S; \ldots)}}{\widetilde{\mathcal{S}}(b^2; \ldots)} \qquad (2)$$

- The gauge link $\mathcal{U}[\mathcal{C}_b]$ brings divergences; so we divide it by soft factor $\tilde{\mathcal{S}}$.

Definition of TMDs

Integrating the correlator over the suppressed momentum component k^- yields

$$\Phi^{[\Gamma]}(x, \boldsymbol{k}_\mathrm{T}; P, S; \ldots) \equiv \int dk^- \Phi^{[\Gamma]}(k, P, S; \ldots)$$
$$= \int \frac{d^2 \boldsymbol{b}_\mathrm{T}}{(2\pi)^2} \int \frac{d(b\cdot P)}{(2\pi) P^+} \; e^{ix(b\cdot P) - i\boldsymbol{b}_\mathrm{T}\cdot \boldsymbol{k}_\mathrm{T}} \; \frac{\frac{1}{2} \langle P,S| \; \bar{q}(0)\, \Gamma \; \mathcal{U}[\mathcal{C}_b] \; q(b) \; |P,S\rangle}{\widetilde{\mathcal{S}}(-\boldsymbol{b}_\mathrm{T}^2; \ldots)}\Bigg|_{b^+=0} . \quad (3)$$

The above correlator can be decomposed into TMDs.

$$\Phi^{[\gamma^+]}(x, \boldsymbol{k}_\mathrm{T}; P, S, \ldots) = f_1 - \left[\frac{\epsilon_{ij}\, \boldsymbol{k}_i\, \boldsymbol{S}_j}{m_N} \; f_{1T}^\perp\right]_{\mathrm{odd}} , \quad (4)$$

$$\Phi^{[\gamma^+\gamma^5]}(x, \boldsymbol{k}_\mathrm{T}; P, S, \ldots) = \Lambda\, g_1 + \frac{\boldsymbol{k}_\mathrm{T} \cdot \boldsymbol{S}_\mathrm{T}}{m_N} \; g_{1T} , \quad (5)$$

$$\Phi^{[i\sigma^{i+}\gamma^5]}(x, \boldsymbol{k}_\mathrm{T}; P, S, \ldots) = \boldsymbol{S}_i \; h_1 + \frac{(2\boldsymbol{k}_i \boldsymbol{k}_j - \boldsymbol{k}_\mathrm{T}^2 \delta_{ij}) \boldsymbol{S}_j}{2m_N^2} \, h_{1T}^\perp$$
$$+ \frac{\Lambda \boldsymbol{k}_i}{m_N} h_{1L}^\perp + \left[\frac{\epsilon_{ij} \boldsymbol{k}_j}{m_N} h_1^\perp\right]_{\mathrm{odd}} . \quad (6)$$

Strategy

- The separation b of the quark field operators has an additional transverse component, $b = nb^- + b_\perp$. Thus, in general, this separation is space-like.

$$\Phi^{[\Gamma]}(k, P, S; \ldots) \equiv \int \frac{d^4b}{(2\pi)^4} e^{ik\cdot b} \frac{\overbrace{\frac{1}{2} \langle P, S| \ \bar{q}(0)\, \Gamma\, \mathcal{U}[\mathcal{C}_b] \ q(b) \ |P, S\rangle}^{\equiv \widetilde{\Phi}^{[\Gamma]}_{\text{unsubtr.}}(b, P, S; \ldots)}}{\widetilde{\mathcal{S}}(b^2; \ldots)} \quad (7)$$

- We parametrized this correlator in terms of Lorentz-invariant amplitudes.
- We choose the Lorentz frame in which this nonlocal operator is defined at one single time as the one most suitable for our calculation.
- In the aforementioned frame, the computation of the nonlocal matrix element can be cast in terms of a Euclidean path integral and performed employing the standard methods of lattice QCD.

Link geometry

The gauge link employed in this work reads

$$\mathcal{U}[\mathcal{C}_b^{(\eta v)}] = \mathcal{U}[0, \eta v, \eta v + b, b], \tag{8}$$

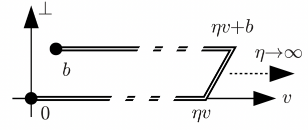


Figure 3: Staple-shaped gauge connection. The four-vectors v and P give the direction of the staple and the momentum, while b defines the separation between the quark operators. *(arXiv:1111.4249v2 [hep-lat])*

The Lorentz-invariant quantity characterizing the direction of v is the Collins-Soper like parameter

$$\boxed{\hat{\zeta} \equiv \zeta/2m_N = \frac{v \cdot P}{\sqrt{|v^2|}\sqrt{P^2}}.} \tag{9}$$

The light-like direction $v = n$ can be approached in the limit $\zeta \to \infty$.

Parametrization of the correlator in position space

$$\widetilde{\Phi}^{[\Gamma]}_{\text{unsubtr.}}(b,P,S,\hat{\zeta},\mu) \equiv \frac{1}{2} \, \langle P,S| \;\, \bar{q}(0)\,\Gamma\, \mathcal{U}[0,\eta v,\eta v+b,b]\; q(b) \;\, |P,S\rangle \tag{10}$$

For the Γ-structures at leading twist, the correlator can be written in the form

$$\frac{1}{2P^+}\widetilde{\Phi}^{[\gamma^+]}_{\text{unsubtr.}} = \widetilde{A}_{2B} + i m_N \epsilon_{ij} \boldsymbol{b}_i \boldsymbol{S}_j \, \widetilde{A}_{12B} \tag{11}$$

$$\frac{1}{2P^+}\widetilde{\Phi}^{[\gamma^+\gamma^5]}_{\text{unsubtr.}} = -\Lambda\, \widetilde{A}_{6B} + i\left\{(b\cdot P)\Lambda - m_N(\boldsymbol{b}_{\mathrm{T}}\cdot\boldsymbol{S}_{\mathrm{T}})\right\}\, \widetilde{A}_{7B} \tag{12}$$

$$\begin{aligned}\frac{1}{2P^+}\widetilde{\Phi}^{[i\sigma^{i+}\gamma^5]}_{\text{unsubtr.}} &= i m_N \epsilon_{ij} \boldsymbol{b}_j\, \widetilde{A}_{4B} - \boldsymbol{S}_i\, \widetilde{A}_{9B} - i m_N \Lambda \boldsymbol{b}_i\, \widetilde{A}_{10B} \\ &\quad + m_N\left\{(b\cdot P)\Lambda - m_N(\boldsymbol{b}_{\mathrm{T}}\cdot\boldsymbol{S}_{\mathrm{T}})\right\}\boldsymbol{b}_i\, \widetilde{A}_{11B}\end{aligned} \tag{13}$$

(Decompositions analogous to work by Metz et al. Phys. Lett. **B618** (2005) 90-96. in momentum space)

TMDs in Fourier space and x-integration

$$\tilde{f}(x, \boldsymbol{b}_\mathrm{T}^2; \ldots) \equiv \int d^2\boldsymbol{k}_\mathrm{T}\, e^{i\boldsymbol{b}_\mathrm{T}\cdot\boldsymbol{k}_\mathrm{T}}\; f(x, \boldsymbol{k}_\mathrm{T}^2; \ldots) \tag{14}$$

$$\tilde{f}^{(n)}(x, \boldsymbol{b}_\mathrm{T}^2 \ldots) \equiv n! \left(-\frac{2}{m_N^2}\partial_{\boldsymbol{b}_\mathrm{T}^2} \right)^n \; \tilde{f}(x, \boldsymbol{b}_\mathrm{T}^2; \ldots) \tag{15}$$

In the limit $|\boldsymbol{b}_\mathrm{T}| \to 0$, one recovers conventional $\boldsymbol{k}_\mathrm{T}$-moments of TMDs:

$$\tilde{f}^{(n)}(x, 0; \ldots) = \int d^2\boldsymbol{k}_\mathrm{T} \left(\frac{\boldsymbol{k}_\mathrm{T}^2}{2m_N^2} \right)^n f(x, \boldsymbol{k}_\mathrm{T}^2; \ldots) \equiv f^{(n)}(x) \ . \tag{16}$$

$\boldsymbol{k}_\mathrm{T}$-moments like $f_1^{(0)}(x)$ and $f_{1T}^{\perp(1)}(x)$ are ill-defined without further regularization, we therefore do not attempt to extrapolate to $\boldsymbol{b}_\mathrm{T} = 0$, but rather state our results at finite $|\boldsymbol{b}_\mathrm{T}|$.

In our studies so far, we only considered the first x-moments (accessible at $b \cdot P = 0$), rather than scanning range of $b \cdot P$

$$f^{[1]}(\boldsymbol{k}_\mathrm{T}^2; \ldots) \equiv \int_{-1}^{1} dx\; f(x, \boldsymbol{k}_\mathrm{T}^2; \ldots) \ . \tag{17}$$

T-even and T-odd TMDs

The T-even distributions f_1, g_1, h_1, g_{1T}, $h_{1L}^{\perp}$ and $h_{1T}^{\perp}$ fulfill

$$\boxed{f^{\text{T-even}}(x, \boldsymbol{k}_{\text{T}}^2; \hat{\zeta}, \ldots, \eta v{\cdot}P) = f^{\text{T-even}}(x, \boldsymbol{k}_{\text{T}}^2; \hat{\zeta}, \ldots, -\eta v{\cdot}P)} \tag{18}$$

while the T-odd distributions, i.e., at leading twist the Sivers function $f_{1T}^{\perp}$ and the Boer-Mulders function $h_1^{\perp}$, fulfill

$$\boxed{f^{\text{T-odd}}(x, \boldsymbol{k}_{\text{T}}^2; \hat{\zeta}, \ldots, \eta v{\cdot}P) = -f^{\text{T-odd}}(x, \boldsymbol{k}_{\text{T}}^2; \hat{\zeta}, \ldots, -\eta v{\cdot}P)} \tag{19}$$

As a result, T-odd distributions must vanish for $\eta = 0$, which corresponds to straight gauge links.

TMDs in Fourier space and invariant amplitudes $\widetilde{A}_i(b^2, b{\cdot}P, (b{\cdot}P)R(\hat{\zeta}^2)/m_N^2, -1/(m_N\hat{\zeta})^2, \eta v{\cdot}P)$

certain x-integrated TMDs in Fourier space directly correspond to the amplitudes $\widetilde{A}_{iB}$ evaluated at $\boxed{b{\cdot}P = 0}$:

$$\begin{aligned}
\tilde{f}_1^{[1](0)}(\boldsymbol{b}_\mathrm{T}^2; \hat{\zeta}, \ldots, \eta v{\cdot}P) &= 2\,\widetilde{A}_{2B}(-\boldsymbol{b}_\mathrm{T}^2, 0, 0, -1/(m_N\hat{\zeta})^2, \eta v{\cdot}P)/\widetilde{\mathcal{S}}(b^2; \ldots)\,, \\
\tilde{f}_{1T}^{\perp[1](1)}(\boldsymbol{b}_\mathrm{T}^2; \hat{\zeta}, \ldots, \eta v{\cdot}P) &= -2\,\widetilde{A}_{12B}(-\boldsymbol{b}_\mathrm{T}^2, 0, 0, -1/(m_N\hat{\zeta})^2, \eta v{\cdot}P)/\widetilde{\mathcal{S}}(b^2; \ldots)\,, \\
\tilde{h}_1^{\perp[1](1)}(\boldsymbol{b}_\mathrm{T}^2; \hat{\zeta}, \ldots, \eta v{\cdot}P) &= 2\,\widetilde{A}_{4B}(-\boldsymbol{b}_\mathrm{T}^2, 0, 0, -1/(m_N\hat{\zeta})^2, \eta v{\cdot}P)/\widetilde{\mathcal{S}}(b^2; \ldots)\,.
\end{aligned} \tag{20}$$

Generalized Sivers shifts from amplitudes

All other renormalization and soft factor related dependences cancel out in the ratio.

- $\langle \boldsymbol{k}_y \rangle^{\text{Sivers}} = \langle \boldsymbol{k}_y \rangle_{TU}$ is T-odd, it describes a feature of the transverse momentum distribution of (unpolarized) quarks in a transversely polarized proton.

$$\begin{aligned}
\langle \boldsymbol{k}_y \rangle_{TU}(\boldsymbol{b}_{\mathrm{T}}^2; \hat{\zeta}, \eta v\cdot P) \;&\equiv\; m_N \frac{\tilde{f}_{1T}^{\perp[1](1)}(\boldsymbol{b}_{\mathrm{T}}^2; \hat{\zeta}, \ldots, \eta v\cdot P)}{\tilde{f}_1^{[1](0)}(\boldsymbol{b}_{\mathrm{T}}^2; \hat{\zeta}, \ldots, \eta v\cdot P)} \\
&= -m_N \frac{\widetilde{A}_{12B}(-\boldsymbol{b}_{\mathrm{T}}^2, 0, 0, -1/(m_N\hat{\zeta})^2, \eta v\cdot P)}{\widetilde{A}_{2B}(-\boldsymbol{b}_{\mathrm{T}}^2, 0, 0, -1/(m_N\hat{\zeta})^2, \eta v\cdot P)} \\
&\xrightarrow{\boldsymbol{b}_{\mathrm{T}}^2=0} \left. \frac{\int dx \int d^2\boldsymbol{k}_{\mathrm{T}}\, \boldsymbol{k}_y \;\; \Phi^{[\gamma^+]}(x, \boldsymbol{k}_{\mathrm{T}}, P, S; \ldots)}{\int dx \int d^2\boldsymbol{k}_{\mathrm{T}} \;\; \Phi^{[\gamma^+]}(x, \boldsymbol{k}_{\mathrm{T}}, P, S; \ldots)} \right|_{\boldsymbol{S}_{\mathrm{T}} = (1,0)}
\end{aligned} \tag{21}$$

Generalized Boer-Mulders shifts from amplitudes

- $\langle \boldsymbol{k}_y \rangle^{\mathrm{BM}} = \langle \boldsymbol{k}_y \rangle_{UT}$ is also T-odd and addresses the distribution of transversely polarized quarks in an unpolarized proton.

$$\langle \boldsymbol{k}_y \rangle_{UT}(\boldsymbol{b}_{\mathrm{T}}^2; \hat{\zeta}, \eta v\cdot P) \equiv m_N \frac{\tilde{h}_1^{\perp[1](1)}(\boldsymbol{b}_{\mathrm{T}}^2; \hat{\zeta}, \ldots, \eta v\cdot P)}{\tilde{f}_1^{[1](0)}(\boldsymbol{b}_{\mathrm{T}}^2; \hat{\zeta}, \ldots, \eta v\cdot P)}$$
$$= m_N \frac{\widetilde{A}_{4B}(-\boldsymbol{b}_{\mathrm{T}}^2, 0, 0, -1/(m_N\hat{\zeta})^2, \eta v\cdot P)}{\widetilde{A}_{2B}(-\boldsymbol{b}_{\mathrm{T}}^2, 0, 0, -1/(m_N\hat{\zeta})^2, \eta v\cdot P)}$$
$$\xrightarrow{\boldsymbol{b}_{\mathrm{T}}^2=0} \left. \frac{\sum_{\Lambda=\pm1} \int dx \int d^2\boldsymbol{k}_{\mathrm{T}}\, \boldsymbol{k}_y \;\; \Phi^{[\gamma^+ + s^j i\sigma^{j+}\gamma^5]}(x, \boldsymbol{k}_{\mathrm{T}}, P, S; \ldots)}{\sum_{\Lambda=\pm1} \int dx \int d^2\boldsymbol{k}_{\mathrm{T}} \;\; \Phi^{[\gamma^+ + s^j i\sigma^{j+}\gamma^5]}(x, \boldsymbol{k}_{\mathrm{T}}, P, S; \ldots)} \right|_{\boldsymbol{s}_{\mathrm{T}} = (1,0)} \tag{22}$$

Lattice Setup

Example: A meson correlation function

$$\langle 0 | \sum_{\vec{x}} \left(\overline{\psi}\gamma_5\psi\right)_{\vec{x},t} \left(\overline{\psi}\gamma_5\psi\right)_{\vec{0},0} |0\rangle$$

$\overline{\psi}\gamma_5\psi$ pseudoscalar quantum numbers

$\sum_{\vec{x}} f(\vec{x}) = \sum_{\vec{x}} e^{-i\vec{p}\cdot\vec{x}} f(\vec{x})\Big|_{\vec{p}=\vec{0}}$ projection into zero momentum

$$= -\sum_{\{U\}} \sum_{\vec{x}} \mathrm{tr}\Big(\big[M^{-1}[U]\big]_{\vec{0}0,\vec{x}t} \gamma_5 \big[M^{-1}[U]\big]_{\vec{x}t,\vec{0}0} \gamma_5 \Big)$$

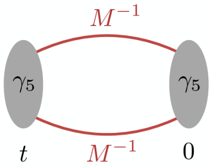
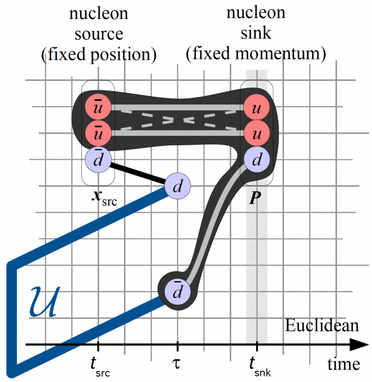


Figure 4: From Dr. Jozef Dudek lecture, HUGS 2023

Lattice Setup



- Evaluate directly
  $\widetilde{\Phi}^{[\Gamma]}_{\text{unsubtr.}}(b, P, S, \hat{\zeta}, \mu) \equiv \frac{1}{2} \langle P, S| \ \bar{q}(0)\, \Gamma\, \mathcal{U}[0, \eta v, \eta v + b, b]\ q(b) \ |P, S\rangle$
- Euclidean time: Place entire operator at one time slice, i.e., b, ηv purely spatial.
- Extrapolate $\eta \longrightarrow \infty$, $\hat{\zeta} \longrightarrow \infty$ numerically.

Numerical Results

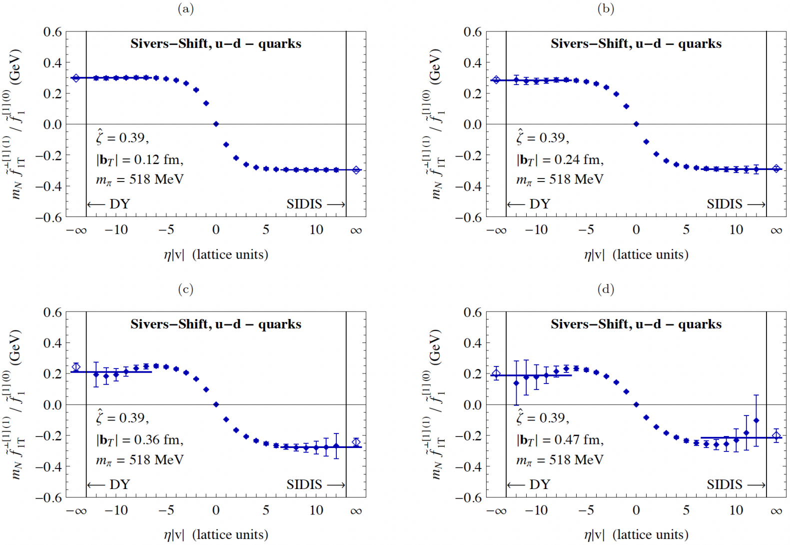


Figure 5: Extraction of the generalized Sivers shift on the lattice with $m_\pi = 518$MeV *(arXiv:1111.4249v2 [hep-lat])*

Numerical Results

Results: Sivers shift

Dependence of SIDIS limit on $|b_T|$

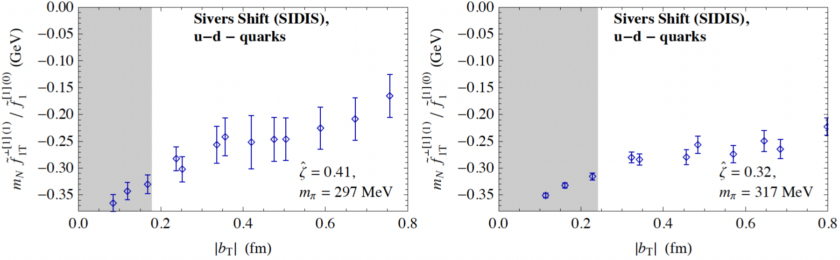
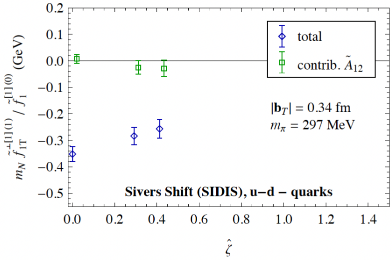


Figure 6: Generalized Sivers shift as a function of the quark separation $|\boldsymbol{b}_T|$ for the SIDIS case ($|\eta v| = \infty$). *arXiv:2301.06118 [hep-lat]*

Numerical Results

Results: Sivers shift

Dependence of SIDIS limit on $\hat{\zeta}$



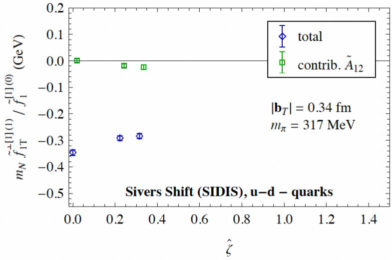


Figure 7: we show the $\hat{\zeta}$-dependence of the generalized Sivers shift, depicting both the full result and the result obtained with just $\tilde{A}_{12}$ in the numerator.
arXiv:2301.06118 [hep-lat]

Few More Numerical Results

- M. Engelhardt, *et al.*, PoS **LATTICE2022**, 103 (2023), [arXiv:2301.06118 [hep-lat]].
- B. Yoon, M. Engelhardt, R. Gupta, T. Bhattacharya, J. R. Green, B. U. Musch, J. W. Negele, A. V. Pochinsky, A. Schäfer and S. N. Syritsyn, Phys. Rev. D **96**, no.9, 094508 (2017), [arXiv:1706.03406 [hep-lat]].
- M. Engelhardt, B. Musch, T. Bhattacharya, J. R. Green, R. Gupta, P. Hägler, S. Krieg, J. Negele, A. Pochinsky and A. Schäfer, *et al.*, EPJ Web Conf. **112**, 01008 (2016)
- M. Engelhardt, B. Musch, T. Bhattacharya, J. R. Green, R. Gupta, P. Haegler, J. Negele, A. Pochinsky, A. Schafer and S. Syritsyn, *et al.*, PoS **QCDEV2015**, 018 (2015)
- M. Engelhardt, B. Musch, T. Bhattacharya, R. Gupta, P. Hägler, S. Krieg, J. Negele, A. Pochinsky, S. Syritsyn and B. Yoon, PoS **LATTICE2015**, 117 (2016)

Challenges and My PhD Work

- In our studies so far, we only considered the first x-moments (accessible at $b \cdot P = 0$), rather than scanning range of $b \cdot P$: $f^{[1]}(\boldsymbol{k}_\mathrm{T}^2; \ldots) \equiv \int_{-1}^{1} dx\ f(x, \boldsymbol{k}_\mathrm{T}^2; \ldots)$.

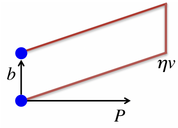


Figure 8: Illustration of the TMD operator with staple-shaped gauge connection

- **The range of accessible $b \cdot P$ is limited by the available b and P, $|b \cdot P| \leq |\mathbf{P}|\sqrt{-b^2}$, leading to an increasing systematic uncertainty at small x.**

$$\boxed{\frac{v \cdot b}{v \cdot P} = b \cdot P \, \frac{R(\hat{\zeta}^2)}{m_N^2} \,,} \tag{23}$$

where

$$R(\hat{\zeta}^2) \equiv 1 - \sqrt{1 + \hat{\zeta}^{-2}} = \frac{m_N^2}{v \cdot P} \frac{v^+}{P^+} \,. \tag{24}$$