Transverse Takahashi identities and their implications for gauge independent dynamical chiral symmetry breaking

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In this article, we employ transverse Takahashi identities to impose valuable nonperturbative constraints on the transverse part of the fermion-photon vertex in terms of new form factors, the so-called Y functions. We show that the implementation of these identities is crucial for ensuring the correct local gauge transformation of the fermion propagator and its multiplicative renormalizability. Our construction incorporates the correct symmetry properties of the Y_i under charge conjugation operation as well as their well-known one-loop expansion in the asymptotic configuration of incoming and outgoing momenta. Furthermore, we make an explicit analysis of various existing constructions of this vertex against the demands of transverse Takahashi identities and the previously established key features of quantum electrodynamics, such as gauge invariance of the critical coupling above which chiral symmetry is dynamically broken. We construct a simple example in its quenched version and compute the mass function as we vary the coupling strength and also calculate the corresponding anomalous dimensions γ_m. There is an excellent fit to the Miransky scaling law, and we find $\gamma_m = 1$ rather naturally in accordance with some earlier results in literature, using arguments based on the Cornwall-Jackiw-Tomboulis effective potential technique. Moreover, we numerically confirm the gauge invariance of this critical coupling.

I. INTRODUCTION

A quantum field theory can be considered completely solved if we are able to compute its full set of n-point Green functions. However, these Green functions are infinite in number. Moreover, these are all intertwined through highly nonlinear coupled integral equations, known as the Schwinger-Dyson equations (SDEs). A brute force method to compute them is a wild goose chase.

It is widely believed that a satisfactory determination of relevant physical observables through a systematic truncation scheme for this infinite tower of equations is achievable if we preserve the key features and symmetries of the underlying theory. Perturbation theory provides an excellent example of such an approximation scheme. However, when the interaction strength grows and can no longer be used as a perturbative expansion parameter, one resorts to truncations which need to be carefully constructed in order to retain the essential features of the original theory, while maintaining contact with experimental data at the same time. Quantum chromodynamics (QCD) is a realization of this scenario in its infrared domain. Considerable progress has been made in the past decades to study its first few

Green functions, e.g., the gluon propagator [1–8] and the quark-gluon vertex [9–19] whose knowledge consequently provides predictions for QCD and hadron physics, e.g., [20–24]; also see reviews [25–29] and references therein.

In several hadronic physics studies, such as electromagnetic and transition form factors [21,23,30–33], probes are generally electromagnetic in nature and many SDE calculations crucially rely on how photons interact with quarks. Thus, quantum electrodynamics (QED) serves as a useful platform to study SDE truncations and provide improvements to preserve its key features, such as its gauge invariance, its renormalizability, and the recuperation of the well-known S-matrix perturbative expansion for its Green functions in the weak coupling regime, which it maintains at all accessible energies. In particular, study of the fermion propagator in QED generally amounts to requiring a physically meaningful and reliable *Ansatz* for the three-point fermion-photon vertex. Gauge invariance provides an essential ingredient in this connection. The gauge technique of Salam, Delbourgo, and collaborators was developed to solve the constraints of the well-known Ward-Fradkin-Green-Takahashi identity (WFGTI) [34–37], writing the

Green functions in terms of spectral representations [38–40]. However, such an approach, despite its elegant and formal results [41,42] is not amicable to straightforward computations [43]. The WFGTI allows us to expand out the vertex in terms of a well-constrained longitudinal part [44] and an undetermined transverse part. Several efforts alternative to the gauge technique start from making an *Ansatz* for this latter part and proceed from thereon.

A natural question to ask is *whether* the transverse part can be constrained through any other symmetry principle?

Whereas the usual WFGTI relates the divergence of the three-point fermion-photon vertex to the inverse fermion propagator, there exist transverse Takahashi identities (TTI) which play a similar role for the curl of the fermion-photon vertex [45–49]. However, in addition to the inverse fermion propagator and the vector vertex, these identities also bring into play a nonlocal axial-vector vertex as well as new inhomogeneous tensor and axial-tensor vertices. Consequently, TTI are richer and more complicated in their structure. In the past, they have been verified to one-loop order [50,51]. More recently, practical implications of TTI have been investigated in [52,53] to get insight into the nonperturbative forms of vector and axial-vector vertices.

In this article, we intend to study constraints of TTI on the transverse part of the fermion-photon vertex. Note that TTI do not modify the usual WFGTI in any way. However, we realize that they are crucially connected to another consequence of local gauge covariance, namely the Landau-Khalatnikov-Fradkin transformations (LKFT), derived in [35,54–56]. LKFT are a well defined set of transformations which describe the response of the Green functions to an arbitrary gauge transformation. These transformations leave the SDEs and the WFGTI form invariant. LKFT potentially play an important role in imposing valuable constraints on the fermion-photon vertex and obtaining gauge invariant chiral symmetry breaking; see, e.g., Refs. [57–72]. More recently, these transformations have also been derived for QCD [73,74].

Both the TTI and the LKFT [through the multiplicative renormalizability (MR) of the fermion propagator] constrain the transverse fermion-photon vertex. Therefore, it is reasonable to seek a combined constraint which would help us converge on pinning down this elusive part of the vertex. The fact that MR constrains the transverse vertex has already been known for some time [42,75–77]. Later works in the literature involving similar considerations in constructing a refined fermion-photon vertex can be found in [12,78–85].

An important issue relevant to our current work concerns the usage of the TTI-constrained vertex to study dynamical chiral symmetry breaking (DCSB) or dynamical mass generation for fermions as a consequence of enhanced interaction strength. This is a strictly nonperturbative phenomenon and a transcendental topic in QCD,

where it induces measurable effects in numerous hadron observables. Therefore, physically meaningful truncations of QCD's SDEs demand incorporation of DCSB through the relevant Green functions, in particular the quark propagator and the quark-gluon vertex. Regarding the latter, valuable progress has been made in both lattice [86–92] and continuum studies. However, due to the non-Abelian nature of QCD, investigating the impact of DCSB on the quark-gluon vertex, and vice versa, from first principles, is still a theoretical challenge. A thorough investigation of the fermion-photon vertex and chiral symmetry breaking in QED is likely to provide a benchmark for the corresponding studies in QCD.

Although QED manifests a perturbative behavior at all observable scales, an intense background electromagnetic field can trigger a transition from perturbative to nonperturbative dynamics, the well-known *magnetic catalysis*; see, e.g., [93–101]. Even a toy QED with an artificially scaled up coupling exhibits this phenomenon. Such a phase transition has long been studied. It is characterized by a critical coupling, α_c, above which DCSB takes place; see [13,78,102] and references therein. Since this critical coupling corresponds to a recognizable phase transition, it is considered to be a physical observable, and hence a gauge invariant parameter. This independence of α_c on the gauge parameter has long been used as a further requirement to constrain the transverse vertex [13], and we follow this argument in the present article.

The TTI connect the transverse structure of the fermion-photon vertex to a set of unknown scalar functions Y_i related to a nonlocal axial-tensor mentioned before. MR of the electron propagator implies that these functions cannot be ignored. Instead, MR constrains their form. This procedure involves an unknown function $W(x)$ of a dimensionless ratio x of the incoming and outgoing fermion momenta. It satisfies an integral constraint which guarantees MR of the electron propagator in the leading logarithm approximation (LLA). Implementing charge conjugation symmetry on the integration kernels involved in the fermion propagator SDE, it is possible to parametrize Y_i in terms of one single scalar, *a priori* unknown function $\mathcal{T}(k^2, p^2)$, which encodes the effect of the fully dressed fermion-photon vertex on the fermion propagator. This general procedure fixes three of the Y functions. An additional constraint comes from demanding gauge independent chiral symmetry breaking. We work in the quenched approximation and obtain a self-consistent solution for $\mathcal{T}(k^2, p^2)$. Following the lead in [12], where MR of both the fermion and the photon propagators are exploited, we expect our analysis and the corresponding study of DCSB to be applicable to full unquenched QED, assuming the nonperturbative realization of MR. This endeavor will be undertaken in the future.

In this article, we work in Euclidean space. Thus, for γ matrices we have $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$ and $\gamma_\mu^\dagger = \gamma_\mu$, where $\delta_{\mu\nu}$ is

the Euclidean metric. Furthermore, we define $\gamma_5 = \gamma_4\gamma_1\gamma_2\gamma_3$, with $\mathrm{Tr}[\gamma_5\gamma_\mu\gamma_\nu\gamma_\alpha\gamma_\beta] = -4\epsilon_{\mu\nu\alpha\beta}$.

This paper is organized as follows: in Sec. II, we review the WFGTI for the three-point vertex in QED, define the longitudinal vertex, write down the transverse part in a general basis, following [44], and highlight the symmetry properties of the transverse form factors under charge conjugation operation. In Sec. III, we introduce Abelian TTI for the vertex and expand out the transverse form factors in terms of the Y functions. We then invert these relations and impose a perturbative constraint on Y_i in the asymptotic limit of $k^2 \gg p^2$. In Sec. IV, SDE for the fermion propagator is presented. We write it in terms of the Y functions and discuss the quenched approximation. In Sec. V, we study the requirement of MR and the power law solution for the wave function renormalization of the fermion propagator within the LLA. We show how the requirement of MR imposes an integral constraint on the form factors of the transverse vertex, in terms of the function $W(x)$. In Sec. VI, we discuss a few examples illustrating the need and importance of the Y functions. In Sec. VII, we construct simple examples to study DCSB and see how it naturally incorporates gauge independence of the critical coupling α_c. In Sec. VIII, we present our conclusions and discuss prospects for future research.

II. VERTEX DECOMPOSITION

In its general decomposition, the three-point fermion-photon vertex can be written in terms of 12 independent spin structures. For the kinematical configuration of Fig. 1, the WFGTI associated with this vertex takes the form

$$iq_\mu\Gamma_\mu(k, p) = S^{-1}(k) - S^{-1}(p), \qquad (1)$$

where $q = k - p$. This identity allows us to split the vertex as a sum of *longitudinal* and *transverse* components, as suggested by Ball and Chiu [44]:

$$\Gamma_\mu(k, p) = \Gamma_\mu^L(k, p) + \Gamma_\mu^T(k, p). \qquad (2)$$

The longitudinal part $\Gamma_\mu^L(k, p)$ alone satisfies the WFGTI (1), and consumes four of the 12 independent spin structures (one of them is zero in QED), so that [44]
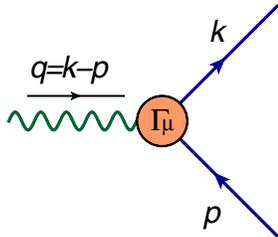
FIG. 1. Diagrammatic representation of the full three-point vertex $\Gamma_\mu(k, p)$, with momentum flow indicated.

$$\Gamma_\mu^L(k, p) = a(k^2, p^2)\gamma_\mu + \frac{b(k^2, p^2)}{2}t_\mu\gamma \cdot t - ic(k^2, p^2)t_\mu \qquad (3)$$

with $t = k + p$, and

$$a(k^2, p^2) = \frac{1}{2}\left[\frac{1}{F(k^2, \Lambda^2)} + \frac{1}{F(p^2, \Lambda^2)}\right],$$

$$b(k^2, p^2) = \left[\frac{1}{F(k^2, \Lambda^2)} - \frac{1}{F(p^2, \Lambda^2)}\right]\frac{1}{k^2 - p^2},$$

$$c(k^2, p^2) = \left[\frac{\mathcal{M}(k^2, \Lambda^2)}{F(k^2, \Lambda^2)} - \frac{\mathcal{M}(p^2, \Lambda^2)}{F(p^2, \Lambda^2)}\right]\frac{1}{k^2 - p^2}, \qquad (4)$$

where Λ is an ultraviolet cutoff regulator. $\mathcal{M}$ and F are the mass function and the wave function renormalization, respectively, related to the fermion propagator $S(k)$ through

$$S(k) = \frac{F(k^2, \Lambda^2)}{i\gamma \cdot k + \mathcal{M}(k^2, \Lambda^2)}. \qquad (5)$$

At the tree level, $F(k^2, \Lambda^2) = 1$ and $\mathcal{M}(k^2, \Lambda^2) = m_0$, where m_0 is the bare mass of the fermion.

The transverse part $\Gamma_\mu^T(k, p)$ of the vertex decomposition (2), which remains undetermined by the WFGTI, is naturally constrained by

$$q_\mu\Gamma_\mu^T(k, p) = 0. \qquad (6)$$

In general, the ultraviolet finite transverse vertex can be expanded out in terms of eight vector structures and their corresponding scalar form factors $\tau_i(k, p)$ [44]:

$$\Gamma_\mu^T(k, p) = \sum_{i=1}^{8}\tau_i(k, p)T_\mu^i(k, p). \qquad (7)$$

Moreover, for the kinematical configuration of Fig. 1, we define

$$T_\mu^1(k, p) = i[p_\mu(k \cdot q) - k_\mu(p \cdot q)],$$

$$T_\mu^2(k, p) = [p_\mu(k \cdot q) - k_\mu(p \cdot q)](\gamma \cdot t),$$

$$T_\mu^3(k, p) = q^2\gamma_\mu - q_\mu(\gamma \cdot q),$$

$$T_\mu^4(k, p) = iq^2[\gamma_\mu(\gamma \cdot t) - t_\mu] + 2q_\mu p_\nu k_\rho\sigma_{\nu\rho},$$

$$T_\mu^5(k, p) = \sigma_{\mu\nu}q_\nu,$$

$$T_\mu^6(k, p) = -\gamma_\mu(k^2 - p^2) + t_\mu(\gamma \cdot q),$$

$$T_\mu^7(k, p) = \frac{i}{2}(k^2 - p^2)[\gamma_\mu(\gamma \cdot t) - t_\mu] + t_\mu p_\nu k_\rho\sigma_{\nu\rho},$$

$$T_\mu^8(k, p) = -i\gamma_\mu p_\nu k_\rho\sigma_{\nu\rho} - p_\mu(\gamma \cdot k) + k_\mu(\gamma \cdot p), \qquad (8)$$

with

$$\sigma_{\nu\rho} = \frac{i}{2}[\gamma_\nu, \gamma_\rho]. \qquad (9)$$

This basis is not exactly the one adopted in [44]. We choose to work with a modification of this initial basis which was put forward in [103] and later employed in [104] as well. This latter choice ensures all transverse form factors of the vertex are independent of any kinematic singularities in one-loop perturbation theory in an arbitrary covariant gauge.

As stated earlier in Sec. I, any *Ansatz* for the full vertex must have the same transformation properties as the bare vertex under charge conjugation operation. This requires all the τ_i's in (7) to be symmetric under the interchange $k \leftrightarrow p$, except τ_4 and τ_6, which are odd:

$$\tau_i(k, p) = \tau_i(p, k), \quad i = 1, 2, 3, 5, 7, 8, \qquad (10)$$

$$\tau_i(k, p) = -\tau_i(p, k), \quad i = 4, 6. \qquad (11)$$

From Eq. (4), it is obvious that $a(k^2, p^2)$, $b(k^2, p^2)$, and $c(k^2, p^2)$ are symmetric under $k \leftrightarrow p$, as they should be, in order to preserve the correct transformation properties under charge conjugation operation for the full vertex.

Although the longitudinal scalar functions (4) are fixed by the WFGTI (1), the transverse scalar functions in decomposition (7) remain unknown. In the next section, we introduce the TTIs for the three-point vertex in QED, which provide a powerful tool in constructing these nonperturbative transverse functions.

III. TRANSVERSE TAKAHASHI IDENTITIES

The TTIs for vector (Γ_μ) and axial-vector (Γ_μ^A) vertices in QED, related to a fermion with bare mass m_0, read [52]

$$q_\mu \Gamma_\nu(k, p) - q_\nu \Gamma_\mu(k, p) = S^{-1}(p)\sigma_{\mu\nu} + \sigma_{\mu\nu}S^{-1}(k)$$
$$+ 2im_0\Gamma_{\mu\nu}(k, p)$$
$$+ t_\alpha \epsilon_{\alpha\mu\nu\beta}\Gamma_\beta^A(k, p) + A_{\mu\nu}^V(k, p), \qquad (12)$$

$$q_\mu \Gamma_\nu^A(k, p) - q_\nu \Gamma_\mu^A(k, p) = S^{-1}(p)\sigma_{\mu\nu}^5 - \sigma_{\mu\nu}^5 S^{-1}(k)$$
$$+ t_\alpha \epsilon_{\alpha\mu\nu\beta}\Gamma_\beta(k, p) + V_{\mu\nu}^A(k, p), \qquad (13)$$

where $\sigma_{\mu\nu}^5 = \gamma_5 \sigma_{\mu\nu}$ and $\Gamma_{\mu\nu}(k, p)$ is an inhomogeneous tensor vertex. The last two tensor structures in Eqs. (12) and (13), $A_{\mu\nu}^V$ and $V_{\mu\nu}^A$, are related to the momentum space expressions for nonlocal axial-vector and vector vertices, whose definitions involve a gauge-field-dependent line integral. These nonperturbative identities are valid for any covariant gauge, and they do not have explicit dependence on the covariant gauge parameter.

The vector and axial-vector TTIs are intricately coupled to each other via the nonlocal terms $A_{\mu\nu}^V(k, p)$ and $V_{\mu\nu}^A(k, p)$, which are complicated even at one-loop order [50,51]. Following the procedure described in Ref. [52], useful progress has been made to disentangle this interdependence. Notice that Eqs. (12) and (13) provide additional constraints on the fermion-photon vertex. We now set out to incorporate these constraints into its nonperturbative construction in terms of conveniently parametrized Y functions.

In order to project out transverse form factors from the TTIs, Eqs. (12) and (13), we introduce the following tensors:

$$T_{\mu\nu}^1 = \frac{1}{2}\epsilon_{\alpha\mu\nu\beta}t_\alpha q_\beta, \qquad (14)$$

$$T_{\mu\nu}^2 = \frac{1}{2}\epsilon_{\alpha\mu\nu\beta}\gamma_\alpha q_\beta. \qquad (15)$$

By contracting the axial-vector identity (13) with tensors (14) and (15), the left-hand sides of the resulting equations reduce to zero, while the right-hand sides yield the following results:

$$(q \cdot t)(t \cdot \Gamma(k, p)) = T_{\mu\nu}^1[S^{-1}(p)\sigma_{\mu\nu}^5 - \sigma_{\mu\nu}^5 S^{-1}(k)]$$
$$+ t^2(q \cdot \Gamma(k, p)) + T_{\mu\nu}^1 V_{\mu\nu}^A, \qquad (16)$$

$$(q \cdot t)(\gamma \cdot \Gamma(k, p)) = T_{\mu\nu}^2[S^{-1}(p)\sigma_{\mu\nu}^5 - \sigma_{\mu\nu}^5 S^{-1}(k)]$$
$$+ (\gamma \cdot t)(q \cdot \Gamma(k, p)) + T_{\mu\nu}^2 V_{\mu\nu}^A. \qquad (17)$$

These expressions only involve the vector vertex $\Gamma_\mu(k, p)$ and do not contain explicit dependence on the fermion mass m_0. Information about the axial-vector vertex $\Gamma_\mu^A(k, p)$ can be obtained through an analogous procedure involving the vector TTI, Eq. (12). Although the terms $T_{\mu\nu}^1 V_{\mu\nu}^A$ and $T_{\mu\nu}^2 V_{\mu\nu}^A$ are still equally unknown, they are Lorentz scalar objects and can thus be conveniently expressed as follows:

$$iT_{\mu\nu}^1 V_{\mu\nu}^A = \mathbf{I}_D Y_1(k, p) + i(\gamma \cdot q)Y_2(k, p)$$
$$+ i(\gamma \cdot t)Y_3(k, p) + [\gamma \cdot q, \gamma \cdot t]Y_4(k, p), \qquad (18)$$

$$iT_{\mu\nu}^2 V_{\mu\nu}^A = i\mathbf{I}_D Y_5(k, p) + (\gamma \cdot q)Y_6(k, p)$$
$$+ (\gamma \cdot t)Y_7(k, p) + i[\gamma \cdot q, \gamma \cdot t]Y_8(k, p), \qquad (19)$$

where $Y_i(k, p)$ are hitherto unconstrained scalar functions and $\mathbf{I}_D$ is the identity matrix. Projections of Eqs. (16) and (17) lead to a set of eight linearly independent, coupled linear equations that fix the eight transverse scalar functions τ_i in terms of the Y functions defined via Eqs. (18) and (19).

From Eqs. (7), (8), and (16)–(19), it is possible to project out the scalar form factors τ_i:

$$\tau_1(k,p) = -\frac{Y_1}{2(k^2-p^2)\nabla(k,p)}, \tag{20}$$

$$\tau_2(k,p) = -\frac{Y_5 - 3Y_3}{4(k^2-p^2)\nabla(k,p)}, \tag{21}$$

$$\tau_3(k,p) = \frac{1}{2}b(k^2,p^2) + \frac{2(k^2-p^2)Y_2 - t^2(Y_3-Y_5)}{8(k^2-p^2)\nabla(k,p)}, \tag{22}$$

$$\tau_4(k,p) = -\frac{(k^2-p^2)(6Y_4+Y_6^A)+t^2Y_7^S}{8(k^2-p^2)\nabla(k,p)}, \tag{23}$$

$$\tau_5(k,p) = -c(k^2,p^2) - \frac{2Y_4+Y_6^A}{2(k^2-p^2)}, \tag{24}$$

$$\tau_6(k,p) = \frac{2q^2Y_2 - (k^2-p^2)(Y_3-Y_5)}{8(k^2-p^2)\nabla(k,p)}, \tag{25}$$

$$\tau_7(k,p) = \frac{q^2(6Y_4+Y_6^A)+(k^2-p^2)Y_7^S}{4(k^2-p^2)\nabla(k,p)}, \tag{26}$$

$$\tau_8(k,p) = -b(k^2,p^2) - \frac{2Y_8^A}{k^2-p^2}, \tag{27}$$

where we have employed the obvious simplifying notation $Y_i \equiv Y_i(k,p)$. Moreover, we have introduced the Gram determinant

$$\nabla(k,p) = k^2 p^2 - (k \cdot p)^2. \tag{28}$$

In addition, the vertex transformation properties under charge conjugation determine the symmetry properties of the Y functions:

$$Y_i(k,p) = Y_i(p,k), \qquad i = 2, 6^S, 7^S, 8^S, \tag{29}$$

$$Y_i(k,p) = -Y_i(p,k), \quad i = 1, 3, 4, 5, 6^A, 7^A, 8^A, \tag{30}$$

where we conveniently introduce the decomposition

$$Y_i(k,p) = Y_i^S(k,p) + Y_i^A(k,p), \tag{31}$$

for $i = 6, 7, 8$, where the superscripts S and A stand for the symmetric and antisymmetric parts of the corresponding Y_i's, under $k \leftrightarrow p$. Note that in Eqs. (20)–(27), there is no contribution of Y_6^S, Y_7^A, and Y_8^S. This is a consequence of the properties (10) and (11), which entail

$$Y_6^S(k,p) = -\frac{(k^2-p^2)Y_1(k,p)}{4\nabla(k,p)}, \tag{32}$$

$$Y_7^A(k,p) = \frac{q^2 Y_1(k,p)}{4\nabla(k,p)}, \tag{33}$$

$$Y_8^S(k,p) = -\frac{q^2 Y_2(k,p) + (k^2-p^2)Y_3(k,p)}{8\nabla(k,p)}. \tag{34}$$

It is also worth noting that the trivial choice $Y_i(k,p) = 0$ for all Y functions completely fixes the transverse vertex, defined through Eqs. (7), (8), (21), and (27), in terms of the fermion wave function renormalization, as reported in Ref. [52]. However, we shall show that *MR of the electron propagator implies that these Y functions cannot all be zero simultaneously.*

We can invert relations (20)–(27) to write out the Y functions in terms of τ_i:

$$Y_1'(k,p) = -2\nabla(k,p)\tau_1(k,p), \tag{35}$$

$$Y_2'(k,p) = \frac{1}{2}(k^2-p^2)[b(k^2,p^2) - 2\tau_3(k,p)] + t^2\tau_6(k,p), \tag{36}$$

$$Y_3'(k,p) = -\frac{1}{2}q^2[b(k^2,p^2) - 2\tau_3(k,p)] + 2\nabla(k,p)\tau_2(k,p) - (k^2-p^2)\tau_6(k,p), \tag{37}$$

$$Y_4'(k,p) = \frac{1}{2}[c(k^2,p^2) + \tau_5(k,p)] + \frac{1}{4}[2(k^2-p^2)\tau_4(k,p) + t^2\tau_7(k,p)], \tag{38}$$

$$Y_5'(k,p) = -\frac{3}{2}q^2[b(k^2,p^2) - 2\tau_3(k,p)] + 2\nabla(k,p)\tau_2(k,p) - 3(k^2-p^2)\tau_6(k,p), \tag{39}$$

$$Y_6'^A(k,p) = -3[c(k^2,p^2) + \tau_5(k,p)] - \frac{1}{2}[2(k^2-p^2)\tau_4(k,p) + t^2\tau_7(k,p)], \tag{40}$$

$$Y_7'^S(k,p) = -[2q^2\tau_4(k,p) + (k^2-p^2)\tau_7(k,p)],$$

$$Y_8'^A(k,p) = -\frac{1}{2}[b(k^2,p^2) + \tau_8(k,p)]. \tag{41}$$

Here, we have conveniently defined

$$Y_i(k,p) = (k^2-p^2)Y_i'(k,p). \tag{42}$$

We expect the study in terms of $Y_i(k,p)$ to be numerically amicable as the additional factor of (k^2-p^2) in the numerator eases out any kinematical singularities in the limit $k^2 \to p^2$.

So far, we have shown that the TTIs relate the transverse vertex form factors to the fermion propagator and a nonlocal tensor vertex, but nevertheless this is not enough to elucidate the analytical behavior of the Y functions. It is

insightful to analyze the asymptotic behavior of the vertex. It has been shown that in the asymptotic limit, defined as the perturbative expansion with $p^2 \gg k^2 \gg m_0^2$, the leading logarithmic term of the transverse vertex reads (in our kinematical configuration) as [79]

$$\Gamma_\mu^T(k,p) \overset{p^2 \gg k^2}{=} \frac{\alpha\xi}{8\pi p^2} \log\left(\frac{k^2}{p^2}\right) T_\mu^{\text{asy}}, \qquad (43)$$

where ξ is the gauge-fixing parameter, and

$$T_\mu^{\text{asy}} \equiv T_\mu^{3\,\text{asy}} = T_\mu^{6\,\text{asy}} = p^2 \gamma_\mu - p_\mu \gamma \cdot p. \qquad (44)$$

On the other hand, from Eqs. (7), (8), (20)–(27), it is straightforward to see that the leading structure of the transverse vertex in the asymptotic limit acquires the following form:

$$\Gamma_\mu^T(k,p) \overset{p^2 \gg k^2}{=} (\tau_3 + \tau_6) T_\mu^{\text{asy}}$$
$$= \frac{\beta}{2p^2} \log\left(\frac{k^2}{p^2}\right) T_\mu^{\text{asy}}$$
$$+ \left\{ \frac{2k \cdot q Y_2 - k \cdot t(Y_3 - Y_5)}{4(k^2 - p^2)\nabla(k,p)} \right\} T_\mu^{\text{asy}}, \qquad (45)$$

where we have used the fact that the one-loop expansion of the wave function renormalization yields $F(k^2) = 1 + \beta \log(k^2/\Lambda^2)$, where β is a constant of order $\mathcal{O}(\alpha)$: we shall show in the next section that $\beta = \alpha\xi/(4\pi)$. Hence, the leading logarithmic expansion for the asymptotic limit of the vertex, Eq. (43), demands

$$2k \cdot q Y_2(k,p) = k \cdot t(Y_3(k,p) - Y_5(k,p)), \qquad (46)$$

which must be fulfilled at least to second order in its perturbative expansion in powers of k^2/p^2, in order to ensure the correct asymptotic limit of the transverse vertex.

Although the TTIs, and in particular the identities (16) and (17), are potentially able to fix the transverse vertex, the construction of an *Ansatz* for this vertex is far from being complete since the Y functions remain unknown. Additional requirements need to be implemented in order to compute them. In this spirit, we shall use the argument of MR for the fermion propagator, in the chirally symmetric limit, in order to derive an integral constraint for these Y functions. We shall also restrict the structure of the vertex by implementing symmetry arguments and demanding a gauge independent breaking of chiral symmetry. To this end, we introduce the SDE for the fermion propagator in the next section.

IV. GAP EQUATION

The SDE for the fermion propagator, also known as the fermion *gap equation*, is diagrammatically represented in Fig. 2.

Mathematically, the gap equation is written as

$$S^{-1}(k) = S_0^{-1}(k) + \frac{\alpha}{4\pi^3} \int_E d^4p \gamma_\nu S(p) \Gamma_\mu(k,p) \Delta_{\mu\nu}(q), \quad (47)$$

where the subscript E indicates that the integral is performed in the Euclidean space, $\alpha = e^2/4\pi$ is the electromagnetic coupling, and $\Delta_{\mu\nu}(q)$ is the fully dressed photon propagator. For an arbitrary covariant gauge, it is defined as

$$\Delta_{\mu\nu}(q) = \Delta(q^2)\left[\delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right] + \xi\frac{q_\mu q_\nu}{q^4}, \qquad (48)$$

where $\Delta(q^2)$ is the photon propagator dressing function. The subscript "0" in the first term of the right-hand side of Eq. (47) denotes the tree level fermion propagator.

Recall from Eq. (5) that the fermion propagator is defined by the wave function renormalization and the mass function, so the gap equation, Eq. (47), can be decomposed into two coupled, integral equations for $\mathcal{M}$ and F, which, in an arbitrary gauge, are, respectively, written as

$$\frac{\mathcal{M}(k^2)}{F(k^2)} = m_0 + \frac{\alpha\xi}{4\pi^3} \int_E \frac{d^4p}{q^4} \frac{F(p^2)}{p^2 + \mathcal{M}^2(p^2)} \frac{1}{F(k^2)}$$
$$\times \{\mathcal{M}(p^2)q \cdot k - \mathcal{M}(k^2)q \cdot p\}$$
$$+ \frac{\alpha}{4\pi^3} \int_E d^4p \frac{F(p^2)}{p^2 + \mathcal{M}^2(p^2)} \mathcal{M}(p^2)G_{\mathcal{M}}(k,p),$$

$$(49)$$

$$\frac{1}{F(k^2)} = 1 - \frac{\alpha\xi}{4\pi^3} \int_E \frac{d^4p}{q^4} \frac{F(p^2)}{p^2 + \mathcal{M}^2(p^2)} \frac{1}{F(k^2)}$$
$$\times \left\{ q \cdot p + \mathcal{M}(k^2)\mathcal{M}(p^2)\frac{q \cdot k}{k^2} \right\}$$
$$+ \frac{\alpha}{4\pi^3} \int_E \frac{d^4p}{k^2} \frac{F(p^2)}{p^2 + \mathcal{M}^2(p^2)} u(k,p)G_F(k,p),$$

$$(50)$$
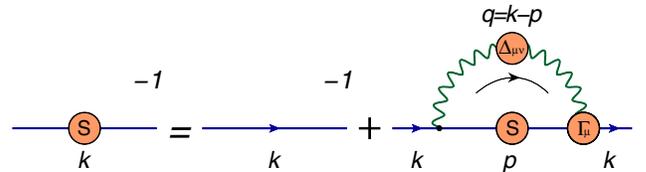
FIG. 2. Gap equation for the fermion propagator. The color-filled blobs labeled with S, $\Delta_{\mu\nu}$, and Γ_μ stand for the fully dressed fermion and photon propagators and the three-point vertex, respectively.

where we have adopted the notation $F(k^2) \equiv F(k^2, \Lambda^2)$ and conveniently defined

$$u(k, p) \equiv 3k \cdot p - 2\frac{\nabla(k, p)}{q^2}. \tag{51}$$

The functions G_F and $G_{\mathcal{M}}$ in Eqs. (49) and (50) encode the effective contribution of the fully dressed fermion-photon vertex to the corresponding equations, and they are defined as

$$\frac{(k^2 - p^2)}{\Delta(q^2)} G_{\mathcal{M}}(k, p)$$
$$= Y_5(k, p) + \frac{\Lambda_M(p, k)}{\mathcal{M}(p^2)}$$
$$+ \left\{ 3k^2 - u(k, p) + [u(k, p) - 3p^2]\frac{\mathcal{M}(k^2)}{\mathcal{M}(p^2)} \right\} \frac{1}{F(k^2)}, \tag{52}$$

$$\frac{(k^2 - p^2)}{\Delta(q^2)} u(k, p) G_F(k, p)$$
$$= -\Lambda_{NM}(k, p) - \mathcal{M}(p^2)\Lambda_M(k, p)$$
$$+ (k^2 - p^2)\{-3k^2 p^2 b(k^2, p^2) + u(k, p)\tilde{b}(k^2, p^2)$$
$$+ \mathcal{M}(p^2)(u(k, p) - 3k^2)c(k^2, p^2)\}, \tag{53}$$

where

$$\tilde{b}(k^2, p^2) = \frac{1}{k^2 - p^2}\left[\frac{k^2}{F(k^2)} - \frac{p^2}{F(p^2)}\right]. \tag{54}$$

Moreover, we have defined *massive* (Λ_M) and *nonmassive* (Λ_{NM}) functions as

$$\Lambda_M(k, p) = \frac{1}{2}Y_1(k, p) + q \cdot kY_6^A(k, p) + t \cdot kY_7^S(k, p), \tag{55}$$

$$\Lambda_{NM}(k, p) = \frac{1}{2}(k^2 - p^2)Y_2(k, p) + \frac{1}{2}t^2 Y_3(k, p)$$
$$- k \cdot pY_5(k, p) + 4\nabla(k, p)Y_8^A(k, p). \tag{56}$$

The reason for referring to Λ_M and Λ_{NM} as massive and nonmassive functions, respectively, is the following: from Eqs. (49) and (52) it is straightforward to see that in the chiral limit, where $m_0 = 0$, a massless solution ($\mathcal{M} = 0$) is trivially achieved if Λ_M has the mass function as a global factor, i.e., $\Lambda_M \sim \mathcal{M}$. On the other hand, the function Λ_{NM} does not follow this argument, and therefore it does not have a dependence on $\mathcal{M}$ as a global factor. In the same spirit, we shall refer to Y_1, Y_6^A, and Y_7^S as massive functions and Y_2, Y_3, Y_5, and Y_8^A as massless functions.

Note that the contribution of Y_4 in Eqs. (49) and (50) cancels out. This is an indication that the vertex cannot be completely extracted solely from the fermion propagator SDE. Nonetheless, Y_4 can be modeled by relying on additional information, e.g., the expected anomalous electromagnetic moment for the corresponding fermion.

A. Quenched QED

The system of Eqs. (49) and (50) has long been studied using different models for the photon propagator; see Ref. [102] and references therein. For the sake of simplicity, we limit this work to the well-known quenched approximation (qQED), where fermion loop contributions to the photon SDE are neglected and the coupling does not run, which in turn yields

$$\Delta(q^2) \equiv \Delta_0(q^2) = 1/q^2. \tag{57}$$

As we mentioned before, one of the goals of the present article is to study the impact of the transverse vertex on the DCSB and vice versa. In particular, we shall investigate the constraints imposed by demanding a gauge independent DCSB in Sec. VII. For this purpose, from now on, we focus our attention on the chiral limit ($m_0 = 0$) since this is the most insightful scenario to elucidate how QED undergoes a phase transition from perturbative to nonperturbative dynamics as we increase the electromagnetic coupling (α) up to the critical value (α_c) where DCSB is triggered. For $\alpha < \alpha_c$ the only possible solution to Eq. (49) in the chiral limit is $\mathcal{M}(k^2) = 0$, but as $\alpha \to \alpha_c$ a second nonzero solution bifurcates away from the trivial one. The theoretical prediction for the critical coupling above which DCSB takes place can be extracted from Eq. (49) through implementing bifurcation analysis.

In the vicinity of the critical coupling $\alpha \sim \alpha_c$, the dynamically generated fermion mass is rather small in comparison with any other mass scale. Therefore, quadratic and higher terms in the mass function can formally be neglected. In this case, Eq. (50) for F and consequently its solution reduce to that of a massless theory. Thus, the survey of the renormalization properties of the fermion propagator in massless QED, and the corresponding implications on the fermion-photon vertex, is mandatory.

In the next section we show that for a massless fermion in quenched QED, the wave function renormalization possesses a power law behavior, which is multiplicatively renormalizable. We also derive a nonperturbative constraint on the nonmassive Y functions that ensures a MR solution for F.

V. MR CONSTRAINTS

It is well known that in QED the gap equation (47) leads to a fermion propagator that is logarithmically divergent. However, we can define renormalized propagators by

absorbing these divergences into the renormalization constants $\mathcal{Z}_i$. For massless QED, this multiplicative renormalization is accomplished by introducing renormalized fields, fermion field $\psi_R = \mathcal{Z}_2^{-1/2}\psi$, photon field $A_\mu^R = \mathcal{Z}_3^{-1/2}A_\mu$, and also renormalized coupling $e_R = \mathcal{Z}_2\mathcal{Z}_3^{1/2}e/\mathcal{Z}_1$. Thus, the MR of the fermion propagator requires renormalized F_R to be related to unrenormalized F through

$$F_R(k^2, \mu^2) = \mathcal{Z}_2^{-1}(\mu^2, \Lambda^2)F(k^2, \Lambda^2), \qquad (58)$$

where μ plays the role of an arbitrary renormalization scale. In order to solve Eq. (58), the functions involved are expanded as perturbative series containing terms of the form $\alpha^n \ln^n$ (called leading logarithmic terms). This is known as the LLA. In the LLA, we then have

$$F(k^2, \Lambda^2) = 1 + \sum_{n=1}^{\infty} \alpha^n A_n \ln^n\left(\frac{k^2}{\Lambda^2}\right), \qquad (59)$$

$$\mathcal{Z}_2^{-1}(\mu^2, \Lambda^2) = 1 + \sum_{n=1}^{\infty} \alpha^n B_n \ln^n\left(\frac{\mu^2}{\Lambda^2}\right), \qquad (60)$$

$$F_R(k^2, \mu^2) = 1 + \sum_{n=1}^{\infty} \alpha^n C_n \ln^n\left(\frac{k^2}{\mu^2}\right), \qquad (61)$$

where A_n, B_n, and C_n are unknown coefficients but can be calculated in perturbation theory to any desired order. However, MR condition (58) restricts the coefficients to be interrelated as follows:

$$A_n = C_n = (-1)^n B_n = \frac{A_1^n}{n!}, \qquad (62)$$

so that the functions F, F_R, and $\mathcal{Z}_2$ obey a power law behavior. Then, the all orders solution of (59) for F can be summed up as follows:

$$F(k^2, \Lambda^2) = \left(\frac{k^2}{\Lambda^2}\right)^\beta, \qquad (63)$$

where we define $\beta = \alpha A_1$. This is the LLA. Beyond it, β would have terms of $\mathcal{O}(\alpha^2)$. Naturally, perturbation theory allows us to evaluate the anomalous dimension β at different orders of approximation.

The one-loop contribution to the fermion propagator can be evaluated by taking the tree level expressions for $S(p)$, $\Gamma_\mu(k, p)$, and $\Delta_{\mu\nu}(q)$ on the right-hand side of Eq. (47). In the massless limit, $\mathcal{M} = 0$ and the resulting expression for F is

$$\frac{1}{F(k^2, \Lambda^2)} = 1 + \frac{\alpha\xi}{4\pi^3}\int_E \frac{d^4p}{p^2}\frac{[p^2 - k\cdot p]}{q^4}$$
$$- \frac{\alpha}{4\pi^3}\int_E \frac{d^4p}{p^2}\frac{[2\nabla(k,p) - 3q^2(k\cdot p)]}{k^2q^4}. \qquad (64)$$

Angular integration of Eq. (64) leads to

$$\frac{1}{F(k^2, \Lambda^2)} = 1 + \frac{\alpha\xi}{4\pi}\int_{k^2}^{\Lambda^2}\frac{dp^2}{p^2}. \qquad (65)$$

Carrying out radial integration in Eq. (65) yields

$$F(k^2, \Lambda^2) = 1 + \frac{\alpha\xi}{4\pi}\log\left(\frac{k^2}{\Lambda^2}\right). \qquad (66)$$

Comparing expression (66) with the perturbative expansion (59) to one-loop order, we see that $A_1 = \xi/4\pi$. Therefore, perturbation theory fixes the $\mathcal{O}(\alpha)$ anomalous dimension in Eq. (63) to be

$$\beta = \frac{\alpha\xi}{4\pi}. \qquad (67)$$

The power law behavior of F in Eq. (63), with β given in Eq. (67), is the solution of

$$\frac{1}{F(k^2, \Lambda^2)} = 1 + \frac{\alpha\xi}{4\pi}\int_{k^2}^{\Lambda^2}\frac{dp^2}{p^2}\frac{F(p^2, \Lambda^2)}{F(k^2, \Lambda^2)}. \qquad (68)$$

Note that Eq. (68) is nonperturbative in nature and serves as a requirement of MR for the wave function renormalization F: any *Ansatz* for the three-point vertex must guarantee that the wave function renormalization F in Eq. (47) satisfies Eq. (68). We shall now proceed to show how the requirement of MR for the fermion propagator, embodied in Eq. (68), constrains the massless Y functions.

In the massless limit, Eq. (50) reduces to

$$\frac{1}{F(k^2)} = 1 + \frac{\alpha\xi}{4\pi^3}\int_E \frac{d^4p}{p^2}\frac{F(p^2)}{F(k^2)}\frac{[p^2 - k\cdot p]}{q^4}$$
$$- \frac{\alpha}{4\pi^3}\frac{1}{k^2}\int_E \frac{d^4p}{p^2}\frac{F(p^2)\Delta(q^2)}{k^2 - p^2}\{\Lambda_{NM}(k,p)$$
$$+ (k^2 - p^2)[3k^2p^2 b(k^2, p^2) - \tilde{b}(k^2, p^2)u(k,p)]\}. \qquad (69)$$

Angular integration of the last term on the right-hand side of the above Eq. (69) vanishes in qQED, since

$$\int_0^\pi d\varphi \sin^2\varphi \frac{u(k,p)}{q^2} = 0, \qquad (70)$$

where φ is the angle between k and p. Bearing in mind the latter result, Eq. (70)), it is straightforward to see from Eq. (69) that if we set, quite generally,

$$\Lambda_{NM}(k, p) = (p^2 - k^2)$$
$$\times [\mathcal{T}(k^2, p^2)u(k, p) + 3k^2 p^2 b(k^2, p^2)], \tag{71}$$

$\mathcal{T}$ being an *a priori* arbitrary, dimensionless function of k^2 and p^2 alone, then Eq. (68) is trivially fulfilled in qQED; i.e., a multiplicatively renormalizable solution for $F(k^2)$ is ensured.

Symmetry properties of the Y functions, Eqs. (29) and (30), restrict $\mathcal{T}(k^2, p^2)$ in Eq. (71) to be fully symmetric under $k^2 \leftrightarrow p^2$. Furthermore, in order to ensure that in perturbation theory the transverse form factors start at $\mathcal{O}(\alpha)$, the perturbative expansion for $\mathcal{T}$ is required to begin at the same order.

The above expression for $\Lambda_{NM}(k, p)$, Eq. (71), provides a nonperturbative *Ansatz* for the corresponding linear combination of the massless Y functions; see Eq. (56). Although Eq. (71) does not fix the Y functions individually, we shall show in Sec. VII that it suffices (along with additional constraints on the remaining, relevant Y functions) to investigate DCSB in the fermion propagator. Moreover, in Eq. (71), we assume that the q^2 dependence of the Y functions involved is effectively incorporated by means of the function $u(k, p)$. However, more realistic *Ansätze* are expected to possess a more complex q^2 dependence.

For an arbitrary q^2 dependence of $\Lambda_{NM}(k, p)$, it seems impossible to proceed any further in integrating Eq. (69) because of the unknown dependence of the Y functions on the angle φ. To circumvent this problem, we shall work with *effective* functions, denoted as $Y_i(k^2, p^2)$, whose relation with the *real* ones, $Y_i(k, p)$, is defined exactly in analogy with [84,105] as follows:

$$Y_2(k^2, p^2) = \frac{1}{f_2(k^2, p^2)} \int_0^\pi d\varphi \sin^2\varphi \frac{Y_2(k, p)}{q^2}, \tag{72}$$

$$Y_3(k^2, p^2) = \frac{1}{f_3(k^2, p^2)} \int_0^\pi d\varphi \sin^2\varphi \frac{t^2 Y_3(k, p)}{q^2}, \tag{73}$$

$$Y_5(k^2, p^2) = \frac{1}{f_5(k^2, p^2)} \int_0^\pi d\varphi \sin^2\varphi \frac{(k \cdot p) Y_5(k, p)}{q^2}, \tag{74}$$

$$Y_8^A(k^2, p^2) = \frac{1}{f_8(k^2, p^2)} \int_0^\pi d\varphi \sin^2\varphi \frac{\nabla(k, p) Y_8^A(k, p)}{q^2}, \tag{75}$$

where we have defined

$$f_2(k^2, p^2) = \int_0^\pi d\varphi \sin^2\varphi \frac{1}{q^2}$$
$$= \frac{\pi}{2}\left[\frac{1}{p^2}\theta(p^2 - k^2) + \frac{1}{k^2}\theta(k^2 - p^2)\right], \tag{76}$$

$$f_3(k^2, p^2) = \int_0^\pi d\varphi \sin^2\varphi \frac{t^2}{q^2}$$
$$= \frac{\pi}{2}\left[\left(1 + 2\frac{k^2}{p^2}\right)\theta(p^2 - k^2)\right.$$
$$\left. + \left(1 + 2\frac{p^2}{k^2}\right)\theta(k^2 - p^2)\right], \tag{77}$$

$$f_5(k^2, p^2) = \int_0^\pi d\varphi \sin^2\varphi \frac{(k \cdot p)}{q^2}$$
$$= \frac{\pi}{4}\left[\frac{k^2}{p^2}\theta(p^2 - k^2) + \frac{p^2}{k^2}\theta(k^2 - p^2)\right], \tag{78}$$

$$f_8(k^2, p^2) = \int_0^\pi d\varphi \sin^2\varphi \frac{\nabla(k, p)}{q^2}$$
$$= -\frac{\pi}{8}\left[\frac{k^2}{p^2}(k^2 - 3p^2)\theta(p^2 - k^2)\right.$$
$$\left. + \frac{p^2}{k^2}(p^2 - 3k^2)\theta(k^2 - p^2)\right], \tag{79}$$

where θ is the usual step function:

$$\theta(x - y) = \begin{cases} 1 & \text{for} \quad x \geq y, \\ 0 & \text{for} \quad x < y. \end{cases} \tag{80}$$

Using aforementioned effective functions, angular integration of Eq. (69) in qQED leads to

$$\frac{1}{F(k^2)} = 1 + \frac{\alpha\xi}{4\pi}\int_{k^2}^{\Lambda^2} \frac{dp^2}{p^2} \frac{F(p^2)}{F(k^2)}$$
$$- \frac{\alpha}{4\pi}\int_0^{k^2} \frac{dp^2}{k^2} F(p^2)\left\{3p^2 b(k^2, p^2)\right.$$
$$+ \frac{1}{2k^2}Y_2(k^2, p^2) + \frac{1}{2}\frac{Y_3(k^2, p^2)}{k^2 - p^2}\left(1 + 2\frac{p^2}{k^2}\right)$$
$$\left. - \frac{p^2}{2k^2}\frac{Y_5(k^2, p^2)}{k^2 - p^2} - p^2\frac{Y_8^A(k^2, p^2)}{k^2 - p^2}\left(\frac{p^2}{k^2} - 3\right)\right\}$$
$$- \frac{\alpha}{4\pi}\int_{k^2}^{\Lambda^2} \frac{dp^2}{k^2} F(p^2)\left\{3k^2 b(k^2, p^2)\right.$$
$$+ \frac{1}{2p^2}Y_2(k^2, p^2) + \frac{1}{2}\frac{Y_3(k^2, p^2)}{k^2 - p^2}\left(1 + 2\frac{k^2}{p^2}\right)$$
$$\left. - \frac{k^2}{2p^2}\frac{Y_5(k^2, p^2)}{k^2 - p^2} - k^2\frac{Y_8^A(k^2, p^2)}{k^2 - p^2}\left(\frac{k^2}{p^2} - 3\right)\right\}. \tag{81}$$

In order to ensure MR of the fermion propagator, we demand $F(k^2)$ on the left-hand side of Eq. (81) to satisfy Eq. (68); this imposes the following restriction:

$$\int_0^{k^2} \frac{dp^2}{k^2} F(p^2) \left\{ 3p^2 b(k^2, p^2) \right.$$

$$+ \frac{1}{2k^2} Y_2(k^2, p^2) + \frac{1}{2} \frac{Y_3(k^2, p^2)}{k^2 - p^2} \left(1 + 2\frac{p^2}{k^2} \right)$$

$$\left. - \frac{p^2}{2k^2} \frac{Y_5(k^2, p^2)}{k^2 - p^2} - p^2 \frac{Y_8^A(k^2, p^2)}{k^2 - p^2} \left(\frac{p^2}{k^2} - 3 \right) \right\}$$

$$+ \int_{k^2}^{\Lambda^2} \frac{dp^2}{k^2} F(p^2) \left\{ 3k^2 b(k^2, p^2) \right.$$

$$+ \frac{1}{2p^2} Y_2(k^2, p^2) + \frac{1}{2} \frac{Y_3(k^2, p^2)}{k^2 - p^2} \left(1 + 2\frac{k^2}{p^2} \right)$$

$$\left. - \frac{k^2}{2p^2} \frac{Y_5(k^2, p^2)}{k^2 - p^2} - k^2 \frac{Y_8^A(k^2, p^2)}{k^2 - p^2} \left(\frac{k^2}{p^2} - 3 \right) \right\} = 0. \tag{82}$$

This requirement encodes the fact that all divergences have already been absorbed in the MR solution for the wave function renormalization F. As a consequence, there is no necessity of regularizing Eq. (82), and we can take $\Lambda^2 \to \infty$ in the integration limit. It is convenient to introduce a dimensionless variable x, defined as

$$x = \frac{p^2}{k^2} \quad \forall \; p^2 \in [0, k^2], \tag{83}$$

$$x = \frac{k^2}{p^2} \quad \forall \; p^2 \in [k^2, \infty], \tag{84}$$

so that Eq. (82) is now expressed as

$$\int_0^1 dx \, W(x) = 0, \tag{85}$$

with

$$W(x) = 6\frac{r(x)}{x-1} + (x^\beta + x^{-2})[h_1(x) + h_2(x)], \tag{86}$$

where we have defined the function

$$r(x) = x(1 - x^\beta) - x^{-1}(1 - x^{-\beta}). \tag{87}$$

The presence of the anomalous dimension β as an exponent in Eqs. (86) and (87) is related to the terms $F(k^2)/F(p^2)$ in (82), which can be expressed as $(k^2/p^2)^\beta$ in light of Eq. (63). Furthermore, in Eq. (86), we have defined

$$h_1(x) = \frac{F(k^2)}{x-1} \frac{1}{k^2} H_1(k^2, xk^2), \tag{88}$$

$$h_2(x) = \frac{F(k^2)}{x-1} \frac{x}{k^2} H_2(k^2, xk^2). \tag{89}$$

These are dimensionless functions, satisfying the properties

$$h_1(x^{-1}) = x^{\beta-1} h_1(x), \tag{90}$$

$$h_2(x^{-1}) = x^{\beta-2} h_2(x). \tag{91}$$

Moreover, in Eqs. (88) and (89), we have conveniently defined scalar functions

$$H_1(k^2, p^2) = \left(\frac{p^2}{k^2} - 1 \right) Y_2(k^2, p^2) - \left(\frac{p^2}{k^2} + 1 \right) Y_3(k^2, p^2)$$

$$- 8p^2 Y_8^A(k^2, p^2), \tag{92}$$

$$H_2(k^2, p^2) = -Y_3(k^2, p^2) + Y_5(k^2, p^2)$$

$$+ 2(k^2 + p^2) Y_8^A(k^2, p^2). \tag{93}$$

Employing $x = p^2/k^2$ in Eq. (86), and using definitions (88), (89), (92), and (93), we have (for $k^2 > p^2$)

$$W\left(\frac{p^2}{k^2} \right) = \frac{S(k^2, p^2)}{p^2 - k^2} \left\{ \left(1 - \frac{k^2}{p^2} \right) Y_2(k^2, p^2) \right.$$

$$- \left(2 + \frac{k^2}{p^2} \right) Y_3(k^2, p^2) + Y_5(k^2, p^2)$$

$$\left. + 2(p^2 - 3k^2) Y_8^A(k^2, p^2) \right\} + 6k^2 \frac{r(p^2/k^2)}{p^2 - k^2}, \tag{94}$$

where we have defined

$$S(k^2, p^2) = F(k^2) \frac{k^2}{p^2} + F(p^2) \frac{p^2}{k^2}, \tag{95}$$

which enters the definition of $r(p^2/k^2)$ through

$$r\left(\frac{p^2}{k^2} \right) = S(k^2, p^2) \left[\frac{1}{F(p^2)} - \frac{1}{F(k^2)} \right]. \tag{96}$$

Equations (85)–(96) constitute nonperturbative constraints on the fermion-photon vertex: for every *Ansatz* for the Y functions, the resulting function W is restricted to guarantee the integral constraint (85), so that the MR of the fermion propagator is ensured. To bring out the applicability and scope of the integral constraint on the massless Y functions, Eq. (85), we now proceed to analyze an existing, rather general transverse vertex *Ansatz*, which was constructed in qQED to implement the requirement of MR for a massless fermion propagator in addition to all other key features of QED mentioned before [26]. Different choices of the free parameters defining this *Ansatz* correspond to numerous

vertices constructed in the past. We will make reference to all these constructions along the way.

VI. EXAMPLES

The Bashir-Bermudez-Chang-Roberts (BBCR) vertex, Ref. [13], is an *Ansatz* for the dressed fermion-photon vertex in QED, whose construction is constrained primarily by two requirements: to provide MR of the fermion propagator and to produce gauge independent critical coupling for DCSB. As it involves projecting the vertex onto the gap equation, it is natural that it is expressed only in terms of the functions which appear in the full fermion propagator, namely $F(k^2)$ and $\mathcal{M}(k^2)$. Moreover, its simplicity lies in the fact that its functional dependence on these entities is solely through the forms which enter the longitudinal vertex, namely, $b(k^2, p^2)$ and $c(k^2, p^2)$. In our kinematical configuration and notation, the transverse form factors for the BBCR vertex read as

$$\tau_1(k^2, p^2) = \frac{a_1}{(k^2 + p^2)} c(k^2, p^2), \tag{97}$$

$$\tau_2(k^2, p^2) = \frac{a_2}{(k^2 + p^2)} b(k^2, p^2), \tag{98}$$

$$\tau_3(k^2, p^2) = a_3 b(k^2, p^2), \tag{99}$$

$$\tau_4(k^2, p^2) = \frac{a_4(k^2 - p^2)}{4k^2 p^2} c(k^2, p^2), \tag{100}$$

$$\tau_5(k^2, p^2) = -a_5 c(k^2, p^2), \tag{101}$$

$$\tau_6(k^2, p^2) = -\frac{a_6(k^2 + p^2)}{(k^2 - p^2)} b(k^2, p^2), \tag{102}$$

$$\tau_7(k^2, p^2) = -\left[\frac{a_4 q^2}{2k^2 p^2} + \frac{a_7}{k^2 + p^2}\right] c(k^2, p^2), \tag{103}$$

$$\tau_8(k^2, p^2) = a_8 \, b(k^2, p^2), \tag{104}$$

where the coefficients a_i are constants. We will consider this example in detail because different choices of a_i correspond to several vertices proposed in the literature (see Ref. [74]), e.g., the Ball-Chiu vertex [44], the Curtis-Pennington vertex [78], and the Qin-Chang vertex [52].

From Eqs. (35)–(41) and (97)–(104), we see that the corresponding Y functions for the BBCR vertex read as

$$Y'_1(k, p) = -2a_1 c(k^2, p^2) \frac{\nabla(k, p)}{k^2 + p^2}, \tag{105}$$

$$Y'_2(k, p) = -b(k^2, p^2)$$
$$\times \left[(k^2 - p^2)\left(a_3 - \frac{1}{2}\right) + a_6\left(\frac{k^2 + p^2}{k^2 - p^2}\right) t^2\right], \tag{106}$$

$$Y'_3(k, p) = b(k^2, p^2)$$
$$\times \left[q^2\left(a_3 - \frac{1}{2}\right) + 2a_2 \frac{\nabla(k, p)}{k^2 + p^2} + a_6(k^2 + p^2)\right], \tag{107}$$

$$Y'_4(k, p) = -\frac{1}{2} c(k^2, p^2)$$
$$\times \left[a_4 \frac{\nabla(k, p)}{k^2 p^2} + (a_5 - 1) + \frac{a_7}{2} \frac{t^2}{k^2 + p^2}\right], \tag{108}$$

$$Y'_5(k, p) = 3b(k^2, p^2)$$
$$\times \left[q^2\left(a_3 - \frac{1}{2}\right) + \frac{2}{3} a_2 \frac{\nabla(k, p)}{k^2 + p^2} + a_6(k^2 + p^2)\right], \tag{109}$$

$$Y'^A_6(k, p) = c(k^2, p^2)$$
$$\times \left[a_4 \frac{\nabla(k, p)}{k^2 p^2} + 3(a_5 - 1) + \frac{a_7}{2} \frac{t^2}{k^2 + p^2}\right], \tag{110}$$

$$Y'^S_7(k, p) = a_7 \, c(k^2, p^2) \frac{k^2 - p^2}{k^2 + p^2}, \tag{111}$$

$$Y'^A_8(k, p) = -\frac{1}{2} b(k^2, p^2)(a_8 + 1). \tag{112}$$

It is mathematically straightforward to show that the asymptotic expansion of Eqs. (106), (107), and (109) in powers of k^2/p^2, for $p^2 \gg k^2 \gg m_0^2$, fulfills the perturbation theory requirement (46) up to second order if [13]

$$a_3 + a_6 = \frac{1}{2}. \tag{113}$$

In order to verify that if, for the massless case, the BBCR vertex satisfies the integral constraint for W, Eq. (85), it is necessary to compute the corresponding massless effective Y functions by means of Eqs. (72)–(75), which in turn yield

$$Y_2(k^2, p^2) = b(k^2, p^2)$$
$$\times \{(1/2 - a_3)(k^2 - p^2)^2$$
$$- a_6(k^2 + p^2)(k^2 + 2p^2)\}, \tag{114}$$

$$Y_3(k^2, p^2) = \frac{1}{2} \frac{k^4 - p^4}{k^2 + 2p^2} b(k^2, p^2)$$
$$\times \{(2a_3 + 2a_6 - 1)k^2 - (a_2 - 4a_6)p^2$$
$$+ a_2 p^2((4k^4 + 6k^2 p^2 - p^4)/((k^2 + p^2)^2))\}, \tag{115}$$

$$Y_5(k^2, p^2) = \frac{k^2 - p^2}{k^2 + p^2} b(k^2, p^2)$$
$$\times \{3a_6(k^2 + p^2)^2 + a_2 p^2(k^2 - p^2/2)\},$$

$$(116)$$

$$Y_8^A(k^2, p^2) = -\frac{1}{2}(a_8 + 1)(k^2 - p^2)b(k^2, p^2). \qquad (117)$$

Using Eqs. (114)–(117) in the expression for $W(p^2/k^2)$, Eq. (94), we find

$$W(x) = -2(a_2 + 2a_3 - 2a_8)\frac{r(x)}{1 - x}$$
$$+ \frac{1}{2}[-3(1 - 2a_6) + (a_2 + 2a_3 - 2a_8)]$$
$$\times \left(\frac{1 + x}{1 - x}\right) r(x). \qquad (118)$$

If we insert the above expression for $W(x)$, Eq. (118), in its corresponding integral restriction, Eq. (85), we find that the integral vanishes if and only if the following two conditions are met:

$$a_2 + 2a_3 - 2a_8 = 0, \qquad (119)$$

$$a_6 = 1/2, \qquad (120)$$

which are the constraints reported in [13] for the coefficients a_i in order to ensure the MR of the fermion propagator.

As a counterexample, we could take all Y functions equal to zero. If we do so, the resulting function $W(x)$ reads as

$$W(x) = -6\frac{r(x)}{1 - x}, \qquad (121)$$

which does not satisfy the integral constraint (85). Therefore, *setting all Y functions equal to zero does not ensure the MR of the fermion propagator.*

Throughout Secs. V and VI, we have investigated the MR solution for the massless fermion propagator in qQED, within the LLA, and derived a consequent nonperturbative, integral constraint for the massless Y functions. In the next section, we study DCSB and implement the argument of a gauge independent critical coupling to impose further constraints on the transverse vertex.

VII. GAUGE INDEPENDENT DCSB

In order to study DCSB through the gap equation by employing a fully dressed fermion-photon vertex, Eqs. (2)–(4), (7), (8), and (20)–(27), we propose an *Ansatz* for the functions Y_i appearing in Eqs. (49)–(56). Naturally, we look for the simplest construction which incorporates

all the key constraints we have enlisted and studied so far. This can be achieved by requiring the following:

(1) The massive Y functions in the gap equation are expressed solely in terms of the fermion dressing functions, F and $\mathcal{M}$.

(2) The antisymmetric contribution of $G_{\mathcal{M}}(k, p)$ and $G_F(k, p)$ vanishes under $k \leftrightarrow p$.

(3) The functions $G_{\mathcal{M}}$ and G_F are the same (up to a constant factor).

These simplifying requirements do not jeopardize the MR of the fermion propagator which can still be ensured in massless qQED.

Assumptions (1) and (2) are fulfilled if we choose

$$Y_1(k, p) = -4\frac{\nabla(k, p)(k^2 - p^2))}{q^2} c(k^2, p^2), \qquad (122)$$

$$Y_6^A(k, p) = -3(k^2 - p^2)c(k^2, p^2), \qquad (123)$$

$$Y_7^S(k, p) = 0. \qquad (124)$$

Moreover, we can implement the simplifying requirement (3) by demanding the MR condition on the massless functions, Eq. (71), and fixing Y_5 as follows:

$$Y_5(k, p) = (k^2 - p^2)[3\mathcal{T}(k^2, p^2) + u(k, p)b(k^2, p^2)]. \qquad (125)$$

In fact, the *Ansatz* for the Y functions, constituted through Eqs. (71) and (122)–(125), yields

$$\frac{1}{3}G_{\mathcal{M}}(k, p) = G_F(k, p) = \Delta(q^2)[\mathcal{T}(k^2, p^2) + \tilde{b}(k^2, p^2)]$$
$$\equiv G(k, p), \qquad (126)$$

which fulfills assumption (2) in addition to (3). Moreover, for the massless limit in qQED, it simplifies Eq. (50) as

$$\frac{1}{F(k^2)} = 1 - \frac{\alpha\xi}{4\pi^3}\int_E \frac{d^4 p}{p^2}\frac{F(p^2)}{F(k^2)}\frac{q \cdot p}{q^4}. \qquad (127)$$

After angular integration, F satisfies Eq. (68) as expected; i.e., it has the power law behavior of Eq. (63) as constrained by MR, with the anomalous dimension β given in Eq. (67).

Let us summarize below the important characteristics of the function $\mathcal{T}$:

(i) It must be a dimensionless function of k^2 and p^2. We assume it to be q^2 independent in order to ensure the MR of $F(k^2)$.

(ii) It must be fully symmetric under $k^2 \leftrightarrow p^2$.

(iii) Its perturbative expansion must start at $\mathcal{O}(\alpha)$.

(iv) It must vanish in the Landau gauge, $\xi = 0$.

Recall from Sec. V that condition (i) is required to ensure a MR solution for the massless fermion propagator in qQED,

while the conditions (ii) and (iii) follow from the symmetry properties of the vertex and its perturbative expansion, respectively. The additional condition (iv) is imposed in order to facilitate the extraction of an ξ-independent critical coupling, and an anomalous dimension for the mass function which, at criticality, is independent of the choice of the vertex, as we shall discuss now.

In the vicinity of the critical coupling, α_c, above which chiral symmetry is broken dynamically, the generated fermion mass is negligible in comparison with any other mass scale. Hence, for $\alpha \sim \alpha_c$, we can formally neglect quadratic and higher powers (if any) of the mass function in Eq. (49). In the limit $m_0 = 0$, it reduces to

$$
\frac{\mathcal{M}(k^2)}{F(k^2)} = \frac{\alpha\xi}{4\pi^3} \int_E \frac{d^4p}{p^2} \frac{F(p^2)}{F(k^2)} \frac{1}{q^4}
$$
$$
\times \{\mathcal{M}(p^2)q \cdot k - \mathcal{M}(k^2)q \cdot p\}
$$
$$
+ \frac{3\alpha}{4\pi^3} \int_E \frac{d^4p}{p^2} F(p^2)\mathcal{M}(p^2)G(k,p), \quad (128)
$$

where our *Ansatz* for the Y functions, Eqs. (71) and (122)–(125), has been implicitly embedded through the function $G(k,p)$, defined in Eq. (126). Moreover, neglecting terms quadratic in $\mathcal{M}$, the equation for F, Eq. (50), reduces to that of a massless theory and decouples from that of the mass function $\mathcal{M}$. In the quenched approximation, it yields Eq. (127). Therefore, from Eqs. (127) and (128) we see that in the vicinity of the critical coupling, in qQED, the mass function satisfies the following equation:

$$
\mathcal{M}(k^2) = \frac{\alpha\xi}{4\pi^3} \int_E \frac{d^4p}{p^2} \frac{F(p^2)}{F(k^2)} \mathcal{M}(p^2) \frac{q \cdot k}{q^4}
$$
$$
+ \frac{3\alpha}{4\pi^3} \int_E \frac{d^4p}{p^2} F(p^2)\mathcal{M}(p^2)G(k,p). \quad (129)
$$

In the neighborhood of α_c, MR forces a power law behavior for the mass function which must hold at all momenta,

$$
\mathcal{M}(k^2) = B_\Lambda (k^2)^{-s}, \quad (130)
$$

where B_Λ is a constant (that depends on Λ) and the exponent $s = 1 - \gamma_m/2$ is defined in terms of the anomalous dimension of the mass function, γ_m. We assume $0 < s \leq 1$ to comply with perturbation theory.

Bardeen *et al.* demonstrated that, at $\alpha = \alpha_c$, the mass anomalous dimension is $\gamma_m = 1$ [106,107]. Some further analyses, based on the Cornwall-Jackiw-Tomboulis effective potential technique, tend to argue that, at criticality, this value holds true regardless of the choice of the vertex [108–110]. In Ref. [82], this value is quite close to unity, though not exactly equal to it. Setting $\gamma_m = 1$ results in the four-fermion interaction operator $(\bar{\psi}\psi)^2$ acquiring dynamical dimension $d = 2(3 - \gamma_m) = 4$ in contrast to its canonical dimension $d = 6$. Therefore, four-fermion interaction

becomes marginal. It must then be included in order to render nonperturbative QED a self-consistent, closed theory [108,111]. Depending upon the nonperturbative details of the fermion-photon interaction, it is plausible to have $\gamma_m > 1$, implying $d < 4$, which would modify the status of the four-point operators from marginal to relevant; see, e.g., the review article [25] and references therein.

At $\alpha = \alpha_c$, the anomalous dimension $\gamma_m = 1$ and its corresponding critical value $s_c = 1/2$ can be obtained by constraining the fermion-photon vertex, a line of action that is followed in [83,108]. In our analysis, this critical value is readily derived from Eq. (128) in Landau gauge, $\xi = 0$, if we demand $\mathcal{T}(k^2, p^2)$ to fulfill condition (iv): from Eq. (127) we see that in the Landau gauge $F(k^2) = 1$, and therefore Eq. (129) reduces to

$$
\mathcal{M}(k^2) = \frac{3\alpha}{4\pi^3} \int_E \frac{d^4p}{p^2} \frac{\mathcal{M}(p^2)}{q^2}. \quad (131)
$$

For the MR solution of the mass function, Eq. (130), the above Eq. (131) results in

$$
s = \frac{1}{2} \pm \frac{1}{2}\sqrt{1 - \frac{\alpha}{\alpha_c}}, \quad (132)
$$

where α_c stands for the critical coupling, which signals the point where the two possible solutions for s match each other, and a nontrivial solution for the mass function[1] bifurcates away from the perturbative one ($\mathcal{M} = 0$): for $\alpha > \alpha_c$, the solution for the mass function enters the complex plane indicating that DCSB has taken place. In this case, the critical coupling is

$$
\alpha_c = \frac{\pi}{3}, \quad (133)
$$

thus revealing a Miransky scaling law for the interaction strength α [113–115], which has been derived using a bare vertex [116,117]. For $\alpha = \alpha_c = \pi/3$ in Eq. (132), the expected critical value for the anomalous mass dimension is obtained, i.e.,

$$
s_c = \frac{1}{2}. \quad (134)
$$

Since the critical coupling pinpoints a phase transition from perturbative to nonperturbative dynamics, it is potentially a physical observable, and hence it is expected to be independent of the gauge parameter. Thus, Eq. (131), and its corresponding MR solution, Eq. (130), must hold in

---

[1]In addition to positive-definite solutions for the mass function (and their corresponding mirrors), an arbitrary vertex may produce spurious oscillatory solutions for $\mathcal{M}$. However, it has been argued that a realistic vertex might only produce monotonically decreasing and increasing nontrivial solutions [112].

all gauges. Therefore, from Eqs. (129) and (131), we see that in order for α_c and s_c to be ξ independent, the mass function and the vertex must satisfy the following equation in qQED:

$$\int_E \frac{d^4 p}{p^2} \frac{\mathcal{M}(p^2)F(p^2)}{q^2}$$
$$\times \left\{ \frac{\xi q \cdot k}{3q^2 F(k^2)} + \mathcal{T}(k^2, p^2) + k^2 b(k^2, p^2) \right\} = 0. \quad (135)$$

After performing angular integration, and introducing the dimensionless variables defined in Eqs. (83) and (84), the above Eq. (135) can be cast in the following form:

$$\int_0^1 \frac{dx}{\sqrt{x}} V(x) = 0, \quad (136)$$

with

$$V(x) = \frac{\xi}{3} x^{\beta - s + \frac{1}{2}} + [x^{\beta - s + \frac{1}{2}} + x^{s - \frac{1}{2}}]g(x)$$
$$+ \left(\frac{x^\beta - 1}{1 - x} \right) x^{-s + \frac{1}{2}} - \left(\frac{x^{-\beta} - 1}{1 - x} \right) x^{s + \frac{1}{2}}, \quad (137)$$

where s and β, defined in Eqs. (67) and (134), appear in the light of the MR solutions for F and $\mathcal{M}$, Eqs. (63) and (130), respectively. Furthermore, in Eq. (137) we have defined

$$g(x) = F(k^2)\mathcal{T}(k^2, xk^2), \quad (138)$$

which is independent of k^2 and satisfies the following property:

$$g(x^{-1}) = x^\beta g(x). \quad (139)$$

It is important to stress the fact that Eq. (136) stands for a nonperturbative constraint on the vertex: any *Ansatz* for $\mathcal{T}(k^2, p^2)$ must provide a function V that should satisfy Eq. (136). Conversely, from a particular solution for V in the latter equation, one can derive the corresponding function $\mathcal{T}$ by means of Eq. (137). However, there exists an infinite number of solutions for $V(x)$ satisfying Eq. (136). In addition, such a solution for V must also satisfy [cf. Eq. (137)]

$$V(x) - V(x^{-1}) = \frac{\xi}{3}(x^{\beta - s + \frac{1}{2}} - x^{-\beta + s - \frac{1}{2}}). \quad (140)$$

A simple choice satisfying Eqs. (136) and (140) at criticality reads

$$V(x) = \frac{\xi}{3} \left\{ x^\beta + \frac{1 - 2\beta}{8\beta^2}(2 - x^\beta - x^{-\beta}) \right\}, \quad (141)$$

valid for $-\frac{1}{2} \leq \beta \leq \frac{1}{2}$ but $\beta \neq 0$. In the Landau gauge, Eqs. (136) and (140) are satisfied with the trivial solution

$$V(x)_{\xi=0} = 0. \quad (142)$$

It is worth reminding the reader that Eq. (136), and consequently Eqs. (137) and (140), are rigorously valid only at criticality. Therefore, for $\alpha_c = \pi/3$ and $s_c = 1/2$, the resulting function $\mathcal{T}(k^2, p^2)$, derived from Eqs. (137) and (141), reads (for $x = p^2/k^2$ and $-6 \leq \xi \leq 6$, but $\xi \neq 0$) as

$$\mathcal{T}(k^2, p^2) = -\frac{1}{2}(k^2 + p^2)b(k^2, p^2)$$
$$+ \frac{(12 - \xi)}{2\xi} \left[\frac{F(k^2) - F(p^2)}{F(k^2) + F(p^2)} \right] \left[\frac{1}{F(k^2)} - \frac{1}{F(p^2)} \right], \quad (143)$$

whereas Eqs. (137) and (142) yield (for Landau gauge)

$$\mathcal{T}(k^2, p^2)_{\xi=0} = -\frac{1}{2}(k^2 + p^2)b(k^2, p^2). \quad (144)$$

The above expressions for $\mathcal{T}$ fulfill conditions (i)–(iv), as expected, but a few observations must be made: (a) MR of the wave function renormalization entails $F(p^2, \Lambda^2)/F(k^2, \Lambda^2) = F(p^2, \mu^2)/F(k^2, \mu^2)$ for some renormalization scale, μ^2, ensuring the Λ^2 independence of the second term on the right-hand side of Eq. (143); and (b) although $F = 1$ for $\xi = 0$ (in the LLA of qQED), leading to $\mathcal{T}_{\xi=0} = 0$, the function $\mathcal{T}_{\xi=0}(k^2, p^2)$ defined in Eq. (144) does not necessarily vanish in the Landau gauge beyond the LLA and the quenched approximation.

Numerical evaluation of the ratio between the Euclidean mass m_E, defined as $m_E = \mathcal{M}(m_E^2)$, and the ultraviolet cutoff Λ is shown in Fig. 3 for different gauges implementing the *Ansätze* introduced in Eqs. (143) and (144). In the plot, points with $\alpha < 1.25$ (highlighted as filled markers) are fitted to the Miransky scaling law [113–117]
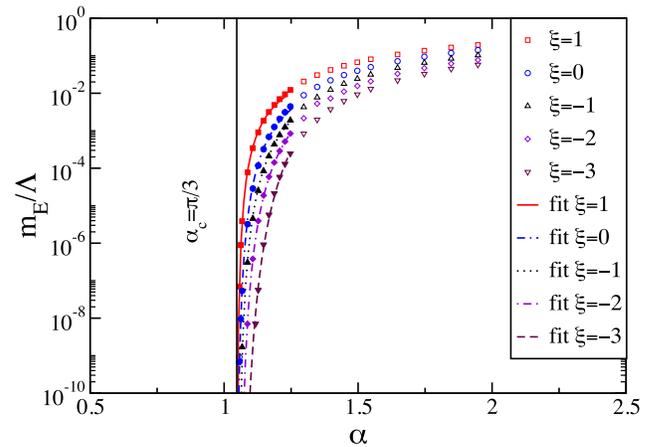
FIG. 3. Ratio of the Euclidean mass and the ultraviolet cutoff in different gauges. $\alpha_c = \pi/3$ within the numerical accuracy of our computation.

TABLE I.   Parameters for different gauges.

ξ	$\alpha_c = \pi/3$	κ	ϕ
1	0.69%	0.695	0.481
0	0.82%	0.970	1.383
-1	0.51%	1.112	1.604
-2	0.51%	1.199	1.679
-3	2.81%	1.283	1.829

$$\frac{m_E}{\Lambda} = \exp\left[-\frac{\pi\kappa}{\sqrt{\frac{\alpha}{\alpha_c}-1}} + \phi\right]. \qquad (145)$$

Each fit yields the critical coupling $\alpha_c = \pi/3$ with a numerical error of less than 1% for $\xi = 1, 0, -1, -2$ (and $\sim2.8\%$ for $\xi = -3$) as indicated in the second column of Table I It is worth reminding the reader that our *Ansatz* reveals $\alpha_c = \pi/3$, just as the bare vertex in the Landau gauge. However, *the bare vertex leads to a highly gauge dependent α_c, including no chiral symmetry breaking for $\xi = -3$. In our case, chiral symmetry is broken in every gauge with the same critical coupling.* The result for $\xi = -3$ particularly emphasizes this point.

It is well known that DCSB manifests itself in the three-point vertex through its massive form factors [118,119], and thus a physically meaningful *Ansatz* for $\mathcal{T}(k^2, p^2)$ should incorporate the mass function. In the present work, we propose a simple, numerically tractable *Ansatz* for $\xi = 0$,

$$\mathcal{T}(k^2, p^2)_{\xi=0} = -\frac{1}{2}(k^2 + p^2)b(k^2, p^2)$$
$$+ \rho\left[\frac{\mathcal{M}(k^2)}{F(k^2)} + \frac{\mathcal{M}(p^2)}{F(p^2)}\right]c(k^2, p^2), \quad (146)$$

which is an extension of Eq. (144), with a mass term weighted by a real constant ρ. The fact that the last term of Eq. (146) contains quadratic powers in $\mathcal{M}$ ensures that this contribution can be neglected at criticality, which in turn yields a gauge independent critical coupling. In addition, characteristics (i)–(iv) remain preserved.

The numerical evaluation of m_E/Λ using Eq. (146) is shown in Fig. 4 for different values of ρ. Again, points with $\alpha < 1.25$ are fitted to Eq. (145), indicating a critical coupling independent of the ρ parameter and equal to $\pi/3$, within a margin of error smaller than 1%.

Numerical results for the Euclidean mass plotted in Figs. 3 and 4 support the argument that any function $\mathcal{T}(k^2, p^2)$ preserving characteristics (i)–(iv) and satisfying conditions defined through Eqs. (136) and (140) will ensure a gauge independent critical coupling $\alpha_c = \pi/3$ in qQED. Equations (143) and (144) define simple, numerically friendly *Ansätze* for the transverse vertex contribution $\mathcal{T}$ to the gap equation. In Landau gauge, an extension of the
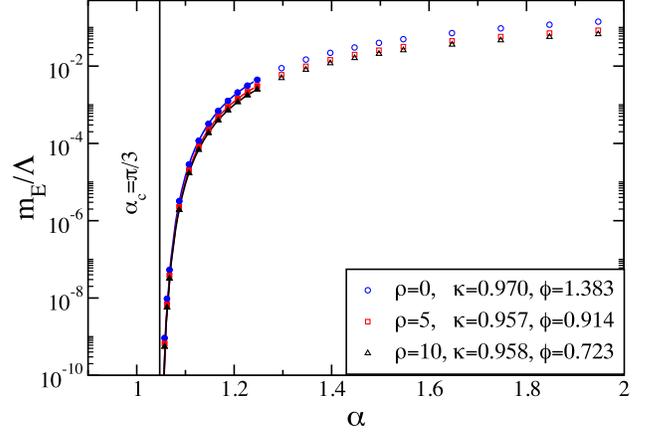
FIG. 4.   Euclidean mass in Landau gauge for $\rho = 0, 5, 10$. ρ stands for the strength of the DCSB in the vertex, Eq. (146).

Ansatz defined through Eq. (146) explicitly incorporates DCSB and still ensures a critical coupling independent of the ρ parameter.

VIII. CONCLUSIONS

In this article, we have investigated combined constraints of TTI, LKFT, MR of the massless fermion propagator, gauge independence of the critical coupling α_c in quenched QED, and one-loop perturbation theory in the asymptotic limit to construct a general fermion-photon vertex. Notice that the TTI involve two new nonlocal vertices. Therefore, the constraints of TTI come at the expense of additional unknown functions. The parametrization of the transverse vertex in terms of Y_i ensures the new constraints, plus the nonlocal vertices are quantitatively integrated into the description of the transverse fermion-photon vertex. We work explicitly with functions Y_i, which arise naturally on the implementation of the TTI, providing, along the way, their symmetry properties under the charge conjugation operation. Through an exact relation, we define effective Y_i for which the angular dependence on the variable q^2 has been integrated out to make their implementation in the gap equation more efficient. As a simplifying consequence of working with Y_i, we observe that the kernel dependence on the Gram determinant $\nabla(k, p)$ for the mass function disappears altogether. Moreover, our study reveals that we cannot force all Y_i to be simultaneously equal to zero. It will violate the LKFT transformation law and the MR of the massless fermion propagator. We work with quite a general vertex construction [13], formulated in terms of Y_i. We also provide simple examples of this fermion-photon vertex and carry out a numerical study of the gap equation to compute the mass function and its variation as a function of the coupling strength. The results clearly follow Miransky scaling law and provide $\alpha_c = \pi/3$. Moreover, anomalous mass dimension $\gamma_m = 1$, as has been advocated in several previous works [83,108–110]. Also, this critical coupling is

gauge independent. As mentioned before, the fermion-photon vertex enters the SDE study of several hadronic observables, such as form factor calculations, where photons interact with quarks. Therefore, an improved understanding of this vertex, such as the one detailed in this article, is very important. Moreover, a natural extension of our work for the quark-gluon vertex in QCD is currently underway.

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[1] P. Boucaud, J. Leroy, A. Le Yaouanc, J. Micheli, O. Pene, and J. Rodríguez-Quintero, J. High Energy Phys. 06 (2008) 099.

[2] A. Aguilar, D. Binosi, and J. Papavassiliou, Phys. Rev. D **78**, 025010 (2008).

[3] M. Pennington and D. Wilson, Phys. Rev. D **84**, 119901 (2011).

[4] A. Ayala, A. Bashir, D. Binosi, M. Cristoforetti, and J. Rodriguez-Quintero, Phys. Rev. D **86**, 074512 (2012).

[5] A. C. Aguilar, D. Binosi, and J. Papavassiliou, Phys. Rev. D **86**, 014032 (2012).

[6] S. Strauss, C. S. Fischer, and C. Kellermann, Phys. Rev. Lett. **109**, 252001 (2012).

[7] A. Bashir, A. Raya, and J. Rodriguez-Quintero, Phys. Rev. D **88**, 054003 (2013).

[8] M. Q. Huber, Phys. Rev. D **91**, 085018 (2015).

[9] R. Bermudez, L. Albino, L. X. Gutiérrez-Guerrero, M. E. Tejeda-Yeomans, and A. Bashir, Phys. Rev. D **95**, 034041 (2017).

[10] R. Alkofer, C. S. Fischer, F. J. Llanes-Estrada, and K. Schwenzer, Ann. Phys. (Amsterdam) **324**, 106 (2009).

[11] L. Chang and C. D. Roberts, Phys. Rev. Lett. **103**, 081601 (2009).

[12] A. Kizilersu and M. Pennington, Phys. Rev. D **79**, 125020 (2009).

[13] A. Bashir, R. Bermudez, L. Chang, and C. Roberts, Phys. Rev. C **85**, 045205 (2012).

[14] M. Hopfer, A. Windisch, and R. Alkofer, Proc. Sci., ConfinementX2012 (**2012**) 073.

[15] E. Rojas, J. de Melo, B. El-Bennich, O. Oliveira, and T. Frederico, J. High Energy Phys. 10 (2013) 193.

[16] A. C. Aguilar, D. Binosi, D. Ibañez, and J. Papavassiliou, Phys. Rev. D **90**, 065027 (2014).

[17] M. Peláez, M. Tissier, and N. Wschebor, Phys. Rev. D **92**, 045012 (2015).

[18] R. Williams, C. S. Fischer, and W. Heupel, Phys. Rev. D **93**, 034026 (2016).

[19] A. C. Aguilar, J. C. Cardona, M. N. Ferreira, and J. Papavassiliou, Phys. Rev. D **98**, 014002 (2018).

[20] R. Alkofer, C. S. Fischer, and R. Williams, Eur. Phys. J. A **38**, 53 (2008).

[21] L. Chang, I. C. Cloët, C. D. Roberts, S. M. Schmidt, and P. C. Tandy, Phys. Rev. Lett. **111**, 141802 (2013).

[22] R. Williams, Eur. Phys. J. A **51**, 57 (2015).

[23] K. Raya, L. Chang, A. Bashir, J. J. Cobos-Martinez, L. X. Gutiérrez-Guerrero, C. D. Roberts, and P. C. Tandy, Phys. Rev. D **93**, 074017 (2016).

[24] M. Gómez-Rocha, T. Hilger, and A. Krassnigg, Phys. Rev. D **93**, 074010 (2016).

[25] A. Bashir and A. Raya, arXiv:hep-ph/0411310.

[26] A. Bashir, L. Chang, I. C. Cloet, B. El-Bennich, Y.-X. Liu, C. D. Roberts, and P. C. Tandy, Commun. Theor. Phys. **58**, 79 (2012).

[27] I. G. Aznauryan et al., Int. J. Mod. Phys. E **22**, 1330015 (2013).

[28] G. Eichmann, H. Sanchis-Alepuz, R. Williams, R. Alkofer, and C. S. Fischer, Prog. Part. Nucl. Phys. **91**, 1 (2016).

[29] M. Q. Huber, Ph. D. thesis, Graz University, 2018.

[30] K. Raya, M. Ding, A. Bashir, L. Chang, and C. D. Roberts, Phys. Rev. D **95**, 074014 (2017).

[31] E. Weil, G. Eichmann, C. S. Fischer, and R. Williams, Phys. Rev. D **96**, 014021 (2017).

[32] G. Eichmann, C. S. Fischer, E. Weil, and R. Williams, Phys. Lett. B **774**, 425 (2017).

[33] M. Ding, K. Raya, A. Bashir, D. Binosi, L. Chang, M. Chen, and C. D. Roberts, Phys. Rev. D **99**, 014014 (2019).

[34] J. C. Ward, Phys. Rev. **78**, 182 (1950).

[35] E. S. Fradkin, Zh. Eksp. Teor. Fiz. **29**, 258 (1955) [Sov. Phys. JETP **2**, 361 (1956)].

[36] H. Green, Proc. Phys. Soc. London Sect. A **66**, 873 (1953).

[37] Y. Takahashi, Nuovo Cimento **6**, 371 (1957).

[38] A. Salam and R. Delbourgo, Phys. Rev. **135**, B1398 (1964).

[39] R. Delbourgo and P. C. West, J. Phys. A **10**, 1049 (1977).

[40] R. Delbourgo, Nuovo Cimento A **49**, 484 (1979).

[41] R. Delbourgo, B. W. Keck, and C. N. Parker, J. Phys. A **14**, 921 (1981).

[42] R. Delbourgo and R. B. Zhang, J. Phys. A **17**, 3593 (1984).

[43] P. Rembiesa, Phys. Rev. D **33**, 2333 (1986).

[44] J. S. Ball and T.-W. Chiu, Phys. Rev. D **22**, 2542 (1980).

[45] Y. Takahashi, in Positano Symp.1985:0019, Alberta 1985, Print-85-0421.

[46] K.-I. Kondo, Int. J. Mod. Phys. A **12**, 5651 (1997).
[47] H.-X. He, F. C. Khanna, and Y. Takahashi, Phys. Lett. B **480**, 222 (2000).
[48] H.-X. He, Commun. Theor. Phys. **46**, 109 (2006).
[49] H.-X. He, Int. J. Mod. Phys. A **22**, 2119 (2007).
[50] M. R. Pennington and R. Williams, J. Phys. G **32**, 2219 (2006).
[51] H.-X. He and F. C. Khanna, Int. J. Mod. Phys. A **21**, 2541 (2006).
[52] S.-X. Qin, L. Chang, Y.-X. Liu, C. D. Roberts, and S. M. Schmidt, Phys. Lett. B **722**, 384 (2013).
[53] S.-X. Qin, C. D. Roberts, and S. M. Schmidt, Phys. Lett. B **733**, 202 (2014).
[54] L. Landau and I. Khalatnikov, Sov. Phys. JETP **2**, 69 (1956).
[55] K. Johnson and B. Zumino, Phys. Rev. Lett. **3**, 351 (1959).
[56] B. Zumino, J. Math. Phys. (N.Y.) **1**, 1 (1960).
[57] C. J. Burden and C. D. Roberts, Phys. Rev. D **47**, 5581 (1993).
[58] A. Bashir, A. Kizilersu, and M. Pennington, arXiv:hep-ph/9907418.
[59] A. Bashir, Phys. Lett. B **491**, 280 (2000).
[60] A. Bashir and A. Raya, Phys. Rev. D **66**, 105005 (2002).
[61] A. Bashir and R. Delbourgo, J. Phys. A **37**, 6587 (2004).
[62] C. S. Fischer, R. Alkofer, T. Dahm, and P. Maris, Phys. Rev. D **70**, 073007 (2004).
[63] A. Bashir and A. Raya, Few-Body Syst. **41**, 185 (2007).
[64] A. Bashir, L. Gutiérrez-Guerrero, and Y. Concha-Sanchez, AIP Conf. Proc. **857**, 279 (2006).
[65] A. Bashir, Y. Concha-Sanchez, and R. Delbourgo, Phys. Rev. D **76**, 065009 (2007).
[66] A. Bashir, A. Raya, and S. Sanchez-Madrigal, J. Phys. A **41**, 505401 (2008).
[67] A. Bashir, A. Raya, S. Sanchez-Madrigal, and C. Roberts, Few-Body Syst. **46**, 229 (2009).
[68] N. Ahmadiniaz, A. Bashir, and C. Schubert, Phys. Rev. D **93**, 045023 (2016).
[69] A. Ahmad, J. J. Cobos-Martinez, Y. Concha-Sánchez, and A. Raya, Phys. Rev. D **93**, 094035 (2016).
[70] S. Jia and M. R. Pennington, Phys. Rev. D **94**, 116004 (2016).
[71] S. Jia and M. R. Pennington, Phys. Rev. D **95**, 076007 (2017).
[72] S. Jia and M. R. Pennington, Phys. Lett. B **769**, 146 (2017).
[73] T. De Meerleer, D. Dudal, S. P. Sorella, P. Dall'Olio, and A. Bashir, Phys. Rev. D **97**, 074017 (2018).
[74] M. J. Aslam, A. Bashir, and L. Gutierrez-Guerrero, Phys. Rev. D **93**, 076001 (2016).
[75] C. Parker, J. Phys. A **17**, 2873 (1984).
[76] J. M. Cornwall, Phys. Rev. D **26**, 1453 (1982).
[77] J. E. King, Phys. Rev. D **27**, 1821 (1983).
[78] D. C. Curtis and M. R. Pennington, Phys. Rev. D **42**, 4165 (1990).
[79] D. Curtis and M. Pennington, Phys. Rev. D **44**, 536 (1991).
[80] D. C. Curtis and M. R. Pennington, Phys. Rev. D **48**, 4933 (1993).
[81] Z.-h. Dong, H. J. Munczek, and C. D. Roberts, Phys. Lett. B **333**, 536 (1994).
[82] A. Bashir and M. Pennington, Phys. Rev. D **50**, 7679 (1994).
[83] A. Bashir and M. Pennington, Phys. Rev. D **53**, 4694 (1996).
[84] A. Bashir, A. Kizilersu, and M. R. Pennington, Phys. Rev. D **57**, 1242 (1998).
[85] L. A. Fernandez-Rangel, A. Bashir, L. X. Gutiérrez-Guerrero, and Y. Concha-Sanchez, Phys. Rev. D **93**, 065022 (2016).
[86] J. I. Skullerud, P. O. Bowman, A. Kizilersu, D. B. Leinweber, and A. G. Williams, J. High Energy Phys. 04 (2003) 047.
[87] J.-I. Skullerud, P. O. Bowman, A. Kizilersu, D. B. Leinweber, and A. G. Williams, Nucl. Phys. B, Proc. Suppl. **141**, 244 (2005).
[88] A. Kizilersu, D. B. Leinweber, J.-I. Skullerud, and A. G. Williams, Eur. Phys. J. C **50**, 871 (2007).
[89] O. Oliveira, A. Kizilersü, P. J. Silva, J.-I. Skullerud, A. Sternbeck, and A. G. Williams, Acta Phys. Pol. B Proc. Suppl. **9**, 363 (2016).
[90] A. Sternbeck, P.-H. Balduf, A. Kizilersü, O. Oliveira, P. J. Silva, J.-I. Skullerud, and A. G. Williams, Proc. Sci., LATTICE2016 (**2017**) 349.
[91] O. Oliveira, T. Frederico, W. de Paula, and J. P. B. C. de Melo, Eur. Phys. J. C **78**, 553 (2018).
[92] O. Oliveira, W. de Paula, T. Frederico, and J. P. B. C. de Melo, Eur. Phys. J. C **79**, 116 (2019).
[93] V. P. Gusynin, V. A. Miransky, and I. A. Shovkovy, Phys. Lett. B **349**, 477 (1995).
[94] V. P. Gusynin, V. A. Miransky, and I. A. Shovkovy, Phys. Rev. D **52**, 4747 (1995).
[95] D. S. Lee, C. N. Leung, and Y. J. Ng, Phys. Rev. D **55**, 6504 (1997).
[96] D. K. Hong, Phys. Rev. D **57**, 3759 (1998).
[97] E. J. Ferrer and V. de la Incera, Phys. Lett. B **481**, 287 (2000).
[98] A. Ayala, A. Bashir, A. Raya, and E. Rojas, Phys. Rev. D **73**, 105009 (2006).
[99] E. Rojas, A. Ayala, A. Bashir, and A. Raya, Phys. Rev. D **77**, 093004 (2008).
[100] P. Watson and H. Reinhardt, Phys. Rev. D **89**, 045008 (2014).
[101] N. Mueller and J. M. Pawlowski, Phys. Rev. D **91**, 116010 (2015).
[102] A. Kizilersu, T. Sizer, M. R. Pennington, A. G. Williams, and R. Williams, Phys. Rev. D **91**, 065015 (2015).
[103] A. Kizilersu, M. Reenders, and M. R. Pennington, Phys. Rev. D **52**, 1242 (1995).
[104] A. I. Davydychev, P. Osland, and L. Saks, Phys. Rev. D **63**, 014022 (2000).
[105] A. Bashir, A. Raya, and S. Sanchez-Madrigal, Phys. Rev. D **84**, 036013 (2011).
[106] W. A. Bardeen, C. N. Leung, and S. T. Love, Phys. Rev. Lett. **56**, 1230 (1986).
[107] C. N. Leung, S. T. Love, and W. A. Bardeen, Nucl. Phys. **B273**, 649 (1986).
[108] F. Akram, A. Bashir, L. X. Gutiérrez-Guerrero, B. Masud, J. Rodríguez-Quintero, C. Calcaneo-Roldan, and M. E. Tejeda-Yeomans, Phys. Rev. D **87**, 013011 (2013).
[109] B. Holdom, Phys. Lett. B **213**, 365 (1988).

[110] B. Holdom, Phys. Rev. Lett. **62**, 997 (1989).

[111] W. A. Bardeen, S. T. Love, and V. A. Miransky, Phys. Rev. D **42**, 3514 (1990).

[112] K. Raya, A. Bashir, S. Hernández-Ortiz, A. Raya, and C. D. Roberts, Phys. Rev. D **88**, 096003 (2013).

[113] P. I. Fomin and V. A. Miransky, Phys. Lett. **64B**, 166 (1976).

[114] P. I. Fomin, V. P. Gusynin, and V. A. Miransky, Phys. Lett. **78B**, 136 (1978).

[115] T. Maskawa and H. Nakajima, Prog. Theor. Phys. **52**, 1326 (1974).

[116] V. A. Miransky, Nuovo Cimento A **90**, 149 (1985).

[117] P. I. Fomin, V. P. Gusynin, V. A. Miransky, and Yu. A. Sitenko, Riv. Nuovo Cimento **6**, 1 (1983).

[118] J. Braun, L. Fister, J. M. Pawlowski, and F. Rennecke, Phys. Rev. D **94**, 034016 (2016).

[119] M. Mitter, J. M. Pawlowski, and N. Strodthoff, Phys. Rev. D **91**, 054035 (2015).