Introduction to Schwinger-Dyson Equations

Institute of Physics and Mathematics
University of Michoacán, Morelia, Michoacán, Mexico

Fulbright Visiting Scientist
Jefferson Laboratory, Newport News, Virginia, USA

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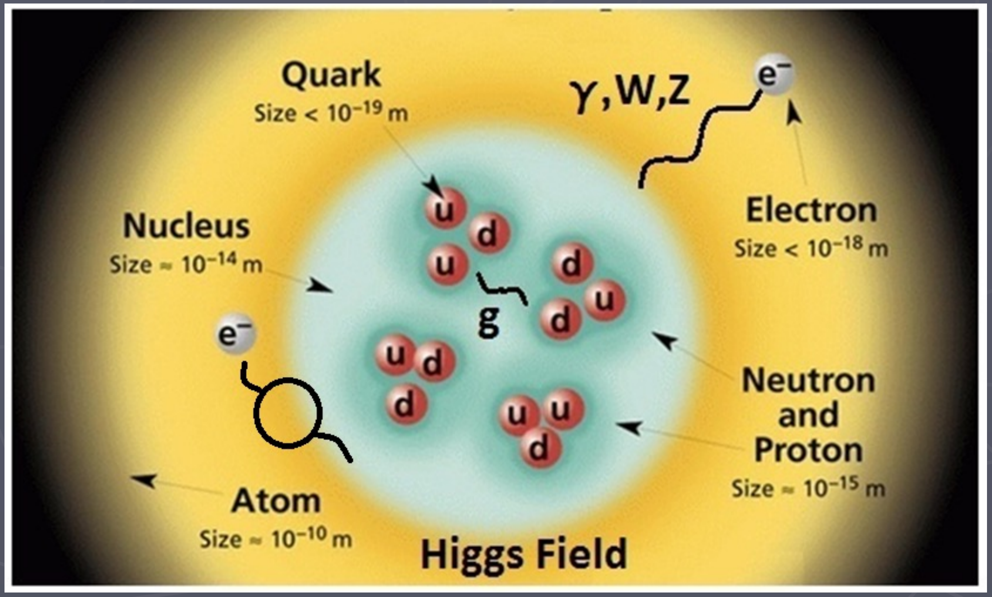
Introduction

A correct description of particles at the fundamental level requires a Poincaré invariant treatment.

It is achieved through a reconciliation of quantum mechanics with special relativity, giving rise to the relativistic quantum mechanics or quantum field theories.

The Standard Model of particle physics provides a unified description of three quantum field theories, QED, weak interactions and QCD.



Introduction

Fundamental equations of motion of every quantum field theory are called Schwinger-Dyson Equations. They encode all the information of a given quantum field theory.

"The S Matrix in Quantum Electrodynamics" F. Dyson, Phys. Rev. 75 (11): 1736 (1949).

"On Green's functions of quantized fields I + II", J. Schwinger, PNAS 37 (7): 452–459, (1951).

Introduction

The fundamental entities in a QFT are fields whose fluctuations represent an infinity of particles.

For example, neutral particles correspond to the following scalar fields allowing for the non-conservation of particles.

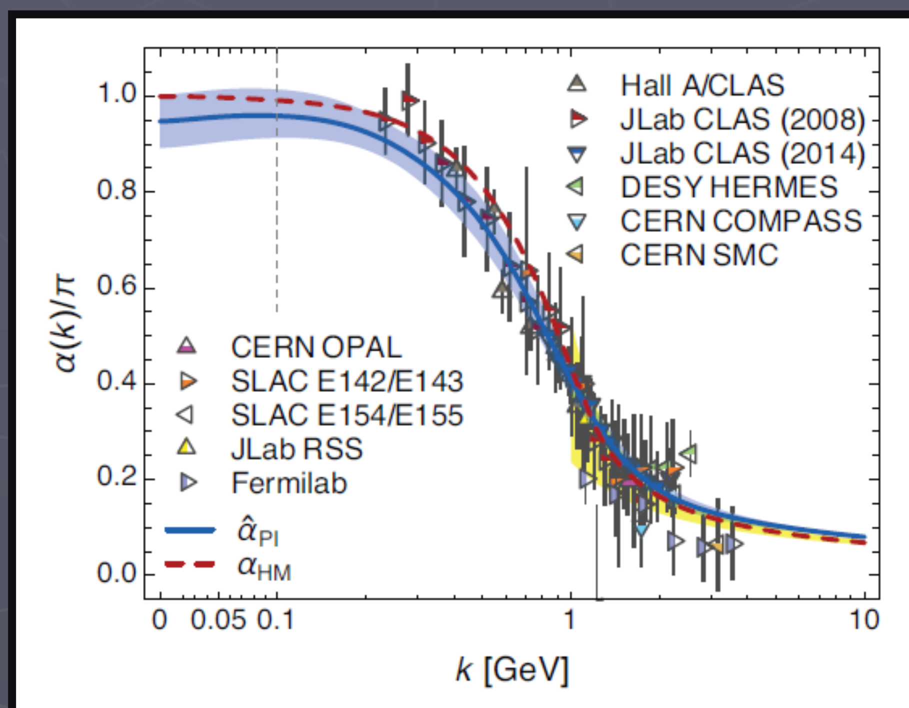
$$\phi(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \left[a(k)e^{-ik\cdot x} + a^\dagger(k)e^{ik\cdot x}\right]$$

This non-conservation of particles is crucial because it is essentially connected with the existence of virtual particles which are the building blocks of the Schwinger-Dyson equations.

Introduction

All relativistic quantum field theories of the Standard Model admit analysis in perturbation theory.

Hadron physics primarily involves QCD. Its perturbative treatment has long been exercised: Gross, Politzer, Wilczek, (Nobel Prize 2004).



What are the Schwinger-Dyson equations?

Non-perturbative QCD is crucial in the study of hadrons. In continuum, SDEs are an ideal tool to study hadron physics.

SDEs are an infinite set of coupled integral equations among the Green functions of a quantum field theory.

The structure of the infinite tower of equations is such that the equation for the 2-point Green function involves the 3-point function, the one for the 3-point function involves the 4-point function and so on ad infinitum.

Formal derivation of SDEs is without any recourse to the smallness of the strength of the interaction involved.

Therefore, within the same formalism, we can study the ultraviolet and infrared dynamics of QCD.

The Dirac equation

The Dirac equation for a free fermion of mass m is:

$$(i\partial\!\!\!/ - m)\psi_0(x) = 0$$

Interaction with an electromagnetic field is incorporated through the usual minimal substitution:

$$\partial_\mu \to D_\mu = \partial_\mu + ieA_\mu(x)$$

$$(i\partial\!\!\!/ - m)\psi(x) \quad = \quad eA\!\!\!/(x)\psi(x)$$

This equation can be solved through the construction of a Green function satisfying:

$$(i\partial\!\!\!/ - m)G_0(x,y) = \delta^4(x-y)$$

The Dirac equation - solution

The formal solution of the Dirac equation is then:

$$\psi(x) = \psi_0(x) + \int d^4y \, G_0(x,y) e\not{A}(y)\psi(y)$$

It is easy to verify:

$$(i\not{\partial} - m)\psi(x) = \int d^4y \, (i\not{\partial} - m) G_0(x,y) e\not{A}(y)\psi(y)$$
$$= \int d^4y \, \delta^4(x-y) e\not{A}(y)\psi(y) = e\not{A}(x)\psi(x)$$

What is $G_0(x,y)$?

The fermion propagator

$G_0(x,y)$ is the position space fermion propagator:

$$(i\partial\!\!\!/ - m)G_0(x,y) = \delta^4(x-y)$$

Fourier transformation of $G_0(x,y)$is:

$$G_0(x,y) = \int \frac{d^4p}{(2\pi)^4} S^0(p) e^{-ip\cdot(x-y)}$$

$$(i\partial\!\!\!/ - m)G_0(x,y) = \int \frac{d^4p}{(2\pi)^4} S^0(p)(i\partial\!\!\!/ - m) e^{-ip\cdot(x-y)}$$

$$\delta^4(x-y) = \int \frac{d^4p}{(2\pi)^4} (p\!\!\!/ - m) S^0(p) e^{-ip\cdot(x-y)}$$

$$\Rightarrow \quad S^0(p) = \frac{1}{p\!\!\!/ - m}$$

The iterative method

Iterative method:

$$\psi(x) = \psi_0(x) + \int d^4y\, G_0(x,y)\, e\not{A}(y)\psi(y)$$

$$\psi(y) = \psi_0(y) + \int d^4y'\, G_0(y,y')\, e\not{A}(y')\psi(y')$$

Beyond the leading order:

$$\psi(x) = \psi_0(x) + \int d^4y G_0(x,y) e\not{A}(y)\, \psi_0(y)$$
$$+ \int d^4y\, d^4y'\, G_0(x,y) e\not{A}(y) G_0(y,y')\, e\not{A}(y')\psi(y')$$

Perturbation theory

In the next iterative step, we have:

$$\psi(x) = \psi_0(x) + \int d^4y\, G_0(x,y) e\not{A}(y)\, \psi_0(y)$$
$$+ \int d^4y\, d^4y'\, G_0(x,y) e\not{A}(y) G_0(y,y')\, e\not{A}(y') \psi_0(y')$$
$$+ \int d^4y\, d^4y'\, d^4y''\, G_0(x,y) e\not{A}(y) G_0(y,y')\, e\not{A}(y')\, G_0(y',y'')\, e\not{A}(y'')\, \psi(y'')$$

This procedure is at the heart of perturbation theory.

Fermion propagator

Recall the equation for the Green function:

$$[i\partial\!\!\!/_{x'} - m]G_0(x',x) = \mathbf{1}\,\delta^4(x'-x)$$

In the presence of an external electromagnetic field:

$$[i\partial\!\!\!/_{x'} - eA\!\!\!/(x') - m]G(x',x) = \mathbf{1}\,\delta^4(x'-x)$$

This yields following perturbative series:

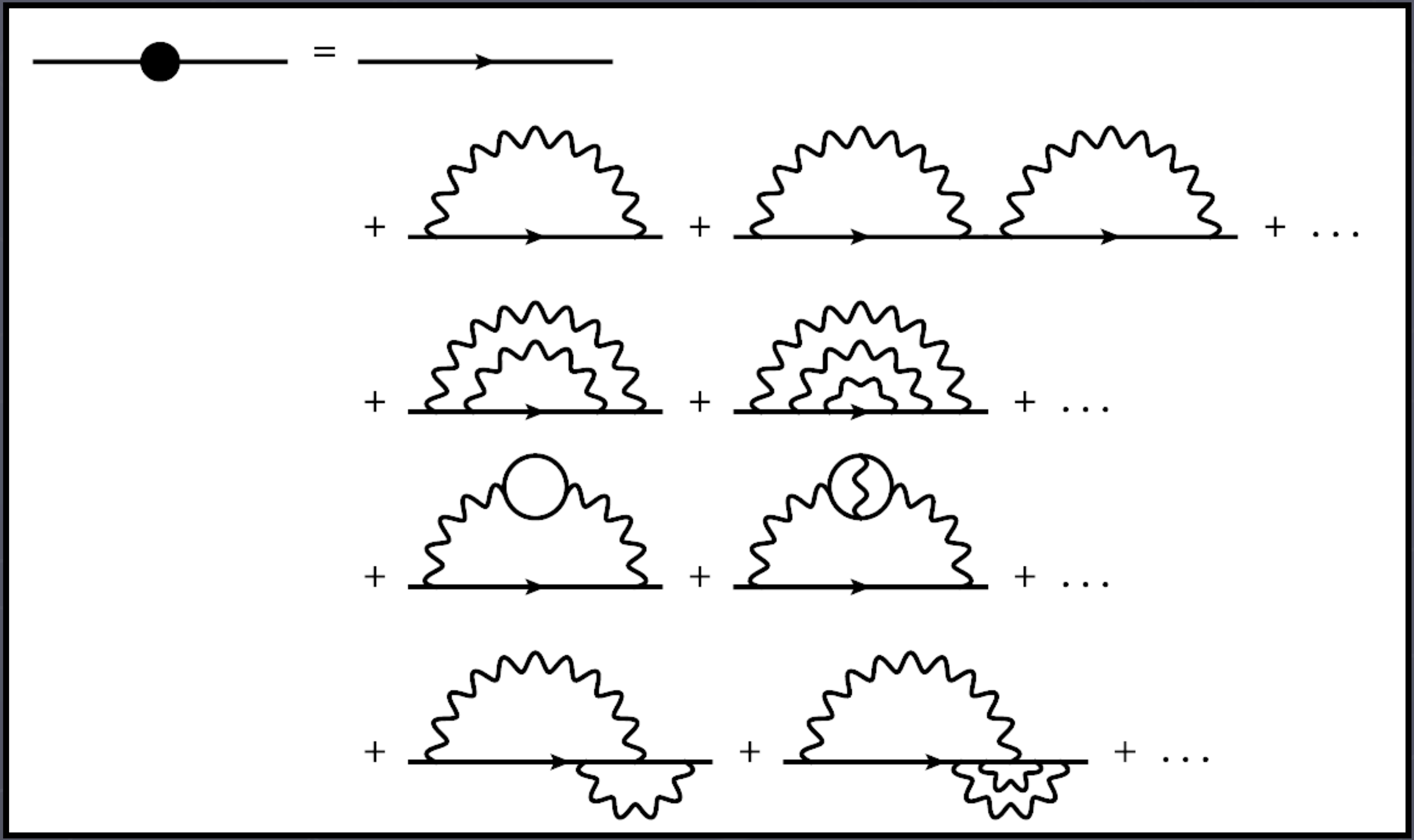
$$\begin{aligned}
G(x',x) &= G_0(x',x) + e\int d^4y_1\, G_0(x',y_1)\, A\!\!\!/(y_1)\, G(y_1,x) \\
&= G_0(x',x) + e\int d^4y_1 G_0(x',y_1) A\!\!\!/(y_1) G_0(y_1,x) \\
&\quad + e^2\int d^4y_1\, d^4y_2\, G_0(x',y_1)\, A\!\!\!/(y_1)\, G_0(y_1,y_2)\, A\!\!\!/(y_2)\, G_0(y_2,x) \\
&\quad + \ldots
\end{aligned}$$

Fermion propagator

Bare propagator:

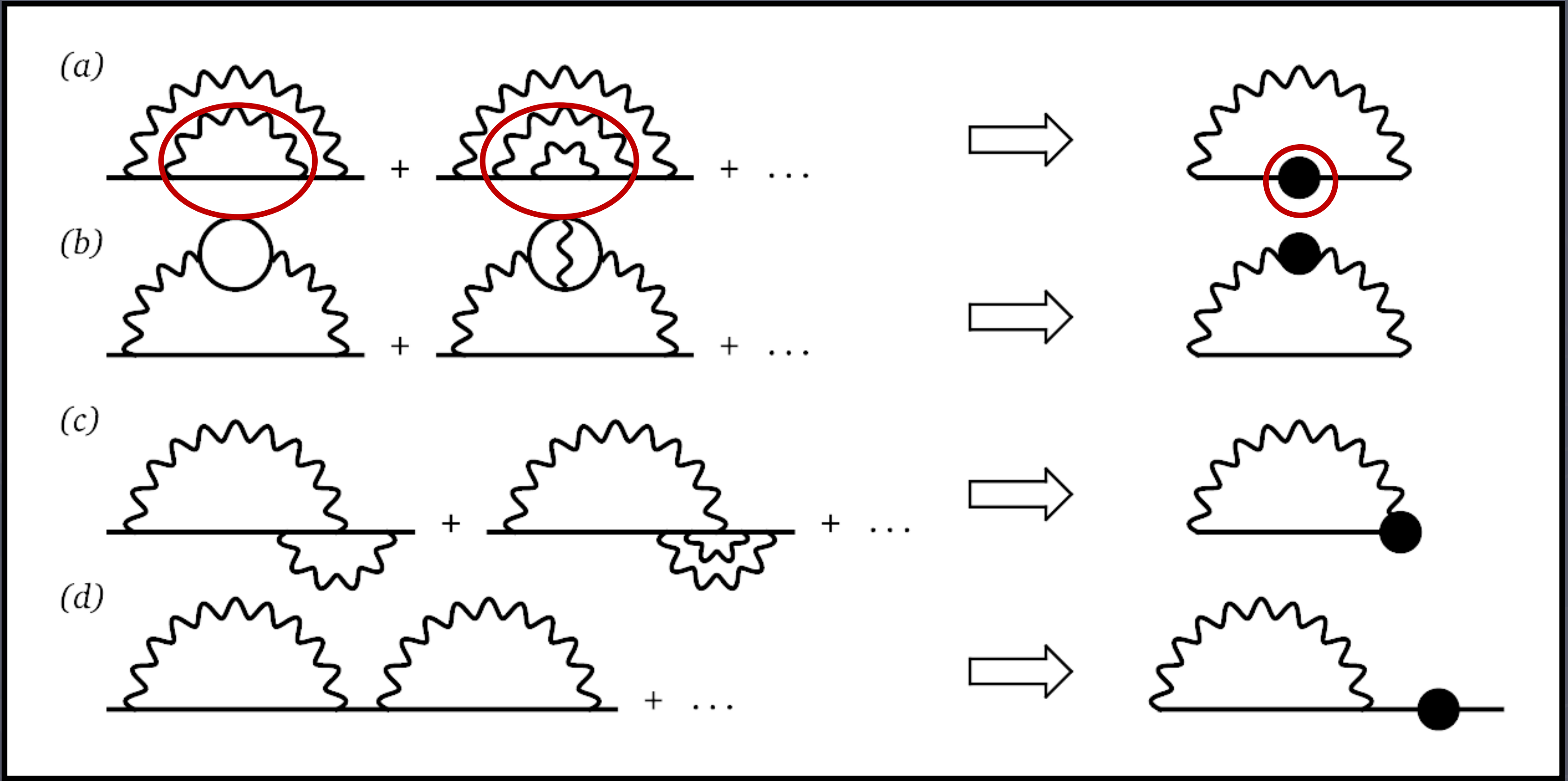
$$S^0(p) = \frac{1}{\not{p} - m} = \longrightarrow$$

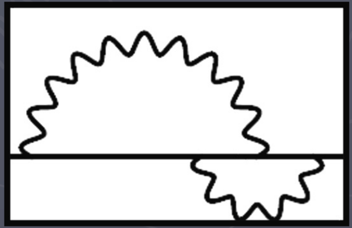
Perturbative series

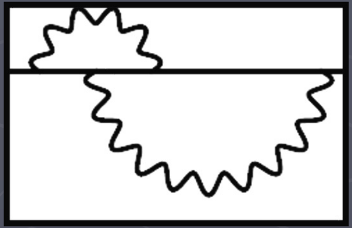


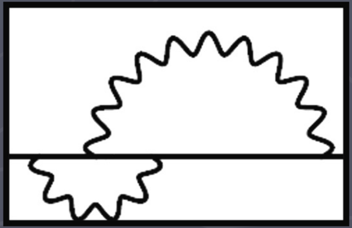
Fermion propagator

Classification of perturbative series:



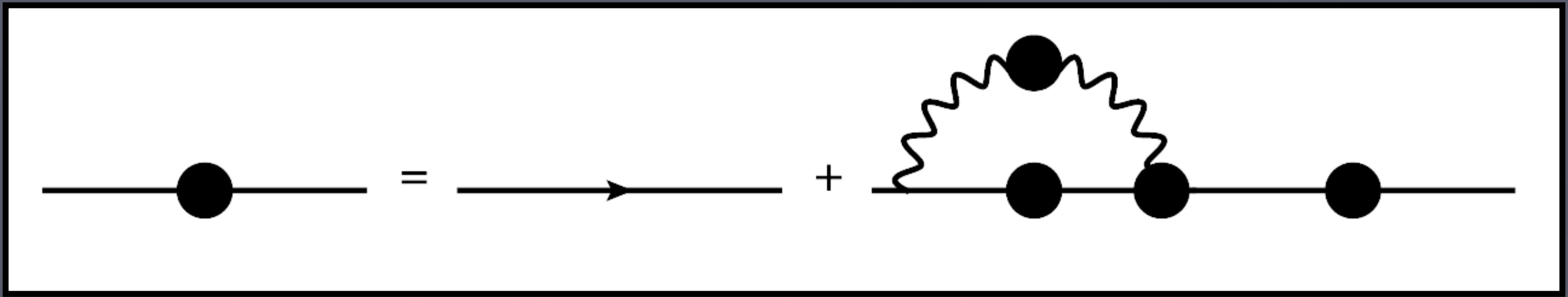




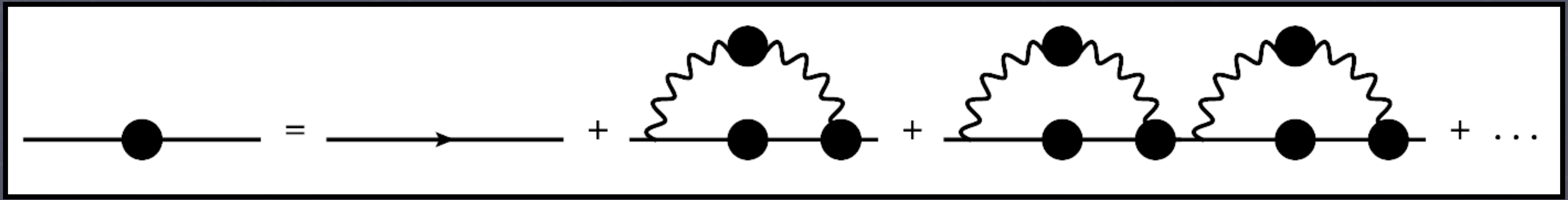


The full Green functions

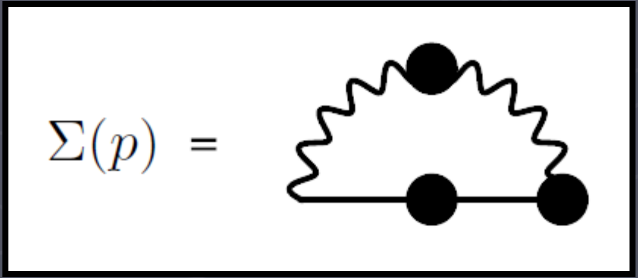
This perturbative series can be summed up:



Expanding out the last blob again, we can draw:

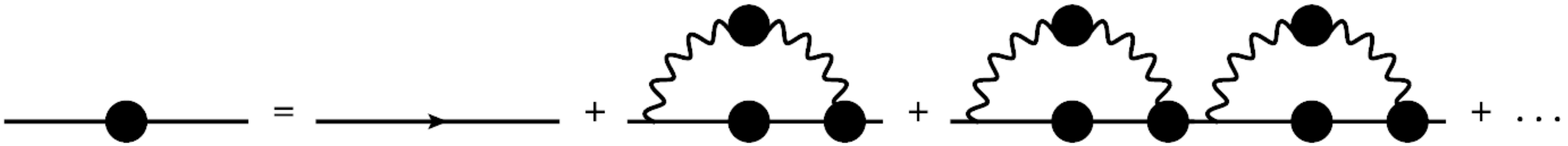


Define self energy as:



The full fermion propagator

Towards mathematical construct:



$$S(p) \;= S^0(p) + S^0(p)\Sigma(p)S^0(p) + S^0(p)\Sigma(p)S^0(p)\Sigma(p)S^0(p) + \ldots$$

This series can be written in a compact manner:

$$\begin{aligned} S(p) &= S^0(p) + S^0(p)\Sigma(p)[\underbrace{S^0(p) + S^0(p)\Sigma(p)S^0(p) + \ldots}_{S(p)}] \\ &= S^0(p) + S^0(p)\Sigma(p)S(p) \end{aligned}$$

Schwinger-Dyson equations

Schwinger-Dyson equation for the fermion propagator:

$$S^{0^{-1}}(p)$$

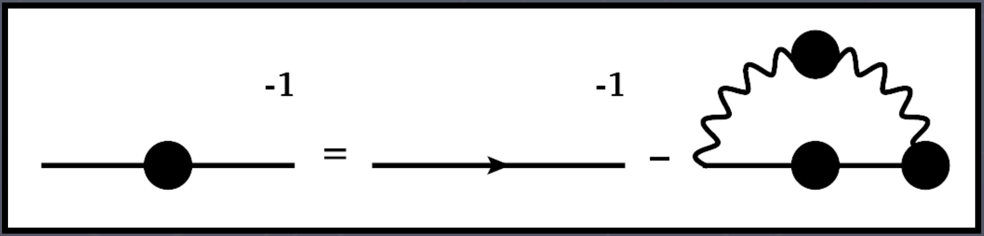
$$S(p) = S^0(p) + S^0(p)\Sigma(p)S(p)$$

$$S^{-1}(p)$$

SDE for the inverse fermion propagator:

$$S^{-1}(p) = S^{0^{-1}}(p) - \Sigma(p)$$

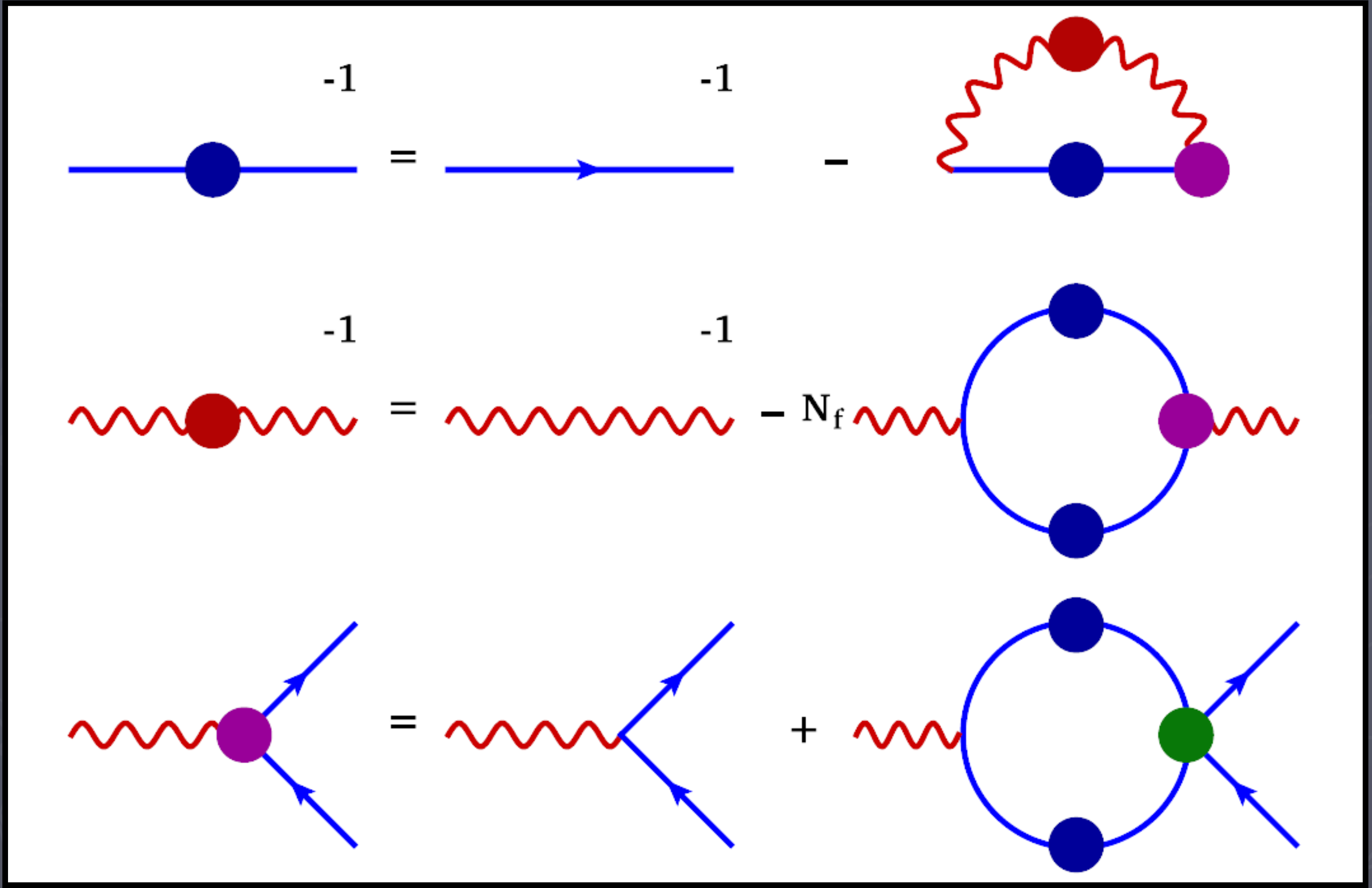
Electron propagator (QED)



Quark propagator (QCD)

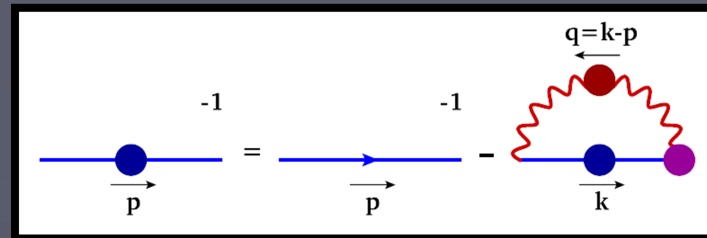
Schwinger-Dyson equations (QED)

First three of an infinite tower of SDEs in QED:



SDE for the fermion propagator

- Fermion propagator SDE:



$$-iS^{-1}(p) = -iS^{0^{-1}}(p) - \int \frac{d^dk}{(2\pi)^d}[-ie\gamma^\mu][iS(k)][-ie\Gamma^\nu(k,p)][-i\Delta_{\mu\nu}(q)]$$

Bare fermion propagator:

$$S^0(p) = \frac{1}{\not{p} - m}$$

Full fermion propagator:

$$S(p) = A(p^2)\,\not{p} + B(p^2) = \frac{F(p^2)}{\not{p} - M(p^2)}$$

$$S(p) = \frac{F(p^2)}{\not{p} - M(p^2)}\frac{\not{p} + M(p^2)}{\not{p} + M(p^2)} = \frac{F(p^2)\left(\not{p} + M(p^2)\right)}{p^2 - M^2(p^2)}$$

$$= \underbrace{\frac{F(p^2)}{p^2 - M^2(p^2)}}_{A(p^2)}\not{p} + \underbrace{\frac{F(p^2)M(p^2)}{p^2 - M^2(p^2)}}_{B(p^2)}$$

On the photon propagator

Bare photon propagator:

$$\Delta^0_{\mu\nu}(q) = \frac{1}{q^2}\left[g_{\mu\nu} + (\xi - 1)\frac{q_\mu q_\nu}{q^2}\right] = \Delta^T_{\mu\nu}(q) + \xi\frac{q_\mu q_\nu}{q^4}$$

Superscript "T" stands for the transversality of the photon propagator to its 4-momentum:

$$q^\mu \Delta^T_{\mu\nu}(q) = \frac{1}{q^2}\left[q^\mu g_{\mu\nu} - \frac{q^\mu q_\mu q_\nu}{q^2}\right] = \frac{1}{q^2}\left[q_\nu - q_\nu\right] = 0$$

Full photon propagator: receives no corrections to its longitudinal part

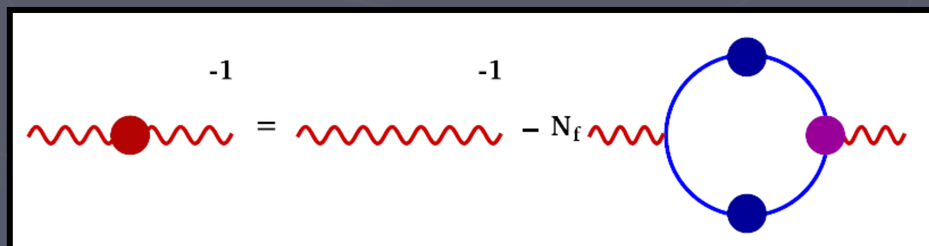
$$\Delta_{\mu\nu}(q) = \frac{G(q^2)}{q^2}\left[g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right] + \xi\frac{q_\mu q_\nu}{q^4}$$

The quenched approximation

Full photon propagator:

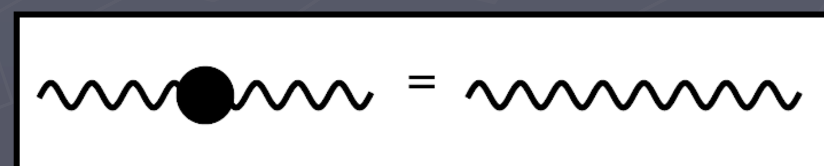
$$\Delta_{\mu\nu}(q) = \frac{G(q^2)}{q^2}\left[g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right] + \xi \frac{q_\mu q_\nu}{q^4}$$

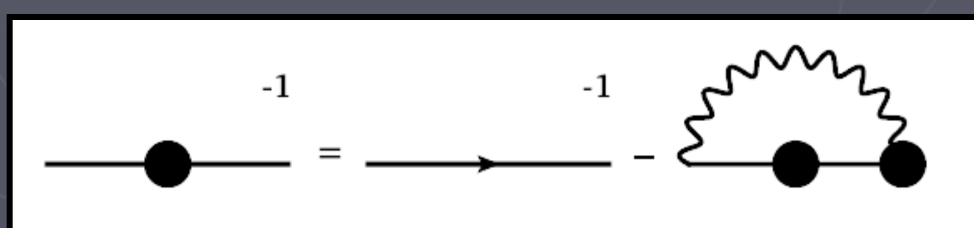
$$G(q^2) = 1 + \mathcal{O}(\alpha N_f)$$



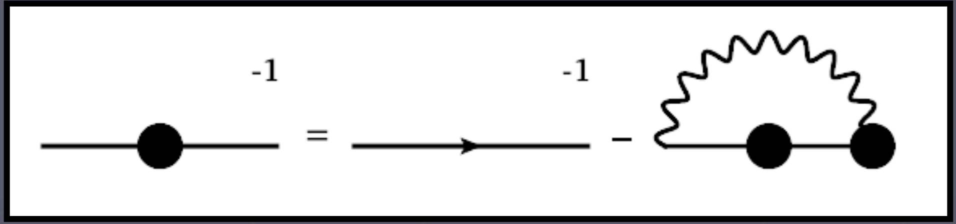
Quenched approximation:

$$N_f = 0$$





The fermion propagator SDE



$$S^{-1}(p) = S^{0^{-1}}(p) + ie^2 \int \frac{d^dk}{(2\pi)^d} \gamma^\mu S(k) \Gamma^\nu(k,p) \left[\Delta^T_{\mu\nu}(q) + \xi \frac{q_\mu q_\nu}{q^4} \right]$$

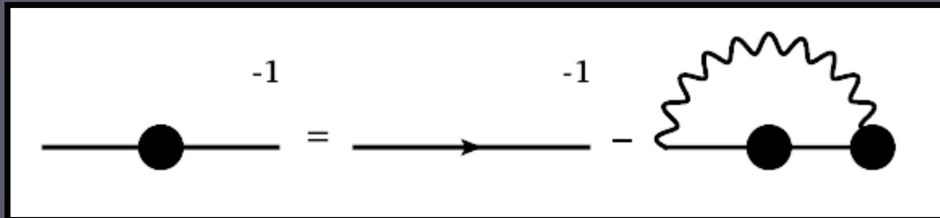
$$= S^{0^{-1}}(p) + ie^2 \int \frac{d^dk}{(2\pi)^d} \gamma^\mu S(k) \Gamma^\nu(k,p) \Delta^T_{\mu\nu}(q)$$

$$+ ie^2 \xi \int \frac{d^dk}{(2\pi)^d} \frac{q_\mu}{q^4} \gamma^\mu S(k) \boxed{[q_\nu \Gamma^\nu(k,p)]}$$

The Ward-Takahashi identity:

$$q_\mu \Gamma^\mu(k,p) = (k-p)_\mu \Gamma^\mu(k,p) = S^{-1}(k) - S^{-1}(p)$$

The fermion propagator SDE



$$S^{-1}(p) = S^{0^{-1}}(p) + ie^2 \int \frac{d^dk}{(2\pi)^d} \gamma^\mu S(k) \Gamma^\nu(k,p) \Delta^T_{\mu\nu}(q)$$
$$+ ie^2 \xi \int \frac{d^dk}{(2\pi)^d} \frac{\not{q}}{q^4} - ie^2 \xi \int \frac{d^dk}{(2\pi)^d} \frac{\not{q}}{q^4} S(k) S^{-1}(p)$$

The odd integral:

$$\int \frac{d^dq}{(2\pi)^d} \frac{\not{q}}{q^4} = 0$$

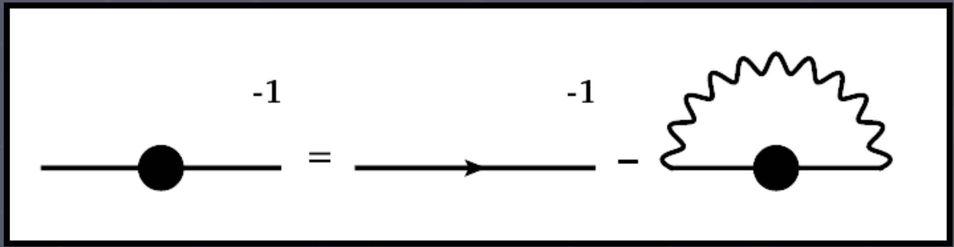
$$S^{-1}(p) = S^{0^{-1}}(p) + ie^2 \int \frac{d^dk}{(2\pi)^d} \gamma^\mu S(k) \Gamma^\nu(k,p) \Delta^T_{\mu\nu}(q) - ie^2 \xi \int \frac{d^dk}{(2\pi)^d} \frac{\not{q}}{q^4} S(k) S^{-1}(p)$$

The rainbow approximation

$$S^{-1}(p) = S^{0^{-1}}(p) + ie^2 \int \frac{d^dk}{(2\pi)^d} \gamma^\mu S(k) \Gamma^\nu(k,p) \Delta^T_{\mu\nu}(q) - ie^2 \xi \int \frac{d^dk}{(2\pi)^d} \frac{\not{q}}{q^4} S(k) S^{-1}(p)$$

The Vertex:

$$\Gamma^\mu(k,p) = \gamma^\mu$$



$$S^{-1}(p) = S^{0^{-1}}(p) + ie^2 \int \frac{d^dk}{(2\pi)^d} \gamma^\mu S(k) \gamma^\nu \frac{1}{q^2} \left[g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right] - ie^2 \xi \int \frac{d^dk}{(2\pi)^d} \frac{\not{q}}{q^4} S(k) S^{-1}(p)$$

Decoupling the equations

$$
\begin{aligned}
\frac{\not{p} - M(p^2)}{F(p^2)} &= \not{p} - m + ie^2 \int \frac{d^d k}{(2\pi)^d} \frac{F(k^2)}{q^2(k^2 - M^2(k^2))} \gamma^\mu (\not{k} + M(k^2)) \gamma_\mu \\
&- ie^2 \int \frac{d^d k}{(2\pi)^d} \frac{F(k^2)}{q^4(k^2 - M^2(k^2))} \not{q} (\not{k} + M(k^2)) \not{q} \\
&- \frac{ie^2 \xi}{F(p^2)} \int \frac{d^d k}{(2\pi)^d} \frac{F(k^2)}{q^4(k^2 - M^2(k^2))} \not{q} (\not{k} + M(k^2)) (\not{p} - M(p^2))
\end{aligned}
$$

It is a matrix equation with two independent equations contained in it:

$i)\ Tr[\not{p}$ Equation $] \rightarrow F(p^2)$

$ii)\ Tr[$ Equation $] \rightarrow M(p^2)$

The details

$$\frac{4p^2}{F(p^2)} = 4p^2 + ie^2 \int \frac{d^4k}{(2\pi)^4} \frac{F(k^2)}{q^2(k^2 - M^2(k^2))} Tr\left[\not{p}\gamma^\mu \not{k}\gamma_\mu\right]$$
$$- ie^2 \int \frac{d^4k}{(2\pi)^4} \frac{F(k^2)}{q^4(k^2 - M^2(k^2))} Tr\left[\not{p}\not{q}\not{k}\not{q}\right]$$
$$- \frac{ie^2\xi}{F(p^2)} \int \frac{d^4k}{(2\pi)^4} \frac{F(k^2)}{q^4(k^2 - M^2(k^2))} Tr\left[\not{p}\not{q}\not{k}\not{p} - \not{p}\not{q}M(k^2)M(p^2)\right]$$

Take traces:

$Tr\left[\gamma^{\alpha_1}\gamma^{\alpha_2} \cdot \ldots \cdot \gamma^{\alpha_n}\right] = 0$ for odd no. of matrices

Thus, e.g., $Tr\left[\not{p}\right] = Tr\left[p_\mu \gamma^\mu\right] = 0$

$\gamma^\mu \not{k} \gamma_\mu = -2\not{k}$, $\quad Tr\left[\not{p}\not{k}\right] = 4p \cdot k$

$Tr\left[\not{a}\not{b}\not{c}\not{d}\right] = 4\left[a \cdot b \; c \cdot d - a \cdot c \; b \cdot d + a \cdot d \; b \cdot c\right]$

From Minkowski to Euclidean space

The Wick rotation:

$$\text{Minkowski } (+,-,-,-), \quad x^\mu = (x^0, \vec{x}), \quad x^2 = {x^0}^2 - \vec{x}^2$$

$$x^2 > 0 \quad \Rightarrow \quad time - like$$

$$x^2 < 0 \quad \Rightarrow \quad space - like$$

$$p^0 \to i p^0$$

$$\Rightarrow \quad p^2 = -{p^0}^2 - \vec{p}^2 = -p_E^2$$

$$p_E^2 > 0 \quad \Rightarrow \quad p^2 \quad \textit{space-like}$$

$$k^2 \quad \to \quad -k_E^2 \equiv -k^2$$

$$p^2 \quad \to \quad -p_E^2 \equiv -p^2$$

$$k \cdot p \quad \to \quad -k \cdot p$$

$$d^4k \quad \to \quad i d^4 k$$

Simplifying the equations

After taking traces and Wick rotation:

$$\frac{1}{F(p^2)} = 1 + \frac{\alpha}{2\pi^3 p^2}\int d^4k \frac{F(k^2)k\cdot p}{q^2(k^2+M^2(k^2))}$$
$$+ \frac{\alpha}{4\pi^3 p^2}\int d^4k \frac{F(k^2)}{q^4(k^2+M^2(k^2))}\left[2p\cdot q\; k\cdot q - k\cdot p\; q^2\right]$$
$$+ \frac{\alpha\xi}{4\pi^3}\frac{1}{p^2F(p^2)}\int d^4k \frac{F(k^2)}{q^4(k^2+M^2(k^2))}\left[p^2\; k\cdot q + M(k^2)M(p^2)p\cdot q\right]$$

Simplify:

$$2p\cdot q\; k\cdot q - k\cdot p\; q^2 = -2k^2p^2 + (k^2+p^2)k\cdot p$$
$$p^2\; k\cdot q + M(k^2)M(p^2)p\cdot q = p^2(k^2 - k\cdot p) + M(k^2)M(p^2)(k\cdot p - p^2)$$

The integrations

Angular integration:

$$k^\mu = \left(k^0, \overrightarrow{k}\right)$$

$$= (k\cos\psi, k\sin\psi\sin\theta\cos\varphi, k\sin\psi\sin\theta\sin\varphi, k\sin\psi\cos\theta)$$

$$0 \le k < \infty,\ 0 \le \psi \le \pi,\ 0 \le \theta \le \pi,\ 0 \le \varphi \le 2\pi$$

$$d^4k = \frac{1}{2}k^2 dk^2 \ \sin^2\psi d\psi \ \sin\theta d\theta \ d\varphi$$

Without loss of generality:

$$p^\mu = (p, 0, 0, 0),\ \ k \cdot p = kp\cos\psi$$

$$\int_0^\pi d\theta \sin\theta \int_0^{2\pi} d\varphi = 4\pi$$

$$I_{nm} = \int_0^\pi d\psi \sin^2\psi \frac{(k \cdot p)^n}{(q^2)^m}$$

The integrations

$$I_{00} = \frac{\pi}{2}\left[\theta(p^2-k^2)+\theta(k^2-p^2)\right]$$

$$I_{01} = \frac{\pi}{2}\left[\frac{1}{p^2}\theta(p^2-k^2)+\frac{1}{k^2}\theta(k^2-p^2)\right]$$

$$I_{02} = \frac{\pi}{2}\frac{1}{k^2-p^2}\left[-\frac{1}{p^2}\theta(p^2-k^2)+\frac{1}{k^2}\theta(k^2-p^2)\right]$$

$$I_{10} = 0$$

$$I_{11} = \frac{\pi}{4}\left[\frac{k^2}{p^2}\theta(p^2-k^2)+\frac{p^2}{k^2}\theta(k^2-p^2)\right]$$

$$I_{12} = \frac{\pi}{2}\frac{1}{k^2-p^2}\left[-\frac{k^2}{p^2}\theta(p^2-k^2)+\frac{p^2}{k^2}\theta(k^2-p^2)\right]$$

$$I_{21} = \frac{\pi}{8}(k^2+p^2)\left[\frac{k^2}{p^2}\theta(p^2-k^2)+\frac{p^2}{k^2}\theta(k^2-p^2)\right]$$

$$I_{02} = \frac{\pi}{8}\frac{1}{k^2-p^2}\left[-\frac{k^2}{p^2}(3k^2+p^2)\theta(p^2-k^2)+\frac{p^2}{k^2}(k^2+3p^2)\theta(k^2-p^2)\right]$$

$$I_{02} = \frac{\pi}{16}\frac{1}{k^2-p^2}\left[\frac{k^4}{p^2}(2p^2+k^2)\theta(p^2-k^2)+\frac{p^4}{k^2}(2k^2+p^2)\theta(k^2-p^2)\right]$$

The coupled equations

Equation for the F-function:

$$F(p^2) = 1 + \frac{\alpha\xi}{2\pi p^4}\int_0^{p^2} dk^2 \frac{F(k^2)M(k^2)M(p^2)}{k^2+M^2(k^2)} - \frac{\alpha\xi}{4\pi}\int_{p^2}^{\Lambda^2} dk^2 \frac{F(k^2)}{k^2+M^2(k^2)}$$

Equation for the M-function:

$$\frac{M(p^2)}{F(p^2)} = m + \frac{3\alpha}{4\pi}\left[\int_0^{p^2} dk^2 \frac{k^2}{p^2}\frac{F(k^2)M(k^2)}{k^2+M^2(k^2)} + \int_{p^2}^{\Lambda^2} dk^2 \frac{F(k^2)M(k^2)}{k^2+M^2(k^2)}\right]$$
$$+ \frac{1}{F(p^2)}\frac{\alpha\xi}{4\pi}\left[\int_0^{p^2} dk^2 \frac{k^2}{p^2}\frac{F(k^2)M(k^2)}{k^2+M^2(k^2)} + \int_{p^2}^{\Lambda^2} dk^2 \frac{F(k^2)M(k^2)}{k^2+M^2(k^2)}\right]$$

In the Landau gauge:

$$F(p^2) = 1$$

$$M(p^2) = m + \frac{3\alpha}{4\pi}\left[\int_0^{p^2} dk^2 \frac{k^2}{p^2}\frac{M(k^2)}{k^2+M^2(k^2)} + \int_{p^2}^{\Lambda^2} dk^2 \frac{M(k^2)}{k^2+M^2(k^2)}\right]$$

The fermion mass function

Equation for the mass function:

$$M(p^2) = m + \frac{3\alpha}{4\pi} \int_0^{\Lambda^2} dk^2 \, \frac{M(k^2)}{k^2 + M^2(k^2)} \left[\frac{k^2}{p^2} \theta(p^2 - k^2) + \theta(k^2 - p^2) \right]$$

Apply bifurcation analysis to linearize the equation by setting M(p)=0 in the chiral limit (m=0).

$$M(p^2) = \frac{3\alpha}{4\pi} \int_0^{\Lambda^2} dk^2 \, \frac{M(k^2)}{k^2} \left[\frac{k^2}{p^2} \theta(p^2 - k^2) + \theta(k^2 - p^2) \right]$$

It gives exact results when the generated mass is so small that its square and higher powers can be neglected.

The fermion mass function

$$M(p^2) = \xcancel{m} + \frac{3\alpha}{4\pi} \int_0^{\Lambda^2} dk^2 \, \frac{M(k^2)}{k^2 + M^2(k^2)} \left[\frac{k^2}{p^2} \theta(p^2 - k^2) + \theta(k^2 - p^2) \right]$$ Original

$$M(p^2) = \frac{3\alpha}{4\pi} \int_0^{\Lambda^2} dk^2 \, \frac{M(k^2)}{k^2} \left[\frac{k^2}{p^2} \theta(p^2 - k^2) + \theta(k^2 - p^2) \right]$$ Linearized

In the chiral limit, Wigner mode solution is obvious. We look for a non perturbative Nambu solution.

Linearization makes the equation scale invariant.

The new integrand contributes extravagantly for $k^2 \to 0$ in contrast with its original parent equation.

To remedy this, we introduce an infrared cut off:

$$\kappa^2 \approx M^2(\kappa^2)$$

Infrared boundary condition

The linearized mass function:

$$p^2 = x \text{ and } k^2 = y$$

$$M(x) \quad = \quad \frac{3\alpha}{4\pi}\left[\frac{1}{x}\int_{\kappa^2}^{x} dy\, M(y) + \int_{x}^{\Lambda^2} dy\, \frac{M(y)}{y}\right]$$

Take the derivative with respect to x:

$$M^{'}(x) \quad = \quad -\frac{3\alpha}{4\pi x^2}\int_{\kappa^2}^{x} dy\, M(y)$$

This imposes the infrared boundary condition:

$$M^{'}(\kappa^2) = 0$$

Differential equation for the mass function

Let us rewrite the first derivative is:

$$x^2 M'(x) = -\frac{3\alpha}{4\pi} \int_{\kappa^2}^{x} dy\, M(y)$$

The second derivative gives the differential equation:

$$x^2 M''(x) + 2x M'(x) + \frac{3\alpha}{4\pi} M(x) = 0$$

The linearized equation can be written as:

$$x M(x) = \frac{3\alpha}{4\pi} \left[\int_{\kappa^2}^{x} dy\, M(y) + x \int_{x}^{\Lambda^2} dy\, \frac{M(y)}{y} \right]$$

The ultraviolet boundary condition

The derivative of the last equation picks out the ultraviolet behaviour of the mass function:

$$[xM(x)]' = \frac{3\alpha}{4\pi} \int_x^{\Lambda^2} dy \, \frac{M(y)}{y}$$

Ultraviolet boundary condition:

$$[xM(x)]' (\Lambda^2) = 0$$

At criticality, the mass function is multiplicatively renormalizable and has solution of the type:

$$M(x) \approx x^{-s}$$

Fermion mass function

The power law solution

$$M(x) \approx x^{-s}$$

substituted in:

$$x^2 M''(x) + 2x M'(x) + \frac{3\alpha}{4\pi} M(x) = 0$$

requires:

$$s^2 - s + \frac{3\alpha}{4\pi} = 0$$

It yields:

$$s_{1,2} = \frac{1}{2} \pm \frac{1}{2}\sqrt{1 - \frac{\alpha}{\alpha_c}} \qquad \alpha_c = \pi/3$$

Fermion mass function

The critical coupling corresponds to the boundary of a phase transition where real and complex solution bifurcate away from each other.

$$s_{1,2} = \frac{1}{2} \pm \frac{i}{2}\tau\,, \quad \tau = \sqrt{\frac{\alpha}{\alpha_c} - 1}$$

$$M(x) = c_1\, x^{-s_2} + c_2\, x^{-s_1}$$

After incorporating boundary conditions, we have

$$\frac{\kappa}{\Lambda} = \mathrm{Exp}\left[-\frac{\pi}{\sqrt{\alpha/\alpha_c - 1}} + 2\right] \Rightarrow \frac{M(\kappa^2)}{\Lambda} = \mathrm{Exp}\left[-\frac{\pi}{\sqrt{\alpha/\alpha_c - 1}} + 2\right]$$

Numerical solution

Recall the equation for the mass function:

$$M(p^2) = m + \frac{3\alpha}{4\pi} \int_0^{\Lambda^2} dk^2 \, \frac{M(k^2)}{k^2 + M^2(k^2)} \left[\frac{k^2}{p^2} \, \theta(p^2 - k^2) + \theta(k^2 - p^2) \right]$$

In the chiral limit m=0, there exist Wigner as well as Nambu modes. Perturbation theory only admits Wigner solution and has no access to Nambu solution.

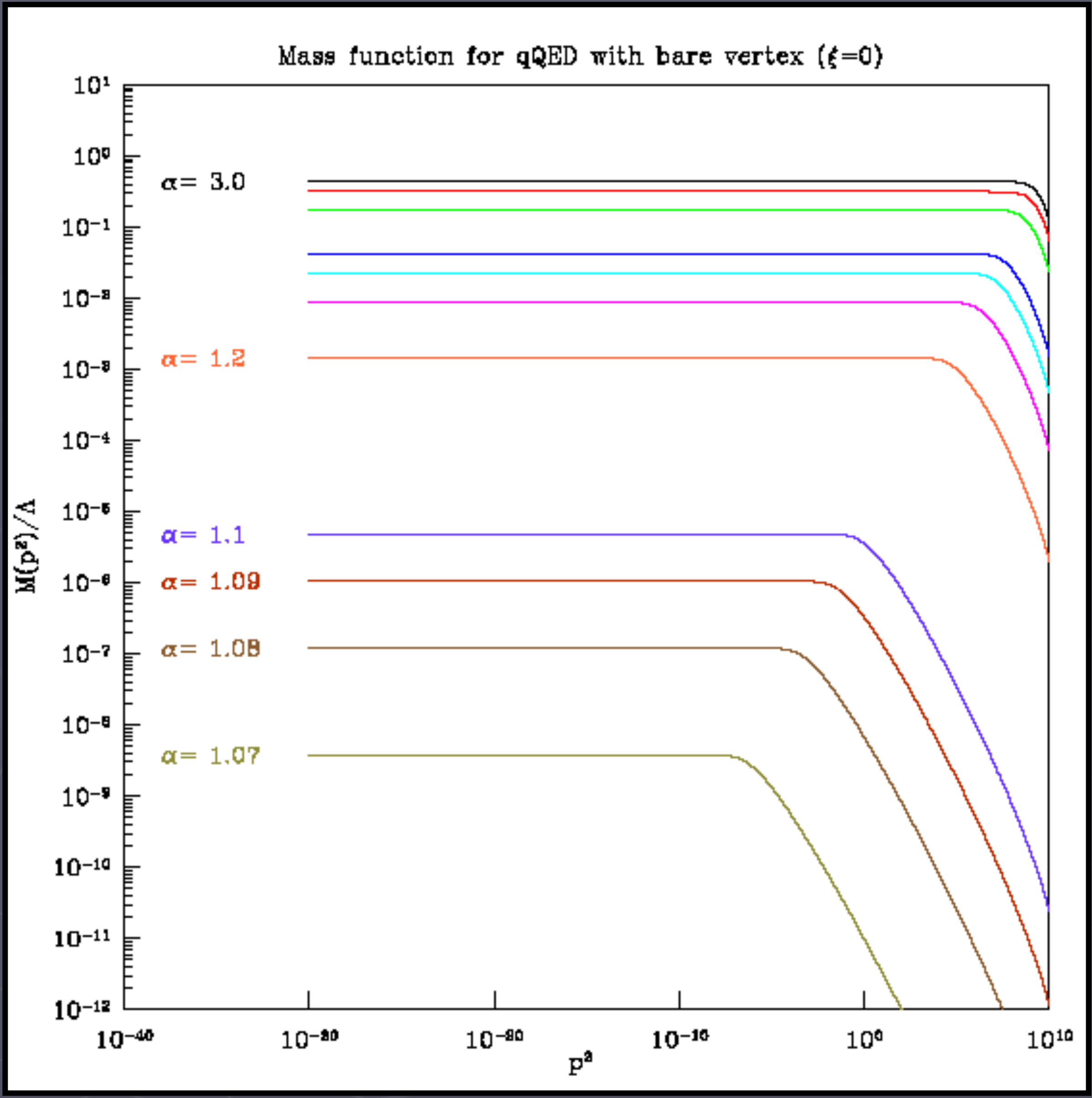
This equation is of the type:

$$X(s) = Y(s) + \lambda \int_a^b dt \, \mathcal{K}(t, s, X(t), X(s))$$

These are non-linear Fredholm equations of the second kind. $Y(s)$ is a known function and $X(s)$ is the function to be determined.

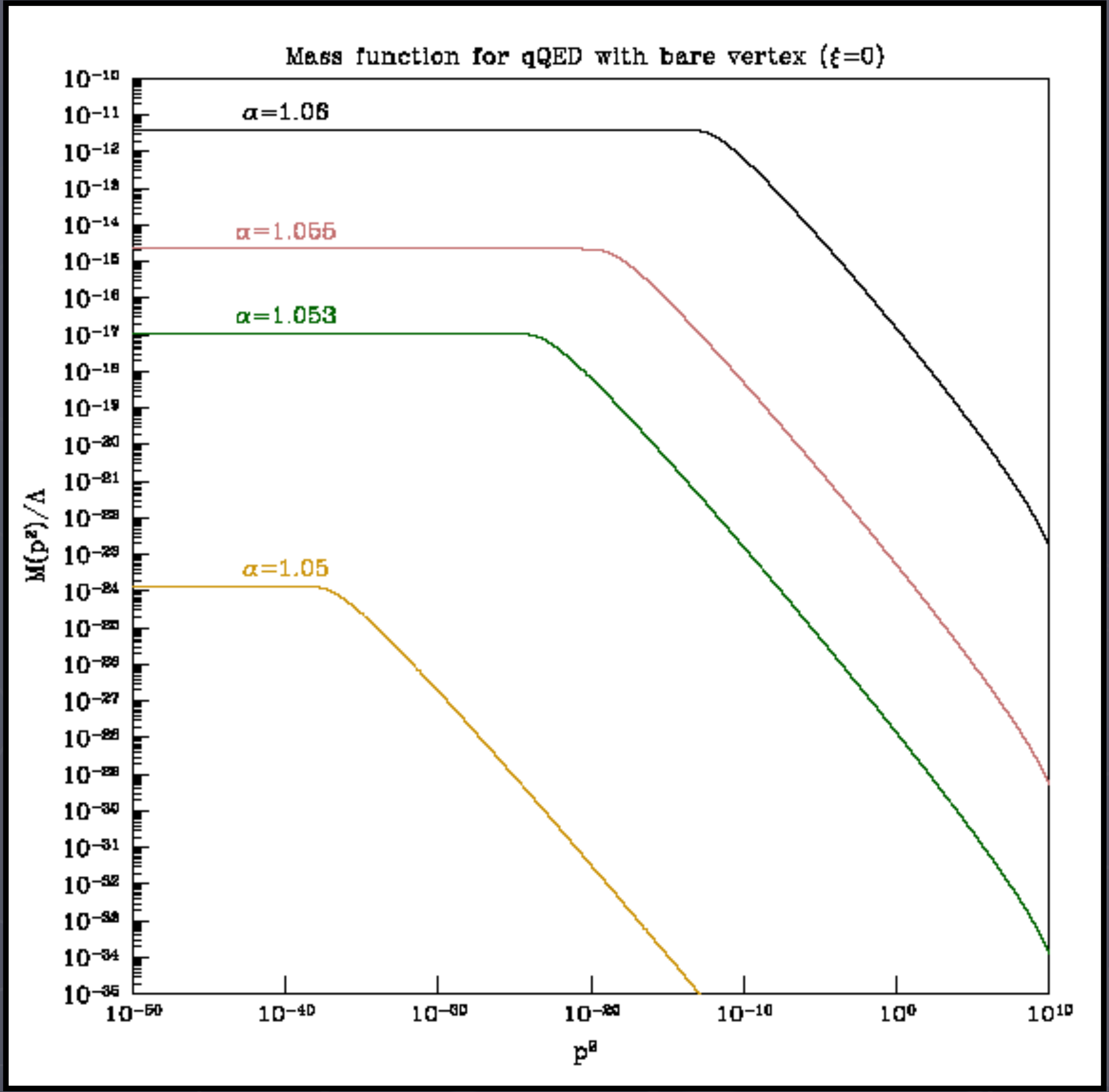
Numerical solution

The **running** fermion **mass function**:



Numerical solution

The **running fermion mass function**:

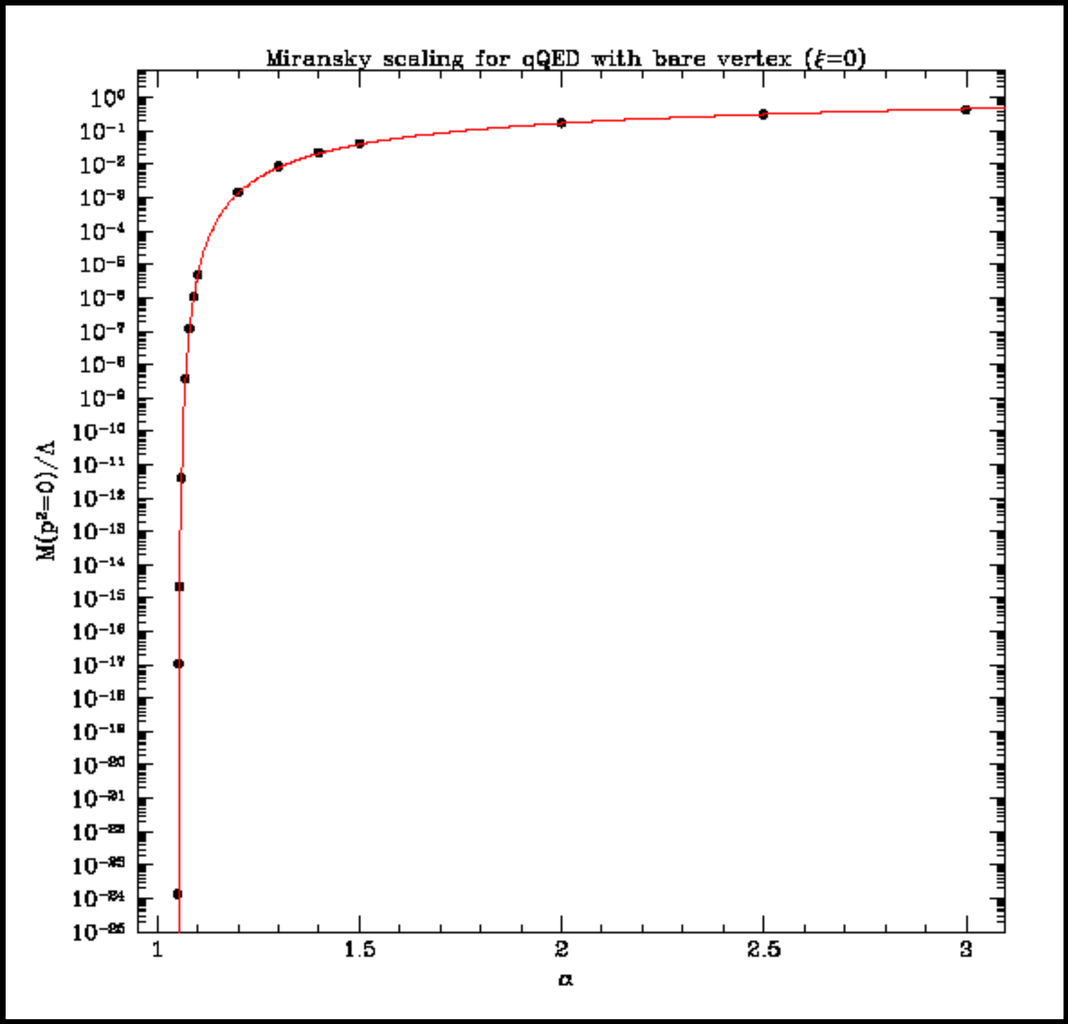


Numerical solution

Miransky scaling

$$M(0) = \Lambda \exp\left[-\frac{A}{\sqrt{\alpha/\alpha_c - 1}} + B\right]$$

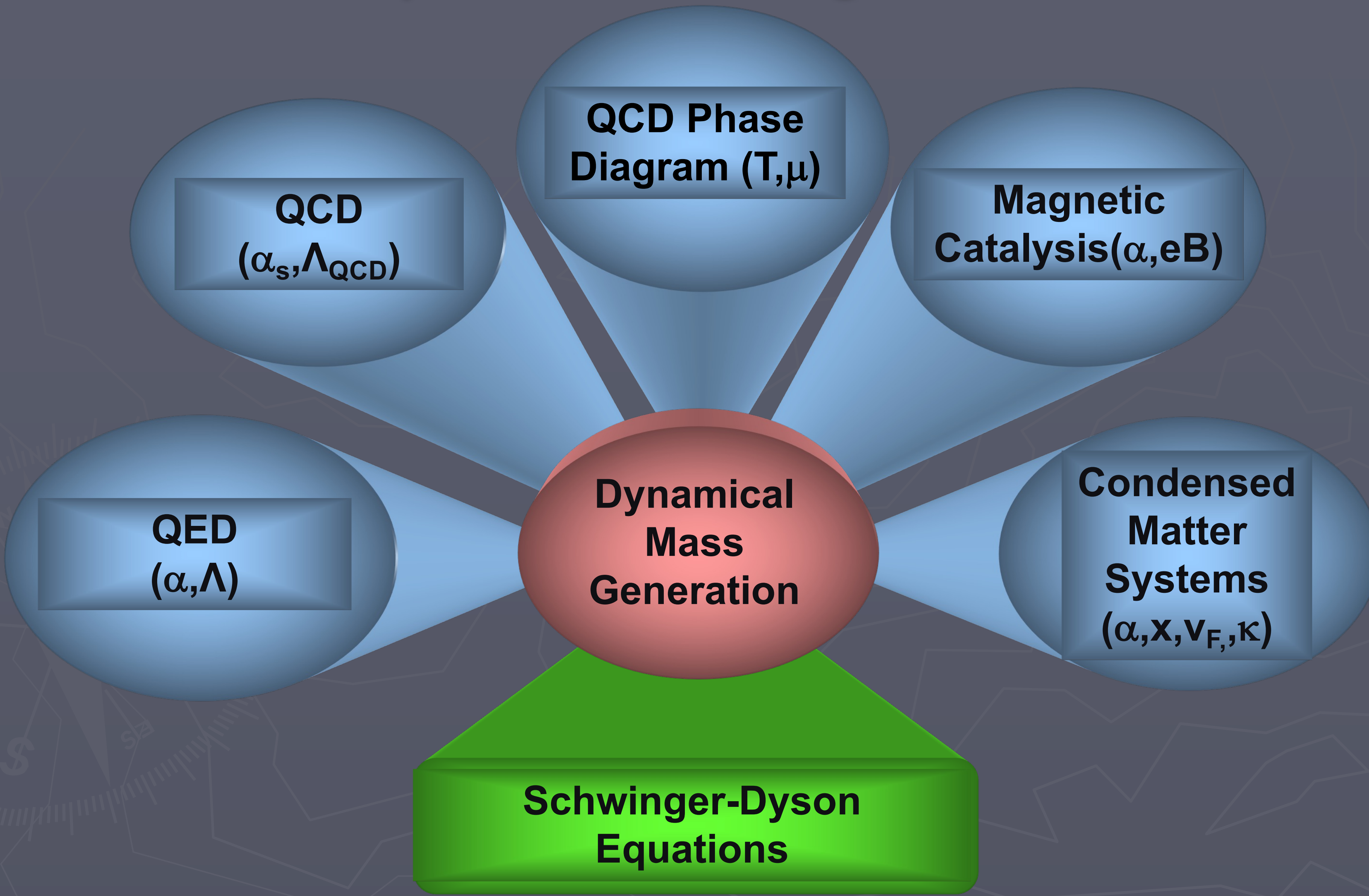
$$\alpha_c = \pi/3$$



This scaling law is also observed in other theories: reduced QED

L. Albino, AB, A.J. Mizher, A. Raya, Phys. Rev. D 106 9 096007 (2022).

On dynamical mass generation



What next?

- We only worked on a simple example of truncating SDEs.
- Can we retain the useful information even on truncating the SDEs at certain Green function?
- How can we ensure symmetries of a QFT are preserved?
- Is our treatment of the SDEs systematically improvable?