On renormalization and running coupling

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Jefferson Lab
Exploring the Nature of Matter

HUGS2023
May 30 - June 16, 2023
Newport News, VA
The 38th Annual Hampton University Graduate Summer Program at Jefferson Lab

Quantum electrodynamics

Let us start with the QED Lagrangian:

$$\mathcal{L} = -\frac{1}{4}\, F_{\mu\nu}\, F^{\mu\nu} - \frac{1}{2\xi}\, (\partial^\mu A_\mu)^2 + \sum_{j=1}^{N_f} \bar{\psi}^j\, (i\gamma^\mu D_\mu - m_j)\, \psi^j$$

1. A_μ is the gauge (photon) field. μ is the Lorentz index.
2. The second term is the gauge fixing term involving gauge parameter ξ. The Landau gauge corresponds to $\xi = 0$.
3. ψ^j are the fermionic fields (spinors). j is the flavor index.
4. The interaction terms involve the coupling constant e.

$$\begin{aligned} F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu \\ D_\mu \psi^j &= \partial_\mu \psi^j - ieA_\mu \psi^j \end{aligned}$$

Quantum Electrodynamics

Employing the QED Lagrangian, when we compute physical observables beyond tree level, they come out infinite.

However, QED is a renormalizable theory. We can add counter terms of the same form as present in the original Lagrangian to come up with a new Lagrangian whose predictions are consistent with experimental results.

As the counter terms are of the same form, it is straight forward to add them to the original Lagrangian to get the modified Lagrangian whose coefficients are infinitely large, namely the bare Lagrangian L_B.

$$\mathcal{L} \Rightarrow \mathcal{L} + \mathcal{L}_{\mathrm{CT}} = \mathcal{L}_{\mathrm{B}}$$

The bare Lagrangian

We choose to write the bare QED Lagrangian as:

$$\mathcal{L}_B = -\frac{1}{4}\, F^B_{\mu\nu}\, F^{\mu\nu}_B - \frac{1}{2\xi_B}\, (\partial^\mu A^B_\mu)^2 + \sum_{j=1}^{N_f} \bar{\psi}^j_B\, (i\gamma^\mu D^B_\mu - m^B_j)\, \psi^j_B$$

$$F^B_{\mu\nu} = \partial_\mu A^B_\nu - \partial_\nu A^B_\mu$$

$$D^B_\mu \psi^j_B = \partial_\mu \psi^j_B - i e_B A^B_\mu \psi^j_B$$

Explicitly, this bare QED Lagrangian is:

$$\mathcal{L}_B = -\frac{1}{4}\, (\partial_\mu A^B_\nu - \partial_\nu A^B_\mu)\, (\partial^\mu A^\nu_B - \partial^\nu A^\mu_B) - \frac{1}{2\xi_B}\, (\partial^\mu A^B_\mu)^2 + \sum_{j=1}^{N_f} \left[i\bar{\psi}^j_B i\gamma^\mu \partial_\mu \psi^j_B + e_B \bar{\psi}^j_B \gamma^\mu A^B_\mu \psi^j_B - m^B_j \bar{\psi}^j_B \psi^j_B \right] ,$$

Renormalization constants

Quantities in this bare Lagrangian are connected to the renormalized quantities of the original Lagrangian through infinite multiplicative renormalization constants for each term:

$$\mathcal{L}_B = -\frac{1}{4} Z_3 \left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right) - \frac{1}{2\xi} Z_6 \left(\partial^\mu A_\mu\right)^2 + \sum_{j=1}^{N_f} \left[i Z_{2F}^j \bar{\psi}^j i\gamma^\mu \partial_\mu \psi^j + Z_{1F}^j e \bar{\psi}^j \gamma^\mu A_\mu \psi^j - Z_4^j m_j \bar{\psi}^j \psi^j \right]$$

This implies certain relations between the bare and the renormalized quantities.

Renormalization constants

$$\mathcal{L}_B = -\frac{1}{4} Z_3 \left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right) - \frac{1}{2\xi} Z_6 \left(\partial^\mu A_\mu\right)^2 + \sum_{j=1}^{N_f} \left[i Z_{2F}^j \bar{\psi}^j i\gamma^\mu \partial_\mu \psi^j + Z_{1F}^j e \bar{\psi}^j \gamma^\mu A_\mu \psi^j - Z_4^j m_j \bar{\psi}^j \psi^j \right]$$

$$\mathcal{L}_B = -\frac{1}{4} \left(\partial_\mu A_\nu^B - \partial_\nu A_\mu^B\right)\left(\partial^\mu A_B^\nu - \partial^\nu A_B^\mu\right) - \frac{1}{2\xi_B} \left(\partial^\mu A_\mu^B\right)^2 + \sum_{j=1}^{N_f} \left[i \bar{\psi}_B^j i\gamma^\mu \partial_\mu \psi_B^j + e_B \bar{\psi}_B^j \gamma^\mu A_\mu^B \psi_B^j - m_j^B \bar{\psi}_B^j \psi_B^j \right] \ ,$$

It implies:

$$A_\mu^B = Z_3^{1/2} \ A_\mu$$
$$\psi_B^j = {Z_{2F}^j}^{1/2} \ \psi^j$$

Renormalization constants

Use:

$$A_\mu^B = Z_3^{1/2}\, A_\mu \qquad \psi_B^j = {Z_{2F}^j}^{1/2}\, \psi^j$$

Define:

$$\xi_B = Z_\xi\, \xi\,,\ \ e_B = Z_e\, e\,,\ \ m_j^B = Z_{m_j}\, m_j$$

$$\mathcal{L}_B = -\frac{1}{4} Z_3\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right) - \frac{1}{2\xi} Z_6 \left(\partial^\mu A_\mu\right)^2$$
$$+ \sum_{j=1}^{N_f}\left[i Z_{2F}^j \bar{\psi}^j i\gamma^\mu \partial_\mu \psi^j + Z_{1F}^j e \bar{\psi}^j \gamma^\mu A_\mu \psi^j - Z_4^j m_j \bar{\psi}^j \psi^j\right]$$

$$\mathcal{L}_B = -\frac{1}{4}\left(\partial_\mu A_\nu^B - \partial_\nu A_\mu^B\right)\left(\partial^\mu A_B^\nu - \partial^\nu A_\mu^B\right) - \frac{1}{2\xi_B} Z_\xi Z_3^{-1} Z_6 \left(\partial^\mu A_\mu^B\right)^2$$
$$+ \sum_{j=1}^{N_f}\Big[i \bar{\psi}_B^j i\gamma^\mu \partial_\mu \psi_B^j + Z_{1F}^j (Z_{2F}^j)^{-1} Z_e^{-1} Z_3^{-1/2}\, e_B \bar{\psi}_B^j \gamma^\mu A_\mu^B \psi_B^j$$
$$- Z_4^j (Z_{2F}^j)^{-1} Z_{m_j}^{-1}\, m_j^B \bar{\psi}_B^j \psi_B^j\Big]$$

Renormalization constants

Now compare the relations:

$$\mathcal{L}_B = -\frac{1}{4}(\partial_\mu A_\nu^B - \partial_\nu A_\mu^B)(\partial^\mu A_B^\nu - \partial^\nu A_\mu^B) - \frac{1}{2\xi_B} Z_\xi Z_3^{-1} Z_6 (\partial^\mu A_\mu^B)^2$$
$$+ \sum_{j=1}^{N_f} \Big[i\bar{\psi}_B^j i\gamma^\mu \partial_\mu \psi_B^j + Z_{1F}^j (Z_{2F}^j)^{-1} Z_e^{-1} Z_3^{-1/2} e_B \bar{\psi}_B^j \gamma^\mu A_\mu^B \psi_B^j$$
$$- Z_4^j (Z_{2F}^j)^{-1} Z_{m_j}^{-1} m_j^B \bar{\psi}_B^j \psi_B^j \Big]$$

$$\mathcal{L}_B = -\frac{1}{4}(\partial_\mu A_\nu^B - \partial_\nu A_\mu^B)(\partial^\mu A_B^\nu - \partial^\nu A_B^\mu) - \frac{1}{2\xi_B}(\partial^\mu A_\mu^B)^2$$
$$+ \sum_{j=1}^{N_f} \left[i\bar{\psi}_B^j i\gamma^\mu \partial_\mu \psi_B^j + e_B \bar{\psi}_B^j \gamma^\mu A_\mu^B \psi_B^j - m_j^B \bar{\psi}_B^j \psi_B^j \right] ,$$

Relations among renormalization constants

The coefficient of each term being unity implies:

$$Z_6 = \frac{Z_3}{Z_\xi}, \qquad Z_e = \frac{Z_{1F}^j}{Z_{2F}^j Z_3^{1/2}}, \qquad Z_{m_j} = \frac{Z_4^j}{Z_{2F}^j}$$

$$Z_i = Z_i(\mu, \epsilon)$$

$$\lim_{\epsilon \to 0} Z_i(\mu, \epsilon) \Rightarrow \infty$$

Z_{2Fj} = electron field renormalization constant

Z_3 = photon field renormalization constant

Z_6 = gauge fixing term renormalization constant

Z_{1Fj} = electron-photon vertex term renormalization constant

Z_{4j} = electron mass term renormalization constant

Z_e = coupling renormalization constant

Z_{m_j} = electron mass renormalization constant

Z_ξ = gauge parameter renormalization constant

Dimensional analysis

Renormalization constants in $\overline{\text{MS}}$ scheme have the structure:

$$Z_i(\alpha) = 1 + \frac{z_1}{\varepsilon}\frac{\alpha}{4\pi} + \left(\frac{z_{22}}{\varepsilon^2} + \frac{z_{21}}{\varepsilon}\right)\left(\frac{\alpha}{4\pi}\right)^2 + \cdots$$

We must define the renormalized coupling in such a way that it remains dimensionless in d dimensions.

The action is dimensionless, because it appears in the exponent in the Feynman path integral. The action is an integral of L over d-dimensional space-time.

Thus the mass dimension of the Lagrangian L is [L]=d.

Show that: (d=4-2ε)

$$[A_B] = 1 - \epsilon \quad [\psi] = 3/2 - \epsilon \quad [e_B] = \epsilon$$

Electromagnetic coupling

To define dimensionless coupling, we introduce a parameter μ called the renormalization scale.

$\overline{\text{MS}}$ scheme:

$$\frac{\alpha(\mu)}{4\pi} = \mu^{-2\epsilon} \frac{e^2}{(4\pi)^{d/2}} \mathrm{e}^{-\gamma\epsilon} = \mu^{-2\epsilon} \frac{e_B^2 Z_e^{-2}}{(4\pi)^{d/2}} \mathrm{e}^{-\gamma\epsilon}$$

where γ is the Euler constant.

Use: $Z_\alpha = Z_e^2$ Thus inversely:

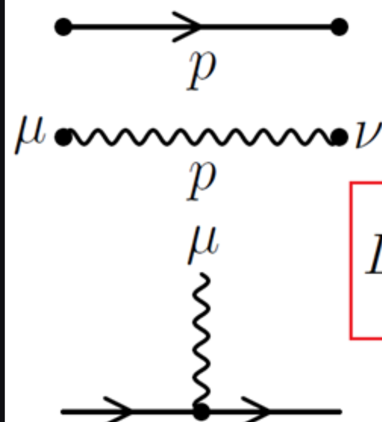
$$\frac{e_B^2}{(4\pi)^{d/2}} = \mu^{2\epsilon} \frac{\alpha(\mu)}{4\pi} Z_e^2(\alpha(\mu))\, \mathrm{e}^{\gamma\epsilon} = \mu^{2\epsilon} \frac{\alpha(\mu)}{4\pi} Z_\alpha(\alpha(\mu))\, \mathrm{e}^{\gamma\epsilon}$$

Thus a physical quantity is first expressed in terms of the bare coupling and then expressed in terms of the renormalized coupling.

QED Feynman rules

Massless QED:

Warning!!! Notation



$= iS_0(p)$

$$S_0(p) = \frac{1}{\not p} = \frac{\not p}{p^2}$$

$= -iD^0_{\mu\nu}(p)$

$$D^0_{\mu\nu}(p) = \frac{1}{p^2}\left[g_{\mu\nu} - (1-\xi\,)\frac{p_\mu p_\nu}{p^2}\right]$$

$= ie\gamma^\mu$

Additional Feynman rules (loops):

A (-1) and trace for every fermion loop.

Integration over undetermined loop momentum $\int d^dk/(2\pi)^d$

Properties of Dirac matrices

$$\gamma_\mu\gamma^\mu = d \qquad \mathrm{Tr}\,1 = 4$$

$$\gamma_\mu \not a \gamma^\mu = -(d-2)\not a$$

$$\gamma_\mu \not a \not b \gamma^\mu = 4a\cdot b + (d-4)\not a \not b$$

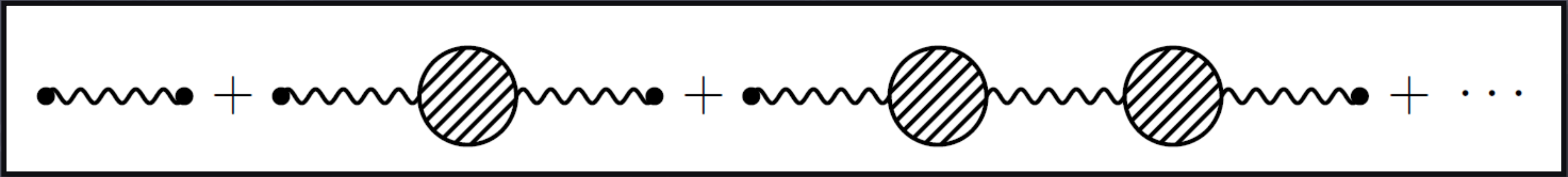
$$\gamma_\mu \not a \not b \not c \gamma^\mu = -2\not c \not b \not a - (d-4)\not a \not b \not c$$

The photon propagator

The photon propagator has the structure:

$$-iD_{\mu\nu}(p) = -iD^0_{\mu\nu}(p) + (-i)D^0_{\mu\alpha}(p)i\Pi^{\alpha\beta}(p)(-i)D^0_{\beta\nu}(p)$$
$$+ (-i)D^0_{\mu\alpha}(p)i\Pi^{\alpha\beta}(p)(-i)D^0_{\beta\gamma}(p)i\Pi^{\gamma\delta}(p)(-i)D^0_{\gamma\nu}(p) + \cdots$$

where the photon self-energy $i\Pi^{\mu\nu}(p)$ (denoted by a shaded blob) is the sum of all one-particle-irreducible diagrams (diagrams which cannot be cut into two disconnected pieces by cutting a single photon line), not including the external photon propagators.



The photon propagator

The series can also be rewritten as:

$$D_{\mu\nu}(p) = D^0_{\mu\nu}(p) + D^0_{\mu\alpha}(p)\Pi^{\alpha\beta}(p)D_{\beta\nu}(p)$$

Thus the inverse of the photon propagator is:

$$D^{-1}_{\mu\nu}(p) = (D^0)^{-1}_{\mu\nu}(p) - \Pi_{\mu\nu}(p)$$

The Ward identity reads:

$$\Pi_{\mu\nu}(p)p^\nu = 0\,, \quad \Pi_{\mu\nu}(p)p^\mu = 0$$

Thus the general form of the photon self energy is:

$$\Pi_{\mu\nu}(p) = (p^2 g_{\mu\nu} - p_\mu p_\nu)\Pi(p^2)$$

Therefore:

$$D^{-1}_{\mu\nu} = p^2\left(1 - \Pi(p^2)\right)\left[g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right] + \frac{p^2}{\xi}\frac{p_\mu p_\nu}{p^2}$$

The photon propagator

$$D_{\mu\nu} = \frac{1}{p^2\left(1-\Pi(p^2)\right)}\left[g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right] + \xi\frac{p_\mu p_\nu}{p^4} = D_\perp(p^2)\left[g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right] + \xi\frac{p_\mu p_\nu}{p^4}$$

Thus the longitudinal part of the full propagator gets no corrections, to any order of perturbation theory.

As the photon propagator involves the product of two photon field vectors, full bare propagator is related to the renormalized one by (watch out for notation!):

$$D_{\mu\nu}(p) = Z_3^{1/2}\, Z_3^{1/2}\; D^r_{\mu\nu}(p;\mu) = Z_3\; D^r_{\mu\nu}(p;\mu)$$

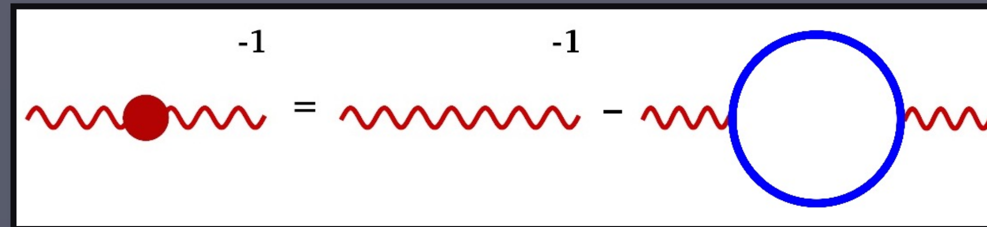
$$D^r_{\mu\nu}(p;\mu) = \frac{Z_3^{-1}}{p^2\left(1-\Pi(p^2)\right)}\left[g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right] + \xi\, Z_3^{-1}\frac{p_\mu p_\nu}{p^4}$$

Thus Ward identity implies:

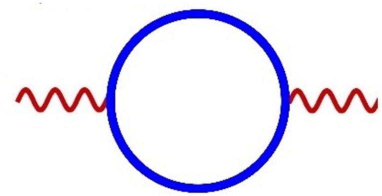
$$Z_3 = Z_\xi \;\Rightarrow\; Z_6 = 1$$

Photon propagator at one loop

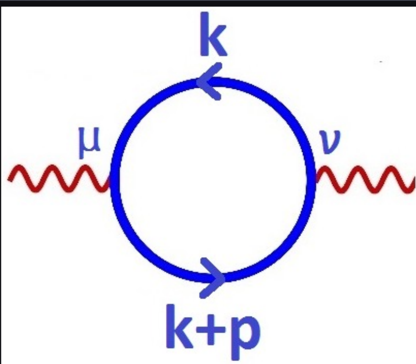
Let us start with one loop photon propagator:



Hence the self energy can be written as:

$\Pi_{\mu\nu} = -\,i$ 

$$\Pi_{\mu\nu} = (p^2 g^{\mu\nu} - p^\mu p^\nu)\, \Pi(p^2)$$



$$= -\int \frac{d^d k}{(2\pi)^d} \mathrm{Tr}\left[ie\gamma^\mu \, iS(k+p)\, ie\gamma^\nu \, iS(k)\right]$$

$$= -e^2 \int \frac{d^d k}{(2\pi)^d} \mathrm{Tr}\left[\gamma^\mu \, \frac{\not{k} + \not{p}}{(k+p)^2}\, \gamma^\nu \, \frac{\not{k}}{k^2}\right]$$

Contract with $g_{\mu\nu}$, take trace and simplify:

$$\Pi(p^2) = -2\frac{(d-2)}{(d-1)}\,\frac{e^2}{(4\pi)^{d/2}}\,(-p^2)^{-\epsilon}\, G(1,1)$$

Photon propagator at one loop

Hence:

$$\Pi(p^2) = \frac{e^2(-p^2)^{-\epsilon}}{(4\pi)^{d/2}} \frac{4(d-2)}{(d-1)(d-3)(d-4)} g_1, \quad g_1 = \frac{\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}$$

Therefore, bare one loop photon propagator is:

$$p^2 D_\perp(p^2) = \frac{1}{1-\Pi(p^2)} = 1 + \Pi(p^2), \; L = \ln\left(-\frac{p^2}{\mu^2}\right)$$

$$p^2 D_\perp(p^2) = 1 - \frac{4}{3}\frac{\alpha(\mu)}{4\pi\epsilon}\left[1 - \left(L - \frac{5}{3}\right)\epsilon + \mathcal{O}(\epsilon^2)\right]$$

Thus:

$$p^2 D^r_\perp(p^2) = p^2 D_\perp(p^2) Z_3^{-1}$$

$$p^2 D^r_\perp(p^2) = Z_3^{-1}\left[1 - \frac{4}{3}\frac{\alpha(\mu)}{4\pi\epsilon}\left(1 - (L - 5/3)\epsilon + \mathcal{O}(\epsilon^2)\right)\right]$$

It requires:

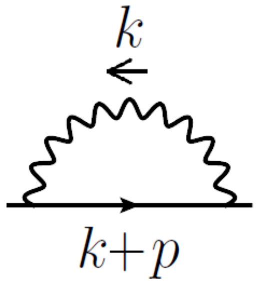
$$Z_3^{-1} = 1 + \frac{\alpha(\mu)}{3\pi\epsilon}$$

rendering:

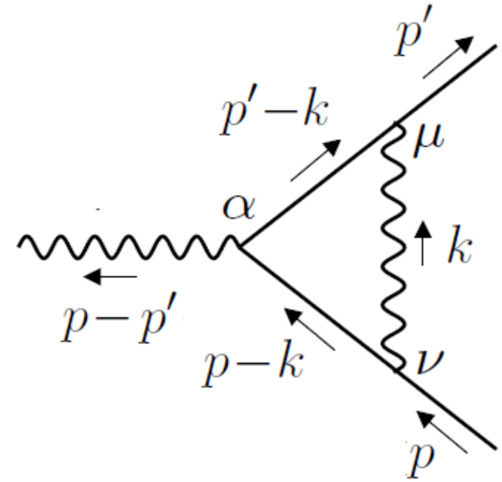
$$p^2 D^r_\perp(p^2) = 1 + \frac{\alpha(\mu)}{3\pi}(L - 5/3)$$

Electron self energy and vertex at one loop

One-loop electron self-energy



$$Z_{1F} = Z_{2F}$$



$$(k-p)_\mu \Gamma^\mu(k,p) = S_F^{-1}(k) - S_F^{-1}(p)$$

$\Gamma^\mu(k,p) \downarrow Z_{1F}$, $S_F^{-1}(k) \downarrow Z_{2F}$

$$Z_\alpha = Z_e^2 = \left(\frac{Z_{1F}}{Z_{2F}}\right)^2 \frac{1}{Z_3}$$

$$Z_{1F} = Z_{2F} \quad \Longrightarrow \quad Z_\alpha = Z_3^{-1}$$

$$Z_\alpha = Z_3^{-1} = 1 + \frac{\alpha(\mu)}{3\pi\epsilon} = 1 + z_1 \frac{\alpha(\mu)}{4\pi\epsilon} \quad \text{where } z_1 = 4/3$$

The β function of QED

Recall the relation between renormalized and bare charge in the Msbar scheme:

$$\frac{e_B^2}{(4\pi)^{d/2}} = \mu^{2\epsilon}\, \frac{\alpha(\mu)}{4\pi}\, \mathcal{Z}_\alpha(\alpha(\mu))\, \mathrm{e}^{\gamma\epsilon}$$

Keeping the μ-dependent quantities on one-side (the right hand side), we can rearrange the above expression as:

$$\mathrm{e}^A \equiv (4\pi)\, \mathrm{e}^{-\gamma\epsilon}\, \frac{e_B^2}{(4\pi)^{d/2}} = \mu^{2\epsilon}\, \alpha(\mu)\, \mathcal{Z}_\alpha(\alpha(\mu))$$

Taking the log of both sides:

$$A = \log\mu^{2\epsilon} + \log\alpha(\mu) + \log\mathcal{Z}_\alpha(\alpha(\mu))$$
$$A = 2\epsilon\, \log\mu + \log\alpha(\mu) + \log\mathcal{Z}_\alpha(\alpha(\mu))$$

Taking the derivative with respect to logμ , we have:

The β function of QED

$$0 = 2\epsilon + \frac{d\log\alpha(\mu)}{d\log\mu} + \frac{d\log Z_\alpha(\alpha(\mu))}{d\log\mu}$$

β-function of QED is defined as:

$$\beta(\alpha(\mu)) = \frac{1}{2}\frac{d\log Z_\alpha(\alpha(\mu))}{d\log\mu}$$

We thus have:

$$\frac{d\log\alpha(\mu)}{d\log\mu} = -2\epsilon - 2\beta(\alpha(\mu))$$

The last equation can be written as:

$$\frac{1}{\alpha(\mu)}\frac{d\,\alpha(\mu)}{d\log\mu} = -2\epsilon - 2\beta(\alpha(\mu))$$

We can rearrange this equation as follows:

The β function of QED

$$\frac{d\,\alpha(\mu)}{d\log\mu} = \left[-2\epsilon - 2\beta(\alpha(\mu))\right]\alpha(\mu)$$

We are interested in it till order α:

$$\frac{d\,\alpha(\mu)}{d\log\mu} = -2\epsilon\,\alpha(\mu) + \mathcal{O}(\alpha^2(\mu))$$

The knowledge of Z_α to one-loop allows us to write:

$$\log Z_\alpha(\alpha(\mu)) = \log\left(1 + z_1\frac{\alpha(\mu)}{4\pi\epsilon}\right) = z_1\frac{\alpha(\mu)}{4\pi\epsilon}$$

Differentiate appropriately:

$$\frac{d\log Z_\alpha(\alpha(\mu))}{d\log\mu} = \frac{z_1}{4\pi\epsilon}\,\frac{d\,\alpha(\mu)}{d\log\mu}$$

The β function of QED

This enables us to evaluate the **1-loop β-function**:

$$\frac{d\log Z_\alpha(\alpha(\mu))}{d\log\mu} = \frac{z_1}{4\pi\epsilon}\left(-2\epsilon\,\alpha(\mu)\right) = -\frac{z_1}{2\pi}\alpha(\mu)$$

Hence the **β-function** is given by:

$$\beta(\alpha(\mu)) = \frac{1}{2}\frac{d\log Z_\alpha(\alpha(\mu))}{d\log\mu} = -\frac{z_1}{4\pi}\alpha(\mu)$$

Therefore the **QED β-function** to **1-loop** is:

$$\beta(\alpha(\mu)) = -\frac{4}{3}\frac{\alpha(\mu)}{4\pi} = \beta_0\frac{\alpha(\mu)}{4\pi} \qquad \beta_0 = -\frac{4}{3}$$

The running coupling of QED

Let us start again from:

$$\frac{d\log\alpha(\mu)}{d\log\mu} = -2\epsilon - 2\beta(\alpha(\mu))$$

And work in the limit $\varepsilon \to 0$:

$$\frac{d\log\alpha(\mu)}{d\log\mu} = -2\beta(\alpha(\mu))$$

It can be re-written as (inserting expansion of β-function):

$$\beta(\alpha(\mu)) = \beta_0 \frac{\alpha(\mu)}{4\pi} \quad \Rightarrow \quad \frac{1}{\alpha(\mu)} \frac{d\,\alpha(\mu)}{d\log\mu} = -2\beta_0 \frac{\alpha(\mu)}{4\pi}$$

The last equation can be simplified as follows:

$$-\frac{4\pi}{(\alpha(\mu))^2} \frac{d\,\alpha(\mu)}{d\log\mu} = 2\beta_0 \quad \Rightarrow \quad \frac{d}{d\log\mu}\left(\frac{4\pi}{\alpha(\mu)}\right) = 2\beta_0$$

The running coupling of QED

$$\frac{d}{d\log\mu}\left(\frac{4\pi}{\alpha(\mu)}\right) = 2\beta_0$$

The solution to this equation can be written as:

$$\frac{4\pi}{\alpha(\mu')} - \frac{4\pi}{\alpha(\mu)} = 2\beta_0 \log\mu' - 2\beta_0 \log\mu$$

The last equation can be simplified as follows:

$$\frac{4\pi}{\alpha(\mu')} = \frac{4\pi}{\alpha(\mu)} + 2\beta_0 \log\left(\frac{\mu'}{\mu}\right)$$

The inverse of this equation is:

$$\frac{\alpha(\mu')}{4\pi} = \frac{1}{4\pi/\alpha(\mu) + 2\beta_0 \log\left(\mu'/\mu\right)}$$

The running coupling of QED

The running coupling of QED is:

$$\alpha(\mu') = \frac{\alpha(\mu)}{1 + \beta_0 \; (\alpha(\mu)/(4\pi)) \; \log\left(\mu'^2/\mu^2\right)}$$

Inserting the calculated value of β_0 (-4/3):

$$\alpha(\mu') = \frac{\alpha(\mu)}{1 - (\alpha(\mu)/(3\pi)) \; \log\left(\mu'^2/\mu^2\right)}$$

In another set of variables:

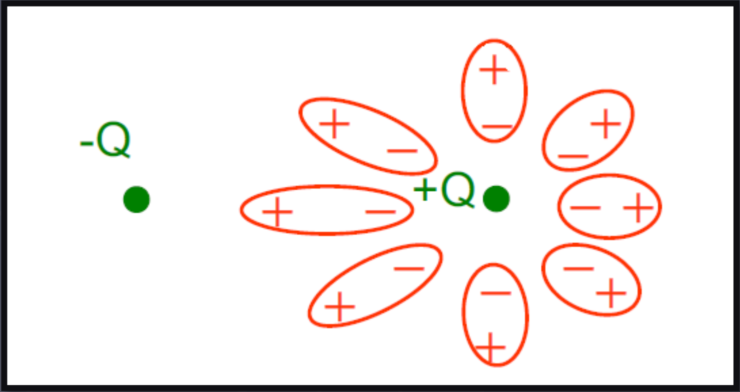
$$\alpha(Q^2) = \frac{\alpha(Q_0^2)}{1 - (\alpha(Q_0^2)/(3\pi)) \; \log\left(Q^2/Q_0^2\right)}$$

The running coupling of QED

Running loop running coupling in QED

$$\alpha(Q^2) \quad = \quad \frac{\alpha(Q_0^2)}{1 - (\alpha(Q_0^2)/(3\pi)) \, \log\left(Q^2/Q_0^2\right)}$$

As we had indicated before, the bare charge of electron is screened by virtual $e^+ e^-$ pairs.



The QED vacuum behaves like a polarizable dielectric.

Recall the expansion:

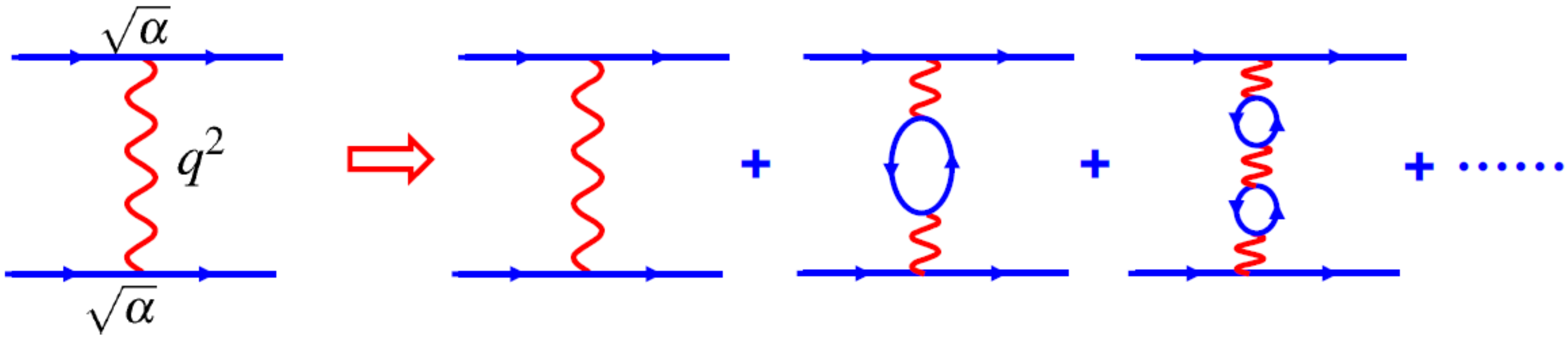
$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots \qquad -1 < x < 1$$

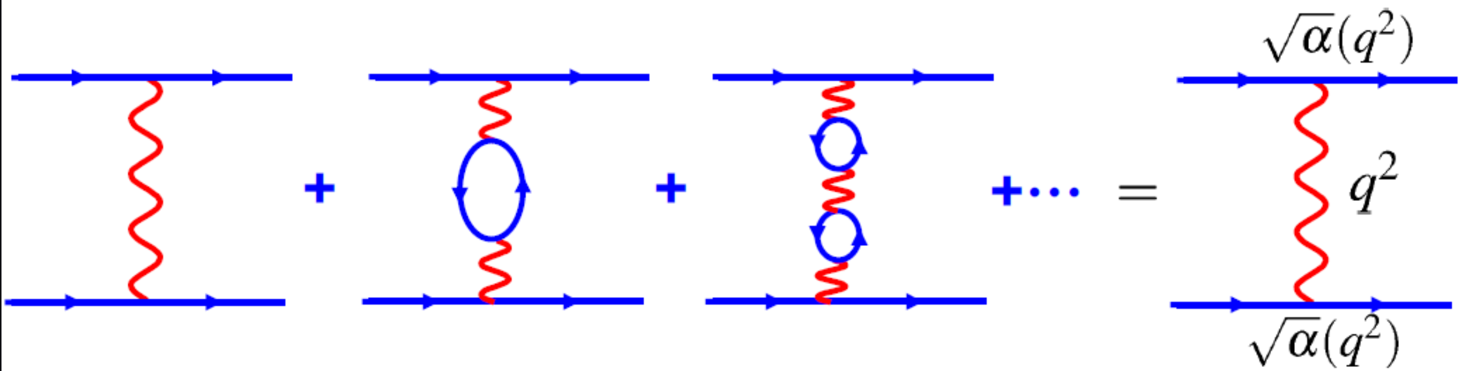
The running coupling of QED

Thus:

$$\alpha(Q^2) = \alpha(Q_0^2)\left[1 + \left(\alpha(Q_0^2)/(3\pi)\right)\log\left(Q^2/Q_0^2\right) + \left(\alpha(Q_0^2)/(3\pi)\right)^2 \log^2\left(Q^2/Q_0^2\right) + \cdots\right]$$

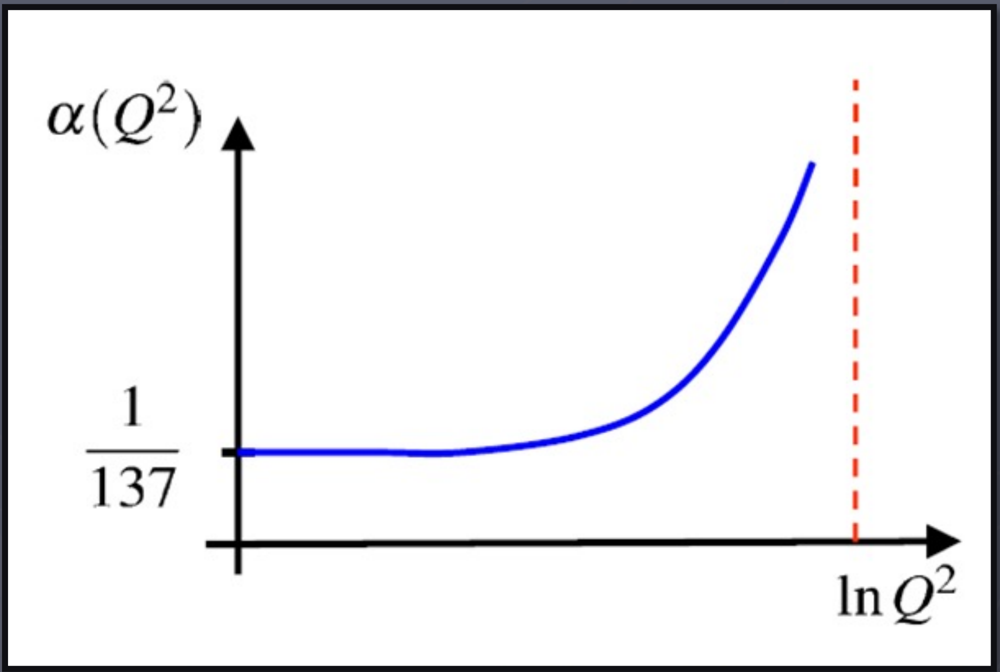
★ **In terms of Feynman diagrams:**





The running coupling of QED

Landau pole:



One might worry about coupling becoming infinite at:

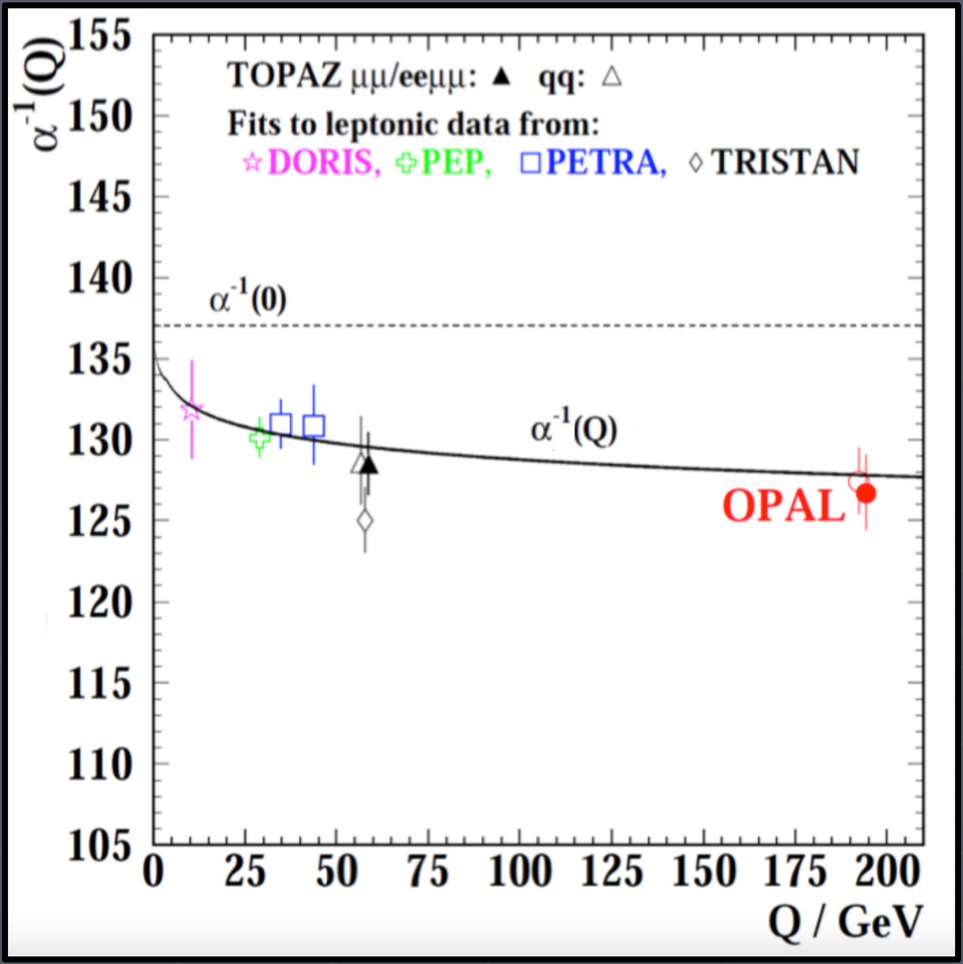
$$\ln\left(\frac{Q^2}{Q_0^2}\right) = \frac{3\pi}{1/137}$$

i.e. at $Q \sim 10^{26}\,\mathrm{GeV}$

But non-perturbative effects would come in way below this energy and is highly unlikely that perturbative QED as is would be valid in this regime.

The running coupling of QED

Experimental Measurement:



In QED, running coupling increases very slowly.

Atomic physics:

$Q^2 \approx 0$

$$1/\alpha = 137.03599976(50)$$

High energy physics:

$$1/\alpha(193\,\mathrm{GeV}) = 127.4 \pm 2.1$$

Quantum chromodynamics

The QCD Lagrangian for n_f massless quark flavors:

$$L = \sum_i \bar{q}_{0i} i \not{D} q_{0i} - \frac{1}{4} G^a_{0\mu\nu} G_0^{a\mu\nu}$$

where q_{0i} are the quark fields. Their covariant derivative is:

$$D_\mu q_0 = (\partial_\mu - i g_0 A_{0\mu})\, q_0 \,, \quad A_{0\mu} = A^a_{0\mu} t^a$$

where $A_{0\mu}{}^a$ are the gluon fields, t^a are the generators of the color group and the field strength tensor, the solution of:

$$[D_\mu, D_\nu] q_0 = -i g_0 G_{0\mu\nu} q_0 \,, \quad G_{0\mu\nu} = G^a_{0\mu\nu} t^a$$

is given by:

$$G^a_{0\mu\nu} = \partial_\mu A^a_{0\nu} - \partial_\nu A^a_{0\mu} + g_0 f^{abc} A^b_{0\mu} A^c_{0\nu} \,, \quad [t^a, t^b] = i f^{abc} t^c$$

Quantum chromodynamic

In covariant gauges, we have to introduce gauge-fixing term and the ghosts

$$\Delta L = -\frac{1}{2a_0}\left(\partial_\mu A_0^{a\mu}\right)^2 + (\partial^\mu \bar{c}_0^a)(D_\mu c_0^a)$$

where a_0 is the gauge parameter, c_0^a is the ghost field, a scalar field obeying Fermi statistics.

Its covariant derivative is:

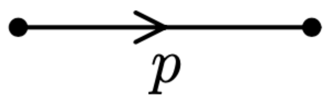
$$D_\mu c_0^a = \left(\partial_\mu \delta^{ab} - i g_0 A_{0\mu}^{ab}\right) c_0^b\,, \qquad A_{0\mu}^{ab} = A_{0\mu}^c (t^c)^{ab}$$

where:

$$(t^c)^{ab} = i f^{acb}$$

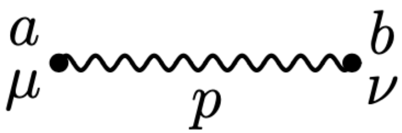
are generators of the color group in adjoint representation.

Feynman rules



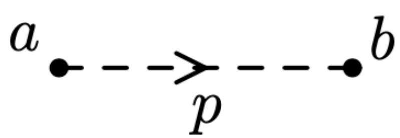
$$= iS_0(p)$$

$$S_0(p) = \frac{1}{\not{p}} = \frac{\not{p}}{p^2}$$



$$= -i\delta^{ab} D^0_{\mu\nu}(p)$$

$$D^0_{\mu\nu}(p) = \frac{1}{p^2}\left[g_{\mu\nu} - (1-a_0)\frac{p_\mu p_\nu}{p^2}\right]$$

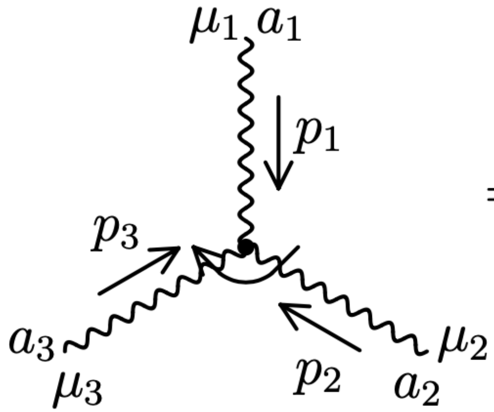


$$= i\delta^{ab} G_0(p)$$

$$G_0(p) = \frac{1}{p^2}$$

Propagators in QCD

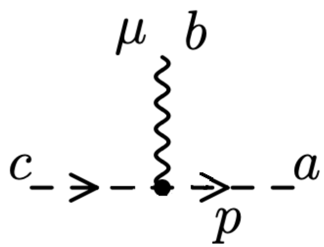
Feynman rules



$$= i f^{a_1 a_2 a_3} \times i g_0 V^{\mu_1 \mu_2 \mu_3}(p_1, p_2, p_3)$$

3–gluon vertex

$$V^{\mu_1 \mu_2 \mu_3}(p_1, p_2, p_3) = (p_3 - p_2)^{\mu_1} g^{\mu_2 \mu_3} + (p_1 - p_3)^{\mu_2} g^{\mu_3 \mu_1} + (p_2 - p_1)^{\mu_3} g^{\mu_1 \mu_2}$$



$$= i f^{abc} \times i g_0 p^{\mu}$$

Ghost–gluon vertex

QCD running coupling

Just as in QED, the renormalized fields and parameters are related to the bare quantities through the renormalization constants:

$$q_{i0} = Z_q^{1/2} q_i\,, \quad A_0 = Z_A^{1/2} A\,, \quad c_0 = Z_c^{1/2} c\,,$$
$$a_0 = Z_A a\,, \quad g_0 = Z_\alpha^{1/2} g$$

Also, analogously, the QCD running coupling is:

$$\frac{\alpha_s(\mu)}{4\pi} = \mu^{-2\varepsilon} \frac{g^2}{(4\pi)^{d/2}} e^{-\gamma\varepsilon}\,, \qquad \frac{g_0^2}{(4\pi)^{d/2}} = \mu^{2\varepsilon} \frac{\alpha_s(\mu)}{4\pi} Z_\alpha(\alpha(\mu)) e^{\gamma\varepsilon}$$

What is needed to be evaluated is:

$$Z_\alpha = (Z_\Gamma Z_q)^{-2} Z_A^{-1}$$

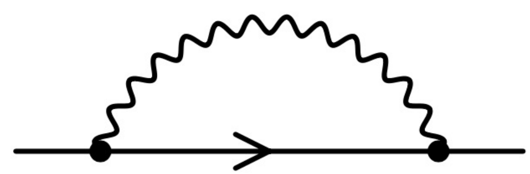
QCD running coupling

$$Z_\alpha = (Z_\Gamma Z_q)^{-2} Z_A^{-1}$$

In QED, we were lucky to have the relation: $$Z_\Gamma Z_q = 1$$

And we only needed to evaluate one-loop photon propagator. We are not so lucky in QCD. So we need to know the one-loop results for quark and gluon propagators as well as the quark-gluon vertex.

For one-loop massless quark propagator:



One-loop quark self-energy

$$\Sigma(p) = \not{p}\,\Sigma_V(p^2)$$

One-loop quark propagator

One-loop massless quark self energy is:

$$\Sigma_V(p^2) = -C_F \frac{g_0^2(-p^2)^{-\varepsilon}}{(4\pi)^{d/2}} \frac{d-2}{2} a_0 G_1$$

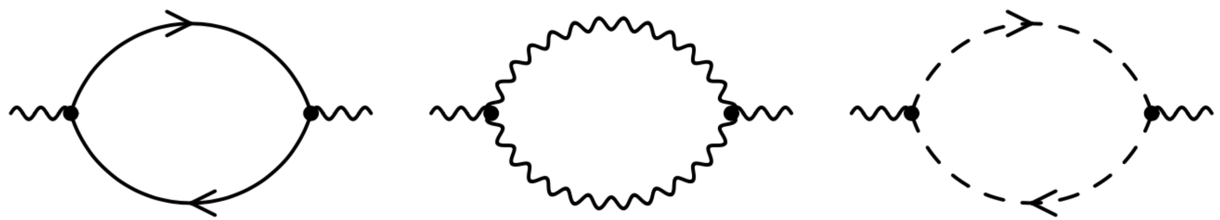
with divergent part:

$$G_1 = -\frac{2\,\Gamma(1+\varepsilon)\Gamma^2(1-\varepsilon)}{(d-3)(d-4)\,\Gamma(1-2\varepsilon)}$$

And we can deduce the quark field renormalization constant:

$$Z_q = 1 - C_F a \frac{\alpha_s}{4\pi\varepsilon} + \cdots$$

One-loop gluon propagator



One-loop gluon self-energy

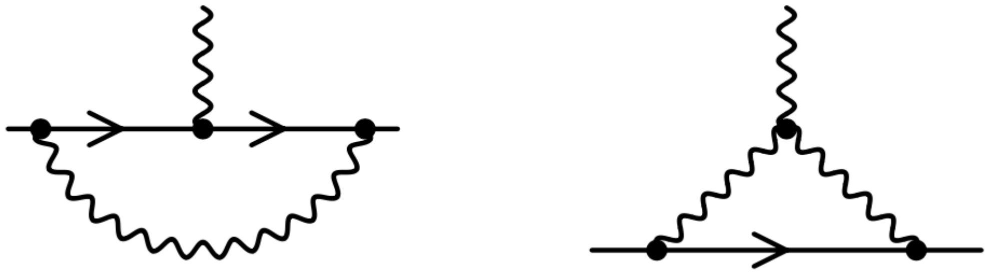
The transverse gluon propagator to one loop accuracy is:

$$p^2 D_\perp(p^2) = 1 + \frac{\alpha_s(\mu)}{4\pi\varepsilon} e^{-L\varepsilon} \left[-\frac{1}{2}\left(a - \frac{13}{3}\right) C_A - \frac{4}{3} T_F n_f + \left(\frac{9a^2 + 18a + 97}{36} C_A - \frac{20}{9} T_F n_f \right) \varepsilon \right], \quad L = \log(-p^2)/\mu^2$$

The gluon field renormalization constant is:

$$Z_A = 1 - \frac{\alpha_s}{4\pi\varepsilon} \left[\frac{1}{2}\left(a - \frac{13}{3}\right) C_A + \frac{4}{3} T_F n_f \right] + \cdots$$

One-loop quark-gluon vertex



Quark–gluon vertex at one loop

The divergent part of the one-loop quark-gluon vertex is:

$$\Lambda^\alpha = \left(C_F a + C_A \frac{a+3}{4} \right) \frac{\alpha_s}{4\pi\varepsilon} \gamma^\alpha$$

The quark-gluon vertex renormalization constant is:

$$Z_\Gamma = 1 + \left(C_F a + C_A \frac{a+3}{4} \right) \frac{\alpha_s}{4\pi\varepsilon} + \cdots$$

One-loop quark-gluon vertex

Recall the coupling renormalization constant:

$$Z_\alpha = (Z_\Gamma Z_q)^{-2} Z_A^{-1}$$

Then difference between QED and QCD:

QED	QCD
$Z_\Gamma Z_q = 1,$	$Z_\Gamma Z_q = 1 + C_A \frac{a+3}{4} \frac{\alpha_s}{4\pi\varepsilon} + \cdots$

The quark-gluon vertex renormalization constant is:

$$Z_\alpha = 1 - \left(\frac{11}{3} C_A - \frac{4}{3} T_F n_f \right) \frac{\alpha_s}{4\pi\varepsilon} + \cdots$$

It is a gauge invariant quantity!

The β-function of QCD

The β-function is:

QCD

$$\beta(\alpha_s) = \beta_0 \frac{\alpha_s}{4\pi} + \cdots \qquad \beta_0 = \frac{11}{3} C_A - \frac{4}{3} T_F n_f$$

QED

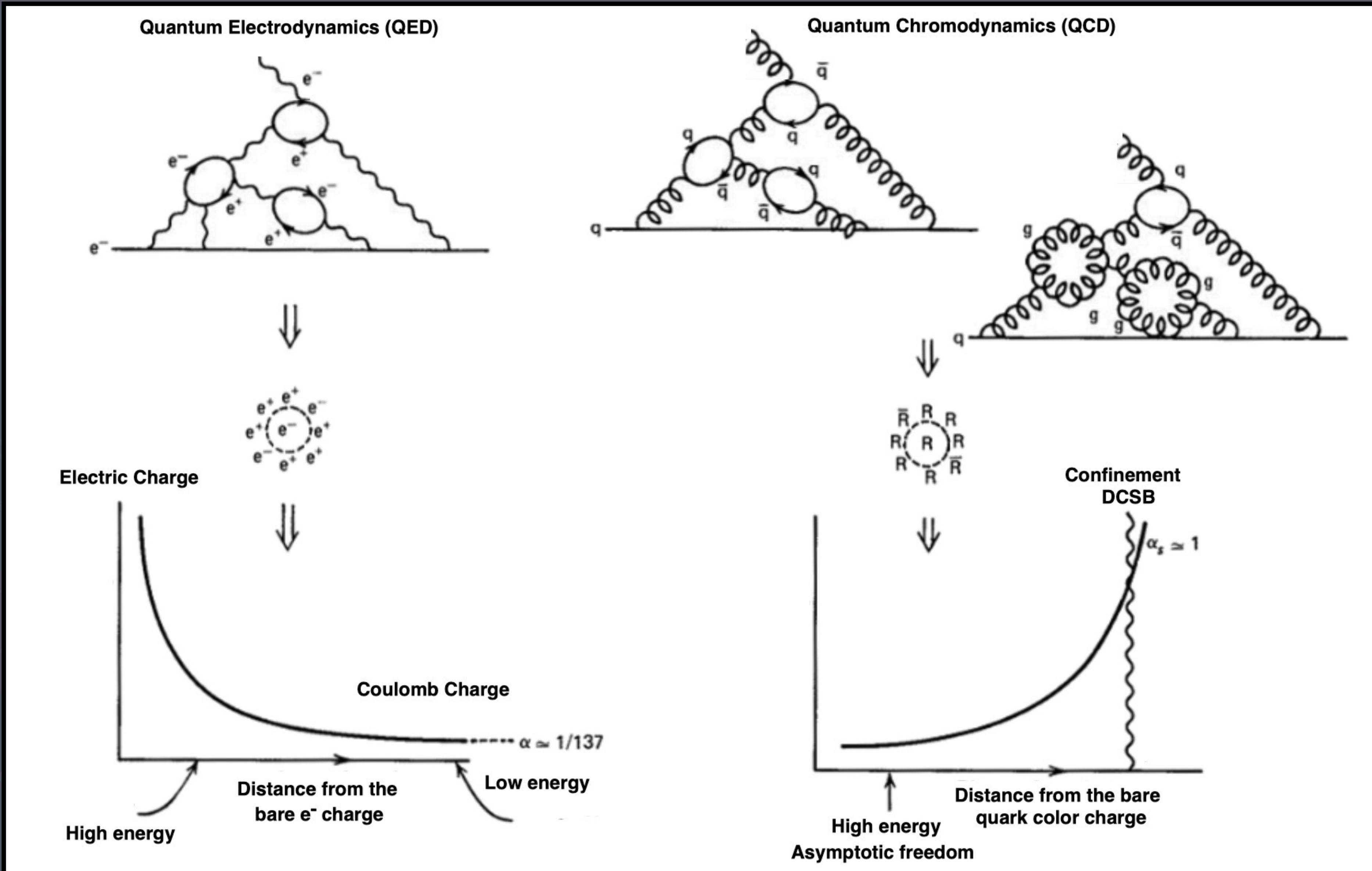
$$\beta(\alpha) = \beta_0 \frac{\alpha}{4\pi} + \cdots \qquad \beta_0 = -4/3.$$

The RG equation is:

$$\frac{d \log \alpha_s(\mu)}{d \log \mu} = -2\beta(\alpha_s(\mu))$$

It shows $\alpha_s(\mu)$ decreases when μ increases. This behavior (opposite to screening) is called asymptotic freedom.

QED vs QCD



Quantum Electrodynamics (QED)
e⁻
e⁻
e⁺
e⁺
e⁻
e⁺
e⁻
Electric Charge
Coulomb Charge
α ≃ 1/137
Distance from the bare e⁻ charge
High energy
Low energy
Quantum Chromodynamics (QCD)
q
q̄
g
R
R̄
Confinement
DCSB
αs ≃ 1
Distance from the bare quark color charge
High energy
Asymptotic freedom

Running coupling of QCD

Competition between color and flavor:

$$\alpha_S(Q^2) = \alpha_S(Q_0^2) \Big/ \left[1 + B\alpha_S(Q_0^2)\ln\left(\frac{Q^2}{Q_0^2}\right)\right]$$

with $B = \dfrac{11N_c - 2N_f}{12\pi}$

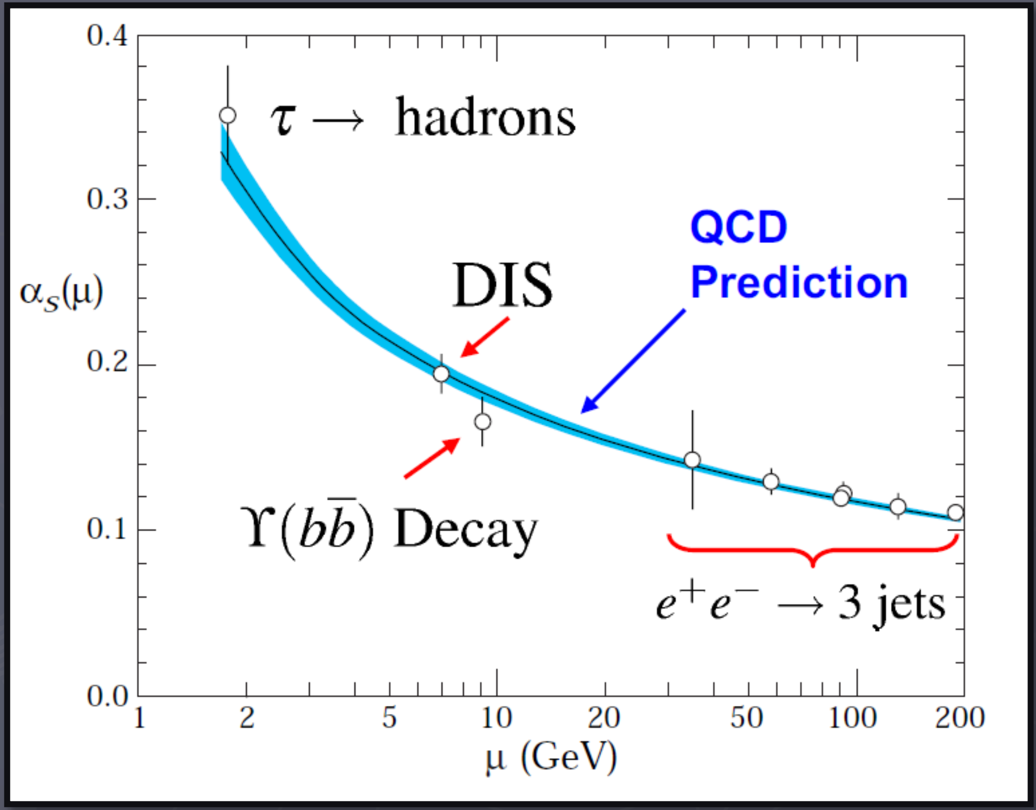
- N_c = no. of colours
- N_f = no. of quark flavours

$N_c = 3;\ N_f = 6$ ➡ $B > 0$

➡ **α_S decreases with Q^2**

Nobel Prize for Physics, 2004
(Gross, Politzer, Wilczek)

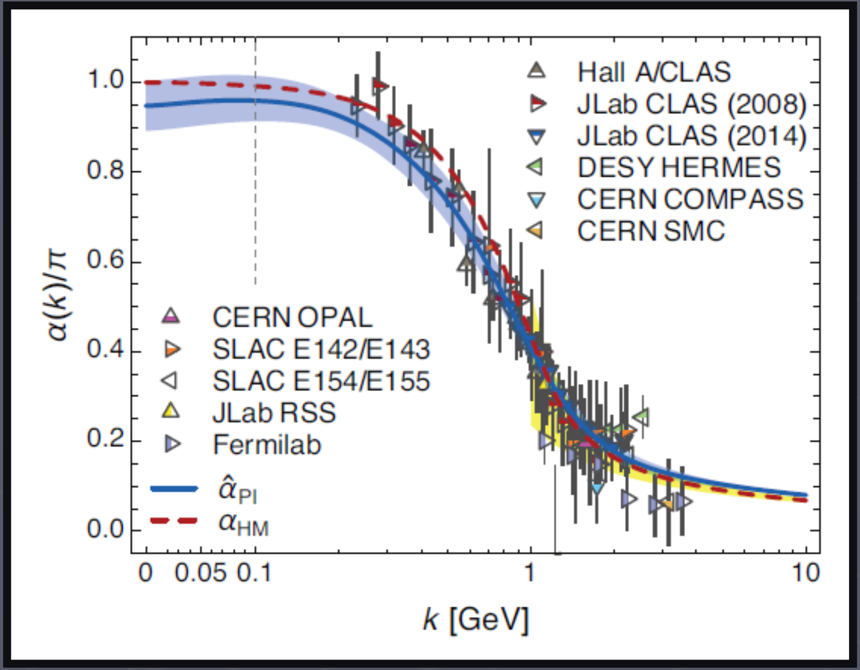
Running coupling of QCD



- At high momentum scale, is rather small, for example at $\mu = M_z$, $\alpha_s \approx 0.12$. This where we can apply perturbation theory.
- That is why in DIS experiments, quarks behave as if they are quasi free confirming asymptotic freedom.

Running coupling of QCD

D. Binosi, C. Mezrag, J. Papavassiliou, C.D. Roberts, J. Rodríguez-Quintero, Phys. Rev. D 96, 054026 (2017)



Process-independent

It is evaluated using the gluon-ghost sector alone.

The gluon mass generation in the infrared causes the running coupling to saturate in the infrared.

What Next?

- How can we study hadron physics starting from QCD?
- Are there fundamental equations of QCD which can allow us to study non-perturbative regimes of QCD?
- How are they derived?
- How are they be solved beyond perturbation theory?